Robust Selection of Equilibria

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ROBUST SELECTION OF EQUILIBRIA

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ABSTRACT

This paper models equilibrium selection for an extensive form game as a special correlation device named a selection device. A selection device selects a strategy profile in a form of consensus. A set of strategy profiles selected by a selection device is called a robust selection if the device has a largest domain of alternative reasonable strategies. A robust selection always exists. A robust selection contains Nash equilibria only and sometimes eliminates unreasonable Nash or even sequential equilibria. Alternatively to forward induction, the robust selection criterion provides a model in which a player views a deviation as a sign of an alternative play. An application to signaling games is presented in comparison with the Cho-Kreps test.

JEL Classification Number: 026

Key Words: extensive form game, equilibrium selection, correlation device, alternative play, forward induction.

1. INTRODUCTION

The problem of equilibrium selection is one of the central issues in noncooperative game theory. In the absence of cooperation, players face a coordination problem of which equilibrium they play. To solve the problem, the players obviously need to share some knowledge. As Bernheim (1984), Pearce (1984), Tan and Werlang (1988) showed, however, the assumption of common knowledge about players' rationality does not ensure attainment of a Nash equilibrium. So the assumption of common knowledge about an equilibrium itself has been a convention to avoid getting into the issue.

In the late 80s, a series of works for extensive form game refinement appeared under the name of forward induction. A short list of these works includes Banks and Sobel (1987), Cho and Kreps (1987), Farrell (1985), and Grossman and Perry (1986). Strangely enough, on the one hand, forward induction attempts to find in a deviation from a presumed equilibrium an explicable intention of the deviation. On the other hand, forward induction follows the convention to assume explicitly or implicitly common knowledge about the presumed equilibrium itself. If an equilibrium is really presumed and common knowledge, there should not be any intention of deviation. This is a severe logical inconsistency in forward induction. The game of Figure 1 taken from van Damme (1989) illustrates the point. The game has two sequential equlibria in pure strategies; $\sigma^1 = (R, S; l, w)$ and $\sigma^2 = (L, W; r, s)$. Both equilibria carry their own forward induction arguments to refute the other one. To refute the equlibrium σ^2 and advocate the equilibrium σ^1, one might argue that, by taking the out-of-equilibrium action R in the equilibrium σ^2, player I can send player II an effective signal claiming that he will play S of σ^1 since a successive play $(S; w)$ is the only sensible hope for player I to be better off than the presumed equilibrium play. To refute the equilibrium σ^1 and advocate the equilibrium σ^2, however, one might argue that, by taking the out-of-equilibrium action r in the equilibrium σ^1, player II can send player I an effective signal claiming that he will play s of σ^2 since a successive play $(L, W; s)$ is the only sensible hope for player II to be better off than the presumed equilibrium play. The arguments contradict each other.

The purpose of this paper is to propose an alternative approach to extensive form game refinement. Instead of pursuing an alternative play given the assumption of common knowledge about an equilibrium itself, we conceive an exogenous random device which selects reasonable equilibria among alternatives for players, and we hypothesize that the random device is common knowledge. The idea of using a random device to build a Bayesian foundation of rational play was originally proposed by Aumann (1974, 1987). Aumann's correlated equilibrium, however, supports a play which Nash equilibrium never generates. We restrict a class of random devices to the class of ones called selection devices, which yield consensus about a play among the players. Then we define the most reasonable device in the restricted class and call a set of strategy profiles selected by the device as a robust selection. The criterion of robust selection is that the strategy profiles are selected from a largest domain of alternative reasonable strategies. We show that a robust selection always exists and that the robust selection criterion ensures attainment of a Nash equilibrium and sometimes eliminates unreasonable Nash or even sequential equilibrium. Without the logical inconsistency of forward induction, the robust selection criterion provides a model in which a player views a deviation as a sign of an alternative play.

The paper is organized as follows. Section 2 redefines Aumann's correlation device for the purpose of extensive form game analysis. Then section 3 restricts the correlation device to a selection device. In section 4, we present the central notion of robust selection. Section 5 provides the main results of the paper concerning characterizations of robust selection. In section 6, we discuss an application of robust selection to simple signaling games. All the proofs except for Lemma 1 are in the appendix.

2. CORRELATION DEVICE

In this section, we redefine Aumann's (1974, 1987) correlation device for extensive form games. The primitive object of our analysis is a finite extensive form game G with perfect recall. Let $\mathbf{I}$ be the set of players in the primitive game G. Due to the Kuhn (1953) theorem, the strategic opportunity for player $i \in \mathbf{I}$ is represented by the set Σ_i of behaviorally mixed strategies. Consider any nonempty finite subset D_i of Σ_i for each player $i \in \mathbf{I}$ and define a product $D \equiv \prod_{i \in \mathbf{I}} D_i \subset \Sigma \equiv \prod_{i \in \mathbf{I}} \Sigma_i$. Consider any probability distribution $q \in \Delta(D)$ where $\Delta(D)$ denotes the set of probability distributions over the set D. (Note that the distribution q is not necessarily a product measure.) Associated with the primitive game G via the pair (D, q), we define a game $G(D, q)$ as follows. First Nature selects a profile $\sigma = (\sigma_i)_{i \in \mathbf{I}} \in D$ randomly according to the probability distribution q. Then Nature informs each player i privately of the part σ_i in the realization σ. Then the players play the primitive game G.

An extended game $G(D,q)$ is a well defined finite extensive form game with perfect recall. The various notions, like a strategy and a belief system, of noncooperative game theory apply to it in the usual fashion. A player i's strategy in the extended game $G(D,q)$ is a mapping $s_i \in \Sigma_i^{D_i}$ which, for each realized $\sigma_i \in D_i$, prescribes a behaviorally mixed strategy $s_i(\sigma_i) \in \Sigma_i$ of the primitive game G. A special strategy s_i^* such that $s_i^*(\sigma_i) = \sigma_i$ for any $\sigma_i \in D_i$ is called an implementation of D_i. That is, in an implememtation, player i adopts Nature's suggestion σ_i.

A belief system is defined as follows. Let X be the set of decision nodes in the primitive game G. Then a decision node of the extended game $G(D,q)$ is a pair $(x,\sigma) \in X \times D$, which means that x is reached when Nature selects σ. A belief system of the extended game $G(D,q)$ is a mapping $\mu : X \times D \to [0,1]$ such that

$$\forall i \in \mathbf{I}, \forall \sigma_i \in D_i, \forall h \in H_i; \quad \sum_{x \in h} \sum_{\sigma_{-i} \in D_{-i}} \mu(x,(\sigma_i,\sigma_{-i})) = 1$$

where H_i denotes the collection of information sets for player i in the primitive game G and $\sigma_{-i} \in D_{-i}$ denotes a profile $(\sigma_j)_{j \neq i} \in \prod_{j \neq i} D_j$. Let $\Phi(D,q)$ denote the set of belief systems of $G(D,q)$.[2]

Now we define a correlation device for an extensive form game as follows.

Definition

For any finite extensive form game G with perfect recall, let D be a nonempty finite product subset of Σ, $\{q_t\}_{t=1}^{\infty}$ be a convergent sequence in $\Delta(D)$, and $\{\mu_t\}_{t=1}^{\infty}$ be a convergent sequence of $\mu_t \in \Phi(D,q_t)$. Let $\wp \equiv \{< q_t, \mu_t >\}_{t=1}^{\infty}$. The pair $(D,\wp)$ is called a correlation device for G if and only if q_t is fully mixed and μ_t is consistent with the profile s^* of implementations in $G(D,q_t)$ in the Kreps and Wilson (1982) sense. The limit $\mu \equiv \lim_{t\to\infty} \mu_t$ is called a consistent belief[3] of $G(D,q)$ where $q \equiv \lim_{t\to\infty} q_t$.

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[2] The set $\Phi(D,q)$ does not depend on a specification of q.

[3] The notion is an extension of the consistency notion introduced by Kreps and Wilson (1982). We use the term without mentioning the associated special strategy profile s^* of implementations. The associated strategy profile is always the profile s^* throughout the paper.

An interpretation is straightforward. The limit probability distribution q is an Aumann's correlation device with a support in D. (Note that the support of q does not necessarily coincide with the set D.) By using the exogenous device q, the players play the extended game $G(D, q)$. Furthermore, since $G(D, q)$ is an extensive form game, the players form their beliefs not only ex ante but also during a play. The consistent belief μ implied by $\wp$ captures this extension of Aumann's correlation device.

3. SELECTION DEVICE

In this section, we model equilibrium selection as a special type of correlation device. We imagine that, given a primitive game G, the players lack common knowledge about how to play so that they need to seek some form of consensus. Imagine that the players somehow have a nonempty finite set $C^0 \subset \Sigma$ of candidate strategy profiles to be played in G. The players try to reach consensus on one of strategy profiles in a nonempty subset $C^* \subset C^0$ by discarding those candidate strategy profiles in $C^0 \setminus C^*$. A correlation device can be seen as a random device to create such a consensus if the following conditions hold.

Definition

For any finite extensive form game G with perfect recall and any nonempty finite subsets $C^* \subset C^0$ of strategy profiles, let $(D, \wp)$ be a correlation device for G such that $C^0 \subset D$. The triple $(C^0, D, \wp)$ is called a selection device of C^* if and only if

(1) $q(\sigma) > 0$ if and only if $\sigma \in C^*$, and

(2) $\lim_{t \to \infty} \frac{q_t(\sigma)}{\sum_{\hat{\sigma}_{-i} \in D_{-i}} q_t(\sigma_i, \hat{\sigma}_{-i})} = 1$ for any $\sigma = (\sigma_i)_{i \in \mathbf{I}} \in C^*$, and

(3) the part $s_i^*(\sigma_i) = \sigma_i$ of implementation of D_i in $G(D, q)$ is sequentially rational with respect to μ for any $i \in \mathbf{I}$ and any $\sigma_i \in D_i \setminus (proj_i(C^0) \setminus proj_i(C^*))$

where $proj_i$ is an operator which projects a set in Σ to the space Σ_i. The set C^* is called a selection by $(C^0, D, \wp)$.

An interpretation of the conditions is straightforward: (1) C^* is an exhaustive set of possible consensus, (2) any realization $\sigma = (\sigma_i)_{i \in \mathbf{I}} \in C^*$ generates each player's posterior given his private information σ_i that the consensus σ has been reached almost surely, and (3) any strategy choice which is made according to the correlation device must be implemented sequentially rationally unless it is the one discarded by the device.

4. ROBUST SELECTION

For a given primitive game, not all the selection devices $(C^0, D, \wp)$ are reasonable since we have not imposed any rationale on the choice of C^0. In this section, we define a set M of reasonable selection devices and then propose a robustness criterion about which selection devices are most reasonable in M.

First of all, if a triple $(C^0, D, \wp)$ is a selection device of C^0 itself, then the triple should be in M since the choice of C^0 is justified as a choice of reasonable consensus. Let M^0 be a set of all triples $(C^0, D, \wp)$ which are selection devices of C^0 themselves. Furthermore, if there are alternative selection devices in M^0 which support different selections, we say that some dispute of equilibrium selection remains unsolved in M^0. Therefore, if another selection device solves the dispute, the device should also be in M. Formally let M^1 be a set of all selection devices $(C^0, D, \wp)$ for which there exists a finite collection $(C^{01}, D^1, \wp^1), \cdots, (C^{0K}, D^K, \wp^K) \in M^0$ such that $C^0 = \cup_{k=1}^K D^k$. We repeat the procedure of defining M^1 from M^0 to create a sequence $\{M^n\}_{n=0}^{\infty}$. Since the sequence is nondecreasing, we set $M = \cup_{n=0}^{\infty} M^n$.

In the set M of reasonable selection devices, we define the most reasonable ones by the following robustness criterion.

Definition

For any finite extensive form game G with perfect recall, let M be the set of reasonable selection devices. Let $(C^0, D, \wp) \in M$ be a reasonable selection device of some set C^* of strategy profiles. The set C^* is called a robust selection if and only if, for any $(C^{0'}, D', \wp') \in M$ with $D \subset D'$, there exists $(C^{0''}, D'', \wp'') \in M$ such that $D' \subset D''$ and $(C^{0''}, D'', \wp'')$ is a selection device of C^*.

An interpretation of the criterion is the following. Consider any selection C^* by a reasonable selection device $(C^0, D, \wp) \in M$. Suppose that there exist alternaitve selections and that there exists a reasonable selection device $(C^{0'}, D', \wp') \in M$ which solves a dispute between the selection C^* and the alternative selections. Then if there is no way to overturn the selection of $(C^{0'}, D', \wp')$ and conclude the selection C^* by solving the despute by some reasonable selection device $(C^{0''}, D'', \wp'') \in M$, the selection C^* is not the most reasonable selection.

5. MAIN RESULTS

This section presents the main results of this paper concerning characterizations of robust selection. The main results consist of three parts; existence, sufficiency criterion, and refinement.

First, the existence of robust selection is far from trivial, since in the recursively defined set M there must exist a selection device for which no more dispute remains unsolved. Let G be any finite extensive form game with perfect recall. To establish the existence of robust selection for G, note first that any sequential equilibrium (σ, μ) in G generates a triple $(C^0, D, \varphi) \in M^0$ by taking $C^* = C^0 = D = \{\sigma\}$, $q_t(\sigma) = 1$ for any $t \in \mathbf{N}$, and $\mu_t = \mu$ for any $t \in \mathbf{N}$. Call such a triple as a primitive selection device. Hence the following nonemptiness of the set M is immediate.

Lemma 1

For any finite extensive form game G with perfect recall taken as the primitive game, any sequential equilibrium strategy profile of game G forms a singleton selection by a primitive selection device in M^0.

Given Lemma 1, crucial for the existence of robust selection is the existence of "largest" selection device in M. We shall develop a sufficiency condition that a selection device is "largest" in M. Some notations are in order. For any noninitial node x in the primitive game G, let $p(x)$ denote the node which immediately precedes node x and let $\alpha(x)$ denote the action which is taken at node $p(x)$ immediately before reaching node x. If there further exists an immediate predecessor $p(p(x))$ of node $p(x)$, we write it as $p^2(x)$. Repeatedly for any $m \in \mathbf{N}$, if there exists an immediate predecessor $p(p^{m-1}(x))$ of node $p^{m-1}(x)$, we write it as $p^m(x)$. Let $m(x)$ be the total number of predecessors of node x. Now consider any $i \in \mathbf{I}$, any $h \in H_i$, and any $x \in h$. For any strategy profile $\sigma_{-i} \in \Sigma_{-i}$ of other players, we define a deviation index[4] of σ_{-i} to x by

$$\delta(x, \sigma_{-i}) \equiv \#\{1 \leq m \leq m(x) \mid i(p^m(x)) \neq i \text{ and } \sigma_{i(p^m(x))}(\alpha(p^{m-1}(x))) = 0\}$$

where $i(p^m(x))$ denotes the player who owns node $p^m(x)$. Let $\delta(h, \sigma_{-i}) \equiv \min_{x \in h} \delta(x, \sigma_{-i})$.

[4] A similar index was proposed by McLennan (1985).

For any (possibly infinite) product subset D_{-i} of Σ_{-i}, let $\delta(x, D_{-i}) \equiv \min_{\sigma_{-i} \in D_{-i}} \delta(x, \sigma_{-i})$. Then the following criterion is the sufficiency condition.

Lemma 2

Let $(C^0, D, \wp) \in M$ be a selection device of a set C^*. Suppose that, for any $(C^{0\prime}, D', \wp') \in M$ with $D \subset D'$, it holds that $\delta(x, D_{-i}) = \delta(x, D'_{-i})$ for any $i \in \mathbf{I}$, any $h \in H_i$, and any $x \in h$. Then C^* is a robust selection.

The criterion of Lemma 2 establishes the following existence result of robust selection.

Theorem 1

For any finite extensive form game with perfect recall taken as the primitive game, there always exists at least one robust selection.

Second, in applying the notion of robust selection, it is useful to know when the criterion of Lemma 2 holds. For this purpose, we shall develop lower bounds to deviation indexes. For this development, we introduce the following variation of the rationalizability concepts proposed by Bernheim (1984) and Pearce (1984).

Definition

Let $D = \prod_{i \in \mathbf{I}} D_i$ be any (possibly infinite) product subset of Σ. Then any $\sigma_i \in D_i$ is said to be sequentially rationalizable in D if and only if there exist a nonempty finite product subset $D' = \prod_{i \in \mathbf{I}} D'_i \subset D$ and a probability distribution $q' \in \Delta(D')$ such that $\sigma_i \in D'_i$ and that $s^*_i(\sigma_i) = \sigma_i$ is sequentially rational with respect to some consistent belief μ' in the game $G(D', q')$. Let $\Re_i(D)$ denote the set of strategies in D_i which are sequentially rationalizable in D. Let $\Re(D) \equiv \prod_{i \in \mathbf{I}} \Re_i(D)$.

Following Bernheim (1984) and Pearce (1984), we conceive of repeated applications of sequential rationalizability. For any $n \in \mathbf{N}$, let $\Re^n$ denote the n time operation of $\Re$. Define a sequence $\{\Re^n(\Sigma)\}_{n=1}^{\infty}$ of subsets in Σ. The sequence is nonincreasing in n. Therefore we define the following counterpart of a Bernheim-Pearce rationalizable set.

Definition

The set $\Re^* \equiv \cap_{n=1}^{\infty} \Re^n(\Sigma)$ is called the sequentially rationalizable set.

The key observation to develop lower bounds to deviation indexes is the following fact that any strategy supported by a reasonable selection device $(C^0, D, \wp) \in M$ is sequentially rationalizable in its domain D.

Lemma 3

$D = \Re(D)$ for any $(C^0, D, \wp) \in M$.

Lemma 3 gives us the following form of upper bound to the domains of selection devices in M.

Lemma 4

$D \subset \Re^*$ for any $(C^0, D, \wp) \in M$.

The upper bound of Lemma 4 serves as a restricted version of the criterion of Lemma 2 as follows.

Lemma 5

Let C^* be a selection by $(C^0, D, \wp) \in M$. If $\delta(x, D_{-i}) = \delta(x, \Re^*_{-i})$ for any $i \in \mathbf{I}$, any $h \in H_i$, and any $x \in h$, then C^* is a robust selection.

As a corollary of Lemma 5, we also have a sufficient condition that a robust selection is unique.

Lemma 6

Let C^* be a robust selection by $(C^0, D, \wp) \in M$ where $D = \prod_{i \in \mathbf{I}} D_i$. Assume that $\delta(x, D_{-i}) = \delta(x, \Re^*_{-i})$ for any $i \in \mathbf{I}$, any $h \in H_i$, and any $x \in h$. Consider any selection $C^{*\prime}$. Suppose that, for any $(C^{0\prime}, D', \wp') \in M$ by which the set $C^{*\prime}$ is a selection where $D' = \prod_{i \in \mathbf{I}} D'_i$, there exists no $(C^{0\prime\prime}, D'', \wp'') \in M$ such that $\prod_{i \in \mathbf{I}}(D_i \cup D'_i) \subset D''$ and $C^{*\prime}$ is a selection by $(C^{0\prime\prime}, D'', \wp'')$. Then $C^{*\prime}$ is not a robust selection. Especially if the supposition is met for any $C^{*\prime} \neq C^*$, then the set C^* is the unique robust selection.

Third, we present a characterization of surviving equilibria in a robust selection. The basic result is the following.

Theorem 2

For any finite extensive form game with perfect recall taken as the primitive game, any strategy profile in a robust selection is a Nash equilibrium of the primitive game.

Note however that, in contrast with the conventional argument for Nash equilibrium, the notion of robust selection does not presume that a rational strategy becomes common knowledge among the players. Actually the opposite is more often the case. Namely, for the case of $|D| \neq 1$, even if it is the fact that a realization $\sigma \in C^*$ assigns each player i a robust selection component $\sigma_i \in proj_i(C^*)$, this fact is not common knowledge among the players, since there is a possibility that another strategy profile $\sigma' \in D$ with $\sigma'_i = \sigma_i$ but $\sigma'_{-i} \neq \sigma_{-i}$ is actually selected even though the possibility is of probability zero.

A virtue of the robust selection criterion as a refinement of Nash equilibrium is to avoid the logical inconsistency of forward induction. For example, consider the game of Figure 1, for which section 1 showed that forward induction suffers from its own conflicting implication of both σ^1 and σ^2 refuting each other. In contrast, the set $C^* = \{\sigma^1\}$ is a robust selection but σ^2 is never an element of a robust selection for the following reason. Since both σ^1 and σ^2 are sequential equilibrium strategy profiles, Lemma 1 guarantees that they are supported by some primitive selection devices in M^0. Then construct a following triple $(C^0, D, \wp)$. Set $C^0 = \{\sigma^1, \sigma^2\}$ and $D = \{(R,S),(L,W)\} \times \{(l,w),(r,s),(r,w)\}$. Take q_t as $q_t(R,S;l,w) = 1 - 3\epsilon_t - \epsilon_t^2 - \epsilon_t^3$, $q_t(R,S;r,w) = \epsilon_t^2$, $q_t(R,S;r,s) = \epsilon_t^3$ and $q_t(\sigma) = \epsilon_t$ for any other $\sigma \in D$ where ϵ_t is a small positive number converging to zero. Take μ_t as a Kreps and Wilson (1982) consistent belief of $G(D,q_t)$. Then $s_I^*(R,S) = (R,S)$, $s_{II}^*(l,w) = (l,w)$, and $s_{II}^*(r,w) = (r,w)$ are all sequentially rational with respect to $\mu = \lim_{t\to\infty}\mu_t$, since $\mu(v,((R,S),(r,w))) = \mu(x,((R,S),(r,w))) = 1$, $\mu(w,((R,S),(l,w))) = \mu(y,((R,S),(l,w))) = 1$, and $\mu(w,((L,W),(r,w))) = \mu(y,((R,S),(r,w))) = 1$. Hence the triple $(C^0, D, \wp)$ is in M, supporting C^*. Note that any strategy σ_I of player I such that $\sigma_I(R) > 0$ and $\sigma_I(W) > 0$ must be $\sigma_I \notin \Re_I(\Sigma)$ and so $\sigma_I \notin \Re_I^*$, since L dominates (R,W). Hence $\delta(y', D_I) = \delta(y', \Re_I^*) = 1$ and all other deviation indexes are zero. By Lemma 5, therefore, we conclude that C^* is a robust selection. Furthermore, Lemma 4 guarantees that no $(C^{0\prime}, D', \wp') \in M$ allows player I to put positive probabilities on R and W simultaneously. Hence Lemma 6 with the above $(C^0, D, \wp) \in M$ applies to conclude that any $C^{*\prime}$ containing σ^2 is not a robust selection, since player II is forced to believe that he is not at y' but at y in any $(C^{0''}, D'', \wp'')$ of Lemma 6.

The refinement power of the robust selection criterion is very limited for some games. An example is the game of Figure 2.[5] The game has the unique subgame perfect equilibrium $\sigma^1 = (D, \frac{1}{2}H + \frac{1}{2}T; \frac{1}{2}h + \frac{1}{2}t)$. However we can support another strategy profile $\sigma^2 = (A, T; t)$ by a robust selection $C^* = \{\sigma^1, \sigma^2\}$.[6] Therefore the robust selection criterion is not nested even in the subgame perfect equilibrium. The example also illustrates that the robust selection criterion does not satisfy the backward induction property. If the players takes out the proper subgame[7] after D and conduct a robust selection, Theorem 2 guarantees that the unique Nash equilibrium $(\frac{1}{2}H + \frac{1}{2}T; \frac{1}{2}h + \frac{1}{2}t)$ is selected. The entire game, however, admits the positive possibility that (T, t) is intended in the proper subgame although the proper subgame is reached with zero probability in this case.[8]

6. APPLICATION TO SIGNALING GAMES

One of the fields in which forward induction arguments have had dramatic success is the refinement for signaling games. An example is the following labor market signaling game examined by Cho and Kreps (1987). There are three players; a worker (player I) of either type t_L or type $t_H \in R_+$ $(t_L < t_H)$ and two symmetric firms (player II and player III). First Nature moves and selects a type of the worker with a probability $\rho \in (0,1)$ of t_H. Being privately informed of the realized type, then, the worker chooses his education level $e \in R_+$. The education level becomes common knowledge. Finally the symmetric firms bid a wage $w \in R_+$ for hiring the worker in a Bertrand competition. The type t worker

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[5] I owe the example to Eric van Damme.

[6] Construct a triple $(C^0, D, \wp)$ by taking $C^0 = \{\sigma^1, \sigma^2\}$, $D = \{(D, \frac{1}{2}H + \frac{1}{2}T), (A, T)\} \times \{\frac{1}{2}h + \frac{1}{2}t, t\}$, $q_t(\sigma) = 1 - 2\epsilon_t$ if $\sigma = \sigma^1$ or σ^2 and $q_t(\sigma) = \epsilon_t$ otherwise, and μ_t as a Kreps and Wilson (1982) consistent belief of $G(D, q_t)$ where ϵ_t is a small positive number converging to zero. All parts of implementations s^* are sequentially rational with respect to $\mu = \lim_{t \to \infty} \mu_t$ especially because $\mu(y, (\frac{1}{2}H + \frac{1}{2}T; t)) = \mu(y', (\frac{1}{2}H + \frac{1}{2}T; t)) = \frac{1}{2}$. Hence the triple is in M^0, supporting C^*. Since all the deviation indexes are zero, Lemma 2 guarantees that C^* is a robust selection.

[7] We are testing the property (BI1) argued by Kohlberg and Mertens (1986).

[8] The backward induction property holds for a certain class of games. Consider a generic perfect information game such that the backward induction procedure selects the unique sequential equilibrium strategy profile σ^*. Then the singleton set $\{\sigma^*\}$ is the unique robust selection. The proof is by showing that $\Re^* = \{\sigma^*\}$, by a similar argument to the one in Suehiro (1992).

gets a payoff $u_t(e, w)$ and the symmetric firms get a payoff $te - w$ from hiring him. The firms have zero reservation payoffs. We assume that $u_t(e, w)$ is strictly decreasing in e, strictly increasing in w, strictly concave in (e, w), and has a maximum $(e_t^*(\beta), w_t^*(\beta))$ on a line $w = \beta e$ for any $t_L \leq \beta \leq t_H$. We also assume the "single crossing property" that $u_{t_L}(e, w)$ has a steeper indifference curve than $u_{t_H}(e, w)$ at any (e, w). Cho and Kreps (1987) examined a separating equilibrium in which the t_L worker chooses $e_{t_L}^*(t_L)$, the t_H worker chooses $e_{t_H}^{**}(t_H)$, which is a maximum of $u_{t_H}(e, w)$ on the line $w = t_H e$ given a constraint $u_{t_L}(e, w) \leq u_{t_L}(e_{t_L}^*(t_L), t_L e_{t_L}^*(t_L))$, and the firms bid a wage $t_L e_{t_L}^*(t_L)$ to the education level $e_{t_L}^*(t_L)$ and a wage $t_H e_{t_H}^{**}(t_H)$ to the education level $e_{t_H}^{**}(t_H)$. They showed that, for any $\rho \in (0, 1)$, the outcome by the separating equilibrium is the only one which survives the Cho-Kreps criterion.

This result is puzzling. Consider a degenerate game in which there is a t_L worker only. Then the only sensible outcome is the one in which the t_L worker chooses $e_{t_L}^*(t_L)$ and the firms bid $t_L e_{t_L}^*(t_L)$. The original signaling game with $\rho \in (0, 1)$ close enough to zero represents a near-by situation to the degenerate game. One will expect that the t_L worker behaves similarly to what he does in the degenerate game and lets the t_H worker do whatever he likes as long as there is no point for the t_L worker to mimic the t_H worker's behavior. This is exactly what Cho and Kreps (1987) predict. But the situation is reversed for the other degenerate game in which there is a t_H worker only. In this degenerate game, the only sensible outcome is the one in which the t_H worker chooses $e_{t_H}^*(t_H)$ and the firms bid $t_H e_{t_H}^*(t_H)$. One would expect that, in the original game with $\rho \in (0, 1)$ close enough to one, the t_H worker behaves similarly to what he does in the degenerate game and lets the t_L worker do whatever he likes. If $u_{t_L}(e_{t_H}^*(t_H), t_H e_{t_H}^*(t_H)) > u_{t_L}(e_{t_L}^*(t_L), t_L e_{t_L}^*(t_L))$, this expectation implies that a pooling equilibrium would prevail. Irrespective of how close to one the probability ρ is, however, no pooling equilibrium survives the Cho-Kreps criterion.

In contrast, the robust selection criterion gives us the following results[9], which fit our intuition better.

[9] A conjecture of the results was originally suggested to the author by John Roberts.

Proposition

For any $\rho \in (0,1)$, there always exists a robust selection which supports the separating outcome of Cho and Kreps (1987). Furthermore, if $u_{t_L}(e^*_{t_H}(t_H), t_H e^*_{t_H}(t_H)) \leq u_{t_L}(e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))$, the separating outcome of Cho and Kreps (1987) is the only outcome supported by a robust selection for any $\rho \in (0,1)$. On the other hand, if $u_{t_L}(e^*_{t_H}(t_H), t_H e^*_{t_H}(t_H)) > u_{t_L}(e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))$, there exist $0 < \rho_0 < \rho_1 < 1$ such that

(1) for any $\rho \in (0, \rho_0)$, the separating outcome of Cho and Kreps (1987) is the only outcome supported by a robust selection,

(2) for any $\rho \in (\rho_1, 1)$, a robust selection also supports the pooling outcome[10] in which the worker chooses $e^*_{t_H}(\beta(\rho))$ and the firms bid $\beta(\rho)e^*_{t_H}(\beta(\rho))$ where we denote $\beta(\rho) \equiv \rho t_H + (1-\rho)t_L$.

The reversed result comes from the following fact. For any ρ close to one in the last case, the Cho-Kreps criterion upsets the pooling outcome $(e^*_{t_H}(\beta(\rho)), \beta(\rho)e^*_{t_H}(\beta(\rho)))$ by an off-equilibrium play $(e', t_H e')$ where e' satisfies $u_{t_L}(e', t_H e') < u_{t_L}(e^*_{t_H}(\beta(\rho)), \beta(\rho)e^*_{t_H}(\beta(\rho)))$ and $u_{t_H}(e', t_H e') > u_{t_H}(e^*_{t_H}(\beta(\rho)), \beta(\rho)e^*_{t_H}(\beta(\rho)))$. The deviation e' is interpreted by forward induction as a signal of the t_H worker. The play of only the t_H worker taking e', however, is not a part of any alternative equilibrium.[11] The robust selection criterion asks if there is a selection device which forces the firms to believe an alternative play in such a way that the firms' best responses to the alternative play force the pooling outcome to be eliminated. There is no such selection device when the t_H worker prefers the pooling outcome to the separating outcome of Cho and Kreps (1987).

[10] More generally, as is apparent from the proof, a pooling outcome is supported if the t_H worker prefers the pooling outcome to the separating outcome of Cho and Kreps (1987).

[11] Given the Stiglitze critique, some authors, e.g. Okuno-Fujiwara and Postlewaite (1987) and Matthews, Okuno-Fujiwara and Postlewaite (1991), have attempted to formalize forward induction as a disequilibrium process to an alternative equilibrium. Without an explicit model of equilibrium selection, however, those attempts have not succeeded in avoiding possible conflicting implications of forward induction.

APPENDIX

Proof of Lemma 2 :

Let C^* be a selection by $(C^{\prime 0}, D, \wp) \in M$ assumed in Lemma 2 where $D = \prod_{i \in \mathbf{I}} D_i, \wp = \{< q_t, \mu_t >\}_{t=1}^{\infty}$. Take any $(C^{\prime 0\prime}, D', \wp') \in M$ with $D \subset D'$ where $D' = \prod_{i \in \mathbf{I}} D'_i, \wp' = \{< q'_t, \mu'_t >\}_{t=1}^{\infty}$. Consider a triple $(C^{0\prime\prime}, D'', \wp'')$ by taking $C^{0\prime\prime} = D'' = D'$. For such a triple $(C^{\prime 0\prime\prime}, D'', \wp'')$ to be a selection device of C^* in M, the triple must satisfy three conditions of selection devices where the third condition is now that the part $s_i^*(\sigma_i) = \sigma_i$ of implementation of D'_i is sequentially rational with respect to a consistent belief implied by $\wp''$ for any $i \in \mathbf{I}$ and any $\sigma_i \in proj_i(C^*)$. We shall construct an appropriate $\wp''$ serving this purpose.

For each $t \in \mathbf{N}$ fixed, let a sequence $\{s^{t,l}\}_{l=1}^{\infty}$ of behavior strategy profiles in $G(D, q_t)$ be such that $s_i^{t,l} \in (Int\Sigma_i)^{D_i}$ for each $i \in \mathbf{I}$ and each $l \in \mathbf{N}$, $\lim_{l \to \infty} s_i^{t,l} = s_i^*$ for each $i \in \mathbf{I}$, and the sequence generates the Kreps and Wilson (1982) consistent belief μ_t in $G(D, q_t)$. A sequence $\{s^{t,l'}\}_{l=1}^{\infty}$ is defined similarly for the consistent belief μ'_t in $G(D', q'_t)$.

For each $t \in \mathbf{N}$ fixed, from the sequences $\{s^{t,l}\}_{l=1}^{\infty}$, $\{s^{t,l'}\}_{l=1}^{\infty}$, we construct $q''_t \in \Delta(D')$ and $\mu''_t \in \Phi(D', q''_t)$ as follows. First we define $s_i^{t,l''} \in (Int\Sigma_i)^{D'_i}$ for each $i \in \mathbf{I}$ and $l \in \mathbf{N}$ fixed. For the fixed l, find $L \in \mathbf{N}$ such that $L \geq l$ and that

$$\max_{i \in \mathbf{I}} \max_{\sigma_i \in D'_i} \max_{a \in A_i} |s_i^{t,L'}(\sigma_i)(a) - \sigma_i(a)| \leq \frac{1}{l} \prod_{i \in \mathbf{I}} \prod_{\sigma_i \in D_i} \prod_{a \in A_i} |s_i^{t,l}(\sigma_i)(a)|$$

where A_i is a set of actions for player i in the primitive game G. It must be possible to find such an L, since the fact that $\lim_{l \to \infty} s_i^{t,l'}(\sigma_i) = s_i^*(\sigma_i) = \sigma_i$ guarantees that

$$\lim_{\hat{l} \to \infty} \left[\max_{i \in \mathbf{I}} \max_{\sigma_i \in D'_i} \max_{a \in A_i} |s_i^{t,\hat{l}'}(\sigma_i)(a) - \sigma_i(a)| \right] = 0$$

whereas the fact that $s_i^{t,l}(\sigma_i) \in Int\Sigma_i$ guarantees that

$$\frac{1}{l} \prod_{i \in \mathbf{I}} \prod_{\sigma_i \in D_i} \prod_{a \in A_i} |s_i^{t,l}(\sigma_i)(a)| > 0.$$

Take any such L and write it as $L(l)$. Then set

$$s_i^{t,l''}(\sigma_i) = \begin{cases} s_i^{t,l}(\sigma_i) & \text{if } \sigma_i \in D_i \\ s_i^{t,L(l)'}(\sigma_i) & \text{if } \sigma_i \in D_i' \setminus D_i. \end{cases}$$

Let $s^{t,l''} \equiv (s_i^{t,l''})_{i\in \mathbf{I}}$. We have a sequence $\{s^{t,l''}\}_{l=1}^{\infty}$ for each fixed $t \in \mathbf{N}$.

Then some notations are in order. Let ρ be the probability distribution over the set of initial nodes in the primitive game G. Fix any $t, l \in \mathbf{N}$. Take any $i \in \mathbf{I}$ and any $h \in H_i$. For any $y \in h$ and any $\sigma_{-i} \in D'_{-i}$, we define

$$\pi''_{t,l}(y,\sigma_{-i}) \equiv \rho(p^{m(y)}(y)) \prod_{\substack{1\leq m\leq m(y)\\ i(p^m(y))\neq i}} s^{t,l''}_{i(p^m(y))}(\sigma_{i(p^m(y))})(\alpha(p^{m-1}(y)))$$

to denote the probability that node y is reached in the primitive game G when player $j \neq i$ plays a strategy $s_j^{t,l''}(\sigma_j) \in Int\Sigma_j$ and player i takes an action leading to node y with probability one whenever necessary. Similarly, we define

$$\pi_{t,l}(y,\sigma_{-i}) \equiv \rho(p^{m(y)}(y)) \prod_{\substack{1\leq m\leq m(y)\\ i(p^m(y))\neq i}} s^{t,l}_{i(p^m(y))}(\sigma_{i(p^m(y))})(\alpha(p^{m-1}(y)))$$

when we replace the mappings $s_j^{t,l''}$ for $j \neq i$ in $\pi''_{t,l}(y,\sigma_{-i})$ by the mappings $s_j^{t,l}$.

Now we define $< q''_t, \mu''_t >$ for each $t \in \mathbf{N}$ fixed. Consider any $i \in \mathbf{I}$ and any $h \in H_i$. Find $(x,\sigma_{-i}) \in h \times D_{-i}$ and $B \in \mathbf{R}$ such that

$$\forall y \in h, \forall \hat{\sigma}_{-i} \in D_{-i};\ \limsup_{l\to\infty} \frac{\pi_{t,l}(y,\hat{\sigma}_{-i})}{\pi_{t,l}(x,\sigma_{-i})} < B,$$

and write it as (x^h,σ^h_{-i}). It must be possible to find such an (x,σ_{-i}) since $h \times D_{-i}$ is finite. Furthermore, we claim that there exists $B' \in R$ such that

$$\forall y \in h, \forall \sigma_{-i} \in D'_{-i};\ \limsup_{l\to\infty} \frac{\pi''_{t,l}(y,\sigma_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})} < B'$$

for the following reason. Consider any $y \in h$ and any $\sigma_{-i} \in D'_{-i} \setminus D_{-i}$. Suppose that there exists $1 \leq m \leq m(y)$ such that $i(p^m(y)) \neq i$, $\sigma_{i(p^m(y))}(\alpha(p^{m-1}(y))) = 0$, and $\sigma_{i(p^m(y))} \in D'_{i(p^m(y))} \setminus D_{i(p^m(y))}$. Then the choice of $L(l)$ in the construction of $s^{t,l''}$ guarantees that

$$\lim_{l\to\infty} \frac{s^{t,l''}_{i(p^m(y))}(\sigma_{i(p^m(y))})(\alpha(p^{m-1}(y)))}{\pi_{t,l}(x^h,\sigma^h_{-i})} = \lim_{l\to\infty} \frac{s^{t,L(l)'}_{i(p^m(y))}(\sigma_{i(p^m(y))})(\alpha(p^{m-1}(y)))}{\pi_{t,l}(x^h,\sigma^h_{-i})} = 0.$$

Therefore we must have

$$\lim_{l\to\infty} \frac{\pi''_{t,l}(y,\sigma_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})} = 0.$$

So, by letting $J(y,\sigma_{-i}) \equiv \{j \in \mathbf{I} \setminus \{i\} | \sigma_j \in D'_j \setminus D_j$ and $j = i(p^m(y))$ for some $1 \le m \le m(y)\}$, suppose that $\sigma_j(\alpha(p^{m-1}(y))) > 0$ for any $j \in J(y,\sigma_{-i})$ and any $1 \le m \le m(y)$ such that $j = i(p^m(y))$. Let $\sigma^y_{-i} \equiv \arg\min_{\hat{\sigma}_{-i} \in D_{-i}} \delta(y,\hat{\sigma}_{-i})$. Then, for any $j \in J(y,\sigma_{-i})$ fixed, it must be also the case that $\sigma^y_j(\alpha(p^{m-1}(y))) > 0$ for any $1 \le m \le m(y)$ such that $j = i(p^m(y))$. Otherwise, by defining $\sigma'_{-i} \equiv (\sigma_j, \sigma^y_{-\{i,j\}}) \in D'_{-i}$, we would be able to have

$$\delta(y, D'_{-i}) \le \delta(y, \sigma'_{-i}) < \delta(y, \sigma^y_{-i}) = \delta(y, D_{-i}),$$

which contradicts the hypothesis of Lemma 2. Therefore

$$\limsup_{l \to \infty} \frac{\pi''_{t,l}(y,\sigma_{-i})}{\pi''_{t,l}(y,(\sigma^y_{J(y,\sigma_{-i})}, \sigma_{-\{i\} \cup J(y,\sigma_{-i})}))}$$

is bounded where $(\sigma^y_{J(y,\sigma_{-i})}, \sigma_{-\{i\} \cup J(y,\sigma_{-i})}) \in D'_{-i}$ is a strategy profile obtained from σ_{-i} by replacing σ_j by σ^y_j for all $j \in J(y,\sigma_{-i})$. Note that the probability $\pi''_{t,l}(y,(\sigma^y_{J(y,\sigma_{-i})}, \sigma_{-\{i\} \cup J(y,\sigma_{-i})}))$ contains the expressions $s^{t,l}_{i(p^m(y))}$ only for any $1 \le m \le m(y)$ such that $i(p^m(y)) \ne i$. Therefore there exists $\sigma^{(y,\sigma_{-i})}_{-i} \in D_{-i}$ such that $\pi''_{t,l}(y,(\sigma^y_{J(y,\sigma_{-i})}, \sigma_{-\{i\} \cup J(y,\sigma_{-i})})) = \pi_{t,l}(y, \sigma^{(y,\sigma_{-i})}_{-i})$. Hence

$$\limsup_{l \to \infty} \frac{\pi''_{t,l}(y,\sigma_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})} = \limsup_{l \to \infty} \left[\frac{\pi''_{t,l}(y,\sigma_{-i})}{\pi''_{t,l}(y,(\sigma^y_{J(y,\sigma_{-i})}, \sigma_{-\{i\} \cup J(y,\sigma_{-i})}))} \cdot \frac{\pi_{t,l}(y,\sigma^{(y,\sigma_{-i})}_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})} \right]$$

is bounded. This establishes the claim that the bound B' exists. Without loss of generality, we can replace $\limsup_{l \to \infty}$ by $\lim_{l \to \infty}$. Therefore we can define a number

$$\theta(t) \equiv t \cdot \max \left[1, \max_{i \in \mathbf{I}} \max_{h \in H_i} \max_{\sigma_i \in D_i} \left(\frac{\sum_{y \in h} \sum_{\sigma_{-i} \in D'_{-i}} q'_t(\sigma_i,\sigma_{-i}) \lim_{l \to \infty} \frac{\pi''_{t,l}(y,\sigma_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})}}{\sum_{y \in h} \sum_{\hat{\sigma}_{-i} \in D_{-i}} q_t(\sigma_i,\hat{\sigma}_{-i}) \lim_{l \to \infty} \frac{\pi_{t,l}(y,\hat{\sigma}_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})}} \right) \right],$$

where the number $\theta(t)$ is well defined since

$$\forall i \in \mathbf{I}, \forall h \in H_i, \forall \sigma_i \in D_i; \sum_{y \in h} \sum_{\hat{\sigma}_{-i} \in D_{-i}} q_t(\sigma_i,\hat{\sigma}_{-i}) \lim_{l \to \infty} \frac{\pi_{t,l}(y,\hat{\sigma}_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})} \ge q_t(\sigma_i,\sigma^h_{-i}) > 0.$$

Now we define $q''_t \in \Delta(D')$ by

$$q_t''(\sigma) = \begin{cases} \frac{1}{\theta(t)} q'_t(\sigma) + (1 - \frac{1}{\theta(t)}) q_t(\sigma) & \text{if } \sigma \in D \\ \frac{1}{\theta(t)} q'_t(\sigma) & \text{if } \sigma \in D' \setminus D. \end{cases}$$

Then, for each $l \in \mathbf{N}$, let $\mu''_{t,l}$ be the belief system constructed from $s^{t,l''}$ by Bayes rule in the game $G(D', q''_t)$. We define $\mu''_t \equiv \lim_{l \to \infty} \mu''_{t,l}$. Obviously $\mu''_t \in \Phi(D', q''_t)$. This completes our construction of the sequence $\wp'' = \{< q''_t, \mu''_t >\}_{t=1}^{\infty}$.

The first two conditions for the triple $(C^{\prime 0''}, D'', \wp'')$ to be a selection device of C^* in M are obviously met. Let $q'' \equiv \lim_{t \to \infty} q''_t$ and $\mu'' \equiv \lim_{t \to \infty} \mu''_t$. we shall prove the third condition that the part $s^*_i(\sigma_i) = \sigma_i$ of implementation of D'_i is sequentially rational with respect to μ'' in game $G(D'', q'')$ for any $i \in \mathbf{I}$ and any $\sigma_i \in proj_i(C^*)$, by showing that the belief system μ'' is essentially identical with the belief system μ. Fix any $i \in \mathbf{I}$ and any $\sigma_i \in proj_i(C^*)$. Consider any $\sigma_{-i} \in D'_{-i}$, any $h \in H_i$, and any $x \in h$. Then

$$
\begin{aligned}
&\mu''_{t,l}(x, (\sigma_i, \sigma_{-i})) \\
&= \frac{q''_t(\sigma_i, \sigma_{-i}) \pi''_{t,l}(x, \sigma_{-i})}{\sum_{y \in h} \sum_{\tilde{\sigma}_{-i} \in D'_{-i}} q''_t(\sigma_i, \tilde{\sigma}_{-i}) \pi''_{t,l}(y, \tilde{\sigma}_{-i})} \\
&= \frac{q''_t(\sigma_i, \sigma_{-i}) \dfrac{\pi''_{t,l}(x, \sigma_{-i})}{\pi_{t,l}(x^h, \sigma^h_{-i})}}{\dfrac{1}{\theta(t)} \sum_{y \in h} \sum_{\tilde{\sigma}_{-i} \in D'_{-i}} q'_t(\sigma_i, \tilde{\sigma}_{-i}) \dfrac{\pi''_{t,l}(y, \tilde{\sigma}_{-i})}{\pi_{t,l}(x^h, \sigma^h_{-i})} + (1 - \dfrac{1}{\theta(t)}) \sum_{y \in h} \sum_{\hat{\sigma}_{-i} \in D_{-i}} q_t(\sigma_i, \hat{\sigma}_{-i}) \dfrac{\pi_{t,l}(y, \hat{\sigma}_{-i})}{\pi_{t,l}(x^h, \sigma^h_{-i})}} \\
&= \frac{\dfrac{q''_t(\sigma_i, \sigma_{-i}) \dfrac{\pi''_{t,l}(x, \sigma_{-i})}{\pi_{t,l}(x^h, \sigma^h_{-i})}}{\sum_{y \in h} \sum_{\hat{\sigma}_{-i} \in D_{-i}} q_t(\sigma_i, \hat{\sigma}_{-i}) \dfrac{\pi_{t,l}(y, \hat{\sigma}_{-i})}{\pi_{t,l}(x^h, \sigma^h_{-i})}}}{\dfrac{1}{\theta(t)} \cdot \dfrac{\sum_{y \in h} \sum_{\tilde{\sigma}_{-i} \in D'_{-i}} q'_t(\sigma_i, \tilde{\sigma}_{-i}) \dfrac{\pi''_{t,l}(y, \tilde{\sigma}_{-i})}{\pi_{t,l}(x^h, \sigma^h_{-i})}}{\sum_{y \in h} \sum_{\hat{\sigma}_{-i} \in D_{-i}} q_t(\sigma_i, \hat{\sigma}_{-i}) \dfrac{\pi_{t,l}(y, \hat{\sigma}_{-i})}{\pi_{t,l}(x^h, \sigma^h_{-i})}} + (1 - \dfrac{1}{\theta(t)})}.
\end{aligned}
$$

Due to the existence of the bound B', when we take a limit of $\mu''_{t,l}(x, (\sigma_i, \sigma_{-i}))$ with respect to l, the limit can be taken for the denominator and for the numerator separately. Then by taking a limit further with respect to t, the limit of the denominator goes to

$$
\lim_{t \to \infty} \left[\frac{1}{\theta(t)} \cdot \frac{\sum_{y \in h} \sum_{\tilde{\sigma}_{-i} \in D'_{-i}} q'_t(\sigma_i, \tilde{\sigma}_{-i}) \lim_{l \to \infty} \dfrac{\pi''_{t,l}(y, \tilde{\sigma}_{-i})}{\pi_{t,l}(x^h, \sigma^h_{-i})}}{\sum_{y \in h} \sum_{\hat{\sigma}_{-i} \in D_{-i}} q_t(\sigma_i, \hat{\sigma}_{-i}) \lim_{l \to \infty} \dfrac{\pi_{t,l}(y, \hat{\sigma}_{-i})}{\pi_{t,l}(x^h, \sigma^h_{-i})}} + (1 - \frac{1}{\theta(t)}) \right] = 1.
$$

Hence we conclude that if $\sigma_{-i} \in D_{-i}$, then

$$\mu''(x,(\sigma_i,\sigma_{-i})) = \lim_{t\to\infty} \frac{q_t''(\sigma_i,\sigma_{-i}) \lim_{l\to\infty} \frac{\pi_{t,l}(x,\sigma_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})}}{\sum_{y\in h}\sum_{\hat{\sigma}_{-i}\in D_{-i}} q_t(\sigma_i,\hat{\sigma}_{-i}) \lim_{l\to\infty} \frac{\pi_{t,l}(y,\hat{\sigma}_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})}}$$

$$= \lim_{t\to\infty} \Bigg[\frac{1}{\theta(t)} \cdot \frac{q_t'(\sigma_i,\sigma_{-i}) \lim_{l\to\infty} \frac{\pi_{t,l}(x,\sigma_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})}}{\sum_{y\in h}\sum_{\hat{\sigma}_{-i}\in D_{-i}} q_t(\sigma_i,\hat{\sigma}_{-i}) \lim_{l\to\infty} \frac{\pi_{t,l}(y,\hat{\sigma}_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})}}$$

$$+ (1-\frac{1}{\theta(t)}) \frac{q_t(\sigma_i,\sigma_{-i}) \lim_{l\to\infty} \frac{\pi_{t,l}(x,\sigma_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})}}{\sum_{y\in h}\sum_{\hat{\sigma}_{-i}\in D_{-i}} q_t(\sigma_i,\hat{\sigma}_{-i}) \lim_{l\to\infty} \frac{\pi_{t,l}(y,\hat{\sigma}_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})}}\Bigg]$$

$$= \lim_{t\to\infty} \frac{q_t(\sigma_i,\sigma_{-i}) \lim_{l\to\infty} \frac{\pi_{t,l}(x,\sigma_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})}}{\sum_{y\in h}\sum_{\hat{\sigma}_{-i}\in D_{-i}} q_t(\sigma_i,\hat{\sigma}_{-i}) \lim_{l\to\infty} \frac{\pi_{t,l}(y,\hat{\sigma}_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})}}$$

$$= \mu(x,(\sigma_i,\sigma_{-i}))$$

and that if $\sigma_{-i} \in D'_{-i} \setminus D_{-i}$, then

$$\mu''(x,(\sigma_i,\sigma_{-i})) = \lim_{t\to\infty} \Bigg[\frac{1}{\theta(t)} \cdot \frac{q_t'(\sigma_i,\sigma_{-i}) \lim_{l\to\infty} \frac{\pi''_{t,l}(x,\sigma_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})}}{\sum_{y\in h}\sum_{\hat{\sigma}_{-i}\in D_{-i}} q_t(\sigma_i,\hat{\sigma}_{-i}) \lim_{l\to\infty} \frac{\pi_{t,l}(y,\hat{\sigma}_{-i})}{\pi_{t,l}(x^h,\sigma^h_{-i})}}\Bigg] = 0.$$

Therefore the sequential rationality of $s_i^*(\sigma_i) = \sigma_i$ with respect to μ'' in the game $G(D',q'')$ reduces to the sequential rationality of $s_i^*(\sigma_i) = \sigma_i$ with respect to μ in the game $G(D,q)$. The latter is guaranteed by the hypothesis of Lemma 2. ||

Proof of Theorem 1 :

We shall construct a sequence $\{(C'^{0n}, D^n, \wp^n)\}_{n=1}^{\infty}$ in M as follows. Since $M \neq \emptyset$ by Lemma 1, we take an arbitrary triple in M and call it $(C'^{01}, D^1, \wp^1)$. Define $\bar{M}(C'^{01}, D^1, \wp^1) \equiv \{(C'^0, D, \wp) \in M | D^1 \subset D\}$. $\bar{M}(C'^{01}, D^1, \wp^1) \neq \emptyset$ since $(C'^{01}, D^1, \wp^1) \in \bar{M}(C'^{01}, D^1, \wp^1)$. By definition it holds for any $(C'^0, D, \wp) \in \bar{M}(C'^{01}, D^1, \wp^1)$ that $\delta(x, D^1_{-i}) \geq \delta(x, D_{-i})$ for any $i \in \mathbf{I}$, any $h \in H_i$, and any $x \in h$. If there exists $(C'^0, D, \wp) \in \bar{M}(C'^{01}, D^1, \wp^1)$

such that $\delta(x, D^1_{-i}) \gneq \delta(x, D_{-i})$ for some $i \in \mathbf{I}$, some $h \in H_i$, and some $x \in h$, take such $(C^{\prime 0}, D, \wp)$ arbitrarily and call it $(C^{\prime 02}, D^2, \wp^2)$. Otherwise take an arbitrary $(C^{\prime 02}, D^2, \wp^2) \in \bar{M}(C^{\prime 01}, D^1, \wp^1)$. By repeating the procedure, we have a sequence $\{(C^{\prime 0n}, D^n, \wp^n)\}_{n=1}^{\infty}$.

Examine the nature of the sequence. Associated with the sequence, we have $\sum_{i \in \mathbf{I}} \sum_{h \in H_i} \#(h)$ sequences $(\{\delta(x, D^n_{-i})\}_{n=1}^{\infty})_{i \in \mathbf{I}, h \in H_i, x \in h}$. For each $i \in \mathbf{I}$, each $h \in H_i$, and each $x \in h$, the sequence $\{\delta(x, D^n_{-i})\}_{n=1}^{\infty}$ is nonincreasing and $\delta(x, D^n_{-i}) \geq 0$ for any $n \in \mathbf{N}$. Since $\sum_{i \in \mathbf{I}} \sum_{h \in H_i} \#(h)$ is a finite number, there exists $N \in \mathbf{N}$ such that $\delta(x, D^N_{-i}) = \delta(x, D^n_{-i})$ for any $n \geq N$, any $i \in \mathbf{I}$, any $h \in H_i$, and any $x \in h$. By the way of constructing the sequence $\{(C^{\prime 0n}, D^n, \wp^n)\}_{n=1}^{\infty}$, this implies that $\delta(x, D^N_{-i}) = \delta(x, D_{-i})$ for any $i \in \mathbf{I}$, any $h \in H_i$, any $x \in h$, and any $(C^{\prime 0}, D, \wp) \in M$ with $D^N \subset D$. Thus the assumption of Lemma 2 is satisfied for $(C^{\prime 0N}, D^N, \wp^N)$. Hence any selection by $(C^{\prime 0N}, D^N, \wp^N)$ is a robust selection. ||

Proof of Lemma 3 :

Fix any $(C^{\prime 0}, D, \wp) \in M$ where $D = \prod_{i \in \mathbf{I}} D_i$. $\Re(D) \subset D$ by definition. We shall show $D \subset \Re(D)$. Take any $\sigma_i \in D_i$. By the following procedure, we can find $(C^{0\prime}, D', \wp') \in M$ with $D' \subset D$ such that either $\sigma_i \in proj_i(C^{*\prime})$ or $\sigma_i \in D'_i \setminus proj_i(C^{0\prime})$ holds where $C^{*\prime}$ is a selection by $(C^{0\prime}, D', \wp')$. Since $(C^{\prime 0}, D, \wp) \in M$, there exists $N \in \mathbf{N}$ such that $(C^{\prime 0}, D, \wp) \in M^N$. Call $(C^{\prime 0}, D, \wp)$ as $(C^{0N}, D^N, \wp^N)$. If $(C^{0N}, D^N, \wp^N)$ can serve as $(C^{0\prime}, D', \wp')$, then we are done. So suppose not, namely $\sigma_i \in proj_i(C^{\prime 0N}) \setminus proj_i(C^{*N})$ where C^{*N} is a selection by $(C^{\prime 0N}, D^N, \wp^N)$. Then, by definition of M^N, there exists $(C^{0N-1}, D^{N-1}, \wp^{N-1}) \in M^{N-1}$ with $D^{N-1} \subset D^N$ such that $\sigma_i \in D_i^{N-1}$. If $(C^{\prime 0N-1}, D^{N-1}, \wp^{N-1})$ can serve as $(C^{0\prime}, D', \wp')$, then we are also done. Repeat the procedure as long as we are not done yet. This creates a sequence $\{(C^{\prime 0n}, D^n, \wp^n)\}$. It is however guaranteed that we can stop at latest at $n = 0$ with $(C^{\prime 00}, D^0, \wp^0) \in M^0$, since $C^{\prime 00} = C^{\prime *0}$ guarantees that either $\sigma_i \in proj_i(C^{*0})$ or $\sigma_i \in D_i^0 \setminus proj_i(C^{\prime 00})$ holds. Hence there must exists $0 \leq n \leq N$ such that $(C^{0n}, D^n, \wp^n)$ can serve as $(C^{0\prime}, D', \wp')$. Then the part $s_i^*(\sigma_i) = \sigma_i$ of implementations is sequentially rational with respect to μ' in game $G(D', q')$ where μ' is a consistent belief implied by $\wp'$ and q' is a probability distribution over the set D' implied by $\wp'$. By the way of constructing $(C^{0\prime}, D', \wp')$, we know that $D' \subset D$. Hence σ_i is sequentially rationalizable in D. ||

Proof of Lemma 4 :

By definition, the operator $\Re$ of sequential rationalizability is monotone in the sense that $\Re(D) \subset \Re(D')$ for any product subsets $D \subset D' \subset \Sigma$. Fix any $(C^0, D, \wp) \in M$. By the monotonicity of the operator $\Re$, we have $\Re(D) \subset \Re(\Sigma)$ from $D \subset \Sigma$. By conducting the operation repeatedly, we have $\Re^n(D) \subset \Re^n(\Sigma)$ for any $n \in \mathbf{N}$. Lemma 3, however, guarantees that $D = \Re(D) = \Re^2(D) = \cdots = \Re^n(D)$ for any $n \in \mathbf{N}$. Hence we have $D \subset \Re^n(\Sigma)$ for any $n \in \mathbf{N}$. This gives us $D \subset \cap_{n=0}^{\infty}\Re^n(\Sigma) = \Re^*$. ||

Proof of Lemma 5 :

Suppose that a triple $(C^0, D, \wp) \in M$ with a selection C^* satisfies the condition of Lemma 5. Consider any $(C^{0'}, D', \wp') \in M$ with $D \subset D'$. By Lemma 4 we know that $\delta(x, D'_{-i}) = \delta(x, \Re^*_{-i}) = \delta(x, D_{-i})$ for any $i \in \mathbf{I}$, any $h \in H_i$, and any $x \in h$. Then Lemma 2 applies to guarantee that the set C^* is a robust selection. ||

Proof of Lemma 6 :

Let $C^*, (C^0, D, \wp)$, and $C^{*'}$ be as assumed in Lemma 6. Take any $(C^{0'}, D', \wp') \in M$ by which the set $C^{*'}$ is a selection. By Lemma 4, then, the assumption of Lemma 6 guarantees that $\delta(x, \prod_{j\neq i}(D_j \cup D'_j)) = \delta(x, \Re^*_{-i}) = \delta(x, D_{-i})$ for any $i \in \mathbf{I}$, any $h \in H_i$, and any $x \in h$. By the same construction as in the proof of Lemma 2, we can construct $\wp'''$ such that C^* is a selection by $(D \cup D', \prod_{i\in\mathbf{I}}(D_i \cup D'_i), \wp''')$. Hence $(D \cup D', \prod_{i\in\mathbf{I}}(D_i \cup D'_i), \wp''') \in M$. Obviously $D' \subset \prod_{i\in\mathbf{I}}(D_i \cup D'_i)$. In order for $C^{*'}$ to be a robust selection by $(C^{0'}, D', \wp')$, there must exist $(C^{0''}, D'', \wp'') \in M$ such that $\prod_{i\in\mathbf{I}}(D_i \cup D'_i) \subset D''$ and $C^{*'}$ is a selection by $(C^{0''}, D'', \wp'')$. This is impossible by the supposition of Lemma 6. Hence $C^{*'}$ is not a robust selection. ||

Proof of Theorem 2 :

Let C^* be a robust selection by $(C^0, D, \wp)$ for a primitive game G where $\wp = \{< q_t, \mu_t >\}_{t=1}^{\infty}$. Suppose to the contrary that some strategy profile $\sigma \in C^*$ is not a Nash equilibrium in game G. Then for some player i, there exists his information set $h \in H_i$ on the equilibrium path of σ such that σ_i is not sequentially rational at h with respect to a Kreps and Wilson (1982) consistent belief given σ. Since h is on the equilibrium path of σ, the consistent belief at h is calculated by Bayes rule as $\frac{\pi(x,\sigma_{-i})}{\sum_{y\in h}\pi(y,\sigma_{-i})}$ being the posterior that node x has been reached where $\pi(y,\sigma_{-i})$ is defined as

$$\pi(y,\sigma_{-i}) \equiv \rho(p^{m(y)}(y)) \prod_{\substack{1\leq m\leq m(y)\\ \iota(p^m(y))\neq i}} \sigma_{i(p^m(y))}(\alpha(p^{m-1}(y)))$$

to denote the probability that node y is reached when player $j \neq i$ plays the strategy σ_j and player i takes an action leading to node y whenever necessary. For each node $x \in h$ and each strategy profile $\tilde{\sigma} \in \Sigma$, let $U_i(\tilde{\sigma}|x)$ denote the conditional expected payoff to player i from a play of $\tilde{\sigma}$ given that node x has been reached. Then there must exist $\sigma_i' \in \Sigma_i$ such that

$$\sum_{x\in h} \frac{\pi(x,\sigma_{-i})}{\sum_{y\in h}\pi(y,\sigma_{-i})} U_i((\sigma_i',\sigma_{-i})|x) > \sum_{x\in h} \frac{\pi(x,\sigma_{-i})}{\sum_{y\in h}\pi(y,\sigma_{-i})} U_i(\sigma|x).$$

Now examine the sequential rationality of $s_i^*(\sigma_i) = \sigma_i$ at the information set $\cup_{x\in h}\cup_{\hat{\sigma}_{-i}\in D_{-i}} \{(x,(\sigma_i,\hat{\sigma}_{-i}))\}$ for player i in game $G(D,q)$ where $q = \lim_{t\to\infty} q_t$. For each $t \in \mathbf{N}$ fixed, let $\{s^{t,l}\}_{l=1}^{\infty}$ be the sequence of strategy profiles in $G(D,q_t)$ which generates μ_t. Let $\pi_{t,l}$ be defined as in the proof of Lemma 2. Since h is on the equilibrium path of σ, we have $\delta(h,\sigma_{-i}) = 0$. This guarantees that

$$\begin{aligned}\lim_{l\to\infty} \sum_{y\in h}\sum_{\tilde{\sigma}_{-i}\in D_{-i}} q_t(\sigma_i,\tilde{\sigma}_{-i})\pi_{t,l}(y,\tilde{\sigma}_{-i}) &= \sum_{y\in h}\sum_{\tilde{\sigma}_{-i}\in D_{-i}} q_t(\sigma_i,\tilde{\sigma}_{-i})\pi(y,\tilde{\sigma}_{-i})\\ &\geq \sum_{y\in h} q_t(\sigma_i,\sigma_{-i})\pi(y,\sigma_{-i})\\ &> 0.\end{aligned}$$

Therefore, for each $x \in h$ and each $\hat{\sigma}_{-i} \in D_{-i}$,

$$\begin{aligned}\mu_t(x,(\sigma_i,\hat{\sigma}_{-i})) &= \lim_{l\to\infty} \frac{q_t(\sigma_i,\hat{\sigma}_{-i})\pi_{t,l}(x,\hat{\sigma}_{-i})}{\sum_{y\in h}\sum_{\tilde{\sigma}_{-i}\in D_{-i}} q_t(\sigma_i,\tilde{\sigma}_{-i})\pi_{t,l}(y,\tilde{\sigma}_{-i})}\\ &= \frac{q_t(\sigma_i,\hat{\sigma}_{-i})\pi(x,\hat{\sigma}_{-i})}{\sum_{y\in h}\sum_{\tilde{\sigma}_{-i}\in D_{-i}} q_t(\sigma_i,\tilde{\sigma}_{-i})\pi(y,\tilde{\sigma}_{-i})}.\end{aligned}$$

Furthermore, since $\sigma \in C^*$, the definition of $(C^0, D, \wp)$ being a selection device implies that $\lim_{t\to\infty} q_t(\sigma_i,\hat{\sigma}_{-i}) > 0$ if and only if $\hat{\sigma}_{-i} = \sigma_{-i}$. Therefore

$$\mu(x,(\sigma_i,\hat{\sigma}_{-i})) = \lim_{t\to\infty} \mu_t(x,(\sigma_i,\hat{\sigma}_{-i})) = \begin{cases} \frac{\pi(x,\sigma_{-i})}{\sum_{y\in h}\pi(y,\sigma_{-i})} & \text{if } \hat{\sigma}_{-i} = \sigma_{-i}\\ 0 & \text{otherwise.}\end{cases}$$

Hence we compare the conditional expected payoffs to player i of changing to σ_i' and of implementing σ_i at the information set $\cup_{x\in h}\cup_{\hat{\sigma}_{-i}\in D_{-i}}\{(x,(\sigma_i,\hat{\sigma}_{-i}))\}$ in the game $G(D,q)$ as

$$\sum_{x\in h}\sum_{\hat{\sigma}_{-i}\in D_{-i}}\mu(x,(\sigma_i,\hat{\sigma}_{-i}))U_i((\sigma_i',\hat{\sigma}_{-i})|x)-\sum_{x\in h}\sum_{\hat{\sigma}_{-i}\in D_{-i}}\mu(x,(\sigma_i,\hat{\sigma}_{-i}))U_i((\sigma_i,\hat{\sigma}_{-i})|x)$$
$$=\sum_{x\in h}\frac{\pi(x,\sigma_{-i})}{\sum_{y\in h}\pi(y,\sigma_{-i})}U_i((\sigma_i',\sigma_{-i})|x)-\sum_{x\in h}\frac{\pi(x,\sigma_{-i})}{\sum_{y\in h}\pi(y,\sigma_{-i})}U_i((\sigma_i,\sigma_{-i})|x)$$
$$>0.$$

Thus the part $s_i^*(\sigma_i)=\sigma_i$ of the implementation is not sequentially rational for player i in $G(D,q)$. This is a contradiction. ||

Proof of Proposition :

First consider the case of $u_{t_L}(e^*_{t_H}(t_H),t_He^*_{t_H}(t_H)) > u_{t_L}(e^*_{t_L}(t_L),t_Le^*_{t_L}(t_L))$. For each $(\tilde{e},\tilde{w})\in R^2_+$, let $I_t(e|\tilde{e},\tilde{w}) : R_+ - R_+$ denote a function which satisfies $u_t(e,I_t(e|\tilde{e},\tilde{w}))=u_t(\tilde{e},\tilde{w})$. Let $(\hat{e},\hat{w})$, $(\hat{e}',\hat{w}')$ be two intersections of a line $w=t_He$ and a curve $w=I_{t_L}(e|e^*_{t_L}(t_L),t_Le^*_{t_L}(t_L))$ such that $(\hat{e},\hat{w})<(e^*_{t_L}(t_L),t_Le^*_{t_L}(t_L))<(\hat{e}',\hat{w}')$. Find ρ_1 such that a line $w=\beta(\rho_1)e$ is tangent to a curve $w=I_{t_H}(e|\hat{e}',\hat{w}')$. The tangent line exists since the curve $w=I_{t_H}(e|\hat{e}',\hat{w}')$ is convex given the assumptions about $u_{t_H}(e,w)$. Furthermore, the assumption of $u_{t_L}(e^*_{t_H}(t_H),t_He^*_{t_H}(t_H)) > u_{t_L}(e^*_{t_L}(t_L),t_Le^*_{t_L}(t_L))$ guarantees $\rho_1<1$, and the "single crossing property" guarantees $\rho_1>0$. Take any $\rho\in(\rho_1,1)$ fixed. We shall show that a robust selection supports the pooling outcome $(e^*_{t_H}(\beta(\rho)),\beta(\rho)e^*_{t_H}(\beta(\rho)))$. Construct the following sequential equilibrium σ^*. The worker chooses the education level $e^*_{t_H}(\beta(\rho))$ with probability one. The firms bid the wage $\beta(\rho)e^*_{t_H}(\beta(\rho))$ to the education level $e^*_{t_H}(\beta(\rho))$, a wage t_He to any education level $e\in[\hat{e}',+\infty)$, and a wage t_Le to any other education level e. Let $(C^0,D,\wp)\in M^0$ be a primitive selection device of $C^0=\{\sigma^*\}$. Let $(C^{0'},D',\wp')\in M$ be any selection device with $D\subset D'$. We shall construct the following triple $(C^{0''},D'',\wp'')$. Let σ^{**} be a sequential equilibrium in which the t_L worker chooses $e^*_{t_L}(t_L)$ with probability one, the t_H worker chooses $\hat{e}'$ with probability one, and the firms bid $min\left[t_He,I_{t_L}(e|e^*_{t_L}(t_L),t_Le^*_{t_L}(t_L))\right]$ to any education level $e\in R_+$. Let E be a set of education levels which are assigned positive probabilities by some worker's strategies in D'. Let $(\acute{e},\acute{w})$ be an intersection of a line $w=t_He$ and a curve $w=I_{t_H}(e|e^*_{t_H}(t_L),t_Le^*_{t_H}(t_L))$ such that $(e,w)<(e^*_{t_H}(t_L),t_Le^*_{t_H}(t_L))$. Take any education level $e\in E\cap[e,\hat{e}')$ and define a worker's strategy σ_{I}^{e} which prescribes the pure action e to the t_L worker and the pure action $\hat{e}'$ to the t_H worker. Now set $C^{0''} = D' \cup \{\sigma^{**}\}$. Let the set of worker's strategies in D'' consist of the worker's strategies in D', the worker's strategy of σ^{**}, and $\bigcup_{e \in E \cap [\bar{e}, \hat{e}')} \{\sigma_{\mathrm{I}}^{e}\}$. Let the set of firms' strategies in D'' consist of the firms' strategies in D' and the firms' strategies of σ^{**}. Finally set $\wp''$ as follows. Let $\{\epsilon_t\}_{t=1}^{\infty}$ be a sequence of small positive numbers converging to zero. For any strategy profile $\sigma \in D''$ such that $\sigma \neq \sigma^*$ and σ assigns to the firms the strategies of σ^*, set $q_t''(\sigma) = \epsilon_t^2$ if and only if σ assigns to the worker either a strategy σ_{I}^{e} for some $e \in E \cap [\bar{e}, \hat{e}')$ or the strategy of σ^{**}. For any strategy profile $\sigma \in D''$ such that $\sigma \neq \sigma^*$ and σ assigns to the worker either a strategy σ_{I}^{e} for some $e \in E \cap [\bar{e}, \hat{e}')$ or the strategy of σ^{**}, set $q_t''(\sigma) = \epsilon_t$ if and only if σ assigns to the firms the strategies of σ^{**}. For any other strategy profile $\sigma \in D''$ except σ^*, set $q_t''(\sigma) = \epsilon_t^3$. The probablity $q_t''(\sigma^*)$ gets all the remaining weight. Let $q'' = \lim_{t \to \infty} q_t''$. Imagine a consistent belief μ'' of $G(D'', q'')$ such that, if a firm is assigned the strategy of σ^*, then his posterior of the t_H worker is zero to any education level $e \in [0, \hat{e}') \setminus \{e_{t_H}^*(\beta(\rho))\}$, ρ to the education level $e_{t_H}^*(\beta(\rho))$, and one to any othe education level. Given q_t'' thus constructed, we can find such a consistent belief of $G(D'', q'')$ for the following reason. Suppose that a firm is assigned the strategy of σ^*. For any $e \notin E \cup \{\hat{e}'\}$, his posterior of the t_H worker can be any point in $[0,1]$. So consider any $e \in E \cup \{\hat{e}'\}$. If $e < \bar{e}$, his posterior of the t_H worker can be zero, since any worker's strategy which assigns such e to the t_H worker with a positive probability is not sequentially rationalizable and, by Lemma 4, is not in D'. If $e > \hat{e}'$, his posterior of the t_H worker can be one by a symmetric argument. Finally consider an education level $e \in [\bar{e}, \hat{e}']$. His posterior to $e = e_{t_H}^*(\beta(\rho))$ is that the worker has chosen this e believing the strategy of σ^*, that is, the worker is t_H with probability ρ. His posterior to $e = \hat{e}'$ is that the worker has chosen this e believing either a strategy $\sigma_{\mathrm{I}}^{\tilde{e}}$ for some $\tilde{e} \in E \cap [\bar{e}, \hat{e}')$ or the strategy of σ^{**}, that is, the worker is t_H with probability one. His posterior to $e \neq e_{t_H}^*(\beta(\rho))$ or $\hat{e}'$ is that the worker has chosen this e believing the strategy σ_{I}^{e}, that is, the worker is t_L with probability one. Thus we have a consistent belief as described above. Now we shall show that the triple $(C^{0''}, D'', \wp'')$ is a selection device of $\{\sigma^*\}$. It is rational for the worker to implement the strategy of σ^* expecting that the firms will implement the strategies of σ^* with probability one. Given the consistent belief μ'', it is also rational for the firms to implement the strategies of σ^*. So examine the sequential rationality of implementing a strategy σ_I^e for each $e \in E \cap [\bar{e}, \hat{e}')$. If the worker is assigned the strategy σ_I^e, he expects that the firms will implement the strategies of σ^{**}. By the definitions of $\bar{e}, \hat{e}'$ and by the assumption of $u_{t_L}(e^*_{t_H}(t_H), t_H e^*_{t_H}(t_H)) > u_{t_L}(e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))$, we know that $\hat{e} < e$ and, therefore, $min\left[t_H e, I_{t_L}(e|e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))\right] = I_{t_L}(e|e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))$. Hence the t_L worker maximizes his utility by choosing the prescribed e, expecting the firms' bid $w = I_{t_L}(e|e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))$. By the "single crossing property" and by the assumption of $u_{t_L}(e^*_{t_H}(t_H), t_H e^*_{t_H}(t_H)) > u_{t_L}(e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))$, on the other hand, the t_H worker maximizes his utility by choosing $\hat{e}'$. Thus the triple $(C^{'0''}, D'', \wp'')$ is a selection device of $\{\sigma^*\}$ in M. Hence the set $\{\sigma^*\}$ is a robust selection.

A similar idea to the above construction of $(C^{0''}, D'', \wp'')$ also applies to support the Cho-Kreps outcome for any $\rho \in (0,1)$ in the case of $u_{t_L}(e^*_{t_H}(t_H), t_H e^*_{t_H}(t_H)) > u_{t_L}(e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))$. Consider a sequential equilibrium σ^{***} in which the t_L worker chooses $e^*_{t_L}(t_L)$ with probability one, the t_H worker chooses $\hat{e}'$ with probability one, and the firms bid a wage $t_L e$ to any education level $e \in [0, \hat{e}')$ and a wage $t_H e$ to any other education level e. By replacing σ^* by σ^{***} in the above construction of $(C^{'0''}, D'', \wp'')$, we can prove that a singleton set $\{\sigma^{***}\}$ is a robust selection for any $\rho \in (0,1)$.

Next continue to consider the case of $u_{t_L}(e^*_{t_H}(t_H), t_H e^*_{t_H}(t_H)) > u_{t_L}(e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))$. Then there exists (e_0, w_0) such that $w_0 = I_{t_L}(e_0|e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L)) = I_{t_H}(e_0|e^*_{t_H}(t_L), t_L e^*_{t_H}(t_L))$. Let $\rho_0 > 0$ be such that $w_0 = \beta(\rho_0)e_0$. The assumption of $u_{t_L}(e^*_{t_H}(t_H), t_H e^*_{t_H}(t_H)) > u_{t_L}(e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))$ guarantees $\rho_0 < 1$. Furthermore, $\rho_0 < \rho_1$ since $u_{t_H}(\hat{e}', \hat{w}') > u_{t_H}(e_0, w_0) = u_{t_H}(e^*_{t_H}(t_L), t_L e^*_{t_H}(t_L))$. Take any $\rho \in (0, \rho_0)$ fixed. We shall show that, for such ρ, the Cho-Kreps outcome supported by the sequential equilibrium σ^{***} is the only outcome supported by any robust selection. Consider any selection device $(C^0, D, \wp)$ of some set C^*. Suppose that some strategy profile $\sigma^0 \in C^*$ prescribes a different outcome from the Cho-Kreps outcome. By Theorem 2, the strategy profile σ^0 must be a Nash equilibrium. This Nash equilibrium should not allow any fully pooling outcome since there is no (e, w) such that $w = \beta(\rho)e$, $w \geq I_{t_L}(e|e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))$, and $w \geq I_{t_H}(e|e^*_{t_H}(t_L), t_L e^*_{t_H}(t_L))$. By the same reason, the equilibrium σ^0 should not be a partially pooling equilibrium in which the t_L worker takes some pure action e^0 and the t_H worker chooses the education level e^0 with a probability in $(0,1)$. Hence σ^0 must be one of the following two types of Nash equilibria. The first type is a partially pooling equilibrium in which the t_H worker takes some pure action $e^0 < \hat{e}'$, the t_L worker takes education levels $e^*_{t_L}(t_L)$ and e^0 with positive probabilities and the firms bid $w^0 = I_{t_L}(e^0|e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))$ to the education level e^0. The other type is a fully separating equilibrium in which the t_L worker takes the pure action $e^*_{t_L}(t_L)$ and the t_H worker takes some pure action $e^0 > \hat{e}'$. Consider the first type equilibrium. By the "single crossing property", we can find (e', w') on the line $w = t_H e$ such that $e' > \hat{e}'$ and $u_{t_H}(e', w') > u_{t_H}(e^0, w^0)$. Let σ^{****} be a sequential equilibrium in which the t_L worker chooses $e^*_{t_L}(t_L)$ with probability one, the t_H worker chooses e' with probability one, and the firms bid a wage $t_L e^*_{t_L}(t_L)$ to the education level $e^*_{t_L}(t_L)$ and a wage $t_H e'$ to the education level e'. By the same argument as for $\rho \in (\rho_1, 1)$, we can prove that the singleton set $\{\sigma^{***}\}$ is a robust selection by some selection device $(C^{0\prime}, D', \wp')$ such that $D \subset D'$ and $\sigma^{***}, \sigma^{****} \in D'$. But there is no selection device $(C^{0\prime\prime}, D'', \wp'')$ of C^* with $D' \subset D''$ since, if a firm is assigned the strategy of σ^0 and sees e', then he is forced to believe that the worker is t_H with probability one and his best response bid $t_H e'$ induces the t_H worker to deviate from σ^0 and choose the education level e'. Thus the set C^* is not a robust selection. Consider the remaining possibility of the second type equilibrium. Define (e', w') such that $e' = \frac{e^0 + \hat{e}'}{2}$ and $w' = t_H e'$. Then $e' > \hat{e}'$ and $u_{t_H}(e', w') > u_{t_H}(e^0, t_H e^0)$. Therefore the same argument as for the first type equilibrium applies. The set C^* is not a robust selection.

Lastly to the case of $u_{t_L}(e^*_{t_H}(t_H), t_H e^*_{t_H}(t_H)) \leq u_{t_L}(e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))$ applies also an argument similar to the one for the case of $u_{t_L}(e^*_{t_H}(t_H), t_H e^*_{t_H}(t_H)) > u_{t_L}(e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))$ with $\rho \in (0, \rho_0)$. Let $\sigma^{**\prime}$ be a sequential equilibrium exactly as σ^{**} except that the t_H worker chooses not $\hat{e}'$ but $e^*_{t_H}(t_H)$ with probability one. Also let $\sigma^{***\prime}$ be a sequential equilibrium exactly as σ^{***} except that the t_H worker chooses not $\hat{e}'$ but $e^*_{t_H}(t_H)$ with probability one. Then let $\sigma^{***\prime}$ and $\sigma^{**\prime}$ replace σ^* and σ^{**} in the construction of $(C^{0\prime\prime}, D'', \wp'')$ for the case of $u_{t_L}(e^*_{t_H}(t_H), t_H e^*_{t_H}(t_H)) > u_{t_L}(e^*_{t_L}(t_L), t_L e^*_{t_L}(t_L))$ with $\rho \in (\rho_1, 1)$. We can prove that a singleton set $\{\sigma^{***\prime}\}$ is a robust selection and that the Cho-Kreps outcome thus supported by $\sigma^{***\prime}$ is the only outcome supported by a robust selection. ||

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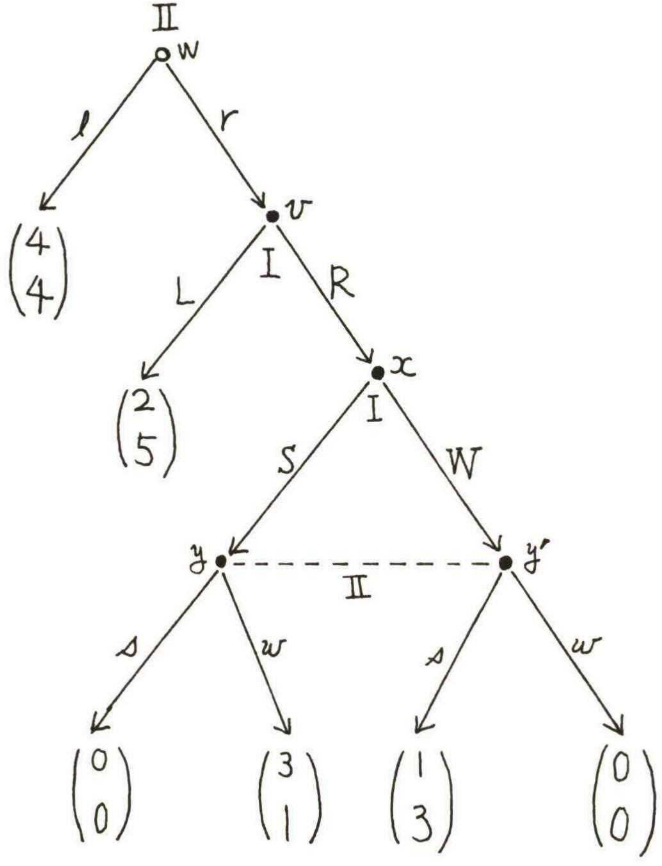

Figure 1

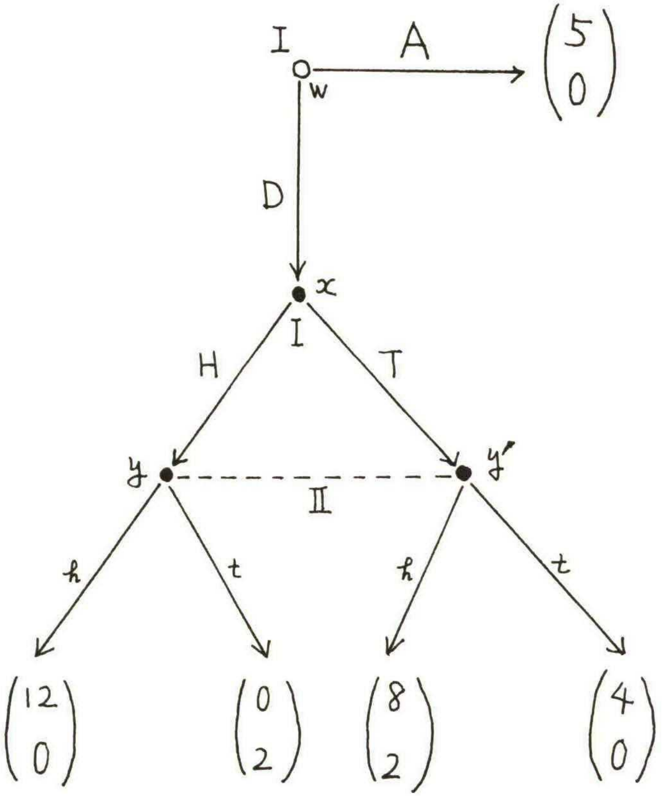

Figure 2

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