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RESEARCH MEMORANDUM

A QUADRATICALLY CONVERGENT PARALLEL JACOBI-PROCESS FOR DIAGONAL DOMINANT MATRICES WITH NONDISTINCT EIGENVALUES

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ABSTRACT

This article presents a new Jacobi-like eigenvalue algorithm for nonhermitian almost diagonal $n \times n$ matrices. In each step $n/2$ submatrices of order 2 are diagonalized. The precautions for the multiple eigenvalues are based on theorems of Fan and Hoffman (1954) and Wilkinson (1961). The proof of the quadratic convergence generalizes our previous result for distinct eigenvalues. The convergence theorem is pessimistic concerning the region of attraction to a diagonal as is shown in examples. The local information structure makes the process ready to parallelization on a hypercube or a systolic array.

Keywords: multiple eigenvalues, diagonal dominancy, nonhermitean matrices, quadratic convergence, Jacobi method, parallel algorithm.

1. INTRODUCTION

In the welknown Jacobi algorithm [7] for the diagonalization of a real *symmetric* matrix each of the successive *orthogonal* similarity transformations is designed to annihilate a symmetric pair of off-diagonal elements. The ultimately quadratic convergence of this algorithm has been investigated by several authors [8,9,15,16].
This report deals with a similar parallel computational method for eigenvalues of a diagonalizable nonhermitean matrix $A \in \mathbb{C}^{n\times n}$. Our Jacobi-like method is asymptotically quadratic convergent. It is assumed that A is almost diagonal. In [12] a parallel algorithm was developed for solving eigenvalues of almost diagonal matrices with distinct eigenvalues. The extension of that method in this article does not exclude multiple eigenvalues. We assume $n \geqq 4$, and in order to avoid inessential difficulties in the description of the algorithm n to be even. Matrix A is the first element $A^{(0)}$ of a recursively constructed sequence $\{A^{(k)}\}$:

$$A^{(k+1)} = S_k^{-1}A^{(k)}S_k , \qquad k \geqq 0 . \tag{1.1}$$

Each S_k is a direct sum of 2×2 unimodular matrices

$$T_{i,k} = \begin{bmatrix} p_{i,k} & q_{i,k} \\ r_{i,k} & s_{i,k} \end{bmatrix} \begin{matrix} \leftarrow \ell(i,k) \\ \leftarrow m(i,k) \end{matrix} , \quad i = 1,\dots,n/2 . \tag{1.2}$$
$$\qquad\quad \uparrow \qquad\quad \uparrow$$
$$\qquad \ell(i,k) \quad m(i,k)$$

Every shear $T_{i,k}$ annihilates the symmetrically placed pair $(a_{\ell(i,k),m(i,k)}, a_{m(i,k),\ell(i,k)})$ or, with $T_{i,k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, let them unchanged. If $a_{\ell(i,k),\ell(i,k)}$ and $a_{m(i,k),m(i,k)}$ are affiliated to the same eigenvalue then the annihiliation is skipped. In (1.12) is the condition for the choice between the annihilator and the identity. Generally the tranformation matrix S_k is not unitary. Consequently monotonic decrease of the Frobenius norms of the non-diagonal parts of $A^{(k)}$ can not be guaran-

teed despite the annihilation of elements. In [13] the same lack of monotonicity caused genuine difficulties in the proof of the quadratic convergence of the Eberlein algorithm [4,10] for the algebraïc eigenproblem.
In each *step* (1.1) of the algorithm the n/2 ordered pivot pairs ($\ell(i,k)$, $m(i,k)$, $i = 1,\ldots,n/2$) are such that $1 \leqq \ell(i,k) < m(i,k) \leqq n$ and

$$\{(\ell(i,k),m(i,k)))|i = 1,\ldots,n/2,\ k = 0,\ldots,n-2\} =$$

$$= \{(i,j)|1 \leqq i < j \leqq n\} . \tag{1.3}$$

Hence in any *sweep* of n-1 steps each index pair (i,j) occurs once and only once as a pivot pair.
For the definition of an adequate concept of almost diagonality and also for the detailed description of process (1.1) we introduce some notation. Let be $D^{(k)}$ and $E^{(k)}$ the diagonal and nondiagonal part of $A^{(k)}$ respectively. We set

$$\varepsilon_k = \|E^{(k)}\|_\infty , \tag{1.4}$$

and

$$\eta = \min\{|\lambda_i - \lambda_j| \ |\lambda_i \neq \lambda_j|\} , \tag{1.5}$$

where λ_i, $i = 1,\ldots,n$ are the eigenvalues of A. Further

$$\tau_k = \varepsilon_k/\eta . \tag{1.6}$$

DEFINITION 1.1. Matrix $A \in \mathbb{C}^{n\times n}$ is *diagonal dominant* with respect to *separation* η of its spectrum if $\varepsilon = \varepsilon_0 \leqq c\eta$ for some $c \in [0,\frac{1}{2})$. □

We assume $A^{(0)}$ to be diagonal dominant relatively η, namely

$$\varepsilon = \varepsilon_0 \leqq \eta/10 \tag{1.7}$$

For the formulation of the algorithm we make use of a theorem of Fan and Hoffman.

THEOREM 1.2. [6] If λ is a t-fold eigenvalue of A then the inequality

$$|a_{i,i} - \lambda| \leqq \sum_{j\neq i} |a_{i,j}| \tag{1.8}$$

holds for at least t indices i. □

As a consequence of assumption (1.7) we get that for a t-fold eigenvalue λ inequality (1.8) holds for exactly t indices i. Hence if $a_{\ell,\ell}$ and $a_{m,m}$ are affiliated with the same eigenvalue λ then $|a_{\ell,\ell} - a_{m,m}| \leqq 2\varepsilon_0$ otherwise $|a_{\ell,\ell} - a_{m,m}| \geqq \eta - 2\varepsilon_0$.

DEFINITION 1.3. The set of index pairs

$$J_k := \{(\ell(i,k), m(i,k)) \,|\, i = 1,\dots,n/2, \quad \left|a^{(k)}_{\ell(i,k),\ell(i,k)} - a^{(k)}_{m(i,k),m(i,k)}\right| \leqq 2\varepsilon_k\}\ , \quad k \geqq 0\ , \tag{1.9}$$

is called the k-th set of *forbidden* pivot pairs. □

By means of J_k we define $\tilde{E}^{(k)}$ as follows:

$$\tilde{e}^{(k)}_{i,j} = \begin{cases} a^{(k)}_{i,j} & ,\ (i,j) \in J_k \text{ or } (j,i) \in J_k \\ 0 & ,\ \text{otherwise}\ . \end{cases} \quad , \quad k \geqq 0 \tag{1.10}$$

For the elements $\hat{e}^{(k)}_{i,j}$ of $\hat{E}^{(k)}$ holds

$$\hat{e}^{(k)}_{i,j} = \begin{cases} e^{(k)}_{i,j} & ,\ \left|a^{(k)}_{i,i} - a^{(k)}_{j,j}\right| \leqq 2\varepsilon_k \\ 0 & ,\ \text{otherwise}\ . \end{cases} \tag{1.11}$$

In our convergence analysis we use Wilkinson's estimate for $\|\hat{E}^{(0)}\|_\infty$, namely

THEOREM 1.4 [17]. Let be $\tilde{E} = \tilde{E}^{(0)}$ and $\hat{E} = \hat{E}^{(0)}$ as defined in (1.10) and (1.11) resp. Then

$$\|\tilde{E}\|_\infty \leqq \|\hat{E}\|_\infty \leqq (1-2\varepsilon/\eta)^{-1}\,\varepsilon^2/\eta \,. \qquad \square$$

In order to control and bound the growth of ε_k, $k = 1,\dots,n-1$, it is advisable to avoid the annihilation of the nondiagonal elements corresponding with the forbidden pivot pairs. So we arrive at a *treshold* strategy for the Jacobi process $A^{(k+1)} = S_k^{-1}A^{(k)}S^{(k)}$, $k \geqq 0$. The shears $T_{i,k}$, $i = 1,\dots,n/2$ in the transformation matrix S_k are defined in the following way

$$\begin{cases} T_{i,k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\ell(i,k),m(i,k)) \in J_k \\ T_{i,k} \text{ annihiliates } a_{\ell(i,k),m(i,k)} \text{ and } a_{m(i,k),\ell(i,k)}, (\ell(i,k),m(i,k) \notin J_k \,. \end{cases} \tag{1.12}$$

As in [12] an annihilating shear $T_{i,k} = T = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, with $ps - qr = 1$ and such that

$$T^{-1}\begin{pmatrix} \alpha & \mu \\ \sigma & \beta \end{pmatrix} T = \begin{pmatrix} s & -q \\ -r & p \end{pmatrix}\begin{pmatrix} \alpha & \mu \\ \sigma & \beta \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} \alpha' & 0 \\ 0 & \beta' \end{pmatrix},$$

is chosen to be

$$\begin{pmatrix} \left(\frac{1}{2} + \frac{1}{2F}\right)^{\frac{1}{2}} & \dfrac{-\mu\sqrt{2}}{\nu(F+F^2)^{\frac{1}{2}}} \\ \dfrac{\sigma\sqrt{2}}{\nu(F+F^2)^{\frac{1}{2}}} & \left(\frac{1}{2} + \frac{1}{2F}\right)^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}. \tag{1.13}$$

In (1.13) we used the notations:

$$\nu = \alpha - \beta \,, \qquad F := (1+4\sigma\mu\nu^{-2})^{\frac{1}{2}} \text{ with } \mathrm{Re}(F) > 0 \,. \tag{1.14}$$

Without loss of generality we assume the pivots to be generated by the *caterpillar* permutation P [3], defined by

$$P(i) = \begin{cases} i+2\ , & i \leqq n-2 \text{ and even,} \\ i-2\ , & 5 \leqq i \text{ and odd} \\ i-1\ , & i = 3,n \\ i\ , & i = 1 \end{cases} \tag{1.15}$$

The pivot strategy is given by

$$\{(\ell(i,k),m(i,k))\,|\,i = 1,\dots,n/2\} =$$

$$\{(P^k(2i-1),P^k(2i))\,|\,i = 1,\dots,n/2\},\ k \geqq 0\ . \tag{1.16}$$

Remark that P^{n-1} = id and that (1.3) has been satisfied. In each step $n/2$ shears are active in a parallel way. Other parallel orderings effectuate the same convergence results as those given below.
The main result of this article is the quadratic convergence property

$$\text{if } \varepsilon_0 \leqq \eta/(10n) \qquad \text{then} \qquad \varepsilon_{n-1} \leqq \left[4n + \frac{9}{2}\right]\varepsilon_0^2/\eta\ . \tag{1.17}$$

The proof of (1.17) requires a precise analysis of the annihilation. In section 2 we investigate the first step: $A^{(1)} = S_0^{-1}A^{(0)}S_0$. Section 3 analyses the first sweep of n-1 steps that terminates with $A^{(n-1)} = S_{n-2}^{-1}A^{(n-2)}S_{n-2}$. There we prove (1.17). Numerical results and their discussion are presented in section 4. In section 5 are concluding remarks.

2. THE FIRST STEP

In this section we investigate the first step $A^{(1)} = S_0^{-1}A^{(0)}S_0$. Basic estimates will be derived for the analysis of the first sweep. The affiliation of the eigenvalues and diagonal elements remains unchanged in the first step, and also as a consequence of (1.7) we prove that $\varepsilon_1 \leqq (1+\frac{46}{11}\tau_0)\varepsilon_0$.
For shortness of notation we write

$$T_i = T_{i,0} = \begin{pmatrix} p_i & q_i \\ r_i & s_i \end{pmatrix}, \quad i=1,\dots,n/2 \tag{2.1}$$

THEOREM 2.1. If $\varepsilon_0 \leqq \eta/10$ then

(i) $$|p_i| \leqq 1 + \frac{9}{11}\tau_0^2, \tag{2.2}$$

(ii) $$|q_i|,|r_i| \leqq \frac{9}{7}\tau_0, \qquad i=1,\dots,n/2 \tag{2.3}$$

(iii) $$\|T_i\|_\infty \leqq 1 + \frac{26}{19}\tau_0. \tag{2.4}$$

PROOF. The nontrivial case concerns the submatrix with $|\nu_i| = |a_{2i-1,2i-1} - a_{2i,2i}| \geqq \eta - 2\varepsilon_0$, for otherwise $|\nu_i| \leqq 2\varepsilon_0$ and $p_i = s_i = 1$, $q_i = r_i = 0$. For reasons of simplicity we omit the index i. We use the formulae (1.13) and (1.14).

(i) $|p| \leqq (\frac{1}{2} + \frac{1}{2}(1-4\varepsilon^2|\nu|^{-2})^{-\frac{1}{2}})^{\frac{1}{2}} \leqq (\frac{1}{2} + \frac{1}{2}(1-4\tau^2/(1-2\tau)^2)^{-\frac{1}{2}})^{\frac{1}{2}} \leqq$
$1 + \frac{9}{11}\tau^2$, for $\tau = \varepsilon/\eta \leqq 1/10$.

(ii) $|q| \leqq |\mu\,\nu^{-1}\,(F+F^2)^{-\frac{1}{2}}|\,\sqrt{2} \leqq \tau\,(1-2\tau)^{-1}\,|F+F^2|^{-\frac{1}{2}}\,\sqrt{2}$.
Since $|F|^2 \geqq 1-4\,\tau^2\,(1-2\tau)^{-2}$ we find
$|q|,|r| \leqq \tau\,(1-2\tau)^{-1}(1-4\tau^2(1-2\tau)^{-2} + (1-4\tau^2(1-2\tau)^{-2})^{\frac{1}{2}})^{-\frac{1}{2}}\,\sqrt{2}$.
With simple but tedious calculations we derive from $\tau \leqq 1/10$: $|q| \leqq \frac{9}{7}\tau$. The some estimate holds for $r = \sigma\,\nu^{-1}\,(F+F^2)^{-\frac{1}{2}}\,\sqrt{2}$.

(iii) $\|T\|_\infty = \max\{|p|+|q|,|r|+|s|\} \le 1 + \frac{9}{11}\tau^2 + \frac{9}{7}\tau \leqq 1 + \frac{26}{19}\tau$ as follows from (2.2) and (2.3). □

With (2.4) we find

THEOREM 2.2. If $\varepsilon_0 \leqq \eta/10$ then

$$\varepsilon_1 \leqq (1+\frac{46}{11}\tau_0)\,\varepsilon_0 . \tag{2.5}$$

PROOF. Consider the partitioned matrix $\hat{A} = (\hat{A}_{i,j})$ with 2×2 blocks $\hat{A}_{i,j}$ defined by

$$\hat{A}_{i,j} = \begin{cases} A_{i,j} = \begin{bmatrix} a_{2i-1,2j-1} & a_{2i-1,2j} \\ a_{2i,2j-1} & a_{2i,2j} \end{bmatrix}, & i \neq j \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & i = j. \end{cases} \tag{2.6}$$

Now $S_0^{-1}\hat{A}S_0 = (T_i^{-1}\hat{A}_{i,j}T_j)$. Since, see (1.11)

$$\max_{1 \leqq i \leqq n/2} \max\{|a_{2i-1,2i}|,|a_{2i,2i-1}|\} \leqq \beta = (1-2\tau_0)^{-1}\tau_0\varepsilon_0,$$

we have with (2.4) and $\tau_0 \leqq 1/10$

$$\varepsilon_1 \leqq \| S_0^{-1}\,\hat{A}S_0\|_\infty + \beta \leqq (1+\frac{26}{19}\tau_0)^2\,\varepsilon_0 + (1-2\tau_0)^{-1}\,\tau_0\varepsilon_0 \leqq (1+\frac{46}{11}\,\tau_0)\varepsilon_0. \qquad \square$$

Finally we show that in the first step no diagonal element changes its affiliated eigenvalue.

THEOREM 2.3. If $\varepsilon_0 \leqq \eta/10$ then the application of the diagonal elements to the eigenvalues remains unchanged in the first step.

PROOF. It is easy to verify that

$$|a^{(1)}_{2i-1,2i-1} - a_{2i-1,2i-1}| = |2\mu\sigma\nu^{-1}(1+F)^{-1}|$$
$$\leqq 2\tau^2(1-2\tau)^{-2}(1+(1-4\tau^2(1-2\tau)^2)^{\frac{1}{2}})^{-1}|\nu| \leqq |\nu|/62, \tag{2.7}$$

since $\tau \leqq 1/10$. Now assume $a_{2i-1,2i-1}$ and $a_{2i,2i}$ are affiliated with distinct eigenvalues λ_{2i-1} and λ_{2i} resp., $|\lambda_{2i}-\lambda_{2i-1}| \geqq \eta$ and that $a^{(1)}_{2i-1,2i}$ and $a^{(1)}_{2i,2i}$ are affilated to λ_{2i} and λ_{2i-1} resp. Then $|a_{2i-1,2i-1}-\lambda_{2i-1}| \leqq \varepsilon_0$ and $|a^{(1)}_{2i-1,2i-1}-\lambda_{2i}| \leqq \frac{156}{110}\varepsilon_0$ as follows from (2.5). By (2.7)

$$|a^{(1)}_{2i-1,2i-1}-a_{2i-1,2i-1}| \leqq |\nu|/62 \leqq (|\lambda_{2i-1}-\lambda_{2i}| + 2\varepsilon_0)/62 \ . \quad (2.8)$$

The change of affiliation implies

$$|a^{(1)}_{2i-1,2i-1}-a_{2i-1,2i-1}| \geqq |\lambda_{2i-1}-\lambda_{2i-1}| - \frac{156}{110}\varepsilon_0-\varepsilon_0 \quad (2.9)$$

By (2.8) and (2.9) $|\lambda_{2i-1}-\lambda_{2i}| \leqq \frac{5}{2}\varepsilon_0$. This contradicts (1.7). □

3. THE EFFECT OF A COMPLETE SWEEP

For the estimation of $\varepsilon_k = \|E^{(k)}\|_\infty$, $k=0,1,\dots, n-1$, we make use of theorem 2.2: if $\varepsilon_k \leqq \eta/10$ then

$$\varepsilon_{k+1} \leqq (1+\frac{46}{11}\tau_k)\,\varepsilon_k\,. \tag{3.1}$$

Firstly we prove that a sufficient small ε_0 quarantees such a slow growth of ε_k that $\varepsilon_k \leqq \eta/10$ for $k=0,\dots, n-1$.

THEOREM 3.1. If $\varepsilon_0 \leqq (10n)^{-1}\,\eta$ then holds for $k=0,\dots, n-1$

$$\varepsilon_k \leqq (n-k)^{-1}\,\eta/10 \tag{3.2}$$

PROOF. (3.2) can be proved by induction with (3.1). □

As in [12] the transform $A^{(k+1)}$ will be considered to be obtained from $A^{(k)}$ in two stages. Let be $H^{(k)}$ the diagonal part of S_k and thus also of S_k^{-1}. Then

$$\begin{cases} A^{(k+\frac12)}=S_k^{-1}A^{(k)}=H^{(k)}A^{(k)}+B^{(k)} & (3.3)\\ A^{(k+1)}=A^{(k+\frac12)}S_k=A^{(k+\frac12)}H^{(k)}+C^{(k)}=H^{(k)}A^{(k)}H^{(k)}+G^{(k)}\,, & (3.4)\end{cases}$$

with

$$G^{(k)} = B^{(k)}H^{(k)}+C^{(k)}\,. \tag{3.5}$$

With (1.2) the elements of $B^{(k)}$ and $C^{(k)}$ can be written als follows

$$\begin{cases} b^{(k)}_{l(i,k),j} = -q_{i,k}a^{(k)}_{m(i,k),j},\; b^{(k)}_{m(i,k),j} = -r_{i,k}a^{(k)}_{\ell(i,k),j},\; i=1,\dots,n/2, & (3.6)\\ c_{j,l(i,k)} = r_{i,k}a^{(k+\frac12)}_{j,m(i,k)},\; c^{(k)}_{j,m(i,k)} = q_{i,k}a^{(k+\frac12)}_{j,\ell(i,k)},\; j=1,\dots,n. & (3.7)\end{cases}$$

A tardy growth of $H^{(k)}A^{(k)}H^{(k)}$ is guaranteerd by

THEOREM 3.2. If $\varepsilon_0 \leqq (10n)^{-1} \eta$ then

$$\prod_{k=2}^{n-2} \|H^{(k)}\|_2^2 \leqq 16/15 \tag{3.8}$$

PROOF. By (2.2) and (3.2) $\|H^{(k)}\|_2^2 \leqq (1+\frac{9}{11}\tau_{k_\infty}^2)^2 \leqq 1 + \frac{5}{3}\tau_k^2 \leqq 1+(n-k)^{-2}/60$. So $\prod_{k=2}^{n-2} \|H^{(k)}\|_2^2 \leqq \prod_{k=2}^{\infty} (1+\frac{1}{60}k^{-2}) \leqq \frac{93}{92}$ for $\sum_{k=2} k^{-2} = \pi^2/6-1$ and $\exp((\pi^2/6-1)/60) \leqq 93/92$. □

Now we consider the genealogy of $a_{i,j}^{(n-1)}$, $i\neq j$ especially its history after the occurence of (i,j) as a pivot pair, say its happen in steps $N(i,j)-1 = N-1$.

If $(i,j) \notin J_{N-1}$ then $a_{i,j}^{(N)} = 0$, $a_{i,j}^{(N+1)} = g_{i,j}^{(N)}$ and

$$a_{i,j}^{(k+1)} = h_{i,i}^{(k)} a_{i,j}^{(k)} h_{j,j}^{(k)} + g_{i,j}^{(k)},\ k = N+1,\dots,n-2 . \tag{3.9}$$

As a consequence of (3.8) we find

$$a_{i,j}^{(n-1)} \leqq \frac{93}{92} \sum_{k=N}^{n-2} |g_{i,j}^{(k)}| \leqq \frac{93}{92} \sum_{\substack{k=0\\k\neq N-1}}^{n-2} |g_{i,j}^{(k)}|,\ (i,j) \notin J_{N-i} \tag{3.10}$$

To the contrary, if $(i,j) \in J_{N-1}$ then $g_{i,j}^{(N-1)} = 0$. With recursion (3.9), now from $k=0$ until $k=n-2$ we get

$$|a_{i,j}^{(n-1)}| \leqq |a_{i,j}^{(0)}| \prod_{m=0}^{n-2} \|H^{(m)}\|_2^2 + \sum_{k=0}^{n-2} |g_{i,j}^{(k)}| \prod_{m=k+1}^{n-2} \|H^{(m)}\|_2^2$$

$$\leqq \frac{93}{92} |a_{i,j}^{(0)}| + \frac{93}{92} \sum_{\substack{k=0\\k\neq N-1}}^{n-2} |g_{i,j}^{(k)}|,\ (i,j) \in J_{N-1} . \tag{3.11}$$

For the investigation of $\varepsilon_{n-1} = \|E^{(n-1)}\|_\infty$ we introduce, compare (2.6)

DEFINITION 3.3. $\hat{G}^{(k)} = (\hat{g}_{i,j}^{(k)}) \in \mathbb{C}^{n\times 2}$, $k=0,\dots,n-2$, is generated from the matrix $G^{(k)}$ in (3.5) by

$$\hat{G}^{(k)}_{i,j} = \begin{cases} 0 \ , & i=j \text{ or } \{i,j\}\in\{\{l(i,k),m(i,k)\}\,|\,i=1,\dots,n/2\} \\ g^{(k)}_{i,j} \ , & \text{otherwise.} \end{cases}$$ □

With definition (3.3) and the notation described in (1.11) the inequalities (3.10) and (3.11) can be written in matricial form

$$|E^{(n-1)}| \leqq \frac{93}{92} |\hat{E}^{(0)}| + \frac{93}{92} \sum_{k=0}^{n-2} |\hat{G}^{(k)}| \ .$$

Hence by theorem 1.4

$$\varepsilon_{n-1} \leqq \frac{93}{92} \varepsilon_0^2/(\eta-2\varepsilon_0) + \frac{93}{92} \sum_{k=0}^{n-2} \|\hat{G}^{(k)}\|_\infty \ . \qquad (3.12)$$

THEOREM 3.4. If $\varepsilon_0 \leqq (10n)^{-1}\eta$ then

$$\|\hat{G}^{(k)}\|_\infty \leqq \frac{18}{7} (1+3(n-k)^{-1}/40) \ \varepsilon_k^2/\eta \ . \qquad (3.13)$$

PROOF. We write $\begin{bmatrix} \alpha & \mu \\ \sigma & \beta \end{bmatrix}$ for the 2×2 block $A^{(k)}_{i,j}$ of $A^{(k)}$ corresponding with the pivot pairs $(l(i,k),m(i,k))$ and $(l(j,k),m(j,k))$ with $i\neq j$. Then

$$A^{(k+1)}_{i,j} = T^{-1}_{i,k} A^{(k)}_{i,j} T_{j,k} = \begin{bmatrix} s_i & -q_i \\ -r_i & p_i \end{bmatrix} \begin{bmatrix} \alpha & \mu \\ \sigma & \beta \end{bmatrix} \begin{bmatrix} p_j & q_j \\ r_j & s_j \end{bmatrix} = p_i p_j \ A^{(k)}_{i,j} + \hat{G}^{(k)}_{i,j}$$

where

$$\hat{G}^{(k)}_{i,j} = \begin{bmatrix} p_i r_j \mu - q_i p_j \sigma - q_i r_j \beta & p_i q_j \alpha - q_i p_j \beta - q_i q_j \sigma \\ p_i r_j \beta - r_i p_j \alpha - r_i r_j \mu & p_i q_i \sigma - r_i p_j \mu - r_i q_j \alpha \end{bmatrix} \begin{matrix} \leftarrow l(i,k) \\ \leftarrow m(i,k) \end{matrix}$$

$$\begin{matrix} \uparrow & & \uparrow \\ l(j,k) & & m(j,k) \end{matrix}$$

By (2.2) and (2.3) we get

$$\|\hat{G}_{i,j}^{(k)}\|_\infty \leqq \frac{9}{7}\tau_k\,(2(1+\frac{9}{11}\tau_k^2)+\frac{9}{7}\tau_k)\|A_{i,j}^{(k)}\|_\infty$$

$$\leqq \frac{18}{7}\tau_k\,(1+\frac{3}{4}\tau_k)\,\|A_{i,j}^{(k)}\|_\infty$$

$$\leqq \frac{18}{7}\tau_k\,(1+3(n-k)^{-1}/40)\,\|A_{i,j}^{(k)}\|_\infty .$$

for $\tau_k \leqq (n-k)/10$.
Hence $\|\hat{G}^{(k)}\|_\infty \leqq \frac{18}{7}\,(1+3(n-k)^{-1}/40)\,\varepsilon_k^2/\eta$. This proves (3.13). □

THEOREM 3.5. If $\varepsilon_0 \leqq (10n)^{-1}\eta$ then

$$\sum_{k=0}^{n-2}\|\hat{G}^{(k)}\|_\infty \leqq \frac{18}{7}\,(\frac{20}{13}n+\frac{21}{16})\ ,\ n \geqq 4\ . \qquad (3.14)$$

PROOF. By (3.1) and (3.2) $\varepsilon_{k+1} \leqq (1+\frac{46}{11}\,(n-k)^{-1}/10)\,\varepsilon_k$. Hence
$\varepsilon_k \leqq \prod_{j=0}^{k-1}\,(1+\frac{23}{55}(n-j)^{-1})\varepsilon_0$.

Together with theorem 3.4 we get

$$\sum_{k=0}^{n-2}\|\hat{G}^{(k)}\|_\infty \leqq \frac{18}{7}\sum_{k=0}^{n-2}\,(1+\frac{3}{40}(n-k)^{-1})\prod_{j=0}^{k-1}\,(1+\frac{23}{55}(n-j)^{-1})^2\,\varepsilon_0^2/\eta$$

$$= \frac{18}{7}\,t_n\varepsilon_0^2/\eta\ ,$$

where $t_n = \sum_{k=0}^{n-2}(1+\frac{3}{40}(n-k)^{-1})\prod_{j=0}^{k-1}(1+\frac{23}{55}(n-j)^{-1})^2$.

It is easy to see that t_n satisfies the reccurence relation $t_{n+1} = 1+3(n+1)^{-1}/40 + (1+23(n+1)^{-1}/55)^2\ t_n$. It is simple to verify that $t_n \leqq \frac{20}{13}\,n + \frac{21}{16}$. Hence

$$\sum_{k=0}^{n-2}\|\hat{G}^{(k)}\|_\infty \leqq \frac{18}{7}\,(\frac{20}{13}n+\frac{23}{16})\,\varepsilon_0^2/\eta\ .$$ □

By (3.12) we obtain the final estimate of ε_{n-1} in

THEOREM 3.6. If $\varepsilon_0 \leqq (10n)^{-1}\eta$ then

$$\varepsilon_{n-1} \leqq (4n+\frac{9}{2})\ \varepsilon_0^2/\eta < \varepsilon_0 \ . \tag{3.15}$$

PROOF. Direct consequence of (3.12) and (3.14). □

4. A NUMERICAL EXAMPLE

As an illustration of the procedure experiments were carried out with a 6×6 complex test matrix [4,13] given in table 1.

$$\begin{bmatrix} 90+96i & 3+4i & 21+22i & 23+24i & 41+42i & -89-94i \\ 182+188i & 13+14i & 15+16i & 33+34i & 35+36i & -139-144i \\ 114+120i & 7+8i & 25+26i & 27+28i & 45+46i & -109-114i \\ 206+212i & 17+18i & 19+20i & 37+38i & 39+40i & -159-164i \\ 138+144i & 11+12i & 29+30i & 31+32i & 49+50i & -129-134i \\ 90+96i & 3+4i & 21+22i & 23+24i & 41+42i & -89-90i \end{bmatrix}$$

Table 1. Matrix A

Matrix A is diagonalizable and has a threefold eigenvalue zero. A's departure of normality $(\|A\|_F^2 - \Sigma|\lambda_j|^2)^{\frac{1}{2}}$ equals 669.89. The initial matrix $A_0 = A$ doesn't satisfy the convergence conditions since $\varepsilon_0 \approx 632.23$ and $\eta \approx 11.937$. The computations were performed on a VAX 8700 computer which has an arithmetic precision of approximately 16 decimals. Contrary to rule (1.12) in our implementation the annihilation of the elements $a_{i,j}^{(k)}$ and $a_{j,i}^{(k)}$ is skipped when

$$\left|a_{i,j}^{(k)}\right| + \left|a_{j,i}^{(k)}\right| \leqq \varepsilon_k/(10n^2) . \tag{4.1}$$

After k steps the computed approximation $\tilde{A}^{(k)}$ of the exact iterate $A^{(k)}$ equals

$$\tilde{A}^{(k)} = A^{(k)} + F^{(k)} = D^{(k)} + E^{(k)} + F_D^{(k)} + F_E^{(k)} = \tilde{D}^{(k)} + \tilde{E}^{(k)}, \; k>0. \tag{4.2}$$

Due to the finite precision $\tilde{A}^{(k)}$ has been afflicted with error $F^{(k)} = F_D^{(k)} + F_E^{(k)}$ where $F_D^{(k)}$ is the diagonalpart of $F^{(k)}$. $\tilde{\varepsilon}_{18} = \|\tilde{E}^{(18)}\|_\infty$, being 0.1871 is the first $\tilde{\varepsilon}_k$ that satisfies $\tilde{\varepsilon}_k \leqq \eta/(10n)$. In table 2 we give the results. $\tilde{\varepsilon}_k$ denotes $\|\tilde{E}^{(k)}\|_\infty$, ζ^k the ∞-norm of the nondiagonal part of the 3×3 matrix $\hat{E}^{(k)}$ associated with the threefold eigenvalue zero, and $\tilde{\sigma}_k = \tilde{\varepsilon}_k^2/\eta$, $k \geqq 12$.

	$\tilde{\varepsilon}_k$	ζ_k	$\tilde{\rho}_k$	k	$\tilde{\varepsilon}_k$	ζ_k	$\tilde{\rho}_k$
0	$6.322_{10}{+2}$			20	$3.641_{10}{-2}$	$1.221_{10}{-11}$	$1.111_{10}{-4}$
5	$1.974_{10}{+2}$			21	$9.595_{10}{-4}$	$3.981_{10}{-13}$	$7.711_{10}{-8}$
10	$1.491_{10}{+2}$			22	$1.453_{10}{-5}$	$1.029_{10}{-13}$	$1.7686_{10}{-11}$
11	$3.954_{10}{+1}$			23	$1.837_{10}{-6}$	$1.922_{10}{-14}$	$2.826_{10}{-13}$
12	$3.773_{10}{+1}$	$4.616_{10}{-1}$	$1.192_{10}{+2}$	24	$1.031_{10}{-9}$	$1.922_{10}{-14}$	$8.905_{10}{-20}$
13	$3.749_{10}{+1}$	$2.242_{10}{-1}$	$1.178_{10}{+2}$	25	$6.807_{10}{-10}$	$1.922_{10}{-14}$	$3.821_{10}{-20}$
14	$2.013_{10}{+1}$	$1.122_{10}{-2}$	$3.396_{10}{+1}$	26	$1.922_{10}{-14}$	$1.922_{10}{-14}$	$3.096_{10}{-29}$
15	$1.898_{10}{+1}$	$7.264_{10}{-3}$	$3.019_{10}{+1}$	30	$1.219_{10}{-14}$	$1.722_{10}{-14}$	$1.245_{10}{-29}$
16	1.253	$3.308_{10}{-3}$	$1.315_{10}{-1}$	35	$4.485_{10}{-16}$	$8.041_{10}{-15}$	$1.685_{10}{-32}$
17	1.177	$1.123_{10}{-3}$	$1.159_{10}{-1}$	40	$4.093_{10}{-21}$	$1.826_{10}{-21}$	$1.404_{10}{-42}$
18	$1.871_{10}{-1}$	$7.727_{10}{-7}$	$2.932_{10}{-3}$	45	$1.453_{10}{-25}$	$6.203_{10}{-26}$	$1.769_{10}{-51}$
19	$4.021_{10}{-2}$	$4.226_{10}{-8}$	$1.354_{10}{-4}$	50	$1.332_{10}{-33}$	0	$1.487_{10}{-67}$
				53	0	0	0

Table 2. $\tilde{\varepsilon}_k$, ζ_k and $\tilde{\rho}_k$ derived from $\tilde{A}^{(k)}$

127.386 670 773 067 7	+	i	132.278 203 200 121 7
7.073 313 248 823 7	-	i	9.558 389 037 045 5
-9.459 984 021 891 4	+	i	7.280 185 836 923 8
0.000 000 000 000 0	-	i	0.000 000 000 000 2
0.000 000 000 000 7	-	i	0.000 000 000 000 8
0.000 000 000 000 8	+	i	0.000 000 000 000 5

Table 3. The computed eigenvalues of A

By roundoff generally $\tilde{A}^{(k)}$, $k > 0$ has three pathologically close eigenvalues instead of the threefold zero of $A^{(k)}$. When the nondiagonal elements of $A^{(k)}$ start to be small, the nondiagonal elements of $A^{(k)}$ associated with the threefold eigenvalue are overruled by the corresponding elements of $F^{(k)}$. The results for $k = 24$ elucidate this phenomenon. The bulk of $\tilde{\varepsilon}_{24} = 1.031_{10}-9$ is built up by the nondiagonal elements of $A^{(24)}$ associated with the really *different* eigenvalues. So we can conclude that the nondiagonal elements of $F^{(24)}$ associated with the *equal* eigenvalues cause ζ_{24} to be $1.922_{10}-14$; the contribution $\tilde{\rho}_{24} = 8.905_{10}-20$ of $E^{(24)}$ of $A^{(24)}$ to that quantity is neglectable.

$1.274_{10}+2$ $1.323_{10}+2$	$2.099_{10}-11$	0	$3.247_{10}-11$	$1.522_{10}-21$	$2.118_{10}-21$
$1.144_{10}-10$	-9.460 7.280	$9.252_{10}-18$	$2.451_{10}-28$	0	$1.947_{10}-17$
0	$1.746_{10}-18$	$7.533_{10}-15$ $6.410_{10}-15$	$4.834_{10}-11$	$8.133_{10}-15$	$2.946_{10}-15$
$6.807_{10}-10$	$4.647_{10}-28$	$3.504_{10}-10$	7.073 -9.558	$2.214_{10}-17$	0
$7.124_{10}-10$	0	$2.588_{10}-15$	$4.839_{10}-15$	$8.544_{10}-15$ $-1.137_{10}-15$	$2.309_{10}-15$
$7.751_{10}-21$	$5.327_{10}-18$	$3.252_{10}-15$	0	$1.597_{10}-14$	$-8.800_{10}-16$ $-2.144_{10}-16$

Table 4. $\tilde{A}^{(24)}$; $\tilde{\varepsilon}_{24} = 1.031_{10}-9$; $\zeta_{24} = 1.922_{10}-14$; $\tilde{\varepsilon}^2_{24}/n = 8.905_{10}-20$.

In the following iterations the small separation of the pathologically close eigenvalues slows down the speed of convergence. The alteration in the diagonal elements of $\tilde{A}^{(k)}$ essentially descend from the roundoff in

$F^{(k)}$ when $\tilde{\varepsilon}_k = \|\tilde{E}^{(k)}\|_\infty$ is of order 10^{-16}. This train of thinking leads to the criterion: iterate until

$$\tilde{\varepsilon}_k \leqq \|\tilde{A}^{(k)}\|_\infty * 10^{-16} . \qquad (4.3)$$

The method has been tested on many testcases. The numerical are very satisfactory accurate. The errors in the computed eigenvalues are in conformity with the precision of the floating point arithmetic and the condition numbers of the individual eigenvalues.

5. CONCLUSIONS

The analysis of the annihilating procedure shares features from [9] and [13]. As in [13] the difficulties in the understanding of the process comes from the non-unitary similarity transformations.

In our process the numerical results are surprising accurate despite the non-unitary transformations. The parallel annihilations effectuate a norm-reduction and so, as a welcome side-effect, the condition of the eigen-value problem improves [5]. This may declare that the accuracy of the computed eigenvalues is in accordance with the floating point precision and the condition of the initial eigenvalue problem.

Our convergence theorem demonstrates that in the final stage of Eberlein-like methods [4,10,13] the complex normreducing similarity transforma-tions, based on global information are unnecessary. In [13] the claim

$$\left|a^{(k+1)}_{\ell(i,k),m(i,k)}\right|, \left|a^{(k+1)}_{m(i,k),\ell(i,k)}\right| = O(\varepsilon_k^2)$$

is of apparent importance. The same reasoning occurs in our analysis con-cerning the diagonal elements associated with multiple roots. The last example illustrates that the annihilating method converges also for matrices far from diagonal dominancy. Evidently the process convergence for Hermitean and more general for normal matrices, since it generalizes the classical Jacobi methods for these matrices. In case the departure of normality of the original matrix is rather large a few parallel normreduc-tions suffice to reach the region of convergence for the annihilating method.

The parallel annihilators can be implemented on a hypercube [1] or on a (systolic) array processor [2,3].

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