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An Extension of the "Folk Theorem" with Continuous Reaction Functions*

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Abstract

In Friedman and Samuelson (1990a), we showed that there exist subgame perfect equilibria for infinitely repeated games in which the equilibrium strategy combinations are continuous. This paper extends these results by providing a counterpart to the Fudenberg and Maskin folk theorem in continuous strategies. We show that any outcome of the stage game which is feasible and strictly individually rational can be supported as an outcome of a subgame perfect equilibrium of the infinitely repeated game with continuous strategies, providing discount factors are sufficiently high.

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An Extension of the "Folk Theorem" with Continuous Reaction Functions

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1. Introduction

In an earlier paper, Friedman and Samuelson (1990a), the authors showed that there exist subgame perfect equilibria for infinitely repeated games in which the equilibrium strategy combinations are continuous. These equilibria support payoff outcomes dominating those associated with a single-shot noncooperative equilibrium of the game. The equilibrium strategies consist of an initial action and a continuous decision rule for choosing actions in later periods.

In Friedman and Samuelson (1990a) we argued that discontinuous decision rules are sometimes implausible. We do not take the position that they are always less plausible than continuous strategies, but only that some circumstances favor one, some the other. In that spirit, this paper extends our earlier results by providing a counterpart to the Fudenberg and Maskin folk theorem in which decision rules are continuous.

It has long been known that outcomes that Pareto dominate a Nash equilibrium of the stage game can be supported in infinitely repeated games by Nash reversion trigger strategies. Let s^* denote the action combination called for by an equilibrium strategy combination and s^c a single-shot Nash equilibrium action combination. Under a trigger strategy player i chooses s_i^* if no deviation from s^* has been encountered, but chooses s_i^c after *any* deviation. The continuous strategies in Friedman and Samuelson (1990a) are analogous to trigger strategy equilibria in two ways: the equilibrium payoffs dominate single-shot Nash payoffs and deviations from equilibrium behavior draw a punishment that does not depend on which player deviates. The strategies of Friedman and Samuelson differ from trigger strategies in that the latter are discontinuous, prescribing the same punishment for all deviations from s^*. In contrast, the Friedman and Samuelson decision rules prescribe an action near s_i^* in response to a small deviation and prescribe actions which move continuously nearer to s_i^c the larger the deviation. These continuous decision rules are akin to the reaction functions of oligopoly theory.

Discontinuous decision rules also appear in the existing formulations of the folk theorem for repeated games. In the discounted folk theorem of Fudenberg and Maskin (1986), which is typical of results in the area, equilibrium strategy combinations are constructed that (1) will support a given payoff outcome if it dominates the minimax payoffs of the players (i.e., if it is *strictly individually rational*), (2) are subgame perfect, (3) tailor punishment for defection to the defecting player, and (4) are based on discontinuous decision rules. The Fudenberg and Maskin equilibria differ from trigger strategy equilibria with respect to (1) and (3), and also differ in requiring significantly more complicated strategies. Much of this complication appears in order to tailor punishments to the identity of the deviator who has triggered the punishment.

In order to achieve a folk theorem, we will work in this paper with strategies that share features (1) to (3) with the Fudenberg and Maskin strategies. In particular, our strategies will yield subgame perfect equilibria that support (virtually) any payoff that is strictly individually rational. Our strategies will also tailor punishments to the identity of the deviator who triggers the punishment. This tailoring of the punishment to the defector appears crucial to supporting arbitrary individually rational outcomes. However, our strategies are continuous, in contrast to those of Fudenberg and Maskin.

Strategies in the folk theorem family generally utilize "reference points." These reference points can be intuitively interpreted as indicating *where the game is*. For (discontinuous) grim trigger strategy equilibria, the reference point takes on one of only two values, indicating whether there has been prior defection.[1] The Fudenberg and Maskin folk theorem requires a more complicated reference point that indicates (1) whether there has been a defection, (2) who is the most recent defector if there has been any defection, and (3) how many periods in the past the most recent defection occurred. This information is needed to determine the current period action prescribed by the equilibrium strategies.

In our previous work, we examined strategies that did not require a reference point and also strategies requiring a scalar reference point β, taking values in the interval [0, 1], to indicate the level of punishment currently being carried out. Zero corresponded to the maximal punishment and one corresponded to no punishment. As with trigger strategies, no account was taken of which player defected.

To achieve a full folk theorem generalization with continuous strategies, this paper abandons the scalar reference point in favor of a vector valued reference point of dimension n + 1. The reference point vector is denoted $\beta = (\beta_0, \beta_1, \ldots, \beta_n)$. The first coordinate ($\beta_0$) has the same meaning as our previous reference point and takes values in [0, 1], indicating the current *extent of cooperative behavior* with $\beta_0 = 1$ being maximal cooperation and $\beta_0 = 0$ being the extreme of punishment. The remaining n coordinates $(\beta_1, \ldots, \beta_n)$ designate the degree to which the various players are *current defectors*. $\Sigma_{i=1}^{n}\beta_i = 1$ at all times, meaning that there is a nominal defector status across players that always sums to unity. When the game begins, $(\beta_1, \ldots, \beta_n) = (1/n, \ldots, 1/n)$, reflecting symmetry in the initial defector status.[2] In any period when no player defects $(\beta_1, \ldots, \beta_n)$ is unchanged. If player i defects and $\beta_i < 1$, then β_i is increased and the remaining positive β_j ($j \neq i$) are decreased in equal proportions to preserve the total of unity. A maximal deviation by player i will cause $\beta_i = 1$ and $\beta_j = 0$, $j \neq i$. Similar to Fudenberg and Maskin, if $\beta_0 = 0$ and $\beta_i = 1$ for some player i, then player i is being held to his minimax payoff. As with our earlier formulation, if $\beta_0 < 1$ and players follow their equilibrium strategies, then β_0 will rise over time, converging to $\beta_0 = 1$ in the limit.

We thus work with a reference point that combines features found in Fudenberg and Maskin (1986) and Friedman and Samuelson (1990a). This allows us to combine characteristics of the Fudenberg and Maskin strategies, supporting individually rational payoffs, with characteristics of the Friedman and Samuelson strategies, yielding equilibria in continuous strategies. The result is an extension of the folk theorem to subgame perfect continuous strategies.

The sense in which our equilibrium strategies are continuous deserves attention. In Friedman and Samuelson (1990a) the decision rule of a player i selected the period t action (s_{it}) as a continuous function of either the action combination of the previous period (s_{t-1}) or the action combination and the reference point (s_{t-1}, β_{t-1}). The equilibrium contruction is illustrated in the payoff space shown in Figure 1. Equilibrium behavior called for selecting s^* at $t = 0$ (for a payoff of $P(s^*)$) and at later times, given that no defections had occurred.

(place Figure 1 about here)

Defections then called for equilibrium choices, parameterized by β, on the line connecting single shot Nash payoffs $P(s^c)$ and $P(s^*)$. The smaller the defection the nearer the indicated point would be to $P(s^*)$.

Not all games permit such decision rules, as Figure 2 illustrates. The results of

(place Figure 2 about here)

Friedman and Samuelson (1990a) require that the connecting path from $P(s^c)$ to $P(s^*)$ must be upward sloping (though it need not be linear). The points $P(s^c)$ and either C or A in Figure 2 cannot be connected by an upward sloping path contained in the set of feasible payoffs. We have been unable to develop general conditions under which such a connecting path can be constructed. We propose two routes around the problem.[3] One route, followed in Friedman and Samuelson (1990b), is to investigate particular classes of games, such as duopolies, in which the problem can be shown to be absent. The other route, followed below, is to adopt the standard convention of allowing the players to choose correlated mixed actions (cf. Fudenberg and Maskin (1986)). In terms of Figure 1, this would give β_t the role of a probability distribution. When following their equilibrium decision rules, the players would choose a correlated mixed action placing probability β_t on s^* and $1 - \beta_t$ on s^c. This paper uses correlated mixtures to develop a class of equilibria which is more general than the type illustrated in Figure 1 and under which virtually any individually rational single shot payoff vector can be supported.

The remainder of the paper is divided into four sections. The model is described in Section 2. The main theorem, in which the players are allowed to choose correlated mixed actions, is presented in Section 3. Section 4 examines cases in which the folk theorem can be achieved without correlated mixed actions. Concluding comments are in Section 5.

2. The Model

2.1 *The Single-Shot Game*

The single-shot game is characterized by (N, S, P), where $N = \{1,..., n\}$ is the set of players, S_i is the single-period pure strategy space of player i (the set of actions available to player i in any period), $S = \times_{i \in N} S_i$ is the single-period pure action space, P_i is the single-period payoff function of player i, and $P = (P_1,..., P_n)$. A pure action combination for the players in

$N\backslash\{i\}$, the n - 1 players other than player i, is denoted $s_{N\backslash\{i\}}$. The corresponding action space is denoted $S_{N\backslash\{i\}} \equiv \times_{j\in N\backslash\{i\}} S_j$. We make the following common assumptions:

ASSUMPTION 1 N is finite.

ASSUMPTION 2 $S_i \subset \mathbf{R}^m$ is compact and convex, $i \in N$.

ASSUMPTION 3 P_i is scalar valued and continuous on S, $i \in N$.

We do not make the usual quasiconcavity assumption, which is used to ensure the existence of a noncooperative equilibrium in the single-shot game, because this equilibrium plays no role in our analysis. The critical punishment payoffs are *minimax* rather than equilibrium payoffs.

2.2 *The Infinitely Repeated Game*

The infinitely repeated game can now be formally expressed. Although the equilibrium we examine has a stationary character, the players can use a succession of different decision rules over time and strategy spaces must be formulated to take this into account. In particular, for $t \geq 1$, player i can use any rule from the set $V_{it} = \{v_{it} \mid v_{it}: S^t \to S_i\}$. As we assume *perfect monitoring* (i.e., that each player knows at time t the actions taken by all players in the past), the history of the game at time t, $h_t = (s_0,..., s_{t-1}) \in S^t$, is known to each player who is then free to select any decision rule from V_{it}. A typical strategy in the repeated game is then $\sigma_i = (s_{i0}, v_{i1}, v_{i2},...) \in S_i \times \Pi_{t=1}^{\infty} V_{it} \equiv \Sigma_i$. Letting $\sigma = (\sigma_1,..., \sigma_n)$, $V_t = \times_{i\in N} V_{it}$, and $\Sigma = \times_{i\in N} \Sigma_i$, note that a strategy combination, $\sigma \in \Sigma$, induces a specific path of action combinations: $u(\sigma) = (u_0(\sigma), u_1(\sigma),...)$ where $u_0(\sigma) \equiv s_0$, $u_1(\sigma) \equiv v_1(s_0) = v_1(u_0(\sigma))$, $u_2(\sigma) \equiv v_2(u_0(\sigma), u_1(\sigma))$, and, in general $u_t(\sigma) \equiv v_t(u_0(\sigma),\ldots, u_{t-1}(\sigma))$.

Letting $\alpha = (\alpha_1,\ldots, \alpha_n) \in (0, 1)^n$ be the players' discount parameters, the repeated game payoff functions are

$$G_i(\sigma) = \sum_{t=0}^{\infty}\alpha_i^t P_i(u_t(\sigma)) \tag{1}$$

Letting $G = (G_1,\ldots, G_n)$, the game is then given by (N, Σ, G).

2.3 *Minimax Payoffs*

The minimax payoff of player i is defined as $\min_{s_{N\setminus\{i\}}\in S_{N\setminus\{i\}}}\max_{s_i\in S_i}P_i(s_{N\setminus\{i\}},s_i)$.[4] Let s^i be the action combination that minimaxes player i, so that $v_i = P_i(s^i)$ and $v = (v_1,\ldots, v_n)$. Let $P(s^i) = (y_1^i,\ldots, y_{i-1}^i, v_i, y_{i+1}^i,\ldots,y_n^i) \equiv y^i$, so that $P(s^i)$ is the vector of payoffs when player i is minimaxed. Note that we do not know, in general, how a given y_i^j (the payoff player i receives when player j is being minimaxed) is related to v_i. When minimaxing player j, player i could receive a payoff either higher or lower than v_i.

2.4 *Reference Points*

Let $\Delta_n = \{x \in \mathbf{R}_+^n \mid \sum_{i=1}^n x_i = 1\}$ be the unit simplex in $\mathbf{R}^n$. The reference point is then defined to be $\beta = (\beta_0, \beta_1,\ldots, \beta_n) \in [0,1]\times\Delta_n \equiv Y$. A payoff vector is associated with each value of the reference point. The payoffs associated with various reference points can be described by a family of line segments in payoff space, one for each n-vector $(\beta_1,\ldots, \beta_n)$. These line segments are intimately related to the minimax payoffs of the players and to the payoffs that are sustained when there is no defection.

It is helpful to intuitively sketch this relationship. Let $v \ll x^*$ and $\epsilon > 0$. For each $i \in N$ let $\omega^i \in S$ satisfy

$$P(\omega^i) = (x_1^* + \epsilon,\ldots, x_{i-1}^* + \epsilon, x_i^*, x_{i+1}^* + \epsilon,\ldots, x_n^* + \epsilon) \equiv x^i.$$

Then the expected payoff vector supported by *cooperative* behavior (i. e., $\beta_0 = 1$) is $\Sigma_{i=1}^n \beta_i P(\omega^i)$, under which player i receives $x_i^* + (1 - \beta_i)\epsilon$. Thus if player i has zero defector status ($\beta_i = 0$) he gets ϵ more than if he is the sole defector ($\beta_i = 1$). At the other extreme, if zero cooperation (i. e., $\beta_0 = 1$) is called for, then the expected payoff vector is $\Sigma_{i=1}^n \beta_i P(s^i)$. Figure 3

(place Figure 3 here)

provides an illustration. Let $\beta_1 = .8$ and $\beta_2 = .2$. If $\beta_0 = 1$, then expected payoffs are at B; however, if $\beta_0 = 0$ expected payoffs are at A. The broken line from A to B traces out the expected payoff points as β_0 goes from zero to one. At C, $\beta_0 = .6$. More generally, given $\beta \in Y$, the players adopt a correlated action combination, denoted $\zeta(\beta)$, under which they play ω^i with probability $\beta_0\beta_i$ and s^i with probability $(1 - \beta_0)\beta_i$ for each $i \in N$. Given β, the expected payoff vector is $\Sigma_{i=1}^n \beta_i[\beta_0 P(\omega^i) + (1 - \beta_0)P(s^i)]$. For example at C in Figure 3 the expected payoff to player 1 is $.8(.6P(\omega^1)) + .8(.4P(s^1)) + .2(.6P(\omega^2)) + .2(.4P(s^2))$.

3. The Continuous, Subgame Perfect Folk Theorem

3.1 *Strategies*

Because each value of the reference point designates a correlated mixed action and has an associated expected payoff vector, specifying strategies is accomplished by specifying how the reference point is determined. We begin by letting the initial value of the reference point be given by $\beta_0 \equiv (1, 1/n,\ldots, 1/n)$. Then the transition of β_t into β_{t+1} can be specified. If no defection took place at time t, then β_{1t} through β_{nt} are unchanged, meaning that the defector status of the players is unchanged. (I. e., $\beta_{i,t+1} = \beta_{it}$ for $i \in N$.) If $\beta_{0t} < 1$, then $\beta_{0,t+1} = \mu\beta_{0t} + 1 - \mu$ for $\mu \in (0,1)$, which is nearer to one. If there was defection in period t, then $\beta_{i,t+1} \geq \beta_{it}$ for the defectors; while, for the nondefectors, $\beta_{i,t+1} \leq \beta_{it}$. As long as $\beta_{jt} > 0$ for some nondefectors, strict inequality will hold for such nondefectors and for all defectors. Furthermore, the β_i of the nondefectors will be reduced in equal proportions. Also, $\beta_{0,t+1}$ will be smaller than it would have been in the absence of a defection.

To see the preceding in detail, let $P_i^D(s) = \max_{u_i \in S_i} P_i(s \backslash u_i)$ denote the maximum deviation payoff for player i from s and let $c(\beta_t)$ denote the realization at time t of the random correlated action mechanism (thus $c(\beta_t) \in \{s^1,..., s^n, \omega^1,..., \omega^n\} \equiv S^c$). Then z_{it}, defined below, is the normalized extra deviation payoff received by player i if he deviates form $c(\beta_t)$ by choosing some $s_{it} \neq c(\beta_t)$:

$$z_{it} = \max\left\{\frac{P_i(c(\beta_t)\backslash s_{it}) - P_i(c(\beta_t))}{P_i^D(c(\beta_t)) - P_i(c(\beta_t))}, 0\right\} \tag{2}$$

Note that z_{it} is normalized to measure extra payoff as a fraction of the largest possible extra deviation payoff that the player could have and, if s_{it} reduces the payoff of player i, the extra deviation payoff is taken to be zero. Let

$$z_t^0 = \sum_{i=1}^n z_{it},\ z_t^M = \max\{1, z_t^0\}$$
$$z_t^m = \min\{1, z_t^0\}.$$

Eqs. (3) and (4) below give the transition from β_t to β_{t+1}.

$$\beta_{i,t+1} = (1 - z_t^m)\beta_{it} + z_{it}/z_t^M,\ i \in N \tag{3}$$

$$\beta_{0,t+1} = \max\{0, \mu\beta_{0t} - z_t^0 + 1 - \mu\} \tag{4}$$

Eq. (3) has the desired properties that (1) a player's defector status must be nonnegative, (2) the sum over players is one, and (3) if one or more players defect when the defectors' combined defector statuses are less than one, then their combined defector status rises while that of each nondefector falls to a fraction of its previous value. Eq. (4) assures that the level of cooperation is between zero and one and that, following defection, the level of cooperation is lower than it would have been in the absence of defection. When there is exactly one defector, then equations (3) and (4) become

$$\beta_{i,t+1} = (1 - z_{it})\beta_{it} + z_{it} \text{ and } \beta_{0,t+1} = \max\{0, \mu\beta_{0t} - z_{it} + 1 - \mu\}. \quad (5)$$

Eqs. (2) to (4), expressing the mapping of (β_t, s_t) to β_{t+1}, define a function $\psi: Y \times S \to Y$.

LEMMA 1 The function ψ, defined by eqs. (2) to (4), is continuous.

PROOF Each z_{it} is continuous in (β_t, s_t); $\beta_{i,t+1}$ is continuous in $(\beta_{it}, z_{1t}, \ldots, z_{nt})$, $i \in N$; and $\beta_{0,t+1}$ is continuous in $(\beta_{0t}, z_{1t}, \ldots, z_{nt})$. □

We now construct equilibrium strategies. Using ψ, the equilibrium decision rule for player i is to choose an action in time t from the set $S_i^\zeta = \{\omega_i^1, \ldots, \omega_i^n, s_i^1, \ldots, s_i^n\}$ according to a *commonly observed* random mechanism with probability distribution $\zeta_i(\beta_t) = (\beta_{0t}\beta_{1t}, \ldots, \beta_{0t}\beta_{nt}, (1 - \beta_{0t})\beta_{1t}, \ldots, (1 - \beta_{0t})\beta_{nt})$ where $\beta_{jt} = \psi_j(\beta_{t-1}, s_{t-1})$, $j = 0, \ldots, n$. Thus $\zeta_i = \zeta_j$ for all players i and j; the players use the same mixed action and they draw their actions by observing the same random variable. This decision rule, along with the period zero action of player i, is the (proposed) equilibrium strategy of the player. Thus the equilibrium strategy of player i is $\sigma_i^* = (\zeta_i(\beta_0), \zeta_i(\psi(\cdot)))$ and the equilibrium strategy combination is denoted σ^*.

3.2 *Equilibrium*

These strategies provide a continuous folk theorem:

THEOREM 1. Under Assumptions 1 to 3, if $x^* >> v$ and an ϵ-neighborhood[5] of x^* is contained in the attainable payoff space of (N, S, P), there exist values of $(\alpha, \mu) \in (0, 1)^{n+1}$ such that σ^* is a continuous subgame perfect equilibrium strategy combination for the repeated game yielding an equilibrium payoff to player i in each period of $x_i^* + (1-1/n)\epsilon$.

PROOF The sense in which the strategies of the players are continuous is that each ζ_i is continuous in β_t and ψ is continuous in (β_{t-1}, s_{t-1}). It remains to show that the strategy combination σ^* is a subgame perfect equilibrium.

Let $\Pi_i = \max_{s \in S} P_i(s)$ denote the global maximum payoff to player i. Then the maximum payoff to a player is bounded above by $\Pi_i - \beta_i(1 - \beta_0)(\Pi_i - v_i)$ for $\beta_i \in [0, 1]$.

Now consider a deviation of relative size $z_{it} = z$ by a single player, i, when the reference point is β. To simplify notation suppose that

$$y = \Sigma_{j \neq i} \beta_j P_i(s^j) / \Sigma_{j \neq i} \beta_j,$$

$$w = \beta_i v_i + \Sigma_{j \neq i} \beta_j P_i(s^j) = \beta_i v_i \; (1 - \beta_i) y$$

$$x = x_i^* + (1 - \beta_i)\epsilon,$$

$$w' = [\beta_i + z(1 - \beta_i)]v_i + (1 - \beta_i)(1 - z)y = w - (1 - \beta_i)z(y - v_i),$$

$$x' = x_i^* + (1 - \beta_i)(1 - z)\epsilon = x - (1 - \beta_i)z\epsilon,$$

Thus for fixed β_j, $j \in N$, w is the expected (one period) payoff to player i at $\beta_0 = 0$ and x is his expected payoff at $\beta_0 = 1$. Using the expression for y, $w = \beta_i v_i + (1 - \beta_i)y$. Following the small deviation by player i of size z, β_i rises and the other β_j ($j \neq 0$, i) fall. The respective counterparts of w and x, after β changes due to the deviation, are w' and x'. The payoff for player i with $z = 0$ (i.e., if he followed the equilibrium prescription and did not deviate) is

$$P_i(c(\beta_0)) + \Sigma_{t=1}^{\infty} \alpha_i^t [(1 - \mu^t(1 - \beta_0))x + \mu^t(1 - \beta_0)w]$$

$$= P_i(c(\beta_0)) + \left[\frac{\alpha_i}{1 - \alpha_i} - \frac{\alpha_i \mu (1 - \beta_0)}{1 - \alpha_i \mu}\right] x + \frac{\alpha_i \mu (1 - \beta_0)}{1 - \alpha_i \mu} w \tag{6}$$

and the maximum payoff player i could receive as a result of deviating is

$$z[\Pi_i - \beta_i(1 - \beta_0)(\Pi_i - v_i)] + (1 - z)P_i(c(\beta_0)) +$$

$$\left[\frac{\alpha_i}{1 - \alpha_i} - \frac{\alpha_i(\mu - \mu\beta_0 + z)}{1 - \alpha_i\mu}\right]x' + \frac{\alpha_i(\mu - \mu\beta_0 + z)}{1 - \alpha_i\mu}w' \tag{7}$$

The following inequality expresses the condition that eq. (6) exceeds eq. (7) (i.e., deviation payoff is less than nondeviation payoff):

$$zP_i(c(\beta_0)) - z[\Pi_i - \beta_i(1 - \beta_0)(\Pi_i - v_i)] - \frac{\alpha_i z}{1-\alpha_i\mu}(w' - x')$$

$$+ \frac{\alpha_i\mu(1 - \beta_0)(w - w')}{1-\alpha_i\mu} + \left[\frac{\alpha_i}{1-\alpha_i} - \frac{\alpha_i\mu(1 - \beta_0)}{1-\alpha_i\mu}\right][x - x'] > 0 \tag{8}$$

To show conditions under which eq. (8) holds, substitute into eq. (8) for x, x', w, and w'. It is then seen that all terms are multiplied by z, which is always positive. Divide by z and separate terms into those multiplied by β_i and those multiplied by $(1 - \beta_i)$. If each of these two groupings is positive then eq. (8) holds. Eq. (9) shows the β_i grouping with the terms rearranged and eq. (10) shows a rearrangement of the $(1 - \beta_i)$ grouping. That the two groupings are positive is equivalent to eqs. (9) and (10) holding; therefore, eqs. (9) and (10) imply eq. (8).

$$\Pi_i - P_i(c(\beta_0)) - (1 - \beta_0)(\Pi_i - v_i) < \frac{\alpha_i}{1 - \alpha_i\mu}(x_i^* - v_i) \tag{9}$$

$$\Pi_i - P_i(c(\beta_0)) - \frac{\alpha_i[(\mu - \mu\beta_0 + z)(y - v_i - \epsilon) + x_i^* - y + \epsilon)}{1 - \alpha_i\mu} < \frac{\alpha_i\epsilon}{1 - \alpha_i} \tag{10}$$

The left side of eq. (9) is bounded while the right side goes to infinity as (α_i, μ) goes to (1, 1); therefore, there is a set of values of $(\alpha_i, \mu) \in (0, 1)^2$ for which eq. (9) holds. In addition, if eq. (9) holds for $(\alpha_i, \mu) \in (0, 1)^2$ and $(\alpha_i', \mu') \in (0, 1)^2$ satisfies $(\alpha_i', \mu') \geq (\alpha_i, \mu)$, then eq. (9) holds for (α_i', μ'). Denote by $A' \subset (0, 1)^{n+1}$ the set of values of (α, μ) for which eq. (9) holds for all $i \in N$.

To see that eq. (10) holds for some $(\alpha_i, \mu) \in (0, 1)^2$ note, first that the left side of eq. (10) is bounded above by $a(\mu)$ for any given value of $\mu \in (0, 1)$. Let $A'' \subset (0, 1)^{n+1}$ be the set of values of (α, μ) for which eq. (10) holds for all $i \in N$. Clearly A" is not empty because, for given μ, the left side is bounded by $a(\mu)$ irrespective of α_i while the right side goes to infinity as α_i goes to one. Finally, let $A = A' \cap A''$. Then A is clearly nonempty and eq. (8) holds for any $(\alpha, \mu) \in A$, proving the theorem.□

To provide some idea of what can happen in this framework, consider Figures 4 and 5. Figure 4 is the situation that is intuitively expected. The payoff to both players is lower when

(place Figures 4 and 5 here)

one is being minimaxed ($\beta_0 = 0$) than when $\beta_0 = 1$, irrespective of the values of β_1 and β_2. To see how the game proceeds, suppose that $\beta_1 = 1$ and $\beta_2 = 0$, so that player 1 is the defector. Then, depending on the value of β_0 the players will randomize between w^1 and s^1, with the expected payoff lying on the edge going from $P(s^1)$ to $P(\omega^1)$. If $\beta_0 = 1$, then "full cooperation" is in effect, given the relative defector status of the players, and play will proceed at w^1 for payofffs of $P(w^1)$. If $\beta_0 = 0$ the players will receive payoffs of $P(s^i)$ and if they continue to play with no defections, the expected payoff will proceed on the straight line to $P(\omega^1)$.

Now suppose $\beta_1 = 2/3$ and $\beta_2 = 1/3$. The expected payoff would then be somewhere on the broken line in Figure 4 running from A to B. The closer is β_0 to unity, the closer will the expected payoffs to A.

Next consider Figure 5. Here, the actions required to minimax player 1 provide player

2 with a higher payoff than player 2 receives at ω^2. To see the potential difficulties which then arise, suppose that $\beta_1 = 1$ and that the players are currently randomizing between ω^1 and s^1 so as to give an expected payoff at A. If player 2 should defect, then β_0 decreases, signaling a reduced degree of cooperation and β_2 increases, signaling an increase in player 2's defector status. It is possible that the net effect of these two changes is to move the game to a point such as B, with 2's payoff rising. It is thus possible for β_0 to fall and β_2 to rise, as shown, with the ensuing payoff to player 2 still going up in the "punishment" phase as a consequence of his defection. It is obvious that such a "punishment" will not deter player 2 from deviating. To constitute an equilibrium, the strategies must be devised so that the payoff to a defector falls after any defection. In Figure 5, this must be accomplished by ensuring that, in the event of a deviation by player 2, β_2 rises sufficiently rapidly relative to the fall in β_0. Much of the proof of Theorem 1 is concerned with establishing this property.

4. Uncorrelated Strategies

The strategies used in the proof of Theorem 1 are constructed with the help of the assumption that players can correlate their mixed strategies. In some games, this may not be required.

Fix an outcome x^* and fix ϵ. Recalling the definitions of ω^i and s^i in (2.4), let

$$H(x^*, \epsilon) = \{x \in \mathbf{R}^n | x = \textstyle\sum_{i=1}^n \beta_i[\beta_0 P(\omega^i) + (1 - \beta_0)P(s^i)], \beta \in Y\}.$$

$H(x^*, \epsilon)$ is thus the set of payoff vectors that could be chosen by the strategy combinations given in (5). Let $H^\beta \in \mathbf{R}^{n+1}$ be the set of n + 1 tuples, (x^*,ϵ), such that $x^* >> v$ and such that there is a connected subset of S, denoted $S(x^*, \epsilon)$, that P maps one-to-one onto $H(x^*, \epsilon)$. Hence, if $(x^*, \epsilon) \in H^\beta$, then every payoff in $H(x^*, \epsilon)$ can be achieved by a pure strategy in

$S(x^*, \epsilon)$. Let $\lambda(\beta) = (\lambda_1(\beta),\dots, \lambda_n(\beta)) \in S(x^*, \epsilon)$ denote the action combination that achieves the payoff vector $\beta \in H(x^*, \epsilon)$. Thus, for $\beta \in Y$, $\lambda(\beta) \in S(x^*, \epsilon)$ satisfies

$$P(\lambda(\beta)) = \Sigma_{i=1}^{n}\beta_i[\beta_0 P(\omega^i) + (1 - \beta_0)P(s^i)] \tag{11}$$

Then let equilibrium strategies be given by

$$\sigma_i^* = (\lambda_i(\beta_0), \zeta_i) \tag{12}$$

where

$$\zeta_i(\beta_t) = \lambda_i(\beta_t) = \lambda_i(\psi(\beta_{t-1}, s_{t-1})). \tag{13}$$

The function λ plays a central role in the construction of the equilibrium strategies and important properties of λ are established in the following lemma.

LEMMA 2 Let $(x^*, \epsilon) \in H^\beta$ and, for $\beta \in Y$, let $\lambda(\beta) \in S(x^*, \epsilon)$ be defined by eq. (11). Then λ is defined, single-valued, and continuous.

PROOF By construction, for each $\beta \in Y$ there is a unique $s \in S(x^*, \epsilon)$ that satisfies eq. (11); thus λ is defined on Y and is single-valued. Continuity of P, connectedness of $S(x^*, \epsilon)$, and $S(x^*, \epsilon)$ being one-to-one onto $H(x^*, \epsilon)$ imply that λ is continuous. □

The statement of the folk theorem extension is now:

THEOREM 2. Under Assumptions 1 to 3, and given $\beta_0 \in Y$, if $(x^*, \epsilon) \in H^\beta$, then there exist values of $(\alpha, \mu) \in (0, 1)^{n+1}$ such that σ^* is a continuous subgame perfect equilibrium strategy combination for the repeated game.

To prove Theorem 2, replace the definition of z_{it} given in eq. (2) by

$$z_{it} = \max\left\{\frac{P_i(\lambda(\beta_t)\backslash s_{it}) - P_i(\lambda(\beta_t))}{P_i^D(\lambda(\beta_t)) - P_i(\lambda(\beta_t))}, 0\right\} \tag{14}$$

and replace $P_i(c(\beta_0))$ in eqs. (6) to (10) with $P_i(\lambda(\beta_0))$. The proof of Theorem 2 then precisely matches that of Theorem 1. Now, however, movements in payoff space are accomplished not by varying the probabilities in a joint mixture over fixed strategies but by changing pure strategies.

The question now naturally arises as to when H^β will be nonempty, so that Theorem 2 is not vacuous. H^β will be nonempty if (1) each payoff function P_i is quasiconcave on S, (2) P^{-1} is single-valued for all x in the range of P, and (3) v is interior to the payoff space. Condition (1) assures that all payoffs in the set $H(x^*, \epsilon)$ are attainable, condition (2) assures that there is a connected subset of S that maps one-to-one onto $H(x^*, \epsilon)$, and condition (3) guarantees that nonempty sets $H(x^*, \epsilon)$ exist. These are clearly quite strong conditions It remains an open question as to what weaker assumptions might imply or be equivalent to the assumption that H^β is nonempty.

5. Concluding Comments

Theorems 1 and 2 present variant versions of the Fudenberg and Maskin (1986) extension of the folk theorem for repeated games. Similar to Fudenberg and Maskin, we give sufficient conditions for the existence of subgame perfect equilibria that support virtually any individually rational payoff vector of the single shot game. Our equilibria differ in two key ways. First, the supported points $(x_1^* + (1 - 1/n)\epsilon, \ldots, x_n^* + (1 - 1/n)\epsilon)$ may not include points on the payoff frontier, although they can get arbitrarily close. Second, the equilibrium strategies utilize continuous decision rules.

Friedman and Samuelson (1990a) examined analogs of trigger strategy equilibria which potentially possessed a particular appeal as strategies that might arise without explicit coordination or preplay communication by the players. It is not so clear that the same justification holds in the present context, because the strategies we develop are much more complicated than their earlier counterparts. This is particularly true for those of Theorem 1, which rely on correlated mixed actions. Nonetheless, we believe the exploration of continuous strategies helps to round out the scope of the results in the *folk theorem* family.

Notes

1. Finite reversion trigger strategies require a reference point that indicates whether a defection has taken place within the preceding K periods where K is the number of punishment periods. Such a reference point would require more than two values; probably K + 1 from the set {0, 1,..., K} where the value indicates the number of punishment periods that remain.

2. Our results do not depend on the initial condition $\beta_i = 1/n$; however, a complete description of the game requires that initial values be specified.

3. Clearly, if each p_i is concave in s and P maps S one-to-one onto the range of P, our construction is assured. Such conditions are extremely restrictive; they even exclude Cournot duopoly with linear demand and constant marginal cost.

4. In defining v_i we implicitly assume N\{i} does not use mixed or correlated actions. If N\{i} did, then the v_i might be smaller, but there would be no material change in the results. The exposition is simpler this way. Correlated actions do play a critical role at another stage of the development, so we use them there.

5. This is our counterpart of the full-dimensionality requirement of Fudenberg and Maskin (1986).

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Fudenberg, Drew, and Maskin, Eric (1986). "The Folk Theorem in Repeated Games with Discounting and with Incomplete Information," *Econometrica, 54*, 533-554.

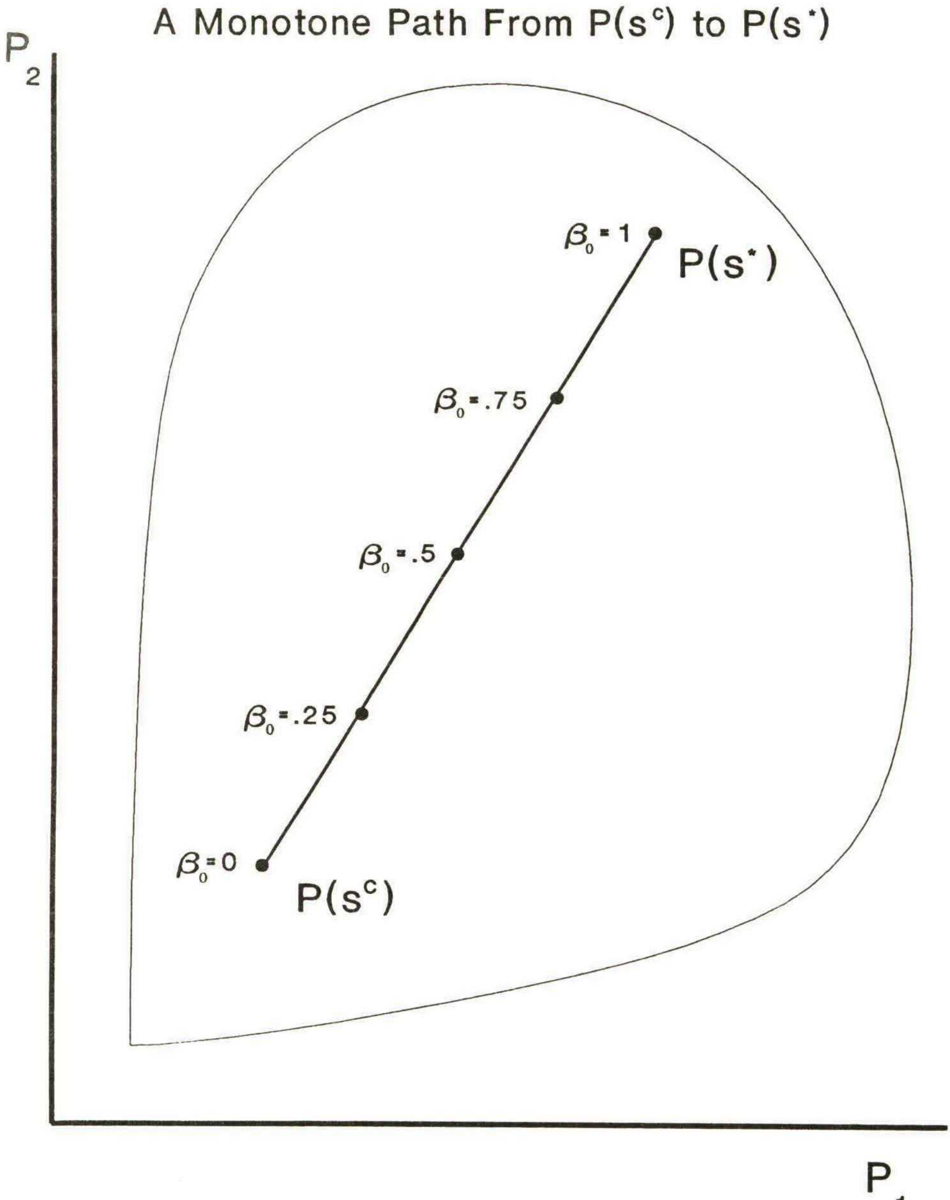
A Monotone Path From P(s^c) to P(s*)
P2
P1
β0 = 1
P(s*)
β0 = .75
β0 = .5
β0 = .25
β0 = 0
P(s^c)

Figure 1

A Payoff Space with "Holes"

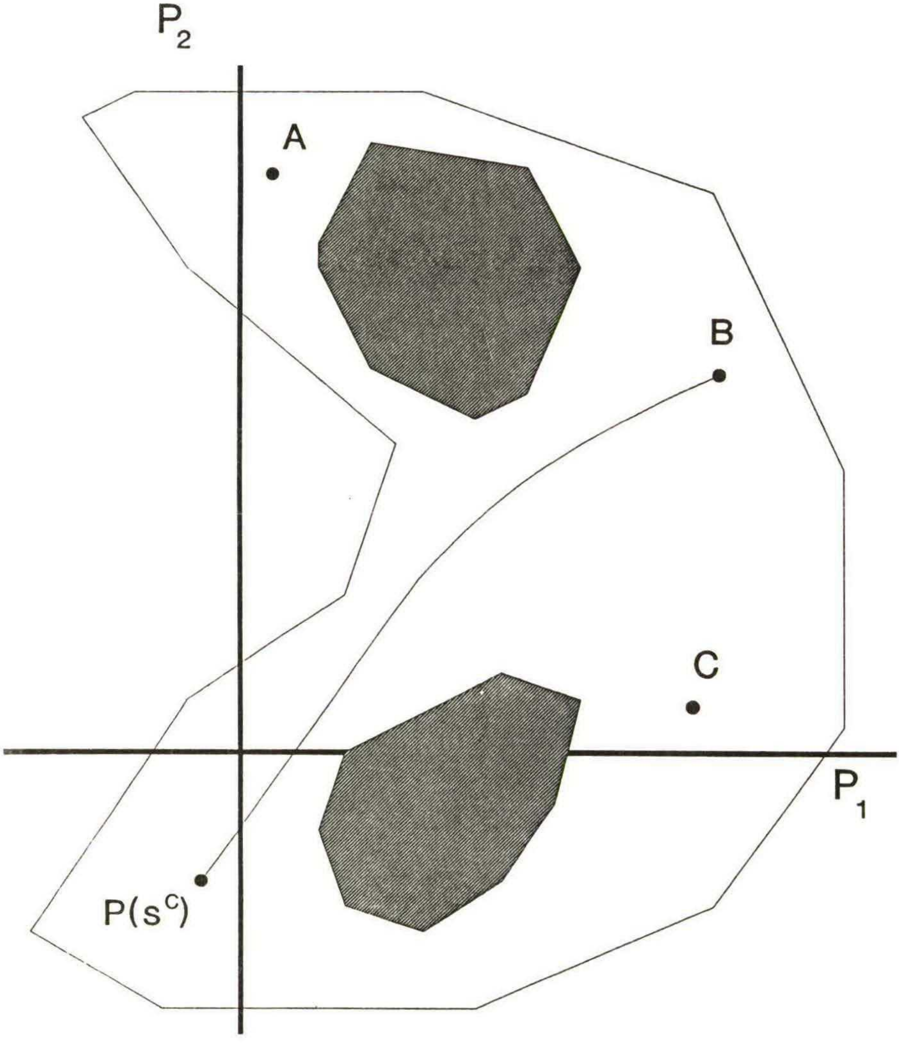

Figure 2

Payoff Space Illustration

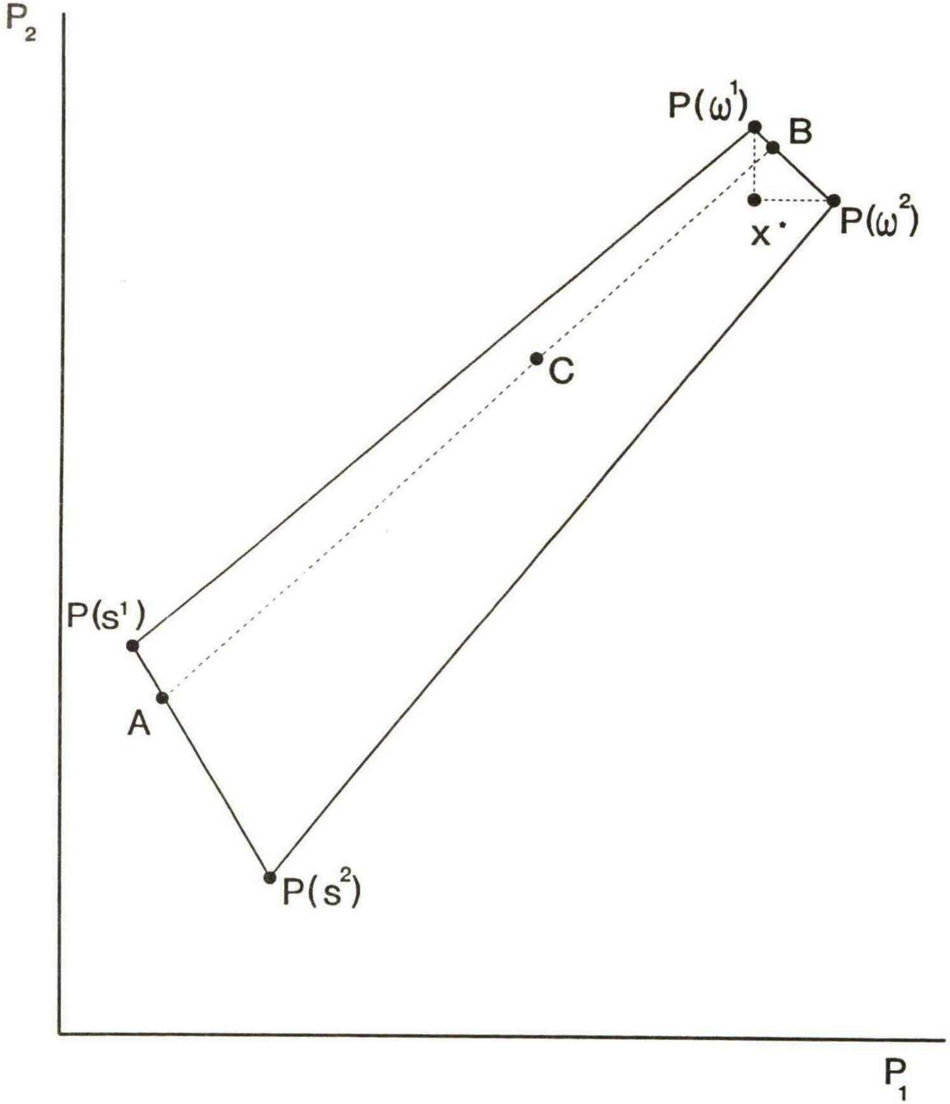

Figure 3

Minimax Payoffs Can Be Below Agreement Payoffs

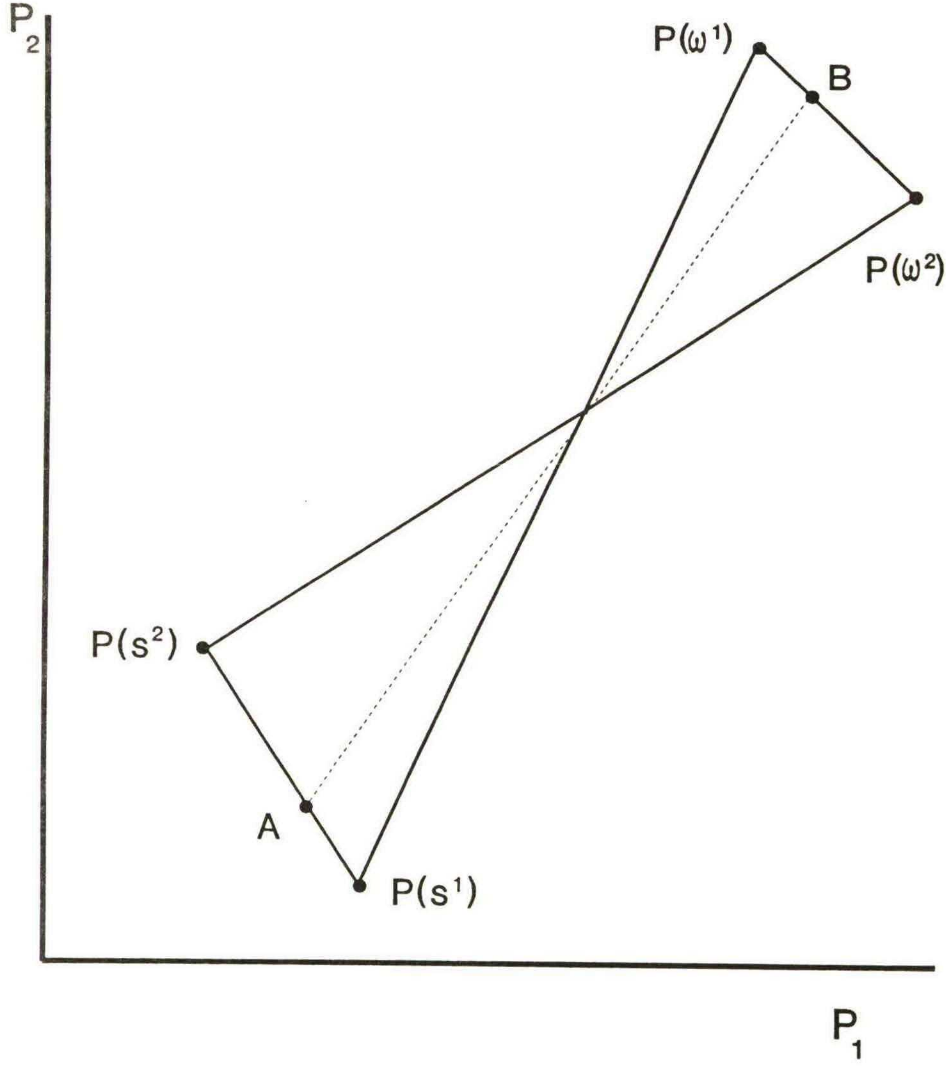

Figure 4

Minimax Payoffs Can Be Above Agreement Payoffs

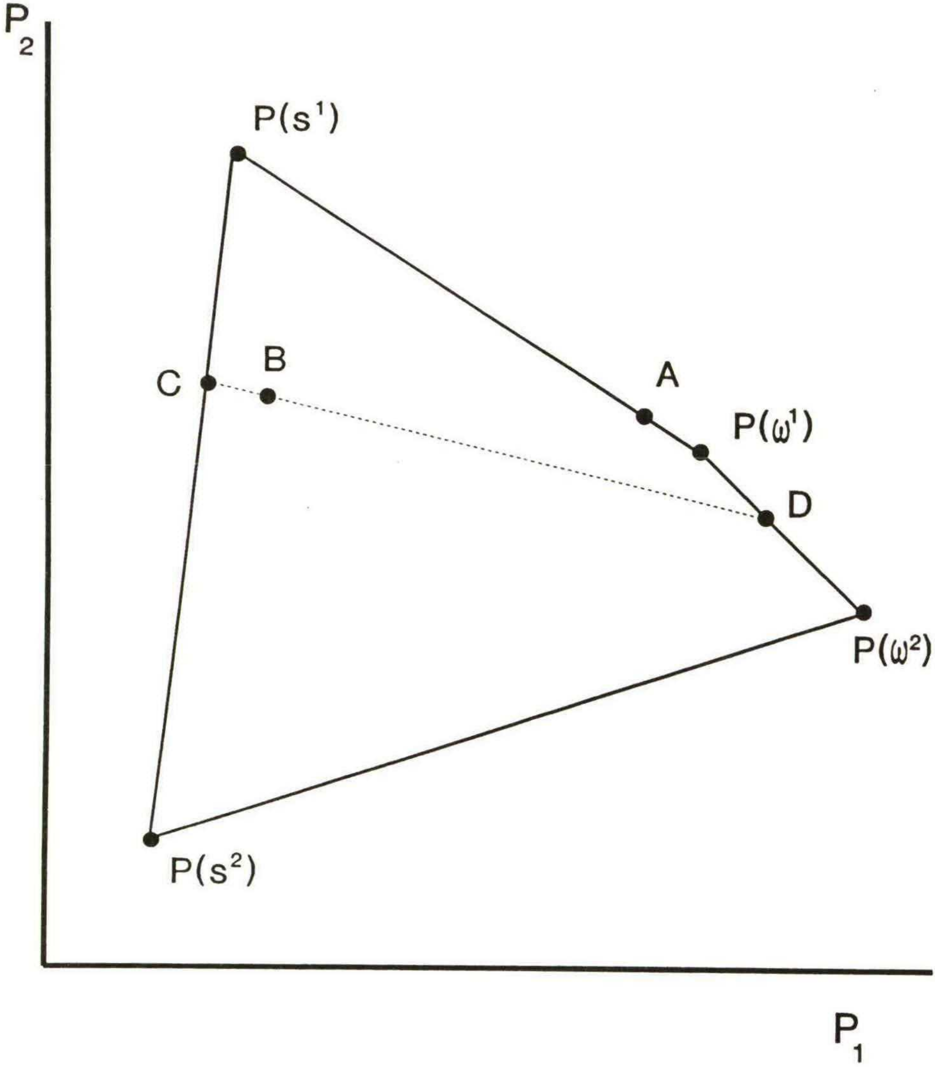

Figure 5

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