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Baye, M.R.; Tian, G.; Zhou, J.

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by Michael R. Baye,
Guoqiang Tian and Jianxin Zhou

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The Existence of Pure-Strategy Nash Equilibrium in Games with Payoffs that are not Quasiconcave*

Michael R. Baye[†]
Department of Economics
Texas A&M University
College Station, Texas 77843

Guoqiang Tian
Department of Economics
Texas A&M University
College Station, Texas 77843

Jianxin Zhou
Department of Mathematics
Texas A&M University
College Station, Texas 77843

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Abstract

This paper characterizes pure-strategy Nash equilibrium in noncooperative games. Conditions which are called 0-generalized quasiconcavity and uniform generalized quasiconcavity, together with some "regular" topological conditions, are shown to be necessary and sufficient for the existence of pure-strategy and dominant-strategy Nash equilibrium. We also provide theorems for existence under weakened topological conditions. Thus our results, which require neither the continuity nor quasiconcavity of individual utility functions, generalize many of the existence theorems on pure-strategy Nash equilibrium in the literature, including those of Nash (1950, 1951), Debreu (1952), Nikaido and Isoda (1955), and Dasgupta and Maskin (1986). **Keywords:** Pure-strategy, Nash Equilibrium, Existence.

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[†]Research Fellow, CentER for Economic Research, Tilburg University.

1 Introduction

Nash equilibrium is "the" equilibrium concept in economics. Its widespread appeal stems from the intuitively appealing notion that, if – given the behavior of other individuals – any individual could improve his well-being by altering his behavior, he would do so. For an economic steady-state to exist in the sense of Nash therefore means that no rational maximizing agent has an incentive to change his behavior, given the behavior of others.

Because of the importance of Nash equilibrium in the study of markets and other games, there has been continued interest in setting forth conditions for the existence of Nash equilibria. Unfortunately, existence theorems invariably set forth only *sufficient* conditions for the existence such an equilibrium. If the conditions of a particular theorem are satisfied, then one knows the particular model has an equilibrium. If the conditions of the theorem are not satisfied, the theorem is of little value; the game may or may not have a Nash equilibrium. It is for this reason that economists continually strive to weaken the conditions that guarantee the existence of Nash equilibrium.

Existence theorems are essentially characterized by the conditions placed on the strategy spaces and payoff functions of the players sufficient to establish a given characterization of equilibrium. The characterization of equilibrium may be the existence of a pure-strategy equilibrium, or a mixed-strategy equilibrium whereby individuals randomize over pure-strategies.

The table below summarizes several of the existence theorems used in economics. The early theorems of Debreu (1952) and Fan (1953) reveal that games possess a pure-strategy Nash equilibrium if (1) the strategy spaces of the agents are nonempty, convex and compact, and (2) players have continuous, quasiconcave utility functions. The theorems, including those of Nash (1950, 1951), say nothing about equilibrium in games with discontinuous payoffs. Accordingly, Dasgupta and Maskin (1986) were motivated to establish an existence theorem valid for discontinuous utility functions. Their results reveal that even games with discontinuous payoffs possess a pure-strategy equilibrium, provided (1) the strategy spaces of the agents are nonempty, convex and compact, and (2) players have utility functions that are quasiconcave, upper semi-continuous, and graph continuous.

	Strategy Space	Payoff Function	Type of Nash Equilibrium
Debreu (1952) Fan (1953)	Convex Compact	Quasiconcave Continuous	Pure-Strategy
Dasgupta-Maskin (Theorem 2, 1986)	Convex Compact	Quasiconcave Discontinuous	Pure-Strategy
Glicksberg (1952)	Compact	Continuous	Mixed-Strategy
Dasgupta-Maskin (Theorem 5*, 1986)	Convex Compact	Discontinuous	Mixed-Strategy

Based on the existing literature, the most that can be said about games with payoff functions that are not quasiconcave is that they posses a mixed-strategy equilibrium, provided that the conditions of either Glicksberg (1952) or Dasgupta-Maskin (1986, Theorem 5*) hold. Each these theorems are applicable only to games with compact strategy spaces. The Glicksberg theorem also requires continuity of the utility functions, while the Dasgupta-Maskin theorem relaxes continuity but requires a convex strategy space.

In summary, given state-of-the-art existence theorems, the most that can be said of games where players do not have quasiconcave utility functions is that they might possess a mixed-strategy equilibrium, provided the strategy spaces and utility functions satisfy some other conditions. Indeed, the central message of Dasgupta-Maskin is that discontinuities in the payoff functions are *not* responsible for the nonexistence of pure-strategy Nash equilibrium, but rather "the blame for nonexistence of pure-strategy equilibrium [is the] lack of quasiconcavity" (p. 3). We show that quasiconcavity can be weakened.

In Section 2 we introduce the basic terminology used in our study of noncooperative games, and then present in Section 3 examples of economic games that have a pure-strategy Nash equilibrium even though the utility functions do not satisfy the continuity and quasiconcavity conditions required by the aforementioned existence theorems. These examples are intended to motivate Section 4, which presents our main theorems on the existence of pure-strategy Nash equilibrium for games in which payoffs are neither quasiconcave nor continuous. The first part of this section deals with games in which strategy spaces are "regular" (eg. convex and compact), while the second part relaxes the assumed regularity conditions. Section 5 applies these theorems to the games used to motivate our analysis, in order to explain why the games possessed pure-strategy equilibria. The proofs of our main results, along with the required additional mathematical terminology and notation, are contained in Section 6. We offer some concluding remarks in Section 7.

2 Non-Cooperative Games

Let I be a countable (possibly infinite) set of players, and suppose that each agent i's strategy set is $Z_i \subset \Re^{L_i}$ (L_i is finite). Denote by Z the (Cartesian) product $\prod_{j \in I} Z_j$ and Z_{-i} the product $\prod_{j \in I \setminus \{i\}} Z_j$. Each player i has a payoff (utility) function u_i: $Z \rightarrow \Re$. Throughout our analysis, variables without subscripts, such as x and y, will be used to denote elements of Z. Subscripts on variables will associate the variable with a particular player or group of players. For example, x_i and y_i will be used to denote elements of Z_i, while x_{-i} and y_{-i} will be used to denote elements of Z_{-i}.

In quasi-games, there exist profiles of strategies that are not *socially feasible*, so let $A \subset Z$ denote the set of *socially feasible actions*. A *game* $\Gamma = (Z_i, A, u_i)_{i \in I}$ is simply a family of ordered triples (Z_i, A, u_i). We seek conditions under which the following types of equilibrium exist for Γ.

Definition 1 (Pure-Strategy Nash Equilibrium) *A pure-strategy Nash equilibrium for Γ is a $y^* \in A$ such that $u_i(y^*) \geq u_i(x_i, y^*_{-i})$ for all $x_i \in Z_i$ with $(x_i, y^*_{-i}) \in A$ and for all $i \in I$.*

Thus a Pure-strategy Nash equilibrium is a vector of actions such that no individual player has an incentive to change his behavior, given the actions of the other players. Clearly, if each player has an action that maximizes his utility for all possible actions of his opponents, then the resulting vector of "best actions" constitutes a Nash equilibrium. Such a vector of actions is termed a *dominant-strategy Nash equilibrium*. More formally,

Definition 2 (Dominant-Strategy Nash Equilibrium) *A dominant-strategy Nash equilibrium for* Γ *is a* $y^* \in A$ *such that for all* $i \in I$, $u_i(y_i^*, x_{-i}) \geq u_i(x_i, x_{-i})$ *for all* $(x_i, x_{-i}) \in A$ *with* $(y_i^*, x_{-i}) \in A$.

Nash (1951) and Debreu (1952) proved that a pure-strategy Nash equilibrium of a game exists if each $Z_i \subset \Re^{L_i}$ is compact, convex, and non-empty, and if u_i is continuous on $A = Z$ and quasiconcave in x_i. Dasgupta and Maskin (1986) extended their results to games where payoff functions are continuous in the following weakened sense:

Definition 3 (Semi-Continuity) *A function* $\phi : X \to \Re$ *is said to be* upper semi-continuous *if for each point* x', *we have*

$$\limsup_{x \to x'} \phi(x) \leq \phi(x'),$$

or equivalently, if $\{(x, a) \in X \times \Re : \phi(x) \geq a\}$ *is a closed subset of* $X \times \Re$. *A function* $\phi : X \to \Re$ *is said to be* lower semi-continuous *if* $-\phi(x)$ *is upper semi-continuous.*

More precisely, Dasgupta and Maskin's Theorem 2 establishes the existence of a pure-strategy Nash equilibrium for games in which each player's strategy set, $Z_i \subset \Re^{L_i}$, is compact, convex, and non-empty, and each player's utility function, $u_i(x_i, x_{-i})$, is quasiconcave in x_i, upper semi-continuous in x and graph-continuous[1]. In what follows we demonstrate that quasiconcavity and upper semi-continuity are not necessary for the existence of a pure-strategy Nash equilibrium. Then we

[1] Dasgupta and Maskin (1986) defined a payoff function to be graph-continuous if for all $\bar{x} \in Z$ there exists a function $F_i : Z_{-i} \to Z_i$ with $F_i(\bar{x}_{-i}) = \bar{x}_i$ such that $u_i(F_i(x_{-i}), x_{-i}))$ is continuous at $x_{-i} = \bar{x}_{-i}$.

present existence theorems based on conditions weaker than quasiconcavity and upper-semicontinuity.

3 Motivation and Examples

Before we present general theorems on the existence of pure-strategy Nash equilibrium in games where payoffs are neither continuous nor quasiconcave, it is instructive to present three examples of games that cannot be analyzed with available existence theorems. The utility functions in the first two examples violate quasiconcavity, while the those in the third violate upper semi-continuity.

Example 1 Consider a two-person game played on the unit square. Thus $Z_1 = Z_2 = [0,1]$. The payoffs $u_i(x_1, x_2)$ $(i = 1,2)$ are given by the functions

$$u_i(x_1,x_2) = \begin{cases} x_i & \text{if } x_i \leq x_{-i} \\ x_i - c & \text{if } x_{-i} < x_i < 1 \\ 1 & \text{if } x_i = 1 \end{cases} \tag{1}$$

for $c > 0$. It is easy to verify that each u_i is upper semi-continuous in x_i but not quasiconcave in x_i. Since the continuity assumptions required by the theorems of Debreu (1952) and Fan (1952) are not satisfied, and the quasiconcavity conditions of Dasgupta-Maskin (1986) are not satisfied, we cannot infer from their theorems that a pure-strategy Nash equilibrium exists. However, it is clear that $x_1 = x_2 = 1$ is a pure-strategy Nash equilibrium; in fact it constitutes a *dominant-strategy Nash equilibrium.*

The next game is a slight modification of the game in Example 1.

Example 2 Consider a two-person game played on the unit square. Thus $Z_1 = Z_2 = [0,1]$. The payoffs $u_i(x_1, x_2)$ $(i = 1,2)$ are given by the functions

$$u_i(x_1,x_2) = \begin{cases} x_i & \text{if } x_i \leq x_{-i} \\ x_i - c & \text{otherwise} \end{cases}, \tag{2}$$

where $1 > c > 0$. Again, these utilities are not quasiconcave, and one cannot use existing theorems to infer the existence of a pure-strategy Nash equilibrium. Unlike the game in Example 1, this game does not have a dominant-strategy Nash equilibrium. However, it does have a pure-strategy Nash equilibrium.[2] In particular, $x_1 = x_2 = 1$ constitutes a *pure-strategy Nash equilibrium.*

Example 3 Consider a two-person game played on the unit square. Thus $Z_1 = Z_2 = [0, 1]$. The payoffs $u_i(x_1, x_2)$ $(i = 1, 2)$ are given by the functions

$$u_i(x_1, x_2) = \begin{cases} \frac{1}{2} & \text{if } x_1 = x_2 = 0 \\ p_i(x_1, x_2) - x_i & \text{otherwise} \end{cases},$$

where $p_i(x_1, x_2) = \frac{x_i^\alpha}{x_1^\alpha + x_2^\alpha}$ and $\alpha > 0$. This game has been proposed by Tullock (1980) to model rent-seeking behavior; p_i is interpreted as the probability player i wins a prize worth \$1 by expending \$$x_i$ in resources. Baye, Kovenock, and de Vries (1989) have shown that the limit of this game as $\alpha \to \infty$ has the same essential structure as Moulin's (1986) all-pay-auction; Varian's (1980) model of sales; Narasimhan's (1988) model of promotional strategies; and Baye and de Vries' (1989) model of trade with brand-loyal consumers. The limit-game is known to have no pure-strategy Nash equilibria, but it does have a unique mixed-strategy equilibrium (see Baye, Kovenock, and de Vries (1990)).[3]

For the purpose of this example, however, suppose $0 < \alpha \leq 1$. Then it is easy to verify that the utility functions are quasiconcave but are not upper-semicontinuous.[4] Thus, the *sufficient* conditions for the existence of a pure-strategy Nash equilibrium set forth in the existing literature *are not satisfied.* However, it is easy to verify that the game has a pure-strategy Nash equilibrium, namely $x_1 = x_2 = \alpha/4$.

Since each of the above games have a pure-strategy Nash equilibrium that the existing theorems do not point out, one might conjecture that the theorems can be

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[2] In fact, it has a continuum of pure-strategy Nash equilibria.

[3] The theorems on mixed-strategy Nash equilibrium presented in Dasgupta and Maskin (1986) are motivated by games such as this limit game.

[4] The problem occurs when $x_i = 0$ for one of the players.

generalized. The next section reveals that this conjecture is indeed correct.

4 Nash Equilibrium with Non-quasiconcave Payoffs

Unless otherwise noted, we assume throughout the remainder of this paper that the topological spaces under consideration are Euclidian spaces, and furthermore, that $A = Z$.[5] In what follows we present theorems that give necessary and sufficient conditions for the existence of pure-strategy and dominant-strategy Nash equilibrium under different topological conditions. We first consider "regular" games, and then demonstrate that the assumed continuity, compactness, and convexity conditions can be mildly weakened. Since the proofs of the theorems presented in this section require additional mathematical baggage, we reserve the proofs for Section 6.

4.1 "Regular" Games

The above examples reveal that quasiconcavity is not essential for the existence of pure-strategy Nash equilibrium. Accordingly, we introduce the following concept.

Definition 4 (Uniform Generalized Quasiconcavity) *A payoff function u_i : $Z \to \Re$ is said to be* uniformly generalized quasiconcave *on Z if for every finite subset $\{x^1, x^2, \dots, x^m\} \subset Z$, there exists a corresponding finite subset $\{y_i^1, y_i^2, \dots, y_i^m\} \subset Z_i$ such that for any subset $\{y_i^{k^1}, y_i^{k^2}, \dots, y_i^{k^s}\} \subset \{y_i^1, y_i^2, \dots, y_i^m\}$, $1 \le s \le m$ and any $y_i^{k0} \in co\,\{y_i^{k^1}, y_i^{k^2}, \dots, y_i^{k^s}\}$ we have*

$$\min_{1 \le l \le s} [u_i(x_i^{k^l}, x_{-i}^{k^l}) - u_i(y_i^{k0}, x_{-i}^{k^l})] \le 0. \tag{3}$$

Note that a sufficient condition for u_i to be uniformly generalized quasiconcave is that player i's utility is independent of the strategies of other players. It is this observation that provides the principal intuition for the following theorem on the existence of dominant-strategy Nash equilibrium.

[5] However, all of our results hold for any Hausdorff topological vector spaces.

Theorem 1 *Suppose that, for each player i, the strategy space Z_i is a nonempty convex compact subset in $\Re^{L_i}$ and u_i: $Z \to \Re$ is a payoff function such that for every $x \in Z$, $u_i(x)$ is upper semi-continuous in x_i. Then Γ has a dominant-strategy Nash equilibrium $y^* \in Z$ if and only if u_i is uniformly generalized quasiconcave on Z for all $i \in I$.*

For games satisfying the "regularity" conditions of convex compact strategy spaces and upper semi-continuous utility functions, Theorem 1 gives necessary and sufficient conditions for the existence of a *dominant-strategy* Nash equilibrium. Since the set of games having a dominant-strategy Nash equilibrium is a proper subset of games having pure-strategy Nash equilibrium, one should be able to weaken the uniform generalized quasiconcavity requirement to obtain necessary and sufficient conditions for the existence of pure-strategy Nash equilibrium. Following Chang and Zhang (1989), we thus consider

Definition 5 (0-Generalized Quasiconcavity) *A function $\phi(x, y)$: $X \times Y \to \Re$ is said to be* 0-generalized quasiconcave *in x, if for any finite subset $\{x^1, \ldots, x^m\} \subset X$, there exists a corresponding finite subset $\{y^1, \ldots, y^m\} \subset Y$ such that for any subset $\{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\} \subset \{y^1, y^2, \ldots, y^m\}$, $1 \leq s \leq m$, and any $y^{k0} \in co\,\{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\}$ we have*

$$\min_{1 \leq l \leq s} \phi(x^{k^l}, y^{k0}) \leq 0.$$

Our next theorem applies this notion of quasiconcavity to the aggregator function, U: $Z \times Z \to \Re \cup \{\pm\infty\}$ given by

$$U(x, y) = \sum_{i \in I} \frac{1}{2^i} [u_i(x_i, y_{-i}) - u_i(y)], \tag{4}$$

for which $U(x, x) = 0$ for all $x \in Z$.

Note that our strategy for using an aggregator function to weaken the quasiconcavity conditions is very similar to that used by Dasgupta and Maskin to prove their Theorem 5*. Unlike their Theorem 5*, however, our choice of an aggregator function allows us to establish the existence of *pure-*, rather than *mixed*-strategy Nash equilibrium.

Theorem 2 *Suppose that the strategy space Z_i is a convex compact subset in $\Re^{L_i}$ (where $L = \sum_{i \in I} L_i$) and $U\colon Z \times Z \to \Re$ is defined by (4) such that for every $x \in Z$, $U(x, y)$ is lower semi-continuous in y. Then Γ has a Nash equilibrium if and only if $U(x, y)$ is 0-generalized quasiconcave in x.*

Note that a sufficient condition for the lower semi-continuity of $U(x, y)$ in y is that u_i is upper semi-continuous in y_i and continuous in y_{-i}.

The following is a direct consequence of Theorem 2.

Corollary 1 *Suppose Γ has a pure-strategy Nash equilibrium, x^*. Then $U(x,y)$ is 0-generalized quasiconcave in x.*

This follows by letting $y^k = x^*$ for $k = 1, 2, \ldots, m$ in the definition of 0-generalized quasiconcavity. Note that the contrapositive of the corollary states that if the aggregator function, $U(x, y)$, is not 0-generalized quasiconcave in x, then the game does not possess a pure-strategy Nash equilibrium. This statement is powerful because it is *not* predicated on individual utilities being (semi) continuous or strategy spaces being compact.

From the above theorems, we conclude that the 0-generalized quasiconcavity of $U(x, y)$ is a necessary and sufficient condition for the existence of Nash equilibrium for games satisfying "regular" topological conditions. It is thus useful to present conditions that imply 0-generalized quasiconcavity, and thus the existence of pure-strategy Nash equilibrium.

Fact 1 *Suppose that the strategy space Z is a convex subset in $\Re^L$. If u_i is uniformly generalized quasiconcave on Z for all $i \in I$, then $U(x, y)$ is 0-generalized quasiconcave in $x \in Z$.*

Fact 2 *For a two person game, if one player's utility function is uniformly generalized quasiconcave on Z, then $U(x, y)$ is 0-generalized quasiconcave in $x \in Z$.*

Fact 2 and Theorem 2 imply the following corollary:

Corollary 2 *For a two-person game, let the strategy space Z be a convex compact subset in $\Re^L$ and let $U: Z \times Z \to \Re$ be defined by (4) such that for every $x \in Z$, $U(x, y)$ is lower semi-continuous in $y \in Z$. If one player's utility function is uniformly generalized quasiconcave, then Γ has a Nash equilibrium.*

Finally, note that if the utility function of every player is continuous and quasiconcave in his own strategy, then there exists a Nash equilibrium by the result of Debreu (1952). Thus if u_i is continuous and quasiconcave in x_i for all $i \in I$, $U(x, y)$ is 0-generalized quasiconcave by Theorem 2.

4.2 "Irregular" Games

In this section we demonstrate that the above results can be extended to games where utilities do not satisfy (semi-) continuity conditions, and in which strategy spaces are neither convex nor compact. We first show that Theorem 1 can be generalized by relaxing the compactness of Z and the upper-semicontinuity of u_i.

Theorem 3 *For each player i, let the strategy space Z_i be a nonempty convex subset in $\Re^{L_i}$. If $u_i: Z \to \Re$ satisfies the following conditions:*

(a) for every $x \in Z$, if $u_i(x) > u_i(y_i, x_{-i})$, then there exist some point $x' \in Z$ and some neighborhood $\mathcal{N}(y_i)$ of y_i such that $u_i(x') > u_i(z_i, x'_{-i})$ for all $z_i \in \mathcal{N}(y_i)$;

(b) there exist $x^1, \ldots, x^n \in Z$ such that $\bigcap_{k=1}^n G_i(x^k)$ is compact on Z_i, where $G_i(x^k) = \{y_i \in Z : u_i(x^k) - u_i(y_i, x^k_{-i}) \leq 0\}$.

Then Γ has a dominant-strategy Nash equilibrium if and only if u_i is uniformly generalized quasiconcave on Z.

Similarly, we can weaken Theorem 2 by relaxing the compactness of Z and the lower-semicontinuity of $U(x, y)$.

Theorem 4 *Suppose that the strategy space Z is a nonempty convex subset of $\Re^L$ and $U: Z \times Z \to \Re$ is defined by (4) such that*

(a) *for every* $x \in Z$, *if* $U(x,y) > 0$, *then there exist some point* $x' \in Z$ *and some neighborhood* $\mathcal{N}(y)$ *of* y *such that* $U(x',z) > 0$ *for all* $z \in \mathcal{N}(y)$;

(b) *there exist* $x^1, \ldots, x^n \in Z$ *such that* $\bigcap_{k=1}^{n} G(x^k)$ *is compact on* Z, *where* $G(x) = \{y \in Z : U(x,y) \leq 0\}$.

Then Γ *has a Nash equilibrium if and only if* $U(x,y)$ *is 0-generalized quasiconcave in* x.

Finally, we can extend our results to games where the set of socially feasible actions (A) is a non-compact, non-convex subset of the Cartesian product of individual strategies (Z).

Theorem 5 *Let the strategy space* Z *be a nonempty convex subset in* $\Re^L$ *and let* $\emptyset \neq A \subset Z$. *Suppose that* U: $Z \times Z \to \Re$ *is defined by (4) such that*

(a) *for every* $x \in A$, *if* $U(x,y) > 0$, *then there exist some point* $x' \in A$ *and some neighborhood* $\mathcal{N}(y)$ *of* y *such that* $U(x',z) > 0$ *for all* $z \in \mathcal{N}(y)$;

(b) *there exist* $x^1, \ldots, x^n \in A$ *such that* $\bigcap_{k=1}^{n} G(x^k)$ *is compact on* Z, *where* $G(x) = \{y \in Z : U(x,y) \leq 0\}$.

(c) *for each* $y \in Z \backslash A$ *there exists* $x \in A$ *such that* $U(x,y) > 0$.

Then Γ *has a Nash equilibrium on* A *if and only if* $U(x,y)$ *is 0-generalized quasiconcave in* x *on* A.

5 Reconsidering the Examples

In this section we demonstrate that the above theorems reveal the existence of equilibrium in the games presented in Examples 1-3.

5.1 Example 1

Consider first the game presented in Example 1. As noted, u_i is upper semicontinuous in x_i but is not quasiconcave in x_i, so the existing results in the literature cannot be applied. However, we will establish the existence of a dominant-strategy equilibrium (and hence, a pure-strategy Nash equilibrium) by verifying that the conditions of our Theorem 1 are satisfied. Since the utility functions are upper semi-continuous and the strategy spaces are nonempty, compact, and convex, we need only show u_i is uniformly generalized quasiconcave on $Z = [0,1] \times [0,1]$.

For any finite subset $\{x^1, x^2, \dots, x^m\} \subset Z$, if we let $y_1^k = 1$ for $k = 1, \dots, m$, then for any subset $\{y_1^{k^1}, y_1^{k^2}, \dots, y_1^{k^s}\} = \{1\} \subset \{y_1^1, y_1^2, \dots, y_1^m\} = \{1\}$, $1 \le s \le m$ and $y_1^{k0} \in co\{y_1^{k^1}, y_1^{k^2}, \dots, y_1^{k^s}\} = \{1\}$, we have

$$\begin{aligned} \min_{1 \le l \le s} [u_1(x_1^{k^l}, x_2^{k^l}) - u_1(y_1^{k0}, x_2^{k^l})] &= u_1(x_1^{k^l}, x_2^{k^l}) - u_1(1, x_2^{k^l}) \\ &\le u_1(x_1^{k^l}, x_2^{k^l}) - 1 \le 0 \end{aligned} \tag{5}$$

for all $1 \le l \le s$. Thus u_1 is uniformly generalized quasiconcave. One can similarly show that u_2 is also uniformly generalized quasiconcave. Thus, by Theorem 1, the game in Example 1 has a dominant-strategy Nash equilibrium.

5.2 Example 2

We now establish the existence of a pure-strategy Nash equilibrium for the game described in Example 2 above by verifying that the conditions of Theorem 4 are satisfied.

We first show that $U(x, y)$ is 0-generalized quasiconcave. Now for any finite subset $\{x^1, x^2, \dots, x^m\} \subset Z$, if we let $y_i^k = 1$ for $i = 1, 2$ and $k = 1, \dots, m$, then for any subset $\{y^{k^1}, y^{k^2}, \dots, y^{k^s}\} = \{(1,1)\} \subset \{y^1, y^2, \dots, y^m\} = \{(1,1)\}$, $1 \le s \le m$ and $y^{k0} \in co\{y_1^{k^1}, y_1^{k^2}, \dots, y_1^{k^s}\} = \{(1,1)\}$ we have

$$\begin{aligned} U(x^{k^l}, y^{k0}) &= [u_1(x_1^{k^l}, y_2^{k0}) - u_1(y_1^{k0}, y_2^{k0})] + [u_2(y_1^{k0}, x_2^{k^l}) - u_2(y_1^{k0}, y_2^{k0})] \\ &= [u_1(x_1^{k^l}, 1) - u_1(1,1)] + [u_2(1, x_2^{k^l}) - u_2(1,1)] \\ &= [u_1(x_1^{k^l}, 1) - 1] + [u_2(1, x_2^{k^l}) - 1] \le 0 \end{aligned} \tag{6}$$

for $1 \le l \le s$. But this means $U(x, y)$ is 0-generalized quasiconcave in x.

Next, we show that $U(x,y)$ satisfies Condition (a).[6] In particular, we must show that if $U(x,y) > 0$ for $x, y \in Z$, then there exists some point $x' \in Z$ and some neighborhood $\mathcal{N}(y)$ of y such that $U(x', z) > 0$ for all $z \in \mathcal{N}(y)$.

Case (i): $y_1 < y_2$.

Case (ia): $1 - c \leq y_2$. Let $x'_1 = y_2 - \epsilon$ and $x'_2 = 1$, where $0 < \epsilon < \min\{1 - y_1, y_2\}$. Then

$$\begin{aligned} U(x', y) &= u_1(x'_1, y_2) - u_1(y_1, y_2) + u_2(y_1, x'_2) - u_2(y_1, y_2) \\ &= (y_2 - \epsilon) - y_1 + (1 - c) - (y_2 - c) = 1 - y_1 - \epsilon > 0. \end{aligned} \tag{7}$$

Case (ib): $1 - c > y_2$. Let $x'_1 = 1$ and $x'_2 = 1$. Then

$$\begin{aligned} U(x', y) &= u_1(x'_1, y_2) - u_1(y_1, y_2) + u_2(y_1, x'_2) - u_2(y_1, y_2) \\ &= (1 - c) - y_1 + (1 - c) - (y_2 - c) \\ &= (1 - c - y_2) + (1 - y_1) > 0. \end{aligned} \tag{8}$$

Case (ii): $y_1 > y_2$.

Case (iia): $1 - c \leq y_1$. Let $x'_1 = 1$ and $x'_2 = y_1 - \delta$, where $0 < \delta < \min\{1 - y_2, y_1\}$.

$$\begin{aligned} U(x', y) &= u_1(x'_1, y_2) - u_1(y_1, y_2) + u_2(y_1, x'_2) - u_2(y_1, y_2) \\ &= (1 - c) - (y_1 - c) + (y_1 - \delta) - y_2 = 1 - y_2 - \delta > 0. \end{aligned} \tag{9}$$

Case (iib): $1 - c > y_1$. Let $x'_1 = 1$ and $x'_2 = 1$. Then

$$\begin{aligned} U(x', y) &= u_1(x'_1, y_2) - u_1(y_1, y_2) + u_2(y_1, x'_2) - u_2(y_1, y_2) \\ &= (1 - c) - (y_1 - c) + (1 - c) - y_2 \\ &= (1 - c - y_1) + (1 - y_2) > 0. \end{aligned} \tag{10}$$

Case (iii): $y_1 = y_2$. In this case, $y_1 = y_2$ must be less than $1 - c$ for otherwise $U(x,y) \leq U(y,y) = 0$, which contradicts the hypotheses that $U(x,y) > 0$. We only

[6]Note that $U(x,y)$ is not lower semi-continuous in y, say, at $\bar{y}$ with $\bar{y}_1 < \bar{y}_2 = x_1 = x_2$ so Theorem 2 cannot be applied.

consider the case that $y_1 = y_2 < 1 - c$. Let $x'_1 = 1$ and $x'_2 = 1$. Then

$$\begin{aligned} U(x', y) &= u_1(x'_1, y_2) - u_1(y_1, y_2)] + [u_2(y_1, x'_2) - u_2(y_1, y_2) \\ &= (1-c) - y_1 + (1-c) - y_2 \\ &= (1-c-y_1) + (1-c-y_2) > 0. \end{aligned} \tag{11}$$

Thus we have $U(x', y) > 0$. Since $U(\cdot)$ is continuous at (x', y) for cases (i) - (ii), $U(x', z) > 0$, provided z is sufficiently close to y. For case (iii), one can directly verify that it is also true that $U(x', z) > 0$ provided z is sufficiently close to y. Hence Condition (a) is satisfied.

Condition (b) is trivially satisfied, since Z is compact. Thus the conditions of Theorem 4 are satisfied, and we conclude that the game in Example 2 has a pure-strategy Nash equilibrium.

5.3 Example 3

Similar to the above examples, one can establish the existence of an equilibrium for the game in Example 3 by showing that the conditions of Theorem 4 are satisfied. To do so, we first establish that $U(x, y)$ is 0-generalized quasiconcave. For any finite subset $\{x^1, x^2, \ldots, x^m\} \subset Z$, if we let $y_i^k = \frac{\alpha}{4}$ for $i = 1, 2$ and $k = 1, \ldots, m$, then for any subset $\{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\} = \{(\frac{\alpha}{4}, \frac{\alpha}{4})\} \subset \{y^1, y^2, \ldots, y^m\} = \{(\frac{\alpha}{4}, \frac{\alpha}{4})\}$, $1 \le s \le m$ and $y^{k0} \in co\,\{y_1^{k^1}, y_1^{k^2}, \ldots, y_1^{k^s}\} = \{(\frac{\alpha}{4}, \frac{\alpha}{4})\}$, we have

$$\begin{aligned} U(x^{k^l}, y^{k0}) &= [u_1(x_1^{k^l}, y_2^{k0}) - u_1(y_1^{k0}, y_2^{k0})] + [u_2(y_1^{k0}, x_2^{k^l}) - u_2(y_1^{k0}, y_2^{k0})] \\ &= [u_1(x_1^{k^l}, \frac{\alpha}{4}) - u_1(\frac{\alpha}{4}, \frac{\alpha}{4})] + [u_2(\frac{\alpha}{4}, x_2^{k^l}) - u_2(\frac{\alpha}{4}, \frac{\alpha}{4})] \\ &= [u_1(x_1^{k^l}, \frac{\alpha}{4}) - \frac{1}{2} + \frac{\alpha}{4}] + [u_2(\frac{\alpha}{4}, x_2^{k^l}) - \frac{1}{2} + \frac{\alpha}{4}] \le 0 \end{aligned} \tag{12}$$

for $1 \le l \le s$, since $u_1(x_1, \frac{\alpha}{4}) \le \frac{1}{2} - \frac{\alpha}{4}$ and $u_2(\frac{\alpha}{4}, x_2) \le \frac{1}{2} - \frac{\alpha}{4}$ for all $0 < \alpha \le 1$ and $0 \le x_i \le 1$. Thus $U(x, y)$ is 0-generalized quasiconcave in x.

We next verify that U satisfies condition (a). To do so, we must show that if $U(x, y) > 0$, then there exists some point $x' \in Z$ and some neighborhood $\mathcal{N}(y)$ of y such that $U(x', z) > 0$ for all $z \in \mathcal{N}(y)$.

Case (i): $y_i > 0$ $(i = 1, 2)$. In this case, $U(x, y)$ is continuous for all x and thus condition (a) is satisfied.

Case (ii): $y_1 = y_2 = 0$. Let $x_1' = x_2' = \delta > 0$. Then for any y_i' with $0 \leq y_i' \leq \frac{\delta}{n}$ $(i = 1, 2)$, we have

$$\begin{aligned} U(x', y') &\geq u_1(\delta, \frac{\delta}{n}) + u_2(\delta, \frac{\delta}{n}) - 1 \\ &= 2\frac{1}{1 + (\frac{1}{n})^\alpha} - 2\delta - 1 > 0, \end{aligned} \tag{13}$$

as δ is a sufficiently small number and n is a sufficiently large number. Thus condition (a) is satisfied in this case.

Case (iii): $y_1 = 0$ and $y_2 > 0$ or $y_1 > 0$ and $y_2 = 0$. We only need to show one of these two cases, say the case of $y_1 = 0$ and $y_2 > 0$. Let $x_1' = x_2' = \frac{y_2}{2} \equiv \delta > 0$. Then for any y_1' with $0 \leq y_1' \leq \frac{\delta}{n}$ and y_2' satisfying $|y_2' - y_2| < \frac{\delta}{n}$, we have

$$\begin{aligned} U(x', y') &\geq u_1(\delta, 2\delta + \frac{\delta}{n}) + u_2(\frac{\delta}{n}, \delta) - 1 + 2\delta \\ &= \frac{1}{1 + (2 + \frac{1}{n})^\alpha} + \frac{1}{1 + (\frac{1}{n})^\alpha} - 1 > 0, \end{aligned} \tag{14}$$

as n is a sufficiently large number.

Hence the example satisfies condition (a) of the theorem. Since Z is compact, condition (b) is trivially satisfied. Thus Theorem 4 establishes the existence of a pure-strategy Nash equilibrium for the game in Example 3.

6 Proofs

Now that we have presented and illustrated our main theorems, we present their proofs. In Section 6.1 we introduce some mathematical concepts that are used to prove five lemmas, which appear in Section 6.2. The proofs of our Theorems 1-5 follow directly from the lemmas and are presented in Section 6.3.

6.1 Mathematical Preliminaries

The proofs require an investment in additional mathematical terminology and notation. Accordingly, let X be a subset of L-dimensional Euclidian space $\Re^L$. Denote the convex hull, closure, and interior of the set D by coD, clD, and $intD$, respectively. Let Z be a convex subset of $\Re^L$ and let $\emptyset \neq X \subset Z$. Denote the set of all subsets of Z by 2^Z. Let S be a subspace (subset) of $\Re^L$ and let $D \subset S$. Denote by $cl_S D \equiv clD \cap S$ and $int_S D \equiv intD \cap S$ the closure and interior of the set D in the subspace S.

In addition to this notation, we introduce

Definition 6 (Generalized KKM Property) *A correspondence* $G : X \rightarrow 2^Y$ *is said to have the* generalized KKM property on X, *if for any finite subset* $\{x^1, x^2, \ldots, x^m \subset X$, *there exists a corresponding finite subset* $\{y^1, y^2, \ldots, y^m\} \subset Y$ *such that for any subset* $\{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\} \subset \{y^1, y^2, \ldots, y^m\}$, $1 \leq s \leq m$, *we have*

$$co\{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\} \subset \bigcup_{l=1}^{s} G(x^{k^l}).$$

The generalized KKM property[7]is a generalization of FS-convexity[8], which has been used elsewhere by Fan (1961) and Sonnenschein (1971). In particular, note that if G is FS-convex, then G has the generalized KKM property. (To see this, let $M = L$ and $y^k = x^k$). However, the generalized KKM property is much weaker than FS-convexity. For example, for any function $g : X \rightarrow \Re \cup \{\pm\infty\}$, define a correspondence $G : X \rightarrow 2^Y$ by $G(x) = \{y \in X : g(y) \geq g(x)\}$ for all $x \in X$. Then G so defined has the generalized KKM property (by letting $y^1 = \cdots = y^m = \max_k g(x^k)$). But, G is not FS-convex if it is not quasiconcave.[9]

6.2 Some Lemmas

Our proofs rely heavily on the following lemma, which generalizes the FKKM lemma of Fan (1961, 1979, 1984) by relaxing FS-convexity and the convexity of X, and generalizes the KKM lemma of Chang and Zhang (1989) by relaxing the convexity

[7]KKM is short for Knaster, Kuratowski, and Mazurkiewicz, whose seminal work in 1929 is the basis for our understanding of fixed-points.

[8]A correspondence $G : X \rightarrow 2^Z$ is said to be *FS-convex* on X if for every finite subset $\{x^1, x^2, \ldots, x^m\}$ of X, $co\{x^1, x^2, \ldots, x^m\} \subset \bigcup_{k=1}^m G(x^k)$.

[9]Tian (1989) shows that the FS-convexity of G is equivalent to the quasiconcavity of g.

of A.[10] The proof is very similar to that of Chang and Zhang (1989) and Tian (1990), and is thus relegated to the Appendix.

Lemma 1 *Let X be a non-empty subset in $\Re^L$ and Y be a nonempty convex subset in $\Re^M$. Let $G : X \rightarrow 2^Y$ be a correspondence such that for each $x \in X$, $G(x)$ is closed in Y. Then, the family of sets $\{G(x) : x \in X\}$ has the finite intersection property if and only if G has the generalized KKM property on X.*

Remark 1 If we define the mapping $G : X \rightarrow 2^Y$ by $G(x) = \{y \in Y : \phi(x,y) \leq 0\}$, it can be easily verified that G has the generalized KKM property if and only if ϕ is 0-generalized quasiconcave in x.

From Lemma 1 and the above remark, we can prove the following lemmas which generalize the results of Fan (1972) and Zhou and Chen (1988) by relaxing the lower semi-continuity and (0-diagonal) quasiconcavity of ϕ.[11]

Lemma 2 *Let X be a nonempty subset in $\Re^L$, let Y be a nonempty compact convex subset in $\Re^M$ and let ϕ: $X \times Y \rightarrow \Re$ be a function such that for every $x \in X$, ϕ is lower semi-continuous in $y \in Y$. Then there exists $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$ if and only if ϕ is 0-generalized quasiconcave in $x \in X$.*

Proof. For each $x \in X$, let $G(x) = \{y \in Y : \phi(x,y) \leq 0\}$. Since ϕ is lower semi-continuous in y, $G(x)$ is closed in Y. Then, by Lemma 1, the family of sets $\{G(x) : x \in X\}$ has the finite intersection property if and only if G has the generalized KKM property on X. Since Y is compact, $\bigcap_{x \in X} G(x) \neq \emptyset$ if and only if G has the generalized KKM property. Also, note that $\phi(x, y^*) \leq 0$ for all $x \in X$ and $y^* \in Y$ is equivalent to $y^* \in \cap_{x \in X} G(x)$. Further note that G has the generalized KKM property if and only if ϕ is 0-generalized quasiconcave in x (cf. Remark 1). Therefore, there exists a point $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$ if and only if ϕ is 0-generalized quasiconcave in $x \in X$. Q.E.D.

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[10] Also, we do not require $A = Z$.

[11] A function $\phi(x,y)$: $X \times Z \rightarrow \Re$ is said to be *0-diagonally quasiconcave* in x, if for any finite subset $\{x^1, \ldots, x^m\} \subset X$ and any $x^0 \in co\,\{x^1, \ldots, x^m\}$, we have $\min_{1 \leq k \leq m} \phi(x^k, x^0) \leq 0$.

If there exists an $x^0 \in X$ such that $\{y \in Y : \phi(y, x^0) \leq 0\}$ is compact on Y, we can relax the compactness of Y.

Lemma 3 *Let X be a nonempty subset in $\Re^L$, let Y be a nonempty convex subset in $\Re^M$ and let ϕ: $X \times Y \to \Re$ be a function such that*

(a) for every $x \in X$, ϕ is lower semi-continuous in $y \in Y$;

(b) there exists an $x^0 \in X$ such that $\{y \in Y : \phi(y, x^0) \leq 0\}$ is compact on Y.

Then there exists a $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$ if and only if ϕ is 0-generalized quasiconcave in $x \in X$.

Proof. For each $x \in X$, let $G(x) = \{y \in Y : \phi(x, y) \leq 0\}$. Thus, to prove the conclusion of the lemma, we only need to show that $\bigcap_{x \in X} G(x) \neq \emptyset$ if and only if G has the generalized KKM property.

Necessity. Suppose $\bigcap_{x \in X} G(x) \neq \emptyset$. Then $\{G(x) : x \in X\}$ has the finite intersection property. Since ϕ is lower semi-continuous in y, $G(x)$ is closed for each $x \in X$. By Lemma 1, G has the generalized KKM property.

Sufficiency. Suppose $G : X \to 2^Y$ has the generalized KKM property. Then, by Lemma 1, G has the finite intersection property and thus $\{G(x) \cap G(x^0) : x \in X\}$ also has the finite intersection property. Now since $\{G(x) \cap G(x^0) : x \in X\}$ is a family of compact sets in $G(x^0)$ we have $\emptyset \neq \bigcap_{x \in X} G(x) \cap G(x^0) = \bigcap_{x \in X} G(x)$. Q.E.D.

More generally, we can replace condition (b) with

(b′) there exist $x^1, \ldots, x^n \in X$ such that $\bigcap_{k=1}^{n} G(x^k)$ is compact.

Similar to Tian (1989), we can also weaken the lower semi-continuity of ϕ. Since Lemma 2 is a special case of Lemma 3, we thus state:

Lemma 4 *Let X be a nonempty subset in $\Re^L$, let Y be a nonempty convex subset in $\Re^M$ and let ϕ: $X \times Y \to \Re$ be a function such that* *(a) for every $x \in X$, if $\phi(x, y) > 0$, then there exist some point $x' \in X$ and some neighborhood $\mathcal{N}(y)$ of y such that $\phi(x', z) > 0$ for all $z \in \mathcal{N}(y)$;*

(b) there exist $x^1, \ldots, x^n \in X$ such that $\bigcap_{k=1}^{n} G(x^k)$ is compact.

Then there exists a $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$ if and only if ϕ is 0-generalized quasiconcave in $x \in X$.

Proof. We only need to prove sufficiency. For each $x \in X$, let $G(x) = \{y \in Y : \phi(x, y) \leq 0\}$. Thus, to prove the conclusion of the lemma, we only need to show that $\bigcap_{x \in X} G(x) \neq \emptyset$ under the above assumptions.

We first prove $\bigcap_{x \in X} cl_Y \ G(x) = \bigcap_{x \in X} G(x)$. It is clear that $\bigcap_{x \in X} G(x) \subset \bigcap_{x \in X} cl_Y G(x)$. So we only need to show $\bigcap_{x \in Y} cl_Y \ G(x) \subset \bigcap_{x \in X} G(x)$. Suppose, by way of contradiction, that there is some y in $\bigcap_{x \in X} cl_Y \ G(x)$ but not in $\bigcap_{x \in X} G(x)$. Then $y \notin G(x)$ for some $x \in X$ and thus $\phi(x, y) > 0$. By condition (a), there is some $x' \in X$ and some neighborhood $\mathcal{N}(y)$ of y such that $\phi(x', z) > 0$ for all $z \in \mathcal{N}(y)$. Thus $y \notin cl_Y G(x')$, a contradiction.

For $x \in X$, let $\bar{G}(x) = cl_Y \ G(x)$. Then $\bar{G}(x)$ is closed and, by the 0-generalized quasiconcavity of ϕ, it has the generalized KKM property. By Lemma 1, $\bar{G}$ has the finite intersection property. Thus, by condition (b), $\bigcap_{x \in X} G(x) = \bigcap_{x \in X} cl_Y \ G(x) \neq \emptyset$. Hence, there exists a $y^* \in Y$ such that $\phi(x, y^*) \leq 0$ for all $x \in X$. Q.E.D.

The final tool we need to prove our theorems is the following lemma, which generalizes the results of Nikaido and Isoda (1955) by relaxing the finiteness of number of players.

Lemma 5 *Suppose that $y^* \in A$ satisfies*

$$\sup_{x \in A} U(x, y^*) \leq 0. \tag{15}$$

Then y^* is an equilibrium for Γ. The converse is true when A is a (Cartesian) product of strategy sets.

Proof. Let $y^* \in A$ be a solution of (15) and let $x = (x_i, y^*_{-i})$. Then we have

$$U(x, y^*) = \frac{1}{2^i}[u_i(x_i, y^*_{-i}) - u_i(y^*)] \leq 0 \tag{16}$$

for any $x_i \in Z_i$ with (x_i, y^*_{-i}). So y^* is an equilibrium of the game.

Conversely, if A is a product of strategy sets, we can obtain (15) by summing up (16) for all $i \in I$. Q.E.D.

6.3 Proofs of Theorems 1-5

Proof of Theorem 1. Define a correspondence $G_i : Z \to 2^{Z_i}$ by $G_i(x) = \{y_i \in Z_i : u_i(x) - u_i(y_i, x_{-i}) \leq 0\}$ for all $x \in Z$. Since u_i is upper semi-continuous in y_i, $G_i(x)$ is a closed subset for all $x \in Z$. Also since Z_i is compact, by Lemma 1, $\bigcap_{x \in Z} G_i(x) \neq \emptyset$ if and only if G_i has the generalized KKM property on Z. So there exists a $y_i^* \in Z_i$ such that $u_i(x) - u_i(y_i^*, x_{-i}) \leq 0$ for all $x \in Z$ if and only if u_i is uniformly generalized quasiconcave on Z. Q.E.D.

Proof of Theorem 2. The proof of Theorem 2 follows directly from Lemma 2 and Lemma 5 by taking $U = \phi$. Q.E.D.

Proof of Theorem 3. The proof follows directly from Lemma 3 and Lemma 5 by taking $U = \phi$. Q.E.D.

Proof of Theorem 4. The proof follows directly from Lemma 4 and Lemma 5 by taking $U = \phi$. Q.E.D.

Proof of Theorem 5. We only need to show the sufficiency. From Lemma 4, we know that there exists a point $y^* \in Z$ such that $\sup_{x \in A} U(x, y^*) \leq 0$. Now we must have $y^* \in A$, for otherwise $U(x, y^*) > 0$ for some $x \in A$ by condition (c). Hence $y^* \in A$. Then by Lemma 5, y^* is a Nash equilibrium. Q.E.D.

7 Concluding Remarks

The examples and theorems presented above reveal that the continuity and quasiconcavity conditions assumed in the literature on the existence of pure-strategy Nash equilibrium can be considerably weakened. Specifically, 0-generalized quasiconcavity and uniform generalized quasiconcavity, together with some "regular" topological conditions, are not only sufficient but also necessary for the existence of pure-strategy Nash equilibrium and dominant-strategy Nash equilibrium, respectively. Similar theorems obtain under weaker topological conditions. Thus our analysis characterizes the existence of pure-strategy Nash equilibrium in games where the payoff functions are discontinuous and non-quasiconcave, the number of players is infinite, and the strategy spaces are neither convex nor compact. Thus the existence theorems given in this paper generalize almost all of the existence theorems on Nash equilibrium in the literature such as those of Nash (1950, 1951), Debreu (1952), Nikaido and Isoda (1955) and Dasgupta and Maskin (1986).

The theorems presented in this paper are based on generalizations of the FKKM theorem of Fan (1961, 1979, 1984) and Chang and Zhang (1989) by relaxing the FKKM convexity and closedness of correspondences, and the compactness and convexity of strategy sets. In the context of the existence theorems appearing in the recent economics literature, perhaps the most notable aspect of our theorems on the existence of pure-strategy Nash equilibrium is that they weaken the quasiconcavity conditions required by the theorem of Dasgupta and Maskin (1986).

Appendix A

Proof of Lemma 1. The proof of this lemma is similar to that given by Chang and Zhang (1989).

Necessity. If $\{G(x) : x \in X\}$ has the finite intersection property, then for any finite subset $\{x^1, x^2, \ldots, x^m\} \subset X$, $\bigcap_{k=1}^m G(x^k) \neq \emptyset$. Taking $y^* \in \bigcap_{k=1}^m G(x^k)$ and letting $y^k = y^*$ for $k = 1, \ldots, m$, we have

$$co\{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\} = \{y^*\} \subset \bigcap_{k=1}^m G(x^k) \subset \bigcup_{l=1}^s G(x^{k^l})$$

for any finite subset $\{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\} \subset \{y^1, y^2, \ldots, y^m\}$. So G has the generalized KKM property on X.

Sufficiency. Let $G : X \to 2^Y$ have the generalized KKM property on X. Suppose, by way of contradiction, that $\{G(x) : x \in X\}$ does not have the finite intersection property, i.e., there exists some finite subset $\{x^1, x^2, \ldots, x^m\} \subset X$ such that $\bigcap_{k=1}^m G(x^k) = \emptyset$.

Since G has the generalized KKM property, for the finite set $\{x^1, x^2, \ldots, x^m\}$, there exists a corresponding subset $\{y^1, y^2, \ldots, y^m\} \subset Y$ such that for any

$$\{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\} \subset \{y^1, y^2, \ldots, y^m\},$$

$$co\{y^{k^1}, y^{k^2}, \ldots, y^{k^s}\} \subset \bigcup_{l=1}^s G(x^{k^l}).$$

In particular, we have $co\{y^1, y^2, \ldots, y^m\} \subset \bigcup_{k=1}^m G(x^k)$. Let $S = co\{y^1, y^2, \ldots, y^m\}$ and $L = span\{y^1, y^2, \ldots, y^m\}$. Then $S \subset L$. Since $G(x)$ is closed, $G(x_i) \cap L$ is a closed set. Let d be the Euclidean metric on L. It is easy to see that

$$d(y, L \cap G(x^k)) > 0 \text{ if and only if } y \notin L \cap G(x^k). \tag{17}$$

Now define a continuous function $f : S \to [0, \infty)$ as follows:

$$f(y) = \sum_{k=1}^m d(y, L \cap G(x^k)) \tag{18}$$

for all $y \in S$. It follows from (17) and $\bigcap_{k=1}^m G(x^k) = \emptyset$ that for each $y \in S$, $f(y) > 0$. Define a continuous function $g : S \to S$ by, for each $y \in S$,

$$g(y) = \sum_{k=1}^m \frac{1}{f(y)} d(y, L \cap G(x^k)) y^k. \tag{19}$$

Then, by the Brouwer fixed point theorem, there exists a $y^* \in S$ such that

$$y^* = g(y^*) = \sum_{k=1}^{m} \frac{1}{f(y^*)} d(y^*, L \cap G(x^k)) y^k. \tag{20}$$

Denote

$$K = \{k \in \{1, \ldots, m\} : d(y^*, L \cap G(x^k)) > 0\}. \tag{21}$$

Then for each $k \in K$, $y^* \notin L \cap G(x^k)$. Since $y^* \in L$, so $y^* \notin G(x^k)$ for any $k \in K$ and thus

$$y^* \notin \bigcup_{k \in K} G(x^k). \tag{22}$$

From (20) - (21), we have

$$y^* = \sum_{k \in K} \frac{1}{f(y^*)} d(y^*, L \cap G(x^k)) y \in co\, \{y^k : k \in K\}. \tag{23}$$

However, since G has the generalized KKM property and maps from X into 2^Y, we have

$$y^* \in co\, \{y^k : k \in K\} \subset \bigcup_{k \in K} G(x^k), \tag{24}$$

which contradicts (22). Hence $\{G(x) : x \in X\}$ has the finite intersection property on X. Q.E.D.

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