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Abstract

This paper introduces a new class of cooperative games arising from cooperative decision making problems in a stochastic environment. Various examples of decision making problems that fall within this new class of games are provided. For a class of games with stochastic payoffs where the preferences are of a specific type, a balancedness concept is introduced. It is proved that the core of a game within this class is non empty if and only if the game is balanced. Further, other types of preferences are discussed. In particular, the effects the preferences have on the core of these games are considered.

KEYWORDS: cooperative games, stochastic variables, core, balancedness, preferences.

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1 Introduction

In general, the payoff of a coalition in cooperative transferable utility games is assumed to be known with certainty. In many cases, however, payoffs to coalitions are uncertain. This would not raise a problem, if the agents can await the realizations of the payoffs before deciding which coalitions to form and which allocations to settle on. But if the formation of coalitions and allocations has to take place before the payoffs will be realized, standard cooperative game theory does no longer apply.

Charnes and *Granot (1973)* considered cooperative games in stochastic characteristic function form. For these games the value $V(S)$ of a coalition S is allowed to be a stochastic variable. They suggested to allocate the stochastic payoff of the grand coalition in two stages. In the first stage, so called prior payoffs are promised to the agents. These prior payoffs are such that there is a good chance that this promise will be realized. In the second stage the realization of the stochastic payoff is awaited and, subsequently, a possibly non feasible prior payoff vector has to be adjusted to this realization in some way. This approach was elaborated in *Charnes* and *Granot (1976)*, *Charnes* and *Granot (1977)*, and *Granot (1977)*. Most of the time the adjustment process is such that objections among the agents are minimized.

In this paper we will not follow the route set out by Charnes and Granot. Instead we will introduce a different and more extensive model. The main reason for this is that the model used by *Charnes* and *Granot (1976)* assumes risk neutral behaviour of all agents. The model we introduce allows different types of behaviour towards risk of the agents. Moreover, each coalition possibly has several actions to choose from, which each lead to a (different) stochastic payoff.

In Section 2 we introduce our model of a game with stochastic payoffs. Furthermore we give examples, arising from linear production problems, financial markets, and sequencing problems, which fall in this class of games with stochastic payoffs. Also the core of such a game is defined. In Section 3 we consider a special class of preferences. The ordering of stochastic payoffs for these preferences is based on the

α-quantile of the stochastic payoff. So, these preferences are determined by the number α. Moreover, different kinds of behaviour towards risk of the agents will result in a different value for α for each agent. For games with these preferences we provide a new balancedness concept, which is an extension of the balancedness concept for standard TU-games. We will show that the core of a game with stochastic payoffs is non empty if and only if this game is balanced.

In Section 4 we look at other types of preferences of the agents. Examples illustrate the effect of the preference relation on the core of the game. Furthermore, we show that for some preferences a similar result as obtained in Section 3 can be derived, if the balancedness concept is slightly adjusted.

2 The model and some examples

In this section we will introduce a general framework to model cooperative games with stochastic payoffs and transferable utilities. Moreover, we will give some examples of situations which can be captured within this framework.

A game with stochastic payoffs is defined as a tuple $(N, (A_S)_{S\subset N}, (X_S)_{S\subset N}, (\succsim_i)_{i\in N})$, where $N = \{1, 2, \ldots, n\}$ is the set of players, A_S is the set of all possible actions coalition S can take, and $X_S : A_S \to L^1(\mathbf{R})$ a function assigning to each action $a \in A_S$ of coalition S a real valued stochastic variable $X_S(a)$ with finite expectation, representing the payoff to coalition S when action a is taken. Finally, $\succsim_i$ describes the preferences of agent i over the set $L^1(\mathbf{R})$ of stochastic variables with finite expectation. For any $X, Y \in L^1(\mathbf{R})$ we denote $X \succsim_i Y$ when the payoff X is at least as good as the payoff Y according to agent i, and $X \succ_i Y$ when agent i strictly prefers X to Y. The set of all games with stochastic payoffs and player set N is denoted by $SG(N)$. An element of $SG(N)$ is denoted by Γ.

If we compare a game with stochastic payoffs to a standard TU-game, we can distinguish two major differences. First, the payoffs can be random variables, which is not allowed in the standard case. Second, in a game with stochastic payoffs the actions a coalition can choose from are explicitly modelled as opposed to the standard case. In the standard case coalitions possibly can choose from several actions, but since the payoff they want to maximize is deterministic there is no doubt about the optimal payoff. Therefore, the actions of a coalition can be omitted in the description of a TU-game.

A first application can be found by modifying linear production games, which were introduced by *Owen (1975)*. In a linear production game each agent $i \in N$ owns a resource bundle $b_i \in \mathbf{R}^r_+$. The resources can be used to produce quantities $x_1, x_2, \ldots, x_m$ of goods $1, 2, \ldots, m$ according to some technology matrix $M \in \mathbf{R}^{r\times m}$, which can be sold for prices $c_1, c_2, \ldots, c_m$. The value of a coalition S of agents then equals the maximal revenue this coalition can obtain given their resources, i.e.

$$v(S) = \max\{\sum_{j=1}^{m} c_j x_j \mid Mx \leq \sum_{i\in S} b_i,\ x = (x_1, x_2, \ldots, x_m) \geq 0\}.$$

Now suppose that the compositions of the resource bundles are not known with certainty, i.e., the resources of agent i are represented by some nonnegative stochastic variable $B^i \in L^1(\mathbf{R}_+)$. Moreover, agents are not allowed to await the realizations of these variables, before deciding upon a (joint) production scheme.

The above situation cannot be modelled as a traditional TU game. However, it can be modelled as a game with stochastic payoffs in the following way. Let N be the set of agents, and define the set of actions of a coalition $S \subset N$ by $A_S = \{a \in \mathbf{R}^m | a_j \geq 0\ , j = 1, 2, \ldots, m\}$, the set of all possible production bundles. Now we define the payoff of a coalition $S \subset N$ with respect to the action $a \in A_S$ as the stochastic variable $X_S(a)$ given by

$$X_S(a) = \begin{cases} c^\top a, & \text{if } Ma \leq \sum_{i\in S} B^i \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the payoff $X_S(a)$ equals $c^\top a$ for any realization of resources for which the production scheme is feasible and it equals zero otherwise. As a consequence coalitions could decide on going for a production plan which is feasible with little probability but yields relatively high revenues when feasible, or, a production scheme which is feasible with high probability but yields relatively low revenues when feasible. Obviously, this decision is highly influenced by the agents' valuation of risk.

In the case considered above, only the resources were assumed to be stochastic. Clearly, one could also assume that prices and/or technology are stochastic. These situations can be modelled as games with stochastic payoffs in a similar way.

The second application concerns financial markets. For a general equilibrium model on financial markets the reader is referred to *Magill* and *Shafer (1991)*. The examples we provide will show some substantial differences with the model considered by *Magill* and *Shafer (1991)*. First, our models focus on cooperation between the agents, and second, the assets we consider are indivisible goods.

In the first example, we have a set N of agents with each agent having an initial endowment m^i of money. Furthermore, we have a set F of assets, where each asset $f \in F$ has a price π_f and stochastic revenues $R^f \in L^1(\mathbf{R})$. Now, each agent can invest his money in a portfolio of assets and obtain stochastic revenues. We allow the set F to contain identical assets, so that we do not need to specify the amounts agents buy of a specific asset. For example, if a firm issues k shares of type f, then all the shares $f_1, f_2, \ldots, f_k$ are contained in F (cf. *Modigliani* and *Miller (1958)*). Instead of buying portfolios individually, agents can also cooperate, combine their endowments of money, and invest in a more diversified portfolio of assets. This behaviour can, on the one hand, result in a less risky investment, but, on the other hand, creates a problem, namely, how to divide the returns and the risk involved over the participating agents. This situation can be modelled as a game with stochastic payoffs by defining for each $S \subset N$, $S \neq \emptyset$

$$A_S = \{A \subset F \mid \sum_{f \in A} \pi_f \leq \sum_{i \in S} m^i\}$$

as the set of all possible portfolios coalition S can afford, and for all $A \in A_S$

$$X_S(A) = \sum_{f \in A} R^f$$

the stochastic revenues with respect to the portfolio A.

In the second example, we assume that each agent i already possesses a portfolio A_i of assets with stochastic revenues $R^i \in L^1(\mathbf{R})$. Again, it is allowed for the agents to combine their portfolios and redistribute risk. In that case, each coalition $S \subset N$ only has one action with stochastic payoff $X_S = \sum_{i \in S} R^i$. Of course, the problem of how to divide the returns and the risk remains, just as in the first example.

The final application we consider arises from sequencing problems. In a one machine sequencing problem a finite number of agents all have exactly one job that has to be processed on a single machine, which can process at most one job at a time. Moreover, each agent incurs costs for every time unit he has to wait for his job to be completed. Further, we assume that there is an initial processing order of the jobs and that each job has a ready time, this means that the processing of a job cannot start before its ready time. Corresponding to such a sequencing problem one can define a cooperative game, where the value of a coalition equals the cost savings this coalition can obtain with respect to the initial order by rearranging their positions in an admissible way; we refer to *Curiel, Pederzoli* and *Tijs (1989)* for the case with affine cost functions and all ready times equal to zero, and *Hamers, Borm* and *Tijs (1993)* for ready times unequal to zero.

However, the results obtained by *Curiel et al. (1989)* and *Hamers et al.(1993)* only apply for the case that processing times are deterministic. When processing times and ready times are uncertain, a sequencing problem can be modelled as a game with stochastic payoffs in the following way. Let N be the set of agents and let $P^i \in L^1(\mathbf{R})$ and $R^i \in L^1(\mathbf{R})$ describe the stochastic processing time and ready time of agent i, respectively. Denote by $\sigma : N \to \{1, 2, \ldots, n\}$ a processing order of the jobs, where $\sigma(i)$ denotes the position of job i in the processing order σ. In particular, σ_0 denotes the initial processing order. Finally, denote by $k^i : \mathbf{R}_+ \to \mathbf{R}$ the cost function of agent i. Then $k^i(t)$ equals the cost agent i incurs when he spends t time units in the system. The set A_S of actions of coalition S will then be the set of all processing orders which are admissible for coalition S. Here, admissible can be defined in several ways, for instance, a processing order σ is admissible for coalition S if no member of S passes an agent outside S (cf. *Curiel et al. (1989)*).

The completion time of agent i in a processing order σ is a stochastic variable $C^i(\sigma) \in L^1(\mathbf{R})$ defined by

$$C^i(\sigma) = \max\{C^j(\sigma), R^i\} + P^i,$$

where j is the agent exactly in front of agent i, that is, $\sigma(i) = \sigma(j) + 1$. Then the stochastic payoff $X_S(\sigma)$ for coalition S with respect to an action $\sigma \in A_S$ becomes

$$X_S(\sigma) = -\sum_{i \in S} k^i(C^i(\sigma)).$$

So the payoff of coalition S equals minus the waiting costs of all members of S. Again, the action taken by a coalition will be influenced by the agents' valuations of risk.

As was the case for traditional TU games, the main issue for games with stochastic payoffs is to find an appropriate allocation of the stochastic payoff of the grand coalition. For this, however, we first need to know how an allocation of a stochastic payoff is defined. For deterministic payoffs, the definition of an allocation is quite obvious. For stochastic payoffs an allocation could be defined in several ways. For instance, let $X \in L^1(\mathbf{R})$ be the payoff and let N be the set of agents. Then an allocation of X can be defined as a vector $(X^1, X^2, \ldots, X^N) \in L^1(\mathbf{R})^N$ such that $\sum_{i \in N} X^i = X$. So, each agent i gets a stochastic payoff X^i such that the total payoff X is allocated. Note that the probability distribution of an agent's payoff need not be of the same type as the probability distribution of the payoff X. Hence, this definition induces a very large class of allocations, which, on the one hand, is nice, but, on the other hand, will give computational difficulties. Therefore we reduce the class of allocations by adopting a more restrictive definition.

Let $S \subset N$, $a \in A_S$ and let $X_S(a) \in L^1(\mathbf{R})$ be the stochastic payoff. An allocation for S can be represented by a tuple $(d, r|a) \in \mathbf{R}^S \times \mathbf{R}^S$ such that

(i) $\sum_{i \in S} d_i = E(X_S(a))$

(ii) $\sum_{i \in S} r_i = 1$ and $r_i \geq 0$ for all $i \in S$,

with the interpretation that the corresponding payoff to agent $i \in S$ equals

$$(d, r|a)_i := d_i + r_i(X_S(a) - E(X_S(a))).$$

So, an allocation of $X_S(a)$ is described by an allocation of the expectation $E(X_S(a))$ and an allocation of the residual $X_S(a) - E(X_S(a))$, which we will call the risk of the payoff $X_S(a)$. The set of all possible allocations for coalition S is denoted by $Z(S)$.

Example 2.1 Consider the following situation, where two agents each possess a portfolio A_i, $i = 1, 2$. The portfolio A_1 consists of riskless bonds, worth \$100 and paying interest of 5%. The portfolio A_2 consists of one asset with stochastic revenue X where $X \sim U(0, 8)$. Hence, we have $R^1 = 5$ and $R^2 = X$. If the agents cooperate and combine their portfolios, their joint payoff equals $R^1 + R^2 = 5 + X$. The expected payoff of this combined portfolio equals 9. Consequently, the risk $R^1 + R^2 - E(R^1 + R^2)$ of this portfolio equals $X - 4 \sim U(-4, 4)$. Then, an allocation (d, r) results in the payoff $d_i + r_i(X - 4)$ for agent $i = 1, 2$. Note that if $d = (5, 4)$ and $r = (0, 1)$ then the payoff to agent 1 equals R^1 and the payoff to agent 2 equals R^2.

Now that we have the definition of an allocation, we can define the core of a game with stochastic payoffs. Let $\Gamma \in SG(N)$. Then the core of this game is defined as the set of all allocations for N for which no coalition S has an action and an allocation of the corresponding stochastic payoff such that all members of S prefer this allocation to the former one. More formally, an allocation $(d, r|a) \in Z(N)$ is a core allocation if there does not exist a coalition S and an allocation $(\hat{d}, \hat{r}|\hat{a}) \in Z(S)$ such that $(\hat{d}, \hat{r}|\hat{a})_i \succ_i (d, r|a)_i$ for all $i \in S$. The core of a game $\Gamma \in SG(N)$ is denoted by $Core(\Gamma)$.

3 Balancedness for games with stochastic payoffs

In this section we introduce a balancedness concept for a special class of games with stochastic payoffs. This class consist of all such games with the following type of preferences. Let $X, Y \in L^1(\mathbf{R})$ with distribution function F_X and F_Y, respectively. Take $\alpha \in (0,1)$. Then $X \succsim_\alpha Y$ if and only if $u_\alpha^X \geq u_\alpha^Y$ with $u_\alpha^X := \sup\{t \mid F_X(t) \leq \alpha\}$ the α-quantile of X. A game where $\succsim_{\alpha_i}$ represents the preferences of agent i for all $i \in N$ is denoted by Γ_α where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (0,1)^n$.

For relating different values of α to different types of risk behaviour we first need to formalize the concepts risk neutrality, risk aversion and risk loving. Therefore, let $\succsim$ describe the preferences of an agent over the set $L^1(\mathbf{R})$ of stochastic variables. Then we say that $\succsim$ implies risk neutral behaviour of the agent if for all $X \in L^1(\mathbf{R})$ we have $X \sim E(X)$. So, the agent is indifferent between the stochastic payoff and its expectation with certainty. Subsequently, we say that $\succsim$ implies risk averse behaviour if $X \precsim E(X)$ holds for all $X \in L^1(\mathbf{R})$ with strict preference for at least one $X \in L^1(\mathbf{R})$, and risk loving behaviour if $X \succsim E(X)$ holds for all $X \in L^1(\mathbf{R})$ with strict preference for at least one $X \in L^1(\mathbf{R})$. So, a risk averse agent prefers the expectation of a stochastic payoff to the stochastic payoff itself, while a risk loving agent rather has the stochastic payoff than its expectation. Moreover, let $\succsim_i$ and $\succsim_j$ be the preferences of agent i and j respectively. Then agent j behaves more risk loving than agent i, or, equivalently, agent i behaves more risk averse than agent j, if for all $X \in L^1(\mathbf{R})$ we have that

$$\{Y \mid Y \succsim_i E(X)\} \subset \{Y \mid Y \succsim_j E(X)\}.$$

Returning to the $\succsim_\alpha$-preferences, we can say that agent i is more risk averse than agent j if and only if $\alpha_i < \alpha_j$. Note, however, that according to the definitions above the $\succsim_\alpha$- preferences cannot be interpreted as either risk averse, risk neutral, or risk loving behaviour in the absolute sense.

Before we introduce the balancedness concept we recall the definition of a balanced map. For that we define for each coalition $S \subset N$ the vector $e_S \in \mathbf{R}^N$ with $(e_S)_i = 1$ if $i \in S$ and $(e_S)_i = 0$ if $i \notin S$. Then, a map $\mu : 2^N \backslash \{\emptyset\} \rightarrow [0, \infty)$ is called balanced if $\sum_{S \subset N} \mu(S) \cdot e_S = e_N$. Subsequently, a game $\Gamma_\alpha \in SG(N)$ is called balanced if for each balanced map μ we have[1]

$$\max_{a \in A_N} \max_{i \in N} u_{\alpha_i}^{X_N(a)} \geq \sum_{S \subset N} \mu(S) \cdot \max_{a \in A_S} \max_{i \in S} u_{\alpha_i}^{X_S(a)}$$

Note that for deterministic TU-games the expression $\max_{a \in A_S} \max_{i \in S} u_{\alpha_i}^{X_S(a)}$ boils down to $\max_{i \in S} u_{\alpha_i}^{v(S)}$ which is equal to $v(S)$. So, for deterministic TU-games this new balancedness concept coincides with the original balancedness concept for such games. In order to prove that the core of Γ is non empty if and only if Γ is balanced, we need the following lemma.

Lemma 3.1 Let $\Gamma_\alpha = (N, (A_S)_{S \subset N}, (X_S)_{S \subset N}, (\succsim_{\alpha_i})_{i \in N}) \in SG(N)$ and let $(d, r|a) \in Z(N)$. Then coalition S has no incentive to split off if and only if

$$\sum_{i \in S} \left(d_i + r_i (u_{\alpha_i}^{X_N(a)} - E(X_N(a))) \right) \geq \max_{\hat{a} \in A_S} \max_{i \in S} u_{\alpha_i}^{X_S(\hat{a})}.$$

PROOF: Let $S \subset N$. We will prove the lemma by showing that the coalition S has an incentive to split off if and only if

$$\sum_{i \in S} \left(d_i + r_i (u_{\alpha_i}^{X_N(a)} - E(X_N(a))) \right) < \max_{\hat{a} \in A_S} \max_{i \in S} u_{\alpha_i}^{X_S(\hat{a})}.$$

[1] We assume that the maximum over the set A_S of actions exists for all $S \subset N$. For the applicability of the forthcoming results, however, this assumption will hardly be any restriction, since often the set of actions will either be finite or can be modified in that way.

We start with the "only if" part. If S has an incentive to split off then there exists an allocation $(\hat{d}, \hat{r}|\hat{a}) \in Z(S)$ such that

$$\hat{d}_i + \hat{r}_i(X_S(\hat{a}) - E(X_S(\hat{a}))) \succ_{\alpha_i} d_i + r_i(X_N(a) - E(X_N(a)))$$

for each $i \in S$. This implies that

$$\hat{d}_i + \hat{r}_i(u_{\alpha_i}^{X_S(\hat{a})} - E(X_S(\hat{a}))) > d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a)))$$

holds for all $i \in S$. Summing over all members of S yields

$$\sum_{i \in S} \left(\hat{d}_i + \hat{r}_i(u_{\alpha_i}^{X_S(\hat{a})} - E(X_S(\hat{a})))\right) > \sum_{i \in S} \left(d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a)))\right).$$

Using $\sum_{i \in S} \hat{d}_i = E(X_S(\hat{a}))$ and $\sum_{i \in S} \hat{r}_i = 1$ results in

$$\sum_{i \in S} \hat{r}_i \cdot u_{\alpha_i}^{X_S(\hat{a})} > \sum_{i \in S} \left(d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a)))\right).$$

Since $0 \leq \hat{r}_i \leq 1$ for all $i \in S$ we have

$$\max_{i \in S} u_{\alpha_i}^{X_S(\hat{a})} > \sum_{i \in S} \left(d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a)))\right).$$

Then the result follows from

$$\max_{\bar{a} \in A_S} \max_{i \in S} u_{\alpha_i}^{X_S(\bar{a})} \geq \max_{i \in S} u_{\alpha_i}^{X_S(\hat{a})}.$$

For the "if" part of the proof, it suffices to show that if

$$\sum_{i \in S} \left(d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a)))\right) < \max_{\bar{a} \in A_S} \max_{i \in S} u_{\alpha_i}^{X_S(\bar{a})},$$

there exists an allocation $(\hat{d}, \hat{r}_i|\hat{a}) \in Z(S)$ with $\hat{a} \in \arg\max_{\bar{a} \in A_S} \max_{i \in S} u_{\alpha_i}^{X_S(\bar{a})}$ such that

$$\hat{d}_i + \hat{r}_i(u_{\alpha_i}^{X_S(\hat{a})} - E(X_S(\hat{a}))) > d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a)))$$

for all $i \in S$. So, it suffices to show that the system of linear equations L1 (see page 23) has a solution for some $\varepsilon > 0$. Without loss of generality we may assume that

$$0 < \varepsilon < \frac{1}{|S|} \left(\max_{i \in S} u_{\alpha_i}^{X_S(\hat{a})} - \sum_{i \in S} \left(d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a)))\right)\right).$$

Applying a variant of Farkas' lemma [2], L1 has a solution if and only if there exist no $(z_i)_{i\in N} \geq 0, p_1, p_2, q_1, q_2 \geq 0, (y_i)_{i\in S} \geq 0$ such that

$$\begin{aligned} z_i &= 0, && \text{for all } i \in N\setminus S \\ y_i(u_{\alpha_i}^{X_S(\hat{a})} - E(X_S(\hat{a}))) + p_1 - p_2 + z_i &= 0, && \text{for all } i \in S \\ y_i + q_1 - q_2 &= 0, && \text{for all } i \in S \end{aligned}$$

$$\sum_{i\in S} y_i \left(d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a))) + \varepsilon\right) + (q_1 - q_2)E(X_S(\hat{a})) + p_1 - p_2 > 0.$$

Or equivalently, there exist no $p, q \in \mathbf{R}$, $(y_i)_{i\in S} \geq 0$ such that

$$\begin{aligned} y_i(u_{\alpha_i}^{X_S(\hat{a})} - E(X_S(\hat{a}))) + p &\leq 0, && \text{for all } i \in S \\ y_i + q &= 0, && \text{for all } i \in S \end{aligned}$$

$$\sum_{i\in S} y_i \left(d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a))) + \varepsilon\right) + q \cdot E(X_S(\hat{a})) + p > 0.$$

From the equalities above we derive $y_i = y$ for all $i \in S$. By combining the two inequalities and substituting $q = -y$, the statement above is equivalent to the non existence of a $y \geq 0$ such that for all $i \in S$ we have

$$y(u_{\alpha_i}^{X_S(\hat{a})} - E(X_S(\hat{a}))) < y\sum_{i\in S}\left(d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a))) + \varepsilon\right) - y \cdot E(X_S(\hat{a})).$$

Equivalently, there is no $y \geq 0$ such that

$$y \max_{i\in S} u_{\alpha_i}^{X_S(\hat{a})} < y\sum_{i\in S}\left(d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a))) + \varepsilon\right).$$

Using $\varepsilon < \frac{1}{|S|}\left(\max_{i\in S} u_{\alpha_i}^{X_S(\hat{a})} - \sum_{i\in S}\left(d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a)))\right)\right)$ yields

$$y \max_{i\in S} u_{\alpha_i}^{X_S(\hat{a})} < y \max_{i\in S} u_{\alpha_i}^{X_S(\hat{a})}.$$

Obviously, such y do not exist. Hence, the system of linear equations L1 has a solution and the proof is finished. □

[2]The variant of Farkas' lemma we use here is: $Ax \geq b$ has a solution if and only if there exists no $y \geq 0$ such that $y^{\top}A = 0$ and $y^{\top}b > 0$.

Theorem 3.2 Let $\Gamma_\alpha = (N, (A_S)_{S\subset N}, (X_S)_{S\subset N}, (\succsim_{\alpha_i})_{i\in N}) \in SG(N)$. The core of Γ_α is non empty if and only if Γ_α is balanced.

PROOF: From Lemma 3.1 we know that an allocation $(d, r|a) \in Z(N)$ is stable against deviations from coalition S if and only if

$$\sum_{i\in S}\left(d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a)))\right) \geq \max_{\tilde{a}\in A_S} \max_{i\in S} u_{\alpha_i}^{X_S(\tilde{a})}.$$

Hence, there exists a core allocation $(d, r|a) \in Z(N)$ if and only if the system of linear equations L2 (see page 24) has a solution.

Applying the same variant of Farkas' lemma as in Lemma 3.1, L2 has a solution if and only if there exist no $(z_i)_{i\in N} \geq 0, p_1, p_2, q_1, q_2 \geq 0, (\mu(S))_{S\subset N} \geq 0$ such that

$$\sum_{S\subset N: i\in S} \mu(S)(u_{\alpha_i}^{X_N(a)} - E(X_N(a))) + p_1 - p_2 + z_i = 0, \quad \text{for all } i \in N$$

$$\sum_{S\subset N: i\in S} \mu(S) + q_1 - q_2 = 0, \quad \text{for all } i \in N$$

$$\sum_{S\subset N} \mu(S) \max_{\tilde{a}\in A_S} \max_{i\in S} u_{\alpha_i}^{X_S(\tilde{a})} + (q_1 - q_2)E(X_N(a)) + p_1 - p_2 > 0.$$

Equivalently, there exist no $p, q \in \mathbf{R}, (\mu(S))_{S\subset N} \geq 0$ such that

$$\sum_{S\subset N: i\in S} \mu(S)(u_{\alpha_i}^{X_N(a)} - E(X_N(a))) + p \leq 0, \quad \text{for all } i \in N$$

$$\sum_{S\subset N: i\in S} \mu(S) = q, \quad \text{for all } i \in N$$

$$\sum_{S\subset N} \mu(S) \max_{\tilde{a}\in A_S} \max_{i\in S} u_{\alpha_i}^{X_S(\tilde{a})} - q \cdot E(X_N(a)) + p > 0.$$

This is equivalent to the existence of $q \in \mathbf{R}$ and $(\mu(S))_{S\subset N} \geq 0$ such that for each $i \in N$

$$\sum_{S\subset N: i\in S} \mu(S)(u_{\alpha_i}^{X_N(a)} - E(X_N(a))) < \sum_{S\subset N} \mu(S) \max_{\tilde{a}\in A_S} \max_{i\in S} u_{\alpha_i}^{X_S(\tilde{a})} - q \cdot E(X_N(a))$$

and

$$\sum_{S\subset N: i\in S} \mu(S) = q, \qquad \text{for all } i \in N. \tag{1}$$

Substituting (1) and rearranging terms yields equivalently that there exist no $q \in \mathbf{R}$ and $(\mu(S))_{S \subset N} \geq 0$ such that for all $i \in N$ we have

$$\sum_{S \subset N : i \in S} \mu(S) u_{\alpha_i}^{X_N(a)} < \sum_{S \subset N} \mu(S) \max_{\bar{a} \in A_S} \max_{i \in S} u_{\alpha_i}^{X_S(\bar{a})} \tag{2}$$

and

$$\sum_{S \subset N} \mu(S) \cdot e_S = e_N \cdot q.$$

Since $\mu(S) = 0$ for all $S \subset N$ is not a solution of (2), we must have that $q > 0$. Hence, we may assume that $q = 1$. Then we have that there exists no balanced map μ such that for all $i \in N$ we have

$$u_{\alpha_i}^{X_N(a)} < \sum_{S \subset N} \mu(S) \max_{\bar{a} \in A_S} \max_{i \in S} u_{\alpha_i}^{X_S(\bar{a})}.$$

Or equivalently, there is no balanced map μ such that

$$\max_{i \in N} u_{\alpha_i}^{X_N(a)} < \sum_{S \subset N} \mu(S) \max_{\bar{a} \in A_S} \max_{i \in S} u_{\alpha_i}^{X_S(\bar{a})}.$$

Again, this is equivalent with the fact that for all balanced maps μ we must have

$$\max_{i \in N} u_{\alpha_i}^{X_N(a)} \geq \sum_{S \subset N} \mu(S) \max_{\bar{a} \in A_S} \max_{i \in S} u_{\alpha_i}^{X_S(\bar{a})}. \tag{3}$$

So, there exists a core allocation $(d, r|a)$ of the payoff $X_N(a)$ if and only if (3) holds for all balanced maps μ. Hence, the core is non empty if and only if for each balanced map μ we have

$$\max_{a \in A_N} \max_{i \in N} u_{\alpha_i}^{X_N(a)} \geq \sum_{S \subset N} \mu(S) \max_{\bar{a} \in A_S} \max_{i \in S} u_{\alpha_i}^{X_S(\bar{a})}.$$ □

Example 3.3 Consider the following three person situation, where agents 1 and 2 possess the same technology and agent 3 possesses some resources. To produce a good out of the resources of agent 3 the technology of agent 1 or 2 is needed. Moreover, the good can be sold for a price, which is not known with certainty beforehand, but is uniformly distributed on the interval $[0,6]$. This situation can be modelled as a game with stochastic payoffs, with $N = \{1,2,3\}$, $|A_S| = 1$ for all coalitions $S \subset N$, $X_S = 0$ for $S = \{3\}$ and all $S \subset N$ with $3 \notin S$, and $X_S = X \sim U(0,6)$ otherwise. Note that since each coalition only has one action to take, the action a is omitted as an argument in X_S. Now, let the preferences of the agents be such that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha \in (0,1)$. Then $(d,r) \in Z(N)$ is a core allocation if and only if no coalition has an incentive to leave the grand coalition. Applying Lemma 3.1 yields that

$$\sum_{i \in S} \left(d_i + r_i(u_\alpha^{X_N} - E(X_N))\right) \geq u_\alpha^{X_S}$$

has to hold for all $S \subset N$. If $S = \{i\}, i \in N$ this results in $d_i + r_i(6\alpha - 3) \geq 0$ for all $i \in N$. Rewriting then gives $d_i \geq r_i(3 - 6\alpha)$. If $S = \{1,2\}$ we get

$$d_1 + d_2 + (r_1 + r_2)(6\alpha - 3) \geq 0.$$

Substituting $d_1 + d_2 = 3 - d_3$ and $r_1 + r_2 = 1 - r_3$ and rearranging terms yield $d_3 \leq 6\alpha + r_3(3 - 6\alpha)$. If $S = \{1,3\}$ we get

$$d_1 + d_3 + (r_1 + r_3)(6\alpha - 3) \geq 6\alpha.$$

Substituting $d_1 + d_3 = 3 - d_2$ and $r_1 + r_3 = 1 - r_2$ and rearranging terms yield $d_2 \leq r_2(3 - 6\alpha)$. Similarly, one derives for $S = \{2,3\}$ that $d_1 \leq r_1(3 - 6\alpha)$. Combining the results above, (d,r) is a core allocation if and only if $d_1 = r_1(3 - 6\alpha)$, $d_2 = r_2(3 - 6\alpha)$ and $d_3 = 6\alpha + r_3(3 - 6\alpha)$.

Now let us try to interpret these results. Because $\alpha_i = \alpha$, $i \in N$ all three agents have the same behaviour towards risk. Let us take $\alpha = \frac{1}{10}$. Next, consider the core allocation with $d = (\frac{12}{5}r_1, \frac{12}{5}r_2, \frac{3}{5} + \frac{12}{5}r_3)$ and $\sum_{i \in N} r_i = 1$, $r_i \geq 0$, $i = 1,2,3$. Then the payoff for agent 1 equals $\frac{12}{5}r_1 + r_1(X - 3)$. Moreover, $u_{0.1}^{\frac{12}{5}r_1 + r_1(X-3)} = 0$. So, for a core allocation, agent 1 is with probability $\frac{1}{10}$ worse off than his initial situation, that is, payoff zero. The same reasoning holds for agent 2. For agent 3, the payoff equals $\frac{3}{5} + \frac{12}{5}r_3 + r_3(X - 3)$. Consequently, $u_{0.1}^{\frac{3}{5} + \frac{12}{5}r_3 + r_3(X-3)} = \frac{3}{5}$. So, agent 3 is worse off than payoff $\frac{1}{5}$ with probability $\frac{1}{10}$. Since all agents have the same behaviour towards risk, we may say that agent 3 is slightly better off than the other two agents. Hence, a more or less similar result is achieved if we consider the core of this same situation with deterministic expected payoffs, i.e. $v(S) = 0$ if $3 \notin S$ or $|S| < 2$ and $v(S) = 3$ otherwise. Then the core equals $\{(0,0,3)\}$, and indeed for that case agent 3 is also better off than the other two agents.

We conclude this section with some remarks. First, note that the action taken by the grand coalition at a core allocation maximizes $\max_{i \in N} u_{\alpha_i}^{X_N(a)}$ with respect to a. Indeed, if $\mu(S) = 1$ when $S = N$ and $\mu(S) = 0$ otherwise, the balancedness condition implies for a core allocation $(d, r|a) \in Z(N)$ that $\max_{i \in N} u_{\alpha_i}^{X_N(a)} \geq \max_{\hat{a} \in A_N} \max_{i \in N} u_{\alpha_i}^{X_N(\hat{a})}$. Moreover, it follows from Lemma 3.1 that for a core allocation $(d, r|a)$ the risk $X_N(a) - E(X_N(a))$ must be allocated over the most risk loving agent(s), i.e. the agents who maximize $\max_{i \in N} u_{\alpha_i}^{X_N(a)}$. For, if this is not the case, we get

$$\begin{aligned} \textstyle\sum_{i \in N} \left(d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a)))\right) &= \textstyle\sum_{i \in N} r_i \cdot u_{\alpha_i}^{X_N(a)} \\ &< \max_{i \in N} u_{\alpha_i}^{X_N(a)} \\ &\leq \max_{\hat{a} \in A_N} \max_{i \in N} u_{\alpha_i}^{X_N(\hat{a})}. \end{aligned}$$

This, however, contradicts the fact that the allocation must be Pareto optimal for coalition N (cf. Lemma 3.1 for $S = N$).

For our final remark we take a closer look at the balancedness condition. If we define for each game $\Gamma \in SG(N)$ a corresponding deterministic TU-game (N, v_Γ) with $v_\Gamma(S) = \max_{a \in A_S} \max_{i \in S} u_{\alpha_i}^{X_S(a)}$ for each $S \subset N$, then Γ is balanced if and only if (N, v_Γ) is balanced. A similar reasoning holds for allocations. An allocation $(d, r|a) \in Z(N)$ is a core allocation for Γ if and only if $(d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a))))_{i \in N}$ is a core allocation for (N, v_Γ). This result follows immediately from Lemma 3.1. Note, however, that the relation between the allocation $(d, r|a)$ and the vector $(d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a))))_{i \in N}$ is not a one-one correspondence.

The results obtained in this section hold for a special class of preferences. In the next section we consider other types of preference relations. Moreover, we show that the results obtained in this section can be extended to some of these preferences.

4 Preferences on stochastic payoffs

A common way of ordering stochastic variables is by the use of von Neumann/Morgenstern utility functions. In that case, an agent prefers one stochastic payoff to another if the expected utility of the first exceeds the expected utility of the latter. More formally, let $X, Y \in L^1(\mathbf{R})$ be stochastic variables and let $u : \mathbf{R} \rightarrow \mathbf{R}$ be the agent's monotonically increasing utility function, then $X \succ Y$ if and only if $E(u(X)) > E(u(Y))$. Moreover, a concave utility function implies that the agent is risk averse, a linear utility function implies that he is risk neutral and, finally, a convex utility function implies that he is risk loving. So, von Neumann/Morgenstern preferences are complete and transitive and can distinguish between different kinds of behaviour of agents towards risk. However, for our game theoretic approach these preferences lead to computational difficulties, as we show in the next example.

Example 4.1 Consider the situation described in example 3.3 but now with von Neumann/Morgenstern preferences instead of $\succsim_\alpha$-preferences. Let the utility function of agent i be given by the concave function $u_i(x) = 6 - \epsilon^{-x}$ and consider an arbitrary allocation (d, r) for N of X. To check whether this allocation is in the core or not, we first have to check whether (d, r) is individually rational, that is, whether $E(u_i(d_i + r_i(X-3)) \geq E(u_i(0))$, for all $i \in N$. Rewriting this inequality gives

$$\begin{aligned} E(u_i(d_i + r_i(X-3)) &= \tfrac{1}{6}\int_0^6 6 - \epsilon^{-d_i - r_i(x-3)}dx \\ &= 6 - \tfrac{1}{r_i}(\epsilon^{-d_i+3r_i} - \epsilon^{-d_i-3r_i}) \geq 5 = E(u_i(0)) \end{aligned}$$

if $r_i > 0$. However, this inequality is difficult to solve. Hence, an explicit condition on the allocation (d, r) is not available. Not surprisingly, the problem will be even more complicated to check whether a two person coalition can improve upon the allocation (d, r).

A natural way of ordering stochastic payoffs is by means of stochastic domination. Let $X, Y \in L^1(\mathbf{R})$ be stochastic variables and denote by F_X and F_Y the distribution functions of X and Y, respectively. Then X stochastically dominates Y, in notation $X \succsim_F Y$, if and only if for all $t \in \mathbf{R}$ it holds that $F_X(t) \leq F_Y(t)$. Moreover, we have $X \succ_F Y$ if and only if for all $t \in \mathbf{R}$ it holds that $F_X(t) \leq F_Y(t)$ and $F_X(t) < F_Y(t)$ for at least one $t \in \mathbf{R}$. Intuitively one may expect that every rationally behaving agent, whether he is risk averse, risk neutral or risk loving, will prefer a stochastic payoff X over Y if $X \succsim_F Y$. However, this preference relation is incomplete. Many stochastic variables will be incomparable with respect to $\succsim_F$. As we will see in the next example, this incompleteness will lead to a relatively large core.

Example 4.2 Consider the situation described in example 3.3, but now with stochastic domination as the preference relation for all agents. One can check[3] that $(d, r) \in$

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[3] These conditions for a core allocation are not obvious. Although it is not difficult to check them, including the proof would lengthen the example with quite a few pages. A sketch of the proof goes as follows. Consider an arbitrary allocation $(d, r) \in Z(N)$ and check for each coalition S separately, if there exists a better allocation $(\tilde{d}, \tilde{r}) \in Z(S)$. For one person coalitions this is straightforward. For two person coalitions it is a bit more difficult. In that case, one has to distinguish nine different cases, namely, $\tilde{r}_i > r_i$ and $\tilde{r}_j > r_j$, $\tilde{r}_i = r_i$ and $\tilde{r}_j > r_j$, etc.... Then, using the same variant of Farkas' lemma as in the proof of Theorem 3.2, one can derive for each case separately conditions on the existence of a better allocation. Then combining these conditions will give the abovementioned result.

$Z(N)$ is a core allocation for this game if and only if for $i = 1, 2$ it holds that

$$d_i \in (-3r_i, 3r_i), \quad \text{if } r_i > 0 \text{ and}$$
$$d_i = 0, \quad \text{if } r_i = 0.$$

and

$$d_3 > -3r_3, \quad \text{if } r_3 > 0 \text{ and}$$
$$d_3 \geq 0, \quad \text{if } r_3 = 0.$$

Now let us compare the core of a game with $\succ_F$- preferences with the core of a game with $\succsim_\alpha$-preferences. First, note that the core of the first one no longer needs to be closed. Second, the core of a game with $\succ_F$-preferences depends on the core of a game with $\succsim_\alpha$-preferences in the following sense. Denote by Γ_F the game Γ with $\succ_F$-preferences, and by Γ_α the game with $\succsim_{\alpha_i}$-preferences, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Since $X \succ_F Y$ implies $X \succsim_\alpha Y$ it follows that $Core(\Gamma_\alpha) \subset Core(\Gamma_F)$. Moreover, this holds for all $\alpha \in (0,1)^N$. Hence, $\cup_{\alpha \in (0,1)^N} Core(\Gamma_\alpha) \subset Core(\Gamma_F)$. The reverse however need not be true, as we will show in the next example.

Example 4.3 Consider again the situation described in example 3.3. From Example 3.3 and Example 4.2 we know that for $d = ((3-6\alpha)r_1, (3-6\alpha)r_2, 6\alpha+(3-6\alpha)r_3)$ both $(d,r) \in Core(\Gamma_\alpha)$ and $(d,r) \in Core(\Gamma_F)$. From the results of example 4.2 we also know that $d = (-1, 1, 3)$ and $r = \frac{1}{12}(7, 4, 1)$ is a core allocation with respect to $\succ_F$-preferences. This allocation, however, cannot be a core allocation for the game with $\succsim_\alpha$-preferences. To see this, suppose that (d,r) is a core allocation. Then $r_1, r_2, r_3 > 0$ implies that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$. Similarly to example 3.3 one can derive that $(\hat{d}, \hat{r})$ is a core allocation if $\hat{d}_1 = (3-6\alpha)\hat{r}_1$, $\hat{d}_2 = (3-6\alpha)\hat{r}_2$ and $\hat{d}_3 = 6\alpha + (3-6\alpha)\hat{r}_3$. However, there exists no α satisfying $-1 = (3-6\alpha)\frac{7}{12}$ and $1 = (3-6\alpha)\frac{4}{12}$. Hence, $(d,r) \notin Core(\Gamma_\alpha)$ for any $\alpha \in (0,1)^N$.

The preference relation $\succsim_F$ yields many incomparabilities of stochastic variables. As a result, the core is a fairly large set and, moreover, counter intuitive core allocations arise. Consider in example 4.2 the allocation with $d = (-3+\varepsilon, 0, 6-\varepsilon)$ with ε small and $r = (1, 0, 0)$. Then agent 1 bears all the risk and, on top of that, he also has to pay an amount almost equal to 3. On the other hand, agent 3 bears no risk and receives an amount of money closely equal to 6. This allocation would only be credible as a core allocation if agent 1 is extremely risk loving, that is, he prefers the payoff $-3+\varepsilon+(X-E(X)) \sim U(-6+\varepsilon, \varepsilon)$, which gives him only a (small) positive payoff with probability $\frac{\varepsilon}{6+2\varepsilon}$, to a payoff zero with probability one. However, the preference relation $\succsim_F$ does not reveal any information about whether the agent is risk loving, risk neutral or risk averse. As a result, the core allocation mentioned above, should also be stable if agent 1 is risk averse, which, from the authors' point of view, seems not very likely. Thus an important difference with the $\succsim_\alpha$- and the von Neumann/Morgenstern preferences is pointed out, namely that different behaviour of agents towards risk cannot be captured when using preferences $\succsim_F$.

A straightforward way of ordering stochastic variables, is looking at the expectation. Then, for two stochastic variables $X, Y \in L^1(\mathbf{R})$ we have $X \succsim_E Y$ if and only if $E(X) \geq E(Y)$. Note that $X \succsim_E Y$ whenever $X \succsim_F Y$. This preference relation is complete and implies risk neutral behaviour of an agent. Hence, risk averse and risk loving attitudes cannot be modelled. If, however, we adapt preferences $\succsim_E$ in the following way, also these types of attitudes can be modelled.

Let $X, Y \in L^1(\mathbf{R})$ be stochastic variables with finite variance and let $b \in \mathbf{R}$ be arbitrary. Then $X \succsim^b Y$ if and only if $E(X) + b\sqrt{V(X)} \geq E(Y) + b\sqrt{V(Y)}$, where $V(X)$ denotes the variance of X. Note that if $b = 0$ the preference relation $\succsim^0$ coincides with $\succsim_E$. For these type of preferences, $b < 0$ implies risk averse behaviour, $b = 0$ risk neutral behaviour and $b > 0$ risk loving behaviour. Moreover, we can derive counterparts of Lemma 3.1 and Theorem 3.2. For this, we replace $u_{\alpha_i}^{X_S(a)}$ by $E(X_S(a)) + b_i\sqrt{V(X_S(a))}$ for all $i \in N$ and all $S \subset N$. The balancedness condition then becomes,

$$\max_{a \in A_N} \max_{i \in N} E(X_N(a)) + b_i \sqrt{V(X_N(a))} \geq$$

$$\sum_{S \subset N} \mu(S) \cdot \max_{a \in A_S} \max_{i \in S} E(X_S(a)) + b_i \sqrt{V(X_S(a))}$$

for all balanced maps μ.

Theorem 4.4 Let $\Gamma = (N, (A_S)_{S \subseteq N}, (X_S)_{S \subseteq N}, (\succsim^{b_i})_{i \in N}) \in SG(N)$. Then the core of Γ is non empty if and only if Γ is balanced.

Although $\succsim^b$ is complete and distinguishes different kinds of behaviour with respect to risk, it is not implied by $\succsim_F$. For example, let $X \sim U(0,6)$ and $Y \sim U(0,2)$. Then $X \succ_F Y$ but $Y \succ^b X$ whenever $b < \underset{3}{=}4$. Although an agent with such preferences is risk averse, it is still natural to expect that he prefers X over Y.

5 Concluding remarks

This paper introduces a new class of cooperative games, with the aid of which various cooperative decision making problems in a stochastic environment can be modelled. Besides a discussion on the applications of the model and the preferences of the agents, our interests were focused on the core of the game. For special classes of games it was shown that the core is non empty if and only if the game is balanced. An interesting question within this framework is if a similar result can be obtained for other preferences, for example the von Neumann/Morgenstern preferences discussed in Section 4.

Other remaining questions concern solution concepts. How to define a Shapley value or nucleolus for games with stochastic payoffs? In answering these questions one has to know what is a marginal vector and how to compare the complaint of one coalition to the complaint of another coalition.

System of linear equations L1:

$$
\begin{array}{l}
(z_i)_{i\in N} \cdots\cdots\cdots \\
p_1 \cdots\cdots\cdots \\
p_2 \cdots\cdots\cdots \\
q_1 \cdots\cdots\cdots \\
q_2 \cdots\cdots\cdots \\
\\
(y_i)_{i\in S} \cdots\cdots \forall i \in S \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right.
\end{array}
\begin{bmatrix}
I_N & 0 \\
e_S^{\mathsf{T}} & 0 \\
-e_S^{\mathsf{T}} & 0 \\
0 & e_S^{\mathsf{T}} \\
0 & -e_S^{\mathsf{T}} \\
\vdots & \vdots \\
(u_{\alpha_i}^{X_S(\hat{a})} - E(X_S(\hat{a}))) \cdot e_i^{\mathsf{T}} & e_i^{\mathsf{T}} \\
\vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
\hat{r}_1 \\ \hat{r}_2 \\ \vdots \\ \hat{r}_n \\ \hat{d}_1 \\ \hat{d}_2 \\ \vdots \\ \hat{d}_n
\end{bmatrix}
\geq
\begin{bmatrix}
0 \\ 1 \\ -1 \\ E(X_S(\hat{a})) \\ -E(X_S(\hat{a})) \\ \vdots \\ d_i + r_i(u_{\alpha_i}^{X_N(a)} - E(X_N(a))) + \varepsilon \\ \vdots
\end{bmatrix}
$$

where I_N denotes the N-dimensional identity matrix and e_S denotes the vector with $(e_S)_i = 1$ if $i \in S$ and $(e_S)_i = 0$ otherwise. The variables on the left denote the dual variables. Note that for notational reasons we have included r_i and d_i for $i \notin S$. Since the corresponding coefficients for these variables are equal to zero, this does not affect the result.

System of linear equations L2:

$$
\begin{array}{l}
(z_i)_{i\in N} \cdots\cdots\cdots \\
p_1 \cdots\cdots\cdots \\
p_2 \cdots\cdots\cdots \\
q_1 \cdots\cdots\cdots \\
q_2 \cdots\cdots\cdots \\
\\
(\mu(S))_{S\subset N} \cdots \ \forall S \subset N \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right.
\end{array}
\begin{bmatrix}
I_N & 0 \\
e_N^\top & 0 \\
-e_N^\top & 0 \\
0 & e_N^\top \\
0 & -e_N^\top \\
\vdots & \vdots \\
\sum_{i\in S}(u_{\alpha_i}^{X_N(a)} - E(X_N(a))) \cdot e_i^\top & e_S^\top \\
\vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
r_1 \\ r_2 \\ \vdots \\ r_n \\ d_1 \\ d_2 \\ \vdots \\ d_n
\end{bmatrix}
\geq
\begin{bmatrix}
0 \\ 1 \\ -1 \\ E(X_N(a)) \\ -E(X_N(a)) \\ \vdots \\ \max_{\bar{a}\in A_S} \max_{i\in S} u_{\alpha_i}^{X_S(\bar{a})} \\ \vdots
\end{bmatrix}
$$

References

CHARNES, A. and D. GRANOT: "Prior Solutions: Extensions of Convex Nucleolus Solutions to Chance- Constrained Games," *Proceedings of the Computer Science and Statistics Seventh Symposium at Iowa State University* (1973), 323-332.

CHARNES, A. and D. GRANOT: "Coalitional and Chance-Constrained Solutions to N-Person Games I," *SIAM Journal of Applied Mathematics*, 31 (1976), 358-367.

CHARNES, A. and D. GRANOT: "Coalitional and Chance-Constrained Solutions to N-Person Games II," *Operations Research*, 25 (1977), 1013-1019.

CURIEL, I., G. PEDERZOLI, and S. TIJS: "Sequencing Games," *European Journal of Operational Research*, 40 (1989), 344-351.

GRANOT, D.: "Cooperative Games in Stochastic Characteristic Function Form," *Management Science*, 23 (1977), 621-630.

HAMERS, H., P. BORM and S. TIJS: "On Games Corresponding to Sequencing Situations with Ready Times," *CentER Discussion Paper*, 9316 (1993) (to appear in Mathematical Programming).

MAGILL, M. and W. SHAFER: "Incomplete Markets," in *Handbook of Mathematical Economics* IV, ed. by W. Hildenbrand and H. Sonnenschein. Amsterdam: Elsevier Science Publishers, 1991.

MODIGLIANI, F. and H. MILLER: "The Cost of Capital, Corporation Finance and the Theory of Investment," *The American Economic Review*, 48 (1958), 261-297.

OWEN, G.: "On the Core of Linear Production Games," *Mathematical Programming*, 9 (1975), 358-370.

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