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The All-Pay Auction with Complete Information

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Publication date:
1990

Link to publication in Tilburg University Research Portal

Citation for published version (APA):
Baye, M., Kovenock, D., & de Vries, C. (1990). *The All-Pay Auction with Complete Information*. (CentER Discussion Paper; Vol. 1990-51). CentER.

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No. 9051

THE ALL-PAY AUCTION WITH COMPLETE INFORMATION

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R20
330.115.2

August 1990

ISSN 0924-7815

THE ALL-PAY AUCTION WITH COMPLETE INFORMATION*

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Preliminary draft : March 1989
Working Paper : June 1990

ABSTRACT

This paper provides an exhaustive and explicit description of the set of Nash equilibria in the n-player, first price sealed bid, all pay auction under complete information. Both the cases of homogeneous and heterogeneous valuations are analyzed. For the common values case with more than two players we show there is a unique symmetric equilibrium and a continuum of asymmetric equilibria. All of the equilibria, however, are payoff and revenue equivalent. With heterogeneous valuations, two new situations can arise. First, if the three highest valuations are strictly unequal then there is a unique asymmetric equilibrium. Second, with a single highest valuation and more than one player with the second highest valuation, there is a continuum of asymmetric equilibria. In both of these latter cases, the expected sum of the bids is below the second highest valuation, and depends on the strategies of the agents with the second highest valuation. Hence, while the equilibria are payoff equivalent, they are not revenue equivalent. The continua of asymmetric equilibria were missed by both the theoretical literature, and the applied literature on e.g. rent seeking and rent dissipation.

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* We are grateful to Chuangyin Dang, Chaim Fershtman, Arthur Robson, Heinrich Ursprung and Ton Vorst for helpful conversations. The paper was completed while Baye was visiting the CentER for Economic Research at Tilburg University.

1. INTRODUCTION

Consider the public auction of a dollar, in which each of n-bidders places money in an envelope. The money in the envelopes is collected and kept by the seller, and the dollar is awarded to the bidder who placed the highest amount of money in his envelope (ties are broken in an arbitrary fashion). This auction, which is called an all-pay auction [cf. Moulin (1986 a,b) and Weber (1985)], is important because many economic problems under complete information have a similar structure. For example, Hillman and Samet (1987) and Hillman (1988) model lobbying as an all-pay auction, where the lobbying parties sweeten the decisionmaker by making a bribe, and the prize (a political favor) is awarded to the party having given the highest bribe. Similarly, much of the contest and principal agent literature under complete information is isomorphic to the all-pay auction; cf. Nalebuff and Stiglitz (1983) and Baye, Kovenock, and de Vries (1989b). Essentially, contests are an all-pay auction in effort : the person putting forth the greatest effort wins the prize, while the effort of other contestants goes unrewarded.

This paper completely characterizes the set of Nash equilibria in the all-pay auction with complete information. We show that the set of equilibria is much larger than had originally been thought. Moreover, in some economically interesting cases, the equilibria are not revenue equivalent.

Before we present a more specific statement of our results and their proofs, it is useful to describe our results vis à vis the existing literature on the all-pay auction. Two cases have been considered in the literature: (1) the case where all players value identically the prize, and (2) the case where some players value the prize more than others. For the case of homogeneous valuations, Hillman and Samet have shown that, in addition to a symmetric Nash equilibrium, there also exist a finite number of asymmetric equilibria. We extend this result by showing that there is actually a <u>continuum</u> of asymmetric equilibria. In each equilibrium, at least two players randomize continuously over the union of the supports of the players' equilibrium mixed-strategies, while up to $n - 2$ players may have a mass point at zero and only randomize over a strict subset of the union of other players' supports. The existence of these additional equilibria has obvious empirical implications. However, for the case of homogeneous valuations, all of these equilibria are payoff and revenue equivalent: The expected sum of the bids equals the value of the prize, and the net expected pay-off to each bidder is zero for all equilibria.

The second case is when several players have heterogeneous valuations of the prize. For the case where the second highest valuation of the prize is strictly greater than the third highest valuation[1], Hillman and Riley (1989) have shown that there is a unique equilibrium and that only the two players with the highest valuations bid a positive amount withe positive probability. Furthermore, they show that if the highest valuation is strictly greater than the second highest, the expected sum of bids is less than the second-highest valuation.

We extend Hillman and Riley's analysis of the heterogeneous valuations case by considering other configurations of individual valuations. One of the more important configurations of valuations is where a single player values most the prize, while all other players value the prize at some common, lower value. This case is economically interesting, because in much of the literature on regulation [cf. Rogerson (1982)] and political contests [cf. Snyder (1989)], one player (usually an incumbent) is modeled as having an advantage over identical challengers. For this case, we show not only that there is a continuum of equilibria, but that the equilibria are not revenue equivalent: the expected sum of bids differs across equilibria.

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1 More precisely, if the players can be ordered in such a way that $v_1 \geq v_2 > v_3 \geq \ldots \geq v_n$, where v_i is the valuation of player i.

The results presented in the present paper are important for two independent reasons. First, and as noted above, our results reveal a wider array of behavior consistent with equilibrium. Second, given the existing literature on the all-pay auction, one might be tempted to take two-ness as a necessary implication of contests. In fact, Hillman (1988, p. 66) claims that, if there are ties for the second highest valuation and a single highest valuation, only two agents will be active. Our results reveal that this is incorrect, and indeed the additional equilibria imply different expected revenues. The fact that there are additional equilibria and that the revenues differ across the equilibria serves as a caveat of Magee, Brock, and Young's (1989, p. 217) argument that two-ness is a general property of political contests. Our results reveal that this is correct only if two contenders value of prize more than all other contenders. This may be why there are typically more than two challengers to an incumbent in, for example, presidential campaigns.

The paper is organized is follows. Section 2 considers the case of homogeneous valuations, while Section 3 examines the situation where some agents have heterogeneous valuations of the prize. The full characterization of the continuum of equilibrium strategies requires several steps which are labelled as lemmas. The main results are collected in six theorems. Interestingly and importantly, we are able to derive closed form expressions for all the equilibrium strategies. Therefore, the reader can easily obtain the intuition behind the main results, e.g. verifying that the strategies satisfy the Nash property, by working out examples on the basis of these expressions. In fact, we employ this strategy to derive Theorem 6 on revenue non equivalence.

2. HOMOGENEOUS VALUATIONS

Suppose first that an object known to be worth $v > 0$ dollars to each of n bidders is to be auctioned. The n bidders simultaneously write down a bid. If player i bids the most he wins the object. All players pay the seller the amount that they bid.

Without loss of generality let the strategy set be $X_i = [0, B]$, where $B > v$ is some large number. The payoff function for player i in this game is

$$u_i(x_1, \dots, x_n) = \begin{cases} -x_i & \text{if } \exists\, j \text{ s.t. } x_j > x_i, \\ \frac{v}{m} - x_i & \text{if i ties for high bid with m-1 others,} \\ v - x_i & \text{if } x_i > x_j \quad \forall\ j \neq i. \end{cases}$$

Let $\underline{s}_i$ and $\bar{s}_i$ denote the lower and upper bound of player i's equilibrium bid distribution G_i. Also, let α_i denote the size of a mass-point in i's distribution. Let u_i^* be player i's equilibrium profit. It is easily shown that with more than one bidder there does not exist a Nash equilibrium in pure strategies. In order to construct the mixed-strategy Nash equilibria, we first obtain the supports of the mixed strategies in Lemmas 1-9.

Lemma 1: $\forall$ i, $v \geq \bar{s}_i \geq \underline{s}_i \geq 0$.

Proof: By setting $x_i = 0$ each player can guarantee at least 0. This rules out bids greater than v. Bids less than 0 are ruled out a priori.

Lemma 2: If $\exists$ i such that $\underline{s}_i \geq \underline{s}_j$ and $\alpha_i(\underline{s}_j) = 0$, then $\underline{s}_j = 0$ and $G_j(0) = \lim_{x \uparrow \underline{s}_i} G_j(x)$. If, in addition, $\alpha_i(\underline{s}_i) = 0$, then $G_j(0) = G_j(\underline{s}_i)$.

Proof: Let $u_j(x_j, G_{-j})$ denote j's payoff to bidding x_j when strategies G_{-j} are employed by the other n-1 players. Now $u_j(\underline{s}_j, G_{-j}) = -\underline{s}_j < 0$ for $\underline{s}_j > 0$. Since the same holds for $u_j(x_j, G_{-j})$ for $x_j < \underline{s}_i$, and $x_j = \underline{s}_i$ if $\alpha_i(\underline{s}_i) = 0$, the claim follows.

Lemma 3: If $\underline{s}_1 = \ldots = \underline{s}_m > \underline{s}_{m+1} \geq \ldots \geq \underline{s}_n$ for $n \geq m \geq 2$, then $\exists$ $i \leq m$ such that $\alpha_i(\underline{s}_i) = 0$.

Proof: Suppose not. Then any $i \leq m$ has incentive to raise $\underline{s}_i$ by ϵ small.

Lemma 4: If $\underline{s}_1 = \ldots = \underline{s}_m > \underline{s}_{m+1} \geq \ldots \geq \underline{s}_n$, for $n \geq m \geq 2$, then $\underline{s}_i = 0 \;\forall\; i$.

Proof: Immediate from Lemmas 2 and 3.

Lemma 5: There exists no player i such that $\underline{s}_i > \underline{s}_j \;\forall\; j \neq i$.

Proof: Suppose such a player did exist. If $\alpha_i(\underline{s}_i) = 0$, from Lemma 2 $G_j(0) = G_j(\underline{s}_i) \;\forall\; j \neq i$, which implies that $u_i(\underline{s}_i, G_{-i}) < \lim_{x_i \downarrow 0} u_i(x_i, G_{-i})$. If the claim held and $\alpha_i(\underline{s}_i) > 0$ then

$\forall\ j \neq i$, $\alpha_j(\underline{s}_i) = 0$, so $G_j(0) = \lim_{x_j \uparrow \underline{s}_i} G_j(x_j)$ leads to a similar contradiction.

Lemma 6: $\underline{s}_i = 0 \ \ \forall\ i$.

Proof: Immediate from Lemmas 4 and 5.

Lemma 7: $u_i^* = u_j^* \ \ \forall\ i,j$.

Proof: Without loss of generality suppose $u_i^* < u_j^*$. Let $\bar{s}_j$ be the upper bound of j's support. $u_i^* < u_j^* = u_j(\bar{s}_j, G_{-j}) \leq \lim_{x_i \downarrow \bar{s}_j} u_i(x_i, G_{-i})$, a contradiction.

Lemma 8: $u_i^* = 0 \ \ \forall\ i$.

Proof: If $\alpha_i(\underline{s}_i) = 0$, $\forall\ i$ we are through. If $\exists\ j$ such that $\alpha_j(\underline{s}_j) > 0$, then $u_j^* = 0$ from Lemmas 3 and 6, and with players receiving equal utility from Lemma 7, $u_i^* = 0 \ \ \forall\ i$.

Lemma 9: $\exists\ i,j$ such that $\bar{s}_i = \bar{s}_j = v$.

Proof: Suppose not. Let $\bar{s}_i$ be the second highest $\bar{s}_j$. The player with the highest $\bar{s}_j$ can bid slightly above $\bar{s}_i$ and earn $u_j = v - \bar{s}_i > u_j^*$.

The nine lemmas above establish that $\underline{s}_i = 0 \ \ \forall\ i$; there exist two i's, say $i = 1,2$, such that $\bar{s}_1 = \bar{s}_2 = v$; and $u_i^* = 0 \ \ \forall\ i$. We now pin down the equilibrium distributions. Let $W(x_i) = v - x_i$, $L(x_i) = -x_i$,

$$A_i = \prod_{\substack{j=1 \\ j\neq i}}^{n} G_j, \quad A_{ij} = \prod_{\substack{k=1 \\ k\neq j,i}}^{n} G_k, \text{ and } A_{ijm} = \prod_{\substack{h=1 \\ h\neq j,i,m}}^{n} G_h.$$

Lemma 10: There are no point masses on the half open inverval $(0, v]$.

Proof : Suppose one of the cumulative distribution functions (c.d.f.s), say G_i, has a mass point at $x_i \epsilon (0, v]$. By Lemma 6, $\forall\, x \epsilon (0, v]$ $A_{ij}G_i > 0$, and hence $A_{ij}G_i$ has an upward jump at x_i, $\forall j \neq i$. This follows directly from the monotonicity of the c.d.f.'s. For $x_i < v$ this implies that it is worthwhile for j to transfer mass from an ϵ-neighborhood below x_i to some δ neighborhood above x_i. At $x_i = v$ it pays for j to transfer mass from an ϵ-neighborhood below x_i to zero. Thus, there would be an ϵ-neighborhood below x_i in which no other player j would put mass. But then it is not an equilibrium strategy for player i to put mass at x_i.

Lemma 11: The integrand

$$B_i(x_i) \equiv W(x_i)A_i(x_i) + L(x_i)(1 - A_i(x_i)) \tag{1}$$

is constant and equal to zero at the points of increase of G_i in the half-open interval $(0, v]$ for all i.

Proof : By Lemma 10, there are no point masses in $(0, v]$. Thus, $B_i(x_i)$ is the expected payoff to player i from bidding $x_i \epsilon (0, v]$. If x_i is a point of increase of G_i, then player i must make its equilibrium payoff at x_i.

Lemma 12: Suppose x is a point of increase of G_i and G_j in (0, v]. Then $G_i = G_j$ at x.

Proof : By Lemma 8, $B_i(x) = B_j(x) = 0$. From (1) we have

$$W(x)\ G_j(x)\ A_{ij}(x) + L(x)[1 - G_j(x)A_{ij}(x)] = 0.$$

This implies $G_j(x)A_{ij}(x) = \frac{-L(x)}{W(x)-L(x)} = G_i(x)A_{ji}(x)$.

Division by $A_{ij}(x) = A_{ji}(x) > 0$ gives $G_j(x) = G_i(x)$. (Note: L(x) is negative in the half-open interval).

Lemma 13: For every i and every point of increase x of G_i in (0, v], there is at least one G_j, $j \neq i$, such that G_j is increasing at x.

Proof : Because $B_i(x)$ is constant in a neighborhood about x by Lemma 11, $dB_i(x) = 0$. Suppose contrary to the hypothesis that $dA_i(x) = 0$. Totally differentiating $B_i(x)$ then gives

$$A_i dW + (1-A_i)dL = 0.$$

However, both dW and dL are negative and $A_i(x) \in (0, 1]$. Hence, for dB_i to be zero, dA_i is necessarily positive. By the monotonicity of the G_j's, at least one has to increase.

Lemma 14: If G_i is strictly increasing on some open interval (a,b), $0 < a < b < v$, then G_i is strictly increasing on (a, v].

Proof : Without loss of generality, suppose, to the contrary, that G_i were constant on (b,c), $b < c \leq v$. Then from Lemma 10, $G_i(b) = G_i(c)$. It is evident that there exists an $\epsilon > 0$ such that on the interval $(b, b+\epsilon)$ there exist at least two players, say h and k, with strictly increasing c.d.f.'s over the interval (otherwise mass would be moved down to b by some player). Thus, for every $x \in (b, b+\epsilon)$, $B_h(x) = B_k(x) = 0$. Furthermore, since there are no mass points in the interval $(0, v]$, $B_h(b) = B_k(b) = B_i(b) = 0$ which, from arguments similar to those used in proving Lemma 12, implies that $G_h(b) = G_k(b) = G_i(b) > 0$. But with $B_i(b) = B_h(b) = B_h(x)\ \forall\ x \in (b, b+\epsilon)$, it must be that $B_i(x) \leq B_h(x)$ $\forall\ x \in (b, b+\epsilon)$, since such values of x do not lie in i's support. But this implies that $A_i(x) \leq A_h(x)$, and hence that $G_h(x) \leq G_i(x)$, a contradiction to the fact that $G_i(b) = G_h(b)$, $G_h(x)$ is increasing on $(b, b+\epsilon)$, and $G_i(x)$ is constant on $(b, b+\epsilon)$.

Lemma 15: At least two players randomize continuously on $[0, v]$.

Proof : Three cases are possible at 0: (i) all players allocate all mass at 0, (ii) all players have $G_i(x_i) = 0$ at some $x_i > 0$, or (iii) there is at least one player with $G_i(x_i) > 0$ for all $x_i > 0$ and $G_i(0) < 1$. Cases (i) and (ii) are easily ruled out by previous lemmas. For the third case, by Lemmas 3 and 6 at least one of the players has $G_i(0) = 0$. Lemmas 12, 13, and 14 then imply that there are at least two players that randomize continuously over $[0, v]$.

Lemma 16: Once G_i is constant on a subset (a,b), $0 < a < b \leq v$, it is constant on $[0,b)$ and has a mass point at 0.

Proof : The first part follows immediately from Lemma 14. The second part follows from Lemma 6.

The above lemmas imply the following result:

Theorem 1: The first price sealed bid all pay common values auction with complete information possesses two types of equilibria. Either all players use the same continuous mixed strategy with support $[0, v]$, or at least two players randomize over $[0, v]$ with each other player i randomizing over $(b_i, v]$, $b_i > 0$,[2] and having a masspoint at 0 equal to $G_i(b_i)$. When two or more players have a positive density over a common interval they play the same continuous mixed strategy over that interval.

Theorem 1 allows one to construct all of the equilibrium strategies explicitly. Suppose, without loss of generality, that players $i = 1,2..,h$, $h \geq 2$, randomize continuously over $[0, v]$ with players $i = h+1, ...,n$ randomizing continuously over $(b_i, v]$, with $b_{h+1} \leq b_{h+2} \leq ... \leq b_n \leq v$. The equilibrium strategies are:

$$\forall\, x \in [b_n, v]: \qquad G_i(x) = \left[\frac{x}{v}\right]^{\frac{1}{n-1}} \qquad i = 1, ..., n;$$

$$\forall\, x \in [b_j, b_{j+1}): \; j \in \{h+1, ..., n-1\} \qquad G_i(x) = \left[\frac{x}{v}\right]^{\frac{1}{j-1}} \left[\prod_{k>j} G_k(b_k)\right]^{-\frac{1}{j-1}} \qquad i = 1, ..., j$$

$$G_k(x) = G_k(b_k) \qquad k = j+1, ..., n;$$

$$\forall\, x \in [0, b_{h+1}): \qquad G_i(x) = \left[\frac{x}{v}\right]^{\frac{1}{h-1}} \left[\prod_{k>h} G_k(b_k)\right]^{\frac{-1}{h-1}} \qquad i = 1, ..., h$$

$$G_k(x) = G_k(b_k) \qquad k = h+1, ..., n.$$

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[2] We could have $b_i > v$, in which case player i places all mass at 0.

The equilibria with $b_{h+1} = v$ are given in Moulin (1986b) for $h = n$. Somewhat more general is the case $b_{h+1} = v$, but $2 \leq h \leq n$, i.e. some agents can be inactive, which is discussed in Hillman and Samet (ibid., p. 72), Hillman (ibid., p. 66) and Hillman and Riley (ibid., ft. 12). Hillman and Samet (ibid., p. 72) claim there are no other equilibria. Also, Proposition 1c in Hillman and Riley which claims that at most one agent spends zero with positive probability is erroneous. The analysis above shows there exists a continuum of asymmetric equilibria. Moreover, with more than two agents, a multitude of different point masses at zero are possible. Importantly, however, it turns out that all of the equilibria are revenue equivalent.

Theorem 2: *(Revenue Equivalence)* In the all-pay <u>common value</u> auction, the expected sum of the bids in <u>any</u> Nash equilibrium equals the value of the prize v.

Proof: By Lemma 8 $E[u_i] = 0$, and hence $E[\sum_{i=1}^{n} u_i] = 0$.

As u_i equals the expected revenue to player i minus the bid x_i, and total expected revenues are v, it follows that

$$v - E[\sum_{i-1}^{n} x_i] = 0.$$

Remark. The same result can be obtained through integration, see Baye, Kovenock and de Vries (1989b) for the n-player case with $b_{h+1} = v$. For the other cases one has to evaluate the sum of integrals of the form

$$(j-1) \int_{b_j}^{b_{j+1}} x \, dG_i(x)$$

In this sum all terms except the first (equal to v) and the last (equal to 0) cancel.

From the bidders' point of view, all the equilibria are also payoff equivalent. This result was established in Lemma 7 above. Apart from these equivalences, all the equilibria share another interesting property. As is shown in Lemma 15, at least two agents have to randomize continuously over the support and have equal c.d.f.'s. Thus it takes at least two players to hold each contender down to the equilibrium payoff of zero. Trivially, with only one contender, he gets everything for nothing. Two seems enough to induce the perfectly competitive outcome, where all rents are competed away. The role of the other n-2 is less important in this sense. Also note the c.d.f.'s of the players who randomize over the entire support strictly first order stochastically dominate the other players' strategies. Whether the perfectly competitive outcome arises generally if there are at least two contenders, is now investigated by considering the case of heterogeneous valuations.

3. HETEROGENEOUS VALUATIONS

Suppose now that the bidders have heterogeneous valuations. Let v_i be the valuation of player i.

A. Unique Highest and Second Highest Valuations

We deal first with the case where $v_1 > v_2 > v_3 \geq \ldots \geq v_n \geq 0$. Cases where one of the strict inequalities adjacent to v_2 is weak require a separate analysis.

Lemma 1': $\forall\ i\ v_i \geq \bar{s}_i \geq \underline{s}_i \geq 0$.

Proof : Insert v_i in place of v in the proof of Lemma 1.

Lemma 2': Same as Lemma 2.

Lemma 3': If $\underline{s}_1 = \ldots = \underline{s}_m > \underline{s}_{m+1}, \ldots, \underline{s}_n$ for $n \geq m \geq 2$ then $\exists\, i \leq m$ such that $\alpha_i(\underline{s}_i) = 0$.

Proof: Suppose not. Then any $i \leq m$ has incentive to raise the bid $\underline{s}_i$ by ϵ small, unless $\underline{s}_i = v_i$, in which case i has an incentive to reduce the bid v_i to 0.

Lemma 4': Same as Lemma 4.

Lemma 5': Same as Lemma 5.

Lemma 6': Same as Lemma 6.

In the analysis that follows let $\bar{s}$ be the upper bound of the union of the supports of the players' equilibrium bid distributions.

Lemma 7': $\bar{s} \leq v_2$.

Proof: Player i would never put mass above v_i since setting the bid equal to 0 strictly dominates such a strategy. Player 1 clearly has no incentive to put mass in the interval $(v_2, v_1]$.

Lemma 8': All players other than player 1 must place a mass point at 0.

Proof: By Lemma 6', $\underline{s}_i = 0 \ \forall \ i$. Since $\bar{s} \leq v_2 < v_1$ player 1 must have an equilibrium payoff u_1^* of at least $v_1 - v_2 > 0$. Thus, player 1 cannot have a mass point at 0. This follows from Lemma 3', i.e. some player must put no mass at 0, in which case player 1 with probability 1 would not submit the high bid at 0, and would have payoff $u_1 = 0$ there. Since $u_1^* > 0$, in every neighborhood above 0 player 1 must outbid every other player with a probability that is bounded away from zero. Thus, every player but player 1 must put a mass point at 0.

Lemma 9': $\forall \ i \neq 1 \ \ u_i^* = 0$.

Proof: Immediate from Lemmas 3' and 8'.

Lemma 10': $\bar{s} = v_2$ and $\bar{s}_1 = \bar{s}_2 = v_2$.

Proof: From Lemma 7' $\bar{s} \leq v_2$. Suppose $\bar{s} < v_2$. By bidding above $\bar{s}$ by an arbitrarily small amount player 2 can earn arbitrarily close to $v_2 - \bar{s} > 0 = u_2^*$, a contradiction. Thus, $\bar{s} = v_2$. The second part of the claim is straightforward.

Lemma 11': There are no point masses on the half open interval $(0, v_2]$.

Proof : Similar to the proof of Lemma 10, inserting v_2 for v the first two times that v appears in the proof, and v_j for v the last two times it appears.

Lemma 12': $B_i(x_i) \equiv (v_i - x_i)\, A_i(x_i) - x_i(1-A_i(x_i))$ is constant and equal to u_i^* at the points of increase of G_i in $(0, v_2]$ for all i. $B_i(x_i) \leq u_i^*$ if x_i is not a point of increase in $(0, v_2]$.

Proof: Similar to Lemma 11.

Lemma 13': $\forall\, x \epsilon (0, v_2]\ \exists\ i_1, i_2$ such that $\forall\ \epsilon > 0$: $G_i(x+\epsilon) - G_i(x-\epsilon) > 0$, $i = i_1, i_2$.

Proof: Immediate.

Lemma 14': $\bar{s}_i = 0 \quad \forall\ i > 2$.

Proof: Without loss of generality assume $\bar{s}_3 = \max_{i \geq 3} \bar{s}_i$. Suppose $\bar{s}_3 \neq 0$. Then there exists an interval of increase $(\bar{s}_3 - \epsilon, \bar{s}_3]$ in which $B_3(x) = u_3^* = 0 = (v_3 - x)A_3(x) - x(1-A_3(x))$. Thus, $v_3 = \frac{x}{A_3(x)}\ \forall\ x \epsilon (\bar{s}_3 - \epsilon, \bar{s}_3]$. But as G_1 and G_2 are increasing on $(\bar{s}_3, v_2]$, $v_2 = \frac{\bar{s}_3}{A_2(\bar{s}_3)}$. Since for $\bar{s}_3 > 0$, $A_2(\bar{s}_3) = \prod_{j \neq 2} G_j(\bar{s}_3) > \prod_{j \neq 3} G_j(\bar{s}_3) = A_3(\bar{s}_3)$, we have a contradiction to the fact that $v_3 < v_2$. Thus, $\bar{s}_3 = 0$.

The above analysis establishes rigorously the following result originally formulated by Hillman (ibid.), and Hillman and Riley (ibid.):

Theorem 3: *(Hillman and Riley)* If $v_1 > v_2 > v_3 \geq \ldots \geq v_n$, then players 1 and 2 will randomize continuously over $(0, v_2]$, with player 2 having a mass point at 0 and all other players bidding 0 with probability one. The c.d.f.'s used by players 1 and 2 over the interval $[0, v_2]$ are $G_1(x) = \frac{x}{v_2}$ and $G_2(x) = \frac{v_1 - v_2}{v_1} + \frac{x}{v_1}$, respectively. Players 2 through n earn a payoff of 0 and player 1 earns a payoff $u_1^* = v_1 - v_2$.

Through integration Corollary 1 of Hillman and Riley (ibid., p. 25) on the expected sum of bids is easily verified.

Theorem 4: If $v_1 > v_2 > v_3 \geq \ldots \geq v_n$, the expected sum of the bids in the all-pay auction is

$$E\,[x_1 + x_2] = \frac{1}{2}\,v_2 + \frac{1}{2}\,v_2\,(\frac{v_2}{v_1}).$$

Note that the average sum of bids is now below the second highest valuation v_2. The intuition behind this result is as follows. With equal valuations, i.e. $(v_2/v_1) = 1$, each player bids half the prize on average. With unequal valuations, player 2 still bids $v_2/2$ conditional upon bidding actively. This happens with probability v_2/v_1.

B. Unique Highest but Multiple Second-Highest Valuations

We now deal with the case where $v_2 = v_3 = \ldots = v_m$, $m \leq n$. These cases again lead to multiple equilibria. We first deal with the case where $v_1 > v_2 = v_3 = \ldots = v_m > v_{m+1} \geq \ldots \geq v_n$ for some $3 \leq m \leq n$. It is easily seen that for this case Lemmas 1' through 9' hold. Lemmas 11' through 13' also continue to hold (with an obvious alteration in the labeling of players in the proof of Lemma 11'). Lemmas 10' and 14' must be slightly altered as follows; the proofs require only a minor change in the labeling of players:

Lemma 10": $\bar{s} = v_2$. There exists at least one player i, $2 \leq i \leq m$, such that $\bar{s}_i = v_2$.

Lemma 14": $\bar{s}_i = 0 \ \forall \ i > m$.

If $n > m$ we may set $\underline{s}_i = \bar{s}_i = 0$ for $i > m$ and proceed with the analysis of the game as if we had an m player game with $v_1 > v_2 = v_3 = \ldots = v_m$. Suppose then that $n = m$, so that $v_1 > v_2 = v_3 = \ldots = v_n$. The following versions of Lemmas 12 and 14 hold for players 2,..., n.

Lemma 15": Suppose x is a point of increase of G_i and G_j in $(0,v_2]$, $i, j \in \{2,\ldots, n\}$. Then $G_i = G_j$ at x.

Proof: Same as proof of Lemma 12.

Lemma 16": If G_i, $i \in \{2,\ldots, n\}$, is strictly increasing on some open subset (a,b), $0 < a < b < v_2$, then G_i is strictly increasing on the whole interval $(a,v_2]$.

Proof: Similar to the proof of Lemma 14 where one of the players h,k must be an element of $\{2,\ldots, n\}$ and this player is used throughout the continuation of the proof.

Lemma 16" together with Lemmas 10" and 13' imply the following:

Lemma 17": At least one of the players 2,..., n must randomize on the interval $[0,v_2]$.

Lemma 18": $\bar{s}_1 = v_2$, and for every bid $0 < x < v_2$ in the support of G_1, $G_1(x) < G_i(x)$, $i \in \{2, \ldots, n\}$.

Proof: From Lemma 17" at least one of the players 2, ..., n has support $[0, v_2]$. Without loss of generality, suppose that player 2 is such a player. From Lemmas 3' and 8' player 1 does not have a mass point at 0, and from Lemma 11" no player has a mass point in $(0, v_2]$. Thus, there exists some point $x \in (0, v_2)$ at which $G_1(x)$ is increasing. At any such point $B_1(x) \geq v_1 - v_2$, since the right hand side is what player 1 can obtain by bidding v_2. Rearranging this expression we obtain

$$A_1(x) \geq \frac{v_1 - v_2 + x}{v_1}.$$

From Lemmas 9' and 12'

$$A_2(x) = \frac{x}{v_2}.$$

Subtracting A_1 from A_2 gives

$$A_2(x) - A_1(x) \leq \frac{(v_2-x)(1 - \frac{v_1}{v_2})}{v_1} < 0,$$

where the strict right hand inequality follows from the assumption that $v_2 > x$ and $v_1 > v_2$. Thus, at any point of increase of G_1 in $(0, v_2)$, $A_1 > A_2$. This directly implies that $G_2 > G_1$ for any such point.

But since G_2 has support $[0, v_2]$ and G_1 has no mass points, this implies $\bar{s}_1 = v_2$. Furthermore, since for any other player $i \in \{2, \ldots, n\}$ and for any $x \in [0, v_2]$, $G_2(x) \leq G_i(x)$, we have the claim.

An immediate consequence of Lemma 18" and the fact that G_1 has no mass points is that $G_i(x) \geq G_1(x)$ for every $x \in [0, v_2]$ $i = 2, \ldots, n$, with strict inequality on the open interval. Thus, each $G_i(x)$ is stochastically dominated by $G_1(x)$.

The next Lemma uses the result that $\bar{s}_1 = v_2$ to show that the support of $G_1(x)$ is $[0, v_2]$.

Lemma 19": The support of $G_1(x)$ is $[0, v_2]$.

Proof: We know that $\bar{s}_1 = v_2$ and $\underline{s}_1 = 0$. Suppose there is a gap (a,b) in which $G_1(x)$ is constant, $0 < a < b < v_2$.

By Lemmas 12' and 13' we know that at $x = a$ there are at least two players $i,k \in \{2,\ldots,n\}$ such that $A_i(x) = A_k(x) = \frac{x}{v_2}$. At $x = b$ this holds as well. In addition, since a and b are in the support of G_1

$$A_1(a) = \frac{v_1 - v_2}{v_1} + \frac{a}{v_1},$$

and

$$A_1(b) = \frac{v_1 - v_2}{v_1} + \frac{b}{v_1}.$$

Thus, we have

(1) $\quad G_1(a)\, G_k(a)\, A_{ik1}(a) = \frac{a}{v_2};$

(2) $\quad G_1(b)\, G_k(b)\, A_{ik1}(b) = \frac{b}{v_2};$

(3) $\quad G_i(a)\, G_k(a)\, A_{ik1}(a) = \frac{v_1 - v_2}{v_1} + \frac{a}{v_1};$ and

$$(4) \qquad G_i(b)\, G_k(b)\, A_{ik1}(b) = \frac{v_1 - v_2}{v_1} + \frac{b}{v_1}.$$

Since $G_1(a) = G_1(b)$ by assumption, and by Lemma 15" $G_i(x) = G_k(x)$ for $x \in [a,b]$, (1) and (2) imply

$\dfrac{G_k(a)}{G_k(b)} \dfrac{A_{ik1}(a)}{A_{ik1}(b)} = \dfrac{a}{b}$, while (3) and (4) imply

$$\frac{[G_k(a)]^2}{[G_k(b)]^2} \frac{A_{ik1}(a)}{A_{ik1}(b)} = \frac{v_1 - v_2 + a}{v_1 - v_2 + b}.$$

Combining this gives $\dfrac{G_k(a)}{G_k(b)} \dfrac{a}{b} = \dfrac{v_1 - v_2 + a}{v_1 - v_2 + b}$. Hence,

$$G_k(a) = \frac{b}{a} \left(\frac{v_1 - v_2 + a}{v_1 - v_2 + b}\right) G_k(b)$$

$$= \frac{b}{a} \left(\frac{\phi + a}{\phi + b}\right) G_k(b), \text{ say, for } \phi > 0.$$

Since $\dfrac{b}{a} > \dfrac{b+\phi}{a+\phi}$, this implies $\dfrac{b(a+\phi)}{a(b+\phi)} > 1$. It follows that $G_k(a) > G_k(b)$, a contradiction to the fact that $b > a$. Thus there cannot be an interval contained in $(0, v_2)$ over which player 1 places no mass.

We have therefore established the following:

Theorem 5: If $v_1 > v_2 = \ldots = v_n$, player 1 randomizes continuously over the interval $[0, v_2]$ and at least one of the players 2, ..., n does the same. Each player i, $i \in \{2,\ldots,n\}$, has a strategy with support contained in $[0, v_2]$; each $i \in \{2,\ldots,n\}$ places a mass point $\alpha_i(0)$ at 0 (the size of which may differ across players); and each $i \in \{2,..n\}$ can be characterized by a number $b_i \geq 0$ such that $G_i(x) = G_i(0) = \alpha_i(0)$ $\forall x \in [0, b_i]$ (where b_i could be greater than v_2, in which case $\alpha_i(0) = 1$) and player i randomizes continuously on $(b_i, v_2]$. Furthermore, when two or more players in the set $\{2,...,n\}$ have a positive density over a common interval, they play the same continuous mixed strategy over that interval. Moreover, $u_1^* = v_1 - v_2$ and $u_j^* = 0$ for $j \neq 1$.

We are now able to provide exact expressions for the equilibrium distributions conditional on the (arbitrary) points, b_i, $i \in \{2,...,n\}$ at which players start randomizing continuously. The distributions may be obtained recursively over $[0, v_2]$.

Suppose without loss of generality that of the players $\{2,...,n\}$ players $i = 2,..., h$, $h \geq 2$ randomize continuously over $(0, v_2]$, with players $i = h+1,...,n$ randomizing continuously over $(b_i, v_2]$, (where $b_i = v_2$ implies $\alpha_i(0) = 1$) with $b_{h+1} \leq b_{h+2} \leq ... \leq b_n \leq v_2$. Then

$$\forall\ x \in [b_n, v_2]: \quad G_i(x) = \left[\frac{v_1 - v_2 + x}{v_1}\right]^{\frac{1}{n-1}} \qquad i = 2,...,n$$

$$G_1(x) = \frac{x}{v_2}\left[\frac{v_1 - v_2 + x}{v_1}\right]^{\frac{2-n}{n-1}};$$

$$\forall\ x \in [b_j, b_{j+1}):\ j \in \{h+1,...,n-1\} \quad G_i(x) = \left[\frac{v_1 - v_2 + x}{v_1}\right]^{\frac{1}{j-1}} \left[\prod_{k>j} G_k(b_k)\right]^{\frac{-1}{j-1}} \qquad i = 2,...,j$$

$$G_k(x) = G_k(b_k) \qquad k = j+1,...,n;$$

$$G_1(x) = \frac{x}{v_2}\left[\frac{v_1 - v_2 + x}{v_1}\right]^{\frac{2-j}{j-1}} \left[\prod_{k>j} G_k(b_k)\right]^{\frac{-1}{j-1}}$$

$$\forall\ x \in [0, b_{h+1}): \quad G_i(x) = \left[\frac{v_1 - v_2 + x}{v_1}\right]^{\frac{1}{h-1}} [\prod_{k>h} G_k(b_k)]^{\frac{-1}{h-1}} \qquad i = 2, \ldots, h$$

$$G_k(x) = G_k(b_k) \qquad k = h+1, \ldots, n$$

$$G_1(x) = \frac{x}{v_2} \left[\frac{v_1 - v_2 + x}{v_1}\right]^{\frac{2-h}{h-1}} [\prod_{k>h} G_k(b_k)]^{\frac{-1}{h-1}}.$$

From Theorem 5 it is immediate that all equilibria are payoff equivalent, as before. Interestingly, in contradistinction with the case of homogeneous valuations, c.f. Theorem 2, in case of heterogeneous valuations the possibility of revenue non equivalence arises.

Lemma 20": If $v_1 > v_2 = \ldots = v_n$, then the expected revenue is

$$\sum_{i=1}^{n} E(x_i) = \{v_2 - (\frac{v_1}{v_2} - 1) \left[(n-1)v_2 - \frac{(n-1)^2}{n} v_1\right]\}$$

$$- (\frac{v_1}{v_2} - 1) \{v_1 \frac{(h-1)^2}{h} (\frac{v_1 - v_2}{v_1})^{\frac{h}{h-1}} \left[\prod_{k=h+1}^{n} G_k(b_k)\right]^{\frac{-1}{h-1}} \}$$

$$- (\frac{v_1}{v_2} - 1) \sum_{j=h}^{n-1} G_{j+1}(b_{j+1}) \left[(1 - \frac{1}{j(j+1)})(v_1 - v_2) - \frac{1}{j(j+1)} b_{j+1}\right].$$

Proof: Evidently $E\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} E(x_i) =$

$$\int_{b_n}^{v_2} x \, dG_1(x) + (n-1) \int_{b_n}^{v_2} x \, dG_i(x) + \ldots +$$

$$\int_{b_j}^{b_{j+1}} x \, dG_1(x) + (j-1) \int_{b_j}^{b_{j+1}} x \, dG_i(x) + \ldots +$$

$$\int_{0}^{b_{h+1}} x \, dG_1(x) + (h-1) \int_{0}^{b_{h+1}} x \, dG_i(x) \; ,$$

where the indices h, i, j, k, n have the same connotation as in in the expressions for the distributions below Theorem 5. Through integration by parts and using the fact that $G_1(x) = x(j-1) v_1 v_2^{-1} \, dG_i(x)$, $i \in \{2,\ldots,n\}$, obtain the following expression for the contribution to the total expected efforts on a particular interval $[b_j, b_{j+1})$:[3]

$$\int_{b_j}^{b_{j+1}} x \, dG_1(x) + (j-1) \int_{b_j}^{b_{j+1}} x \, dG_i(x) =$$

$$\frac{1}{v_2} \left[\prod_{k>j} G_k(b_k)\right]^{\frac{-1}{j-1}} \{ b_{j+1}^2 \left[\frac{v_1 - v_2 + b_{j+1}}{v_1}\right]^{\frac{2-j}{j-1}}$$

$$- \; b_j^2 \left[\frac{v_1 - v_2 + b_j}{v_1}\right]^{\frac{2-j}{j-1}}$$

[3] The derivation is lengthy but straightforward, and analogous to the case h = n as presented in Baye, Kovenock and De Vries [1989b].

$$- (j-1)(v_1-v_2)\ \{b_{j+1} \left[\frac{v_1-v_2+b_{j+1}}{v_1}\right]^{\frac{1}{j-1}} - b_j \left[\frac{v_1-v_2+b_j}{v_1}\right]^{\frac{1}{j-1}}$$

$$- v_1 \frac{j-1}{j} \left[\frac{v_1-v_2+b_{j+1}}{v_1}\right]^{\frac{j}{j-1}} + v_1 \frac{j-1}{j} \left[\frac{v_1-v_2+b_j}{v_1}\right]^{\frac{j}{j-1}} \}\}.$$

To obtain the total expected effort, take the sum over j, where $j = h+1, \ldots, n-1$, plus the first and last interval as well. The resulting expression can be considerably simplified by noting that consecutive terms in the sum do cancel or can be combined, c.f. the Remark below Theorem 2. Note that

$$\prod_{k>j} G_k(b_k) = \left[\frac{v_1-v_2+b_{j+1}}{v_1}\right]^{\frac{1}{j}} \left[\prod_{k>j+1} G_k(b_k)\right]^{\frac{j-1}{j}}.$$

Use this to show that

$$- [\prod_{k>j+1} G_k(b_k)]^{\frac{-1}{j}}\ b^2_{j+1}\ [\frac{v_1-v_2+b_{j+1}}{v_1}]^{\frac{1-j}{j}}$$

$$+ [\prod_{k>j} G_k(b_k)]^{\frac{-1}{j-1}}\ b^2_{j+1}\ [\frac{v_1-v_2+b_{j+1}}{v_1}]^{\frac{2-j}{j-1}} = 0.$$

Which implies that the first element of the j-th summand cancels against the second element of the (j+1)-th summand. For the third and fourth elements a similar procedure gives terms

$$- G_{j+1}\ (b_{j+1})\ b_{j+1},$$

and for the last two elements we get terms

$$v_1 \ (1 - \frac{1}{j(j+1)}) \ G_{j+1} \ (b_{j+1}).$$

Taking care of what happens at 0 and v_2, then gives

$$\Sigma_i \ E(x_i) = v_2 - (\frac{v_1}{v_2} - 1) \ [(n-1)v_2 - \Sigma_j \ G_{j+1} \ (b_{j+1})b_{j+1}]$$

$$- (\frac{v_1}{v_2} - 1) \ [v_1 \ \frac{(h-1)^2}{h} \ (\frac{v_1 - v_2}{v_1})^{\frac{h}{h-1}} \ (\prod_k G_k(b_k))^{\frac{-1}{h-1}} - v_1 \ \frac{(n-1)^2}{n}$$

$$+ v_1 \ \Sigma_j \ (1 - \frac{1}{j(j+1)}) \ G_{j+1} \ (b_{j+1}) \ (\frac{v_1 - v_2 + b_{j+1}}{v_1})].$$

Some further manipulations gives the expression stated in the theorem. This expression consists of three terms. For given n, the first term is fixed. The second and third terms, however, are generally nonzero and depend on h and b_k. Therefore total expected revenue varies with h and b_k for given n.

Theorem 5 and Lemma 20" generalize the two player heterogeneous valuation case discussed in Theorems 3 and 4. Hillman (ibid., p. 66) erroneously claims that the equilibrium bid distributions given in Theorem 3 also constitute the unique equilibrium strategies if $v_1 > v_2 = \ldots = v_n$. Moreover, Hillman and Riley explicitly rule out a tie between the second and third agent, c.f. their Proposition 4. Hence, both the equilibrium where all 2,...,n players employ the symmetric strategy and most of the asymmetric equilibrium strategies are missed. This is not innocuous because of our Lemma 20". From the expression for the expected sum of bids Theorem 2 immediately follows as a special case by setting $v_1 = v_2$. But revenue non equivalence arises whenever $v_1 > v_2 = \ldots = v_n$. To see this, consider the case with 3 players such that $v_1 = 2$, $v_2 = v_3 = 1$ and calculate the expected sum of bids. For the completely asymmetric equilibrium with strategies $G_1(x) = x$, $G_2(x) = (1+x)/2$ and $G_3(x) = 1$, $0 \leq x \leq 1$, we get that $\Sigma_i E(x_i) = 5/3 - 1/4 - 4/6 = 3/4$. For the case where players 2 and 3 play symmetrically, the equilibrium strategies are

$$G_1(x) = x\left(\frac{1+x}{2}\right)^{-1/2}, \quad G_2(x) = G_3(x) = \left(\frac{1+x}{2}\right)^{1/2},$$

and hence

$$\Sigma_i E(x_i) = \frac{5}{3} - \frac{2}{3}\sqrt{2}.$$

For the case where players 2 and 3 employ identical strategies on [1/2, 1], but player 2 places probability mass of $\sqrt{(3/4)}$ at 0, the distributions are defined on [1/2, 1] as in the symmetric case. On [0, 1/2) they are defined by

$$G_1(x) = x\sqrt{(4/3)}, \qquad G_2(x) = \frac{1+x}{2}\sqrt{(4/3)}, \qquad G_3(x) = \sqrt{(3/4)}.$$

In this case, $\Sigma_i \; E(x_i) = \frac{5}{3} - \frac{1}{6}\sqrt{3} - \frac{3}{8}\sqrt{3} = \frac{5}{3} - \frac{13}{24}\sqrt{3}$.

Thus we have shown :

Theorem 6: (Revenue Non Equivalence) : If $v_1 > v_2 = \ldots = v_n$, not all Nash equilibria to the all-pay auction yield the same revenue.

The conclusion is that with ties for second, the expected sum of bids can be different from the value calculated in Theorem 4. Hence, two-ness is an assumption rather than a property of heterogeneous valuation all pay contests; c.f. Magee, Brock, and Young (ibid., p. 217) who take two-ness as a general property in the setting of a political context.

The final case to be covered is when $v_1 = v_2 = \ldots = v_m > v_{m+1} \geq \ldots \geq v_n$ for $2 \leq m \leq n-1$. The analysis of this case is trivial, as it can be shown that players 1 through m play an m-player equilibrium of the type outlined in Theorem 1, while players m+1 through n put all mass at 0.

4. CONCLUSION

Contrary to what has been conjectured in the literature and used in applications, the n-person all-pay auction has an infinity of equilibria. We examined both the case of homogeneous and heterogeneous valuations, and in each case we explicitly derived the equilibrium strategies and the expected sum of bids. For the common values case with more than two players, there is a unique symmetric equilibrium, but a continuum of asymmetric equilibria. However, all of the equilibria are payoff and revenue equilvalent. With heterogeneous valuations, a single agent with the highest valuation, and more than one agent with the second highest valuation, there is a continuum of asymmetric equilibria. These equilibria are not revenue equivalent.

Our results appear important since, in light of the rent-seeking and tournament literature, which examine games with a structure isomorphic to the all pay auction, one is tempted to take twoness as a general property of a contest. Our results warn against this practice, not only because larger numbers of players may actively participate in the bidding process, but because the number of active bidders may affect revenues.

Finally, our finding of a continuum of equilibria has important ramifications for applications of the model. As in other contexts, the multiplicity is troublesome. It is not clear how rational players *should* play the game, let alone how real-life players do play the game. An experimental investigation of the model remains a task for future research.

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