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The optimal number of suppliers in an (s, Q) inventory system with order splitting

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Abstract

In this paper we present an (s, Q) inventory model with order splitting. Replenishment orders are split equally among n suppliers. Demand is modelled as a compound renewal process, and we consider independent identically distributed lead times for the suppliers. By extending results for the standard (s, Q) inventory model, we derive approximate expressions for the expected average physical stock level, the expected average backlog level, and the fraction of the time the physical stock is positive. The optimal values of the decision variables, the reorder point s, the replenishment quantity Q, and the number of suppliers n, are determined by minimizing the sum of ordering, holding, and shortage costs, where the emphasis is on the optimal number of suppliers.

1 Introduction

Order splitting is a vendor management strategy. This strategy can be applied in combination with many inventory replenishment strategies, such as the (s, S) and (s, Q) strategy.

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Order splitting or multiple sourcing is the partitioning of a replenishment order among two or more suppliers.

Order splitting is advocated for the purpose of reducing lead time uncertainties, whereby safety stocks are reduced In Sculli and Wu (1981), Kelle and Silver (1990a, 1990b), and Guo and Ganeshan (1995) order statistics are used to derive analytical expressions for some characteristics of the first arriving partial delivery. Typically the number of suppliers, n, is allowed to be larger than two. In these papers the demand rate is assumed to be constant over time. The optimal value of n is determined based on the reduction in the safety stock.

Other papers focus on another advantage of order splitting, namely the decrease of the inventory holding cost due to the delayed replenishments (see, for example, Zhao and Lau (1992), Lau and Zhao (1993), Lau and Lau (1994), and Chiang and Chiang (1996)). In these papers the number of suppliers is mostly restricted to two, and demand is assumed to be stochastic. The papers focus either on minimizing the sum of holding, ordering, and shortage costs, or on minimizing the sum of ordering and holding costs subject to a service level constraint.

It has been shown that the profitability of order splitting depends on the ratio between the inventory holding cost and the extra transhipment or ordering cost when using more than one supplier (see, for example, Larson (1989), Ramasesh et al. (1991), and Hong and Hayya(1992)).

In this paper we consider an (s, Q) replenishment policy in which a replenishment order is split equally among n suppliers. We focus on minimizing the sum of holding, ordering, and shortage costs. By extending results from the standard (s, Q) inventory model, we derive approximate expressions for the expected length of a replenishment cycle, the average physical stock level, and the average backlog level.

The contribution of this paper is twofold. The expressions for the average physical stock level and the expected average backlog level are derived under general assumptions for the demand and lead time process. Demand is modelled as a compound renewal process, and lead times of the suppliers are independent and identically mixed Erlang distributed random variables. The compound renewal process is suitable for modeling real life demand processes, and the mixtures of Erlang distribution is able to model a wide variety of lead time distributions. Regarding the literature, most papers on order splitting consider constant demand models or consider at most two suppliers. In that sense these models are special cases of the model presented in this paper.

Secondly we present an approximation algorithm for computing the optimal values of s, Q and n, given the first two moments of the underlying demand and lead time process. We note that in practice only the first two moments of the underlying processes can be accurately estimated from the available data, The algorithm developed in this article can be applied in many different practical settings. We concentrate on the impact of the problem parameters on the optimal number of suppliers. For this purpose it is sufficient to consider the case of identically distributed lead times.

The paper is organized as follows. In section 2 the model assumptions are discussed and expressions for the performance measures are derived. In section 3 a method for finding values for the optimal control variables is discussed. Section 4 deals with some computational aspects of the performance measures derived, and an algorithm is presented to actually calculate these measures. In section 5 the algorithm is validated by simulation and in section 6 we used the proposed algorithm to investigate the optimal values of the control parameters. Finally, in section 7 conclusions and future research are discussed.

2 The model description

In this single echelon inventory model with order splitting we assume that the demand process is a compound renewal process. I.e, the interarrival times of customers can be described by the sequence $\{A_i\}_{i=1}^{\infty}$ of independent and identically distributed (i.i.d.) random variables with a common distribution function F_A, where A_i represents the time between the arrival of the i-th and $(i-1)$-th customer after time zero. We assume a customer arrives at time zero. The demand sizes of the customers are described by the sequence $\{D_i\}_{i=1}^{\infty}$ of i.i.d. random variables with a common distribution function F_D, where D_i represents the demand size of the i-th customer after time zero. The sequence $\{D_i\}_{i=1}^{\infty}$ is independent of $\{A_i\}_{i=1}^{\infty}$.

Shortages are backordered, and replenishment decisions are based on the inventory position, being defined as the total stock on hand plus on order minus the total stock backordered. The replenishment strategy that is considered is the continuous review (s, Q) policy. I.e., as soon as the inventory position drops below the reorder point s an amount of Q is ordered, such that the inventory position after ordering is between s and $s+Q$. Hence we implicitly assume that always an amount of exactly Q is ordered. A replenishment order is equally split among n different suppliers. The suppliers have independent and identically distributed lead times with a common distribution function G. If we rearrange the realisations of the lead times of the n partial deliveries in an increasing order, we get the order statistics. These order statistics are denoted by
$L_{1:n} \leq L_{2:n} \leq \ldots \leq L_{n:n}$, with distribution functions $G_{k:n}$ for $k = 1, \ldots, n$.
It is assumed that deliveries of two successive replenishment orders (each consisting of n

partial deliveries) do not cross in time. Thus, the last partial delivery of a replenishment order arrives before any partial delivery of a subsequent replenishment order.

The values of the control parameters s, Q and n are determined such that the total sum of long-run ordering, holding and backordering cost per unit time are minimized. A well-known approach for deriving expressions for the long-run performance measures is to consider an arbitrary replenishment cycle. The renewal reward theorem (see, e.g., Tijms (1994)) enables us to compute expressions for the long-run performance measures by deriving expression related to an arbitrary replenishment cycle.

Let $TRC(s,Q,n)$ denote the total of ordering, holding and backordering cost per unit time incurred during an arbitrary replenishment cycle. The holding costs are proportional to the expected average physical stock level: stocking one unit of product costs h \$ per day. Hong and Hayya (1990) investigated the effects of the ordering costs on models with order splitting. In particular they considered ordering costs that depend on the number of suppliers (denoted by $A(n)$). They showed that the optimal number of suppliers is very sensitive to the shape of $A(n)$. We use the following simple function for the ordering costs,

$$A(n) = n^c K, \quad n \in I\!N, c \in I\!R, \tag{1}$$

where K is a fixed cost component, and c determines the shape of $A(n)$. By varying c we can model a convex, concave, or a linear ordering cost function. Backordering costs are proportional to the number of units short, which coincides with the so-called B_3 criterion in Silver and Peterson (1985): each unit short is charged with an amount of say b \$ per time unit. Hence,

$$TRC(s,Q,n) = \frac{A(n)}{\xi(s,Q,n)} + h\phi(s,Q,n) + b\psi(s,Q,n), \tag{2}$$

where

$\xi(s,Q,n)$ denotes the expected length of an arbitrary replenishment cycle;

$\phi(s,Q,n)$ denotes the average physical stock level during an arbitrary replenishment cycle;

$\psi(s,Q,n)$ denotes the average backlog level during an arbitrary replenishment cycle.

Towards this end we defined a replenishment cycle as the time period between two successive **last** arrivals of partial deliveries of a replenishment. Consider now an arbitrary replenishment cycle, then we define the k-th sub-cycle as the time period between the arrival of the $(k-1)$-th partial delivery and the k-th partial delivery ($k \in \{2,\ldots,n\}$). The first sub-cycle is defined as the time period between the arrival of last partial delivery of the replenishment cycle which preceded the arbitrary replenishment cycle and the arrival of the first partial delivery of the tagged replenishment cycle.

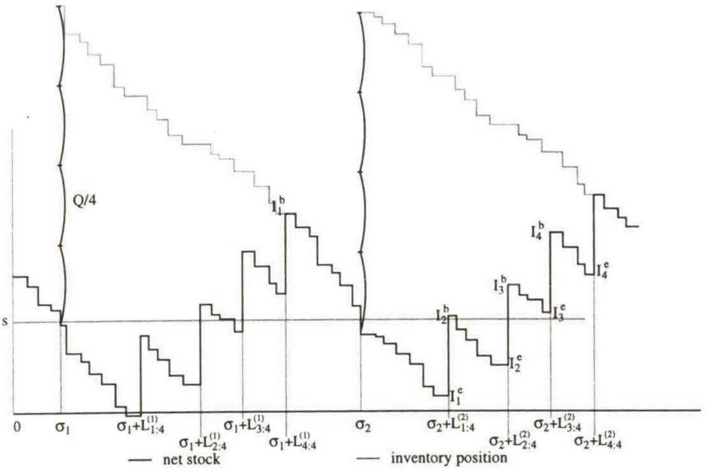

Figure 1: Evolution of the net stock and inventory position during a replenishment cycle for $n = 4$.

Let zero be an arbitrary customer arrival moment. Denote the j-th ordering epoch after zero by σ_j. Let $D(t_1, t_2)$ be the total demand during $(t_1, t_2]$, and U_j the undershoot under s at σ_j. $L^{(j)}_{k:n}$ denotes the lead time of the k-th partial delivery in the j-th replenishment cycle after zero. Consider the second replenishment cycle after zero, see Figure 1. Define the net stock as the stock on hand minus the total stock backordered. Then we denote for $k \in \{1, 2, \dots, n\}$ I_k^b as the net stock at the beginning of the k-th sub-cycle in the second replenishment cycle after zero (just after the partial delivery arrived), and I_k^e as the net stock at the end of the k-th sub-cycle in the second replenishment cycle (just before the partial delivery arrives). Then it can be seen that (see Figure 1):

$$\begin{aligned}
I_1^b &= s - U_1 + Q - D(\sigma_1, \sigma_1 + L^{(1)}_{n:n}); \\
I_1^e &= s - U_2 - D(\sigma_2, \sigma_2 + L^{(2)}_{1:n}): \\
I_k^b &= s - U_2 + \tfrac{k-1}{n} Q - D(\sigma_2, \sigma_2 + L^{(2)}_{k-1:n}), \quad k \in \{2, 3, \dots, n\}: \\
I_k^e &= s - U_2 + \tfrac{k-1}{n} Q - D(\sigma_2, \sigma_2 + L^{(2)}_{k:n}), \quad k \in \{2, 3, \dots, n\}.
\end{aligned}$$

Since the demand process is a compound renewal process and the lead times are i.i.d., it can be seen that $U_1 \stackrel{d}{=} U_2$, and $D(\sigma_1, \sigma_1 + L^{(1)}_{n:n}) \stackrel{d}{=} D(\sigma_2, \sigma_2 + L^{(2)}_{n:n})$, where $\stackrel{d}{=}$ denotes equality in distribution. Hence,

$$I_1^b \stackrel{d}{=} s - U_2 + Q - D(\sigma_2, \sigma_2 + L_{n:n}^{(2)}).$$

For ease of notation we will suppress the indices 2 in σ_2, U_2 and $L_{k:n}^{(2)}$. As in the standard (s, Q) inventory model, it can be shown that the expected demand during a replenishment cycle is equal to Q. Then it is easy to see that

$$\xi(s, Q, n) = \frac{Q \mathbb{E} A}{\mathbb{E} D}. \tag{3}$$

Note that $\xi(s, Q, n)$ is independent of s and n.

In order to derive an expression for $\phi(s, Q, n)$ we need the expected surface between the physical stock level and the zero level during a replenishment cycle. By using results from renewal theory we can derive the following approximate expression for $\phi(s, Q, n)$ (see Appendix 1),

$$\begin{aligned} \phi(s, Q, n) &= \gamma \sum_{k=1}^{n} \frac{\mathbb{E}(I_k^b)^+ - \mathbb{E}(I_k^e)^+}{Q} \\ &\quad + \sum_{k=1}^{n} \frac{\mathbb{E}((I_k^b + U)^+)^2 - \mathbb{E}((I_k^e + U)^+)^2}{2Q}, \end{aligned} \tag{4}$$

where $\gamma = 1/2\big(c_A^2 - 1\big)\mathbb{E} D$. In a similar way the following approximate expression for $\psi(s, Q, n)$ can be derived,

$$\begin{aligned} \psi(s, Q, n) &= \sum_{k=1}^{n} \frac{\mathbb{E}((-(I_k^e + U))^+)^2 - \mathbb{E}((-(I_k^b + U))^+)^2}{2Q} \\ &\quad - \gamma \sum_{k=1}^{n} \frac{\mathbb{E}(-I_k^e)^+ - \mathbb{E}(-I_k^b)^+}{Q}. \end{aligned} \tag{5}$$

We did not use the fact that the lead times of the partial deliveries are identically distributed. Hence, (4) and (5) are also valid for non-identically distributed lead times. However, it is well-known that the distribution function of the order statistics of non-identically distribution random variables is quite complex, see Balakrishnan (1988). In principle it is possible to compute $\phi(s, Q, n)$ and $\psi(s, Q, n)$ for independent and non-identically distributed lead times. Yet the computational complexity is of order $n!$ For computational convenience we therefore restrict ourselves to identically distributed lead times for the different suppliers. The assumptions of identical suppliers is justified by the fact that suppliers of the same product should provide more or less the same prices and lead times to the customers.

3 The optimization problem

In this section we consider the problem of determining values for the cost-optimal control parameters s, Q, and n. The objective is to minimize the sum of the holding, ordering, and backordering costs. Hence, we want to

$$\begin{aligned} \text{minimize} \quad & TRC(s,Q,n) = \tfrac{A(n)}{\xi(s,Q,n)} + h\phi(s,Q,n) + b\psi(s,Q,n) \\ \text{s.t.} \quad & Q \geq 0,\ n \in I\!N. \end{aligned}$$

When n is fixed, we can find the optimal values for s and Q, denoted by $s^*(n)$ and $Q^*(n)$ respectively, in the following way. For given values of n and Q, the optimal value of s can be determined by solving the equation $\frac{\partial TRC(s,Q,n)}{\partial s} = 0$, presuming a unique solution exists. By using relations (4) and (5) it can be derived that

$$\frac{\partial TRC(s,Q,n)}{\partial s} = (h+b)\tau(s,Q,n) - b, \tag{6}$$

where

$$\begin{aligned} \tau(s,Q,n) \;=\; & \gamma \sum_{k=1}^{n} \frac{I\!P(I_k^b < 0) - I\!P(I_k^e < 0)}{Q} \\ & + \sum_{k=1}^{n} \frac{I\!E(I_k^b + U)^+ - I\!E(I_k^e + U)^+}{Q}. \end{aligned} \tag{7}$$

Moreover, it can be shown that $\tau(s,Q,n)$ is equal to the long-run fraction of the time the net stock is positive (see Janssen and de Kok (1997a, 1997b)). Hence, for given values of n and Q the optimal value of s (denoted by $s^*(Q,n)$), can be determined by solving

$$\tau(s,Q,n) = \frac{b}{b+h}. \tag{8}$$

Since $\tau(s,Q,n)$ is increasing in s, and can take all values on (0,1), a unique solution indeed exists. Note the resemblance with the newsboy problem (see Silver and Peterson (1985, p 265)). So, we can find $s^*(n)$ and $Q^*(n)$ by solving the following one-dimensional optimization problem

$$\begin{aligned} \text{minimize} \quad & TRC(s^*(Q,n),Q,n) \\ \text{s.t.} \quad & Q \geq 0. \end{aligned}$$

If we assume that $TRC(s^*(Q,n),Q,n)$ is convex in Q, we can determine $Q^*(n)$ by using for example Golden Section search, and $s^*(n) = s^*(Q^*(n),n)$.

For practical situations we may restrict ourself to a limited number of suppliers (n_{max}). For each n, $1 \leq n \leq n_{max}$, we determine $s^*(n)$ and $Q^*(n)$, and select that n for which the

$TRC(s^*(n), Q^*(n), n)$ is minimal.

4 Computational aspects

A versatile class of distribution functions is the class of mixtures of two Erlang distributions (denoted by ME distributions), i.e.

$$f(x) = \sum_{j=1}^{2} p_j \mu_j^{k_j} \frac{x^{k_j - 1}}{(k_j - 1)!} e^{-\mu_j x}, \quad x \geq 0, \tag{9}$$

where $p_1 \geq 0$, $p_2 \geq 0$, $p_1 + p_2 = 1$, $k_1, k_2 \in I\!N$.

In Tijms (1994, p.358) formulas are given to fit a ME distribution on a positive random variable based on the first two moments of that variable. When X and Y are ME distributed and $z \in I\!R$, then closed form expressions for $I\!E(X-z)^+$, $I\!E((X-z)^+)^2$, $I\!E(X-Y)^+$ and $I\!E((X-Y)^+)^2$ exist.

In the model presented in section 2, we assumed that F_A, F_D and G are known. Expressions (4) and (5) contain the distributions of $D(\sigma, \sigma + L_{k:n})$ $(k = 1, \ldots, n)$ and the distribution of the undershoot U. In general these distribution functions are hard to obtain from F_A, F_D and G. To avoid this problem, we assume that $D(\sigma, \sigma + L_{k:n}) + U$ and $D(\sigma, \sigma + L_{k:n})$ $(k = 1, \ldots, n)$ are ME distributed. So, we only need the first two moments of $D(\sigma, \sigma + L_{k:n}) + U$ and $D(\sigma, \sigma + L_{k:n})$ $(k = 1, \ldots, n)$ to calculate the expressions (4) and (5) for given values of s, Q, and n. Since U is independent of $D(\sigma, \sigma + L_{k:n})$ it is sufficient to find expressions for the moments of U and $D(\sigma, \sigma + L_{k:n})$ separately.

Now we use the fact that the distribution function of the undershoot is approximately equal to the stationary residual lifetime distribution with respect to F_D, when $Q \geq Cond(D)$, (see Tijms (1994, p.14)). For a positive random variable X with finite moments $I\!EX$, $I\!EX^2$, and where c_X represents the coefficient of variation of X, $Cond(X)$ is defined as

$$Cond(X) = \begin{cases} \frac{3}{2} c_X^2 I\!EX & \text{if } \ c_X^2 > 1; \\ I\!EX & \text{if } \ 0.2 < c_X^2 \leq 1; \\ \frac{1}{2c_X} I\!EX & \text{if } \ 0 < c_X^2 \leq 0.2. \end{cases} \tag{10}$$

Then using results from renewal theory gives

$$I\!EU \simeq \frac{I\!ED^2}{2I\!ED}, \tag{11}$$

$$I\!EU^2 \simeq \frac{I\!ED^3}{3I\!ED}. \tag{12}$$

The first two moments of $D(\sigma, \sigma + L_{k:n})$ are given by the well-known results

$$\mathbb{E}D(\sigma,\sigma+L_{k:n}) = \mathbb{E}N(\sigma,\sigma+L_{k:n})\mathbb{E}D, \tag{13}$$

$$\mathbb{E}D^2(\sigma,\sigma+L_{k:n}) = \mathbb{E}N(\sigma,\sigma+L_{k:n})\sigma^2(D) + \mathbb{E}N^2(\sigma,\sigma+L_{k:n})(\mathbb{E}D)^2, \tag{14}$$

where $N(\sigma, \sigma + L_{k:n})$ denotes the number of customer arrivals during $(\sigma, \sigma + L_{k:n}]$. Since σ is an order epoch it follows that a customer arrives at epoch σ. Therefore we can derive the following approximations from asymptotic expressions from renewal theory (see, for example, Cox (1962))

$$\mathbb{E}N(\sigma,\sigma+L_{k:n}) \simeq \frac{\mathbb{E}L_{k:n}}{\mathbb{E}A} + \frac{\mathbb{E}A^2}{2\mathbb{E}A} - 1 \tag{15}$$

$$\begin{aligned}\mathbb{E}N^2(\sigma,\sigma+L_{k:n}) &\simeq \frac{\mathbb{E}(L_{k:n})^2}{(\mathbb{E}A)^2} + \mathbb{E}L_{k:n}\left(\frac{2\mathbb{E}A^2}{(\mathbb{E}A)^3} - \frac{3}{\mathbb{E}A}\right) \\ &\quad + \frac{3(\mathbb{E}A^2)^2}{2(\mathbb{E}A)^4} - \frac{2\mathbb{E}A^3}{3(\mathbb{E}A)^3} - \frac{3\mathbb{E}A^2}{2(\mathbb{E}A)^2} + 1\end{aligned} \tag{16}$$

These asymptotic relations hold for $k = 1, \ldots, n$ only when $\mathbb{P}(L_{k:n} \leq A)$ are negligible. In case this probability is larger than a certain treshold value, we propose a Gamma approximation presented by Smeitink and Dekker (1990) to compute the first two moments of the renewal function.

What remains to compute are the moments of the order statistics $L_{k:n}$. Using an analogous approach as described in Balakrishnan and Cohen (1991, p.44), $\mathbb{E}L_{k:n}^m$ can be computed for $m \in \mathbb{N}$, and $k = 1, \ldots, n$, in case G is ME distributed.

Summarizing, to compute values for the expressions (4) and (5) for given values of s, Q, and n, we have to go through the following three steps

- Compute the moments of the order statistics $\mathbb{E}L_{k:n}^m$ for $m \in \{1, 2\}$ and $k = 1, \ldots, n$.
- Compute the first two moments of U_2 and $D(\sigma, \sigma + L_{k:n})$ for $k = 1, \ldots, n$, by using relations (11) to (16).
- Compute $\xi(s, Q, n)$, $\phi(s, Q, n)$ and $\psi(s, Q, n)$ by fitting ME distributions on $D(\sigma, \sigma + L_{k:n})$ and $D(\sigma, \sigma + L_{k:n}) + U$ $(k = 1, \ldots, n)$, and using relations (3) to (5) respectively.

5 Validation of the algorithm

By simulation we first validate the proposed algorithm for computing the values of $\phi(s, Q, n)$, $\psi(s, Q, n)$, and $\tau(s, Q, n)$. The algorithm yields approximations for the values for the optimal decision parameters, because we assume that

1. Replenishment orders do not cross
2. Exactly Q is ordered at a time.
3. $\tilde{A}$ is distributed as the residual lifetime distribution associated with F_A (see Appendix 1).
4. U is distributed as the residual lifetime distribution associated with F_D.
5. The moments of $N(\sigma, \sigma + L_{k:n})$ are approximated by (15) and (16), which are asymptotic relations.
6. The distribution functions of $D(\sigma, \sigma + L_{k:n}) + U$ and $D(\sigma, \sigma + L_{k:n})$ $(k = 1, \ldots, n)$ are approximated by ME distributions;
7. $TRC(s^*(Q, n), Q, n)$ is convex in Q.

We will show that is spite of these all these assumptions, our calculation scheme provides excellent approximations for the relevant performance characteristics given s, Q and n. Thereby the algorithm given in section 3 yields near-optimal values for s^*, Q^* and n^*. We distinguish between assumptions made for deriving expressions for $\phi(s, Q, n)$ and $\psi(s, Q, n)$ (assumptions 1, 2 and 3), computing the first two moments of $D(\sigma, \sigma + L_{k:n})$ and $D(\sigma, \sigma + L_{k:n}) + U$ $(k = 1, \ldots, n)$ (assumptions 4, 5 and 6), and for selecting the optimal values for the decision variables (assumption 7). It is well-known that for small values of Q with respect to $\mathbb{E}D(\sigma, \sigma + L_{n:n})$ assumption 1 is violated, see, for example, Kelle and Silver (1991b). Assumptions 2 and 4 are violated only when Q is small with respect to $\mathbb{E}D$, i.e. $Q < Cond(D)$. Assumptions 3 and 5 are violated when $\mathbb{E}L < Cond(A)$.

In practice assumption 1 may be violated when the number of suppliers is large. Therefore, we will investigate the effect of assumption 1 on the quality of the computation of the expected average physical stock level, expected average backlog level, and the fraction of the time the net stock is positive by the proposed algorithm of section 3.

We used discrete event simulation to validate the quality of the approximations in terms of the deviation of the calculated performance measures by the algorithm described in section 3, and the performance measures computed by simulation. These experiments are done for a wide range of parameter values. The input values of the system parameters are given in Table 1. For each of these 3240 experiments we calculated s by solving $\tau(s, Q, n) = \tau_{an}$ via a numerical search routine, where τ_{an} represents the ratio $\frac{b}{b+h}$. The number of sub-runs which where performed in the simulation experiment is fixed to 5 (exclusive the initialisation run), and the sub-run length is 100.000 time units. Furthermore, the

Table 1: basic setting parameters for the experiments

n	$\mathbb{E}D$	c_D	$\mathbb{E}A$	c_A	$\mathbb{E}L$	c_L	Q	τ_{an}
1	5	$\frac{1}{2}$	$\mathbb{E}D/5$	$\frac{1}{2}$	5	$\frac{3}{10}$	50	0.90
2	10	1		1	10	$\frac{1}{2}$	100	0.99
3		2		2		1	250	
5								
10								

Table 2: The deviations of simulation and the values calculated with the algorithm.

Q	τ_{an}	c_L	$\lvert\phi_{an} - \phi_{sim}\rvert$	$\lvert\psi_{an} - \psi_{sim}\rvert$	$\lvert\tau_{an} - \tau_{sim}\rvert$	Crossing	$b = h$	$b = 10h$	$b = 20h$
50	0.50	0.3	0.0532	0.0331	0.0017	0.4889	0.0372	0.0320	0.0316
50	0.99	0.3	0.0014	0.5184	0.0000	0.4892	0.0026	0.0139	0.0261
100	0.50	0.3	0.0248	0.0203	0.0002	0.1446	0.0205	0.0194	0.0194
100	0.99	0.3	0.0012	0.3364	0.0000	0.1447	0.0017	0.0082	0.0155
250	0.50	0.3	0.0061	0.0054	0.0000	0.0048	0.0046	0.0051	0.0052
250	0.99	0.3	0.0012	0.2022	0.0000	0.0047	0.0013	0.0035	0.0064
50	0.50	0.5	0.0804	0.0480	0.0041	0.5376	0.0616	0.0510	0.0501
50	0.99	0.5	0.0022	0.7672	0.0001	0.5374	0.0042	0.0222	0.0421
100	0.50	0.5	0.0224	0.0214	0.0006	0.1987	0.0201	0.0203	0.0204
100	0.99	0.5	0.0015	0.5224	0.0000	0.1987	0.0026	0.0142	0.0269
250	0.50	0.5	0.0065	0.0064	0.0001	0.0085	0.0051	0.0059	0.0061
250	0.99	0.5	0.0012	1.1435	0.0000	0.0085	0.0014	0.0049	0.0091
50	0.50	1.0	0.2627	0.0973	0.0188	0.5878	0.1662	0.1154	0.1120
50	0.99	1.0	0.0040	0.9804	0.0001	0.5878	0.0070	0.0338	0.0637
100	0.50	1.0	0.0801	0.0608	0.0037	0.3094	0.0703	0.0628	0.0623
100	0.99	1.0	0.0039	0.9254	0.0001	0.3091	0.0068	0.0333	0.0626
250	0.50	1.0	0.0075	0.0157	0.0001	0.0377	0.0095	0.0137	0.0143
250	0.99	1.0	0.0031	0.4538	0.0000	0.0377	0.0037	0.0181	0.0344

demand sizes, interarrival times, and the lead times, are ME distributed. We computed $\phi(s, Q, n)$, $\psi(s, Q, n)$ and $\tau(s, Q, n)$ by formulas (4), (5) and (7) which are denoted by ϕ_{an}, ψ_{an} and τ_{an} respectively. Simulation was used to verify whether ϕ_{an}, ψ_{an}, and τ_{an} are equal to the related values computed by simulation, denoted by ϕ_{sim}, ψ_{sim}, and τ_{sim}, respectively. Furthermore, we calculated by simulation the fraction of the partial deliveries that crossed any partial deliveries of previously placed replenishment orders, which is denoted by Crossing.

The results of these experiments are aggregated in Table 2, in which each line represents the average of the absolute deviations of the performance measures over 180 experiments. Since the mean absolute deviations of ϕ and ψ have to be related to the absolute values of ϕ and ψ, we also computed the relative errors of the sum of inventory and backordering costs. That is, for $h = 1$ and $b = \{1, 10, 20\}$ we computed $\frac{|h(\phi_{an} - \phi_{sim}) + b(\psi_{an} - \psi_{sim})|}{h\phi_{sim} + b\psi_{sim}}$ (see columns $b = h$, $b = 10h$, and $b = 20h$).

From these experiments we can conclude the following about the quality of the expressions for the performance measures computed by the proposed algorithm in this section.

- For the situations in which $Q = 100$ or $Q = 250$, the algorithm performs good in all cases that are considered. Both the determination of s via $\tau(s, Q, n) = \tau_{an}$ and the computation of $\phi(s, Q, n)$ and $\psi(s, Q, n)$ yield accurate results.

- For the situations where $Q = 50, c_L = 1$, and $\tau_{an} = 0.50$, we see discrepancies between the target and achieved τ-level. The explanation for this deviation is expressed by the fraction of partial deliveries that cross, which is in these situations up to 59 % of the partial deliveries.

- For high values of τ_{an} we note that ψ_{an} deviates from ψ_{sim}. This has only a small impact on the computation of the sum of ordering and holding costs, which follows from the last columns in Table 2. This can be explained by the fact that for large values of b, the determination of the optimal values for the decision variables is basically a trade off between the ordering and holding costs.

- Interestingly, the crossing of orders does not influence the quality of the approximations for high values of τ_{an}, that is, high values of b.

These results point out that the proposed algorithm performs very well. We have to be careful only in situations where crossing of orders frequently occurs, or cases with low values of b.

In the following experiment we checked numerically whether assumption 7 is valid ($TRC(s^*(Q,n),Q,n)$ is convex in Q). Of course this is not the appropriate way of validating the convexity assumption. However, we have not been able to derive conditions for convexity. Therefore, we resort to a numerical investigation into the convexity of $TRC(s^*(Q,n),Q,n)$. For these experiments we fixed the following input values, $(\mathbb{E}D, c_D) = (10,1)$, $(\mathbb{E}A, c_A) = (1,1)$, $\mathbb{E}L = 10$, and $h = 0.01$. In Figures 2 to 5 we plotted $TRC(s^*(Q,n),Q,n)$ as function of Q. The authors did not find any numerical counter examples of the conjecture that $TRC(s^*(Q,n),Q,n)$ is convex. These figures show also that $Q^*(n)$ is increasing in n. Moreover, the optimal number of suppliers is depending on the input parameters. The cost parameters K, c, h, and b indeed influence n^* (compare Figures 2, 4 and 5). But also the parameters of the underlying lead time process do influence n^* (compare Figure 2 with Figure 3).

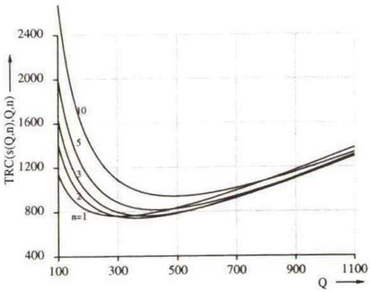

Figure 2: $TRC(s(Q,n),Q,n)$ as function of Q, where $c_L = 0.3$, $K = 30$, $c = 0.5$, and $b = 0.1$.

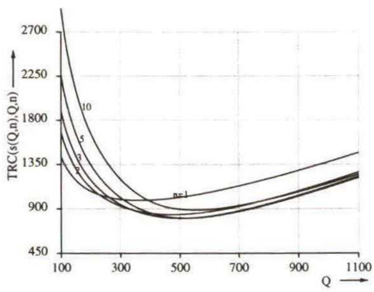

Figure 3: $TRC(s(Q,n),Q,n)$ as function of Q, where $c_L = 1$, $K = 30$, $c = 0.5$, and $b = 0.1$.

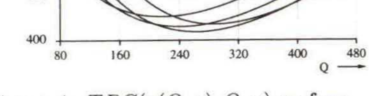

Figure 4: $TRC(s(Q,n),Q,n)$ as function of Q, where $c_L = 0.5$, $K = 5$, $c = 0.5$, and $b = 0.1$.

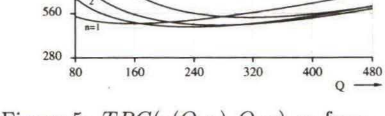

Figure 5: $TRC(s(Q,n),Q,n)$ as function of Q, where $c_L = 0.5$, $K = 5$, $c = 1$, and $b = 0.1$.

6 The optimal number of suppliers

From Figures 2 to 5, it is clear that n^* depends on the values of the input parameters. Therefore, we designed a number of experiments to get some insight into the optimal number of suppliers.

First of all we compared our results with the results presented by Ramasesh et al.(1991). Ramasesh et al. consider the same objective, under constant demand and with at most two suppliers. Hence the model differs from the model discussed in this paper. We fit the parameters of our model to the parameters of the model in Ramasesh et al.(1991) as follows. By considering small interarrival time of customers and low coefficient of variations of D and A, we can approximate the model considered by Ramasesh et al.(1991). Moreover, the ordering costs in Ramasesh et al. (1991) for the two supplier situation are given by $A(2) = \alpha K$, where $\alpha \in [1, 2]$. Hence, the appropriate choice of c is $\log_2 \alpha$. The results for $n = 1$ and $n = 2$ are similar, see Table 3, where (J) denotes the results of our model and (R) the results of the model of Ramasesh et al. (1991). The optimal values of s and Q and the value of the total relevant costs are almost equal. For the situation that $\alpha = 1$ (i.e. $c = 0$) the optimal number of suppliers is infinity, as was already noted by Larson(1989). Furthermore, we note that for values of $\alpha > 1$, using two suppliers can be advantageously, but is not optimal (see $c = 0.263$).

Table 3: Comparison results from Ramasesh et al. (1991) with our results

c		$n=1$ s_1^*	Q_1^*	TRC	$n=2$ s_2^*	Q_2^*	TRC	$n=3$ s_3^*	Q_3^*	TRC	n^*	TRC
0.00	(J)	188.6	1275.2	1222.4	33.7	1334.1	977.7	-6.8	1309.2	889.9	∞	-
	(R)	191.3	1271.2	1220.3	35.6	1333.7	981.2					
0.263	(J)	188.6	1275.2	1222.4	22.6	1407.0	1050.6	-22.0	1418.9	1012.7	4	1010.30
	(R)	191.3	1271.2	1220.3	24.0	1408.0	1054.1					
0.678	(J)	188.6	1275.2	1222.4	5.2	1539.6	1186.3	-51.7	1651.4	1264.1	2	1186.30
	(R)	191.3	1271.2	1220.3	5.7	1542.2	1189.6					
0.761	(J)	188.6	1275.2	1222.4	1.8	1569.1	1216.9	-58.6	1708.6	1324.1	2	1216.92
	(R)	191.3	1271.2	1220.3	2.0	1573.4	1220.3					

In the experiments that follow we take one day as the basic time unit, and one year equal to 250 (working) days. We investigate the effect of the cost parameters K, c, b and h on the optimal number of suppliers. We fixed the following values for the system parameters: $(\mathbb{E}D, c_D) = (10, 1)$, $(\mathbb{E}A, c_A) = (1, 1)$, and $(\mathbb{E}L, c_L) = (10, 0.5)$. We fixed c equal to 0.5 and the inventory holding cost h equal to 0.04. This represents an article with purchase price of \$40 and a opportunity factor of 0.25 /\$/\$/year. First we varied b between 1, 10, 100, and 1000 times h, and for each setting we calculated the optimal number of suppliers as function of K (see Figures 6 and 7). The number of values chosen for K is equal to 100 for each value of b. To generate Figures 6 and 7 required about 14 minutes CPU time on a SUNSPARC-station 4. We see that $n^* \to \infty$ when $K \downarrow 0$, and $n^* = 1$ when $K \to \infty$, which is also intuitively clear. Moreover, n^* increases when $\frac{b}{b+h}$ increases. And n^* decreases when c increases (compare Figures 6 with 7), which is intuitively clear, as well.

In the final experiments we investigate the effect of the parameters of the underlying stochastic processes $(\mathbb{E}D, c_D)$, $(\mathbb{E}A, c_A)$, and $(\mathbb{E}L, c_L)$. We considered situations in which $K = 20$, $c = 0.5$, $h = 0.04$, and $b = 0.4$. We started with $(\mathbb{E}D, c_D) = (10, 1)$, $(\mathbb{E}A, c_A) = (1, 1)$, and $(\mathbb{E}L, c_L) = (10, 0.5)$, as in the basic situation, however in each experiment we varied one or two of these system parameters.

In Figure 8 we computed n^* as function of $\mathbb{E}D$, for various values of $\mathbb{E}L$. We note that n^* is almost linear in both $\mathbb{E}D$ and $\mathbb{E}L$. In Figure 9 we varied $\mathbb{E}A$. Similar to the effect of K, we see that $n^* \to \infty$ when $\mathbb{E}A \downarrow 0$, and $n^* = 1$ when $\mathbb{E}A \to \infty$.

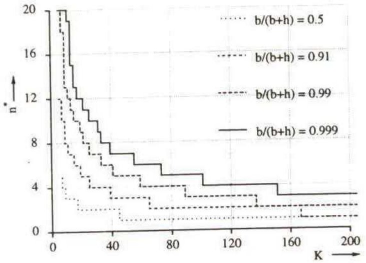

Figure 6: The optimal number of suppliers as function of K with $c = 0.5$.

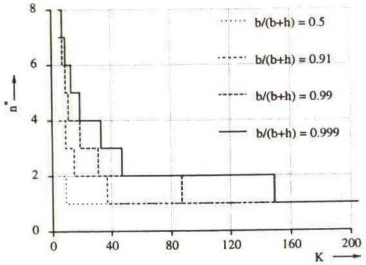

Figure 7: The optimal number of suppliers as function of K with $c = 1$.

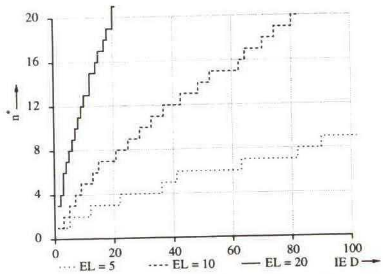

Figure 8: The optimal number of suppliers as function of $\mathbb{E}D$.

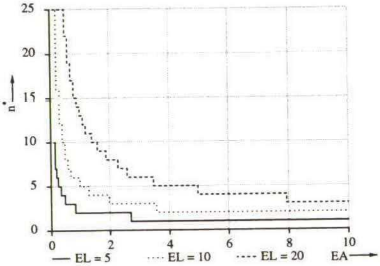

Figure 9: The optimal number of suppliers as function of $\mathbb{E}A$.

In case the coefficients of variation of D and A are varied, we only find minor effects on the optimal number of suppliers. In Figure 10 we varied c_A. It is important to note that higher values of c_A can lead to both lower and higher values of n^*. A detailed investigation of the solutions is given for $\mathbb{E}L = 20$ and c_A is varied between 1.1 and 1.2 (see Table 4). The differences between $TRC(s^*(12), Q^*(12), 12)$ and $TRC(s^*(13), Q^*(13), 13)$ are very small, and for some values of c_A the $TRC(s^*(12), Q^*(12), 12)$ is smaller than $TRC(s^*(13), Q^*(13), 13)$ and for other values the other way around. When n increases, the optimal reorder point will decrease, however, the optimal reorder quantity will increase. Hence, the inventory holding costs may increase or decrease.

The impact on n^* of c_D are similar to the effects of c_A. In contrast with this, n^* turns out to be very sensitive to the value of c_L. In Figure 11 we varied both $\mathbb{E}L$ and c_L. This

Table 4: Detailed investigation of the solutions

c_A	n	$TRC(s^*(n), Q^*(n), n)$	$s^*(n)$	$Q^*(n)$
1.1	1	3051.58	290.78	184.21
	12	2091.28	95.36	445.31
	13	2090.65	93.26	447.99
	14	2091.41	91.41	450.39
1.2	1	3091.85	291.86	185.81
	12	2146.90	97.01	452.29
	13	2146.91	94.88	454.77
	14	2147.94	91.35	457.08

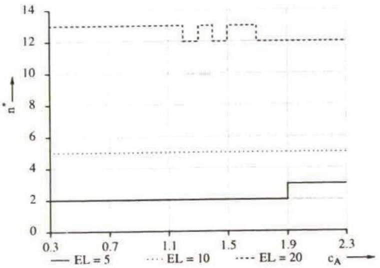

Figure 10: The optimal number of suppliers as function of c_A.

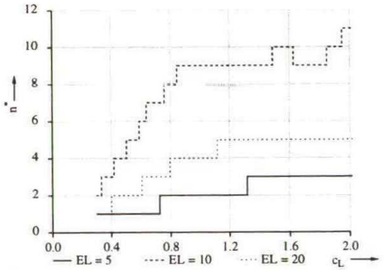

Figure 11: The optimal number of suppliers as function of c_L.

sensitivity can be explained by considering effects of c_L that interfere. Namely when c_L increases, the first orders will arrive earlier, which leads to lower values of the reorder point. But due to the earlier arrival of the partial deliveries the expected average physical stock will slightly increase. Finally, it is noteworthy that often there are only minor differences in the total relevant cost for two successive values of n (see, for example Table 4).

7 Conclusions and future research

In this paper an (s, Q) inventory model is presented with order splitting, where the demand is modelled as a compound renewal process, and lead times of the suppliers are independent and identically distributed random variables. This model can be applied to many practical situations.

We derived expressions for the expected average physical stock, the expected average backlog level, and the fraction of the time that the physical stock is positive. Furthermore, an algorithm is derived to compute these performance measures based only on moments of the underlying demand and lead time process. The algorithm turned out to perform very good for situations in which the number of order crossings was not too high. Although the performance measures are derived for non-identically distributed lead times of suppliers, the algorithm is only developed for identically distributed lead times. Clearly this is a topic of future research.

We considered the problem of determining the appropriate values for the control parameters s, Q, and n. We minimized the sum of ordering, holding, and backordering costs. The optimal number of suppliers turned out to be very sensitive for the combination of input parameters. A striking observation was that n^* is not always increasing when the coefficient of variation of the lead times does. The algorithm can be used to generate graphical support instantaneously for a wide range of input values.

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Appendix 1: Proof of relation (4) and (5)

Given a random variable X with distribution function F and finite mean, and a random variable Y with distribution function G and finite mean, then the distibution function of the convolution of X and Y will be denoted by $(F * G)$, the distribution of the $n-$fold convolution of X with itself is denoted by F^{n*}, and the renewal function, M, associated with F is defined as $M(x) = \sum\limits_{n=0}^{\infty} F^{n*}(x)$.

Define $H(x)$ (and $\tilde{H}(x)$) as the expected area between the physical inventory level and the zero level, in case the physical stock level on epoch 0 equals x ($x \geq 0$), there are no outstanding replenishment orders, and time epoch 0 is an arrival moment of a customer (for $\tilde{H}(x)$ time epoch zero is an arbitrary moment in time). By conditioning on the first arriving customer after time epoch 0, we find

$$H(x) = x\mathbb{E}A + \int_0^x H(x-y)dF_D(y). \tag{A.1.1}$$

Let M be the renewal function associated with F_D, then writing out recurrence relation (A.1.1) yields

$$H(x) = \mathbb{E}A \int_0^x (x-y)dM(y) \tag{A.1.2}$$

Consider the situation that zero is an arbitrary point in time, and let $\tilde{A}$ be the arrival time of the first customer after zero. Then $\tilde{A}$ is the excess life at time epoch zero with respect to the arrival process of customers. Since zero is an arbitrary point in time, and using standard renewal theory, yields the well-known result

$$\mathbb{P}(\tilde{A} \leq x) \simeq \frac{1}{\mathbb{E}A} \int_0^x (1 - F_A(y))dy, \tag{A.1.3}$$

where

$$\mathbb{E}\tilde{A} = \frac{\mathbb{E}A^2}{2\mathbb{E}A}, \tag{A.1.4}$$

$$\mathbb{E}\tilde{A}^2 = \frac{\mathbb{E}A^3}{3\mathbb{E}A}. \tag{A.1.5}$$

By conditioning on the first arriving customer after time epoch 0, results into

$$\tilde{H}(x) = x\mathbb{E}\tilde{A} + \int_0^x H(x-y)dF_D(y). \tag{A.1.6}$$

Using relations (A.1.1) and (A.1.2) gives

$$\tilde{H}(x) = (\mathbb{E}\tilde{A} - \mathbb{E}A)x + \mathbb{E}A\int_0^x (x-y)dM(y) \tag{A.1.7}$$

Lemma A.1.1.
Let M be the renewal function associated with F_D, and let U be the equilibrium excess distribution of D, then

$$(M * U)(x) = \frac{x}{\mathbb{E}D} \tag{A.1.8}$$

Proof:
Let $\tilde{F}_D(y)$ be the Laplace transform of F_D, thus $\tilde{F}_D(y) = \int_0^\infty e^{-yx}dF_D(x)$. Since $\tilde{U}(y) = (1-\tilde{F}_D(y))/(y\mathbb{E}D)$ and $\tilde{M}(y) = 1/(1-\tilde{F}_D(y))$, it follows that the Laplace transform of the convolution equals $1/(y\mathbb{E}D)$. Hence, taking the inverse Laplace transform of $1/(y\mathbb{E}X)$ yields $(M*U)(x) = x/\mathbb{E}D$. □

Lemma A.1.2.
Let M be the renewal function associated with F_D, and let U be the equilibrium excess distribution of D. Furthermore, let Y be a positive random variable with distribution function G. Then, for $s > 0$,

$$\int_0^s \tilde{H}(s-x)d(G*U)(x) = (\mathbb{E}\tilde{A} - \mathbb{E}A)\int_0^s (s-x)d(G*U)(x) + \mathbb{E}A\int_0^s \frac{(s-x)^2}{2\mathbb{E}D}dG(x) \tag{A.1.9}$$

Proof:
Using lemma A.1.1. it is easily seen that

$$\int_0^s \tilde{H}(s-x)d(G*U)(x)$$

$$= (\mathbb{E}\tilde{A} - \mathbb{E}A)\int_0^s (s-x)d(G*U)(x) + \mathbb{E}A\int_0^s\int_0^{s-x}(s-x-y)dM(y)d(G*U)(x)$$

$$= (\mathbb{E}\tilde{A} - \mathbb{E}A)\int_0^s (s-x)d(G*U)(x) + \mathbb{E}A\int_0^s\int_0^{s-x}(s-x-y)d(M*U)(y)dG(x)$$

$$= (\mathbb{E}\tilde{A} - \mathbb{E}A)\int_0^s (s-x)d(G*U)(x) + \mathbb{E}A\int_0^s \frac{(s-x)^2}{2\mathbb{E}D}dG(x)$$

□

Now, consider the k-th sub-cycle ($k \in \{1,\ldots,n\}$). The physical stock at the beginning of the k-th sub-cycle (just after the replenishment arrived) is equal $(I_k^b)^+$, whereas the physical stock at the end of the k-th sub-cycle (just before the replenishment arrives) is equal to $(I_k^e)^+$. Then it is easy to see that the expected area between the physical inventory level and the zero level within the k-th sub-cycle is given by $\mathbb{E}\tilde{H}((I_k^b)^+) - \mathbb{E}\tilde{H}((I_k^e)^+)$. By using (A.1.7), Lemma A.1.2, and by conditioning on I_k^b, we find

$$\begin{aligned}
\mathbb{E}\tilde{H}((I_k^b)^+) &= \int_0^{s+\frac{k-1}{n}Q} \tilde{H}(s+\frac{k-1}{n}Q - x)dF_{D(\sigma,\sigma+L_{k-1:n})+U}(x) \\
&= (\mathbb{E}\tilde{A} - \mathbb{E}A)\int_0^{s+\frac{k-1}{n}Q} (s+\frac{k-1}{n}Q - x)dF_{D(\sigma,\sigma+L_{k-1:n})+U}(x) \\
&\quad + \frac{\mathbb{E}A}{2\mathbb{E}D}\int_0^{s+\frac{k-1}{n}Q} (s+\frac{k-1}{n}Q - x)^2 dF_{D(\sigma,\sigma+L_{k-1:n})}(x) \\
&= (\mathbb{E}\tilde{A} - \mathbb{E}A)\mathbb{E}((I_k^b)^+) + \frac{\mathbb{E}A\mathbb{E}((I_k^b+U)^+)^2}{2\mathbb{E}D}
\end{aligned}$$

and for $\mathbb{E}\tilde{H}((I_k^b)^+)$ an analogue expression can be derived.

Finally using that the length of a replenishment cycle equals $\frac{Q\mathbb{E}A}{\mathbb{E}D}$ and summing up the expected area's of the n sub-cycles, yields

$$\begin{aligned}
\phi(s,Q,n) &= \sum_{k=1}^n \left(\mathbb{E}\tilde{H}((I_k^b)^+) - \mathbb{E}\tilde{H}((I_k^e)^+)\right) \\
&= \gamma\sum_{k=1}^n \frac{\mathbb{E}(I_k^b)^+ - \mathbb{E}(I_k^e)^+}{Q} \\
&\quad + \sum_{k=1}^n \frac{\mathbb{E}((I_k^b+U)^+)^2 - \mathbb{E}((I_k^e+U)^+)^2}{2Q} \qquad \text{(A.1.10)}
\end{aligned}$$

where $\gamma = 1/2\left(c_A^2 - 1\right)\mathbb{E}D$.

Note that (A.1.10) is approximate since the distribution functions of $\tilde{A}$ and U are approximated by the associated residual life time distributions.

For the proof of expression (5) for the average backlog we will use the well-known relation that the inventory position equals the physical stock plus on order minus the backlog (see, for example, Hadley and Whitin (1963, p. 187)). The expected inventory position is equal to $s+Q/2$, and that the expected amount on order is given by $\sum_{k=1}^{n} \frac{\mathbb{E}D\mathbb{E}L_{k:n}}{n\mathbb{E}A}$. The latter equality can be shown analogously to the arguments of Hadley and Whitin. Imagine that orders flow into one end of a pipeline and procurements flow out of the other end. For $k \in \{1, \ldots, n\}$ the k-th partial delivery remains on average $\mathbb{E}L_{k:n}$ time units in the pipeline. A single demand unit has equal probability to be delivered from the k-th ($k \in \{1, \ldots, n\}$) partial delivery, and the expected flow out of the pipeline equals $\frac{\mathbb{E}D}{\mathbb{E}A}$. Therefore, the expected number of units in the pipeline should be $\sum_{k=1}^{n} \frac{\mathbb{E}D\mathbb{E}L_{k:n}}{n\mathbb{E}A}$.

Hence,

$$\psi(s,Q,n) = \phi(s,Q,n) - (s+Q/2) + \sum_{k=1}^{n} \frac{\mathbb{E}D\mathbb{E}L_{k:n}}{n\mathbb{E}A}. \tag{A.1.11}$$

Note that for $k \in \{1, \ldots, n\}$ $\frac{\mathbb{E}D\mathbb{E}L_{k:n}}{\mathbb{E}A} = \mathbb{E}D(\sigma, \sigma + L_{k:n}) - \gamma$. Substitution of (A.1.10) into (A.1.11) yields

$$\begin{aligned}
\psi(s,Q,n) &= \gamma \sum_{k=1}^{n} \frac{\mathbb{E}(I_k^b)^+ - \mathbb{E}(I_k^e)^+}{Q} \\
&\quad + \sum_{k=1}^{n} \frac{\mathbb{E}((I_k^b+U)^+)^2 - \mathbb{E}((I_k^e+U)^+)^2}{2Q} - (s+Q/2) + \sum_{k=1}^{n} \frac{\mathbb{E}D\mathbb{E}L_{k:n}}{n\mathbb{E}A} \\
&= \gamma\Big(\sum_{k=1}^{n} \frac{\mathbb{E}(I_k^b)^+ - \mathbb{E}(I_k^e)^+}{Q} - 1\Big) \\
&\quad + \sum_{k=1}^{n} \frac{\mathbb{E}((I_k^b+U)^+)^2 - \mathbb{E}((I_k^e+U)^+)^2}{2Q} \\
&\quad - \frac{2sQ+Q^2}{2Q} + \sum_{k=1}^{n} \frac{\frac{2Q}{n}\mathbb{E}D(\sigma,\sigma+L_{k:n})}{2Q} \\
&= -\gamma \sum_{k=1}^{n} \frac{\mathbb{E}(-I_k^e)^+ - \mathbb{E}(-I_k^b)^+}{Q} \\
&\quad + \sum_{k=1}^{n} \frac{\mathbb{E}(-(I_k^e+U)^+)^2 - \mathbb{E}(-(I_k^b+U)^+)^2}{2Q} \\
&\quad + \frac{1}{2Q}\Big[(s+Q)^2 - 2(s+Q)\mathbb{E}D(\sigma,\sigma+L_{n:n}) + \mathbb{E}D(\sigma,\sigma+L_{n:n})^2
\end{aligned}$$

$$
\begin{aligned}
& - s^2 + 2s\mathbb{E}D(\sigma, \sigma + L_{1:n}) - \mathbb{E}D(\sigma, \sigma + L_{1:n})^2 - 2sQ - Q^2 \\
& + \sum_{k=2}^{n} \Big((s + \frac{k-1}{n}Q)^2 - 2(s + \frac{k-1}{n}Q)\mathbb{E}D(\sigma, \sigma + L_{k-1:n}) + \mathbb{E}D(\sigma, \sigma + L_{k-1:n})^2 \\
& \qquad - (s + \frac{k-1}{n}Q)^2 + 2(s + \frac{k-1}{n}Q)\mathbb{E}D(\sigma, \sigma + L_{k:n}) - \mathbb{E}D(\sigma, \sigma + L_{k:n})^2 \Big) \Big] \\
& + \sum_{k=1}^{n} \frac{\frac{2Q}{n}\mathbb{E}D(\sigma, \sigma + L_{k:n})}{2Q} \\
= & \sum_{k=1}^{n} \frac{\mathbb{E}((-(I_k^e + U))^+)^2 - \mathbb{E}((-(I_k^b + U))^+)^2}{2Q} \\
& - \gamma \sum_{k=1}^{n} \frac{\mathbb{E}(-I_k^e)^+ - \mathbb{E}(-I_k^b)^+}{Q}.
\end{aligned}
$$

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