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RESEARCH MEMORANDUM

CYCLIC POLLING SYSTEMS: LIMITED SERVICE VERSUS BERNOULLI SCHEDULES

J.P.C. Blanc

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Abstract

The power-series algorithm, an iterative numerical technique for the evaluation of the joint queue length distributions for a broad class of multi-queue systems, is applied to cyclic polling systems with a single server. The performances of two service disciplines are compared: limited service and Bernoulli schedules. To a certain extent the properties of the two service disciplines are quite similar, but some striking differences have also been found. The numerical results suggest some general properties of the mean waiting times at the various stations, which may be helpful in deriving better approximations for these quantities.

Keywords: cyclic polling, limited service, Bernoulli schedules, waiting time, power-series algorithm, heavy traffic.

1. Introduction

The power-series algorithm is a powerful means for evaluating performance measures of systems consisting of a moderate number of queues. This algorithm is based on power-series expansions of the state probabilities as functions of the load of a system in light traffic. The coefficients of these expansions are computed according to a recursive scheme. The ε-algorithm, cf. Wynn [20], is used to improve the convergence of the series in heavy traffic. We refer to Blanc [2,3] for a discussion of the power-series algorithm. The purpose of this paper is to apply this algorithm to multi-queue models for computer-communication systems, in which a token is passed for access to a single communication channel. In particular, we will compare performance measures of cyclic polling systems with limited service disciplines with those of systems with Bernoulli schedules. On the one hand, limited service seems to be the more preferable discipline since it is a deterministic rule which leads to a bounded number of services during a cycle of the server along the queues. It may be expected that Bernoulli schedules will lead to stronger fluctuations in waiting times. On the other hand, models with Bernoulli schedules seem to be easier to analyse and to optimize, because the parameters of this rule are real-valued, and because this rule requires no memory. Because the power-series algorithm is based on representations of queueing models as Markov-processes, it is indeed easier, in the sense of computing time and storage requirement, to obtain results for systems with Bernoulli schedules than for systems with limited service by means of this algorithm.

There exist only a few analytical results for models with the above mentioned service disciplines. The most general result is the conservation law for systems without switching times, cf. Kleinrock [11], and the pseudo-conservation law for systems with Bernoulli schedules and with switching times, cf. Tedianto [18]. Cooper [8] derived a set of linear equations, of which the solution determines the mean waiting times in systems with exhaustive service (a limiting case of both disciplines considered in this paper). Models with two queues and 1-limited service have been solved with the technique which uses Riemann-Hilbert boundary value problems, cf., e.g., Boxma and Groenendijk [6]. Several authors have proposed approximate approaches. Approximations for models with 1-limited service can be found in, e.g., Kühn [12], Arndt and Sulanke [1], Boxma and Meister [4,5] and Srinivasan [15]. Fuhrmann and Wang [9] derive approximations for the mean waiting times in models with (K-)limited service. Servi [13] discusses an iterative scheme for approximating the mean waiting times for models with Bernoulli schedules. Servi and Yao [14] derive upper and lower bounds for M/G/1 vacation models with (K-)limited service and with Bernoulli schedules. Finally, we refer to Takagi [16,17] and Boxma [7] for surveys on polling systems.

The discussion in the present study will be restricted to models with zero switching times, Poisson arrival processes, exponential service times, and infinite buffers. The paper contains several numerical examples and discusses some light and heavy traffic properties of the mean waiting times. The organisation of the paper is as follows. The multi-queue models with cyclic polling will be described in section 2. Section 3 contains the balance equations for the state probabilities for models with limited service (those for models with Bernoulli schedules can be found in Blanc [3]). Waiting times will be discussed in section 4. Light traffic asymptotes of the mean waiting times have been derived from the recursive scheme of the power-series algorithm; heavy traffic asymptotes have been found with the aid of numerical experiments. Examples of cyclic polling systems for which data are listed in tables will be described in section 5. The final section 6 contains some conclusions based on numerical results. More detailed information on the examples can be found in the appendix.

2. The multi-queue model

The system consists of s queues and a single server. Jobs arrive at queue j according to a Poisson process with rate λ_j, $j = 1,\dots,s$. Each queue may contain an unbounded number of jobs. Service times of jobs at queue j are assumed to be negative exponentially distributed with mean $1/\mu_j$, $j=1,\dots,s$. The arrival and service processes are assumed to be independent. The server inspects the queues in a cyclic order $(1,2,\dots,s,1,2,\dots)$. At each queue jobs are served in order of arrival. The number of jobs which are served during a visit of the server to a certain queue depends on the service discipline at that queue. Performance measures for two service disciplines will be compared in this paper. The first discipline is called limited service. This rule can be described by a vector of positive integers $\bar{K} = (K_1,\dots,K_s)$. When the server visits queue j, at most K_j jobs are served at this queue. The server proceeds to the next queue when either K_j jobs have been served or queue j has become empty, j=1,...,s. The second discipline uses Bernoulli schedules. This rule consists of a vector of probabilities $\bar{q} = (q_1,\ldots,q_s)$. When the server arrives at a queue, at least one job is served, unless this queue is empty (in this case the server immediately proceeds to the next queue). After the completion of a service at queue j the server starts serving another job at this queue with probability q_j if queue j has not yet been emptied; otherwise the server proceeds to the next queue (j=1,...,s). The times which are needed for switching from one queue to another will be neglected in the present study. It should be noted that the power-series algorithm can be used to study a much broader class of polling systems, e.g. with Coxian switching times, with Coxian service times, with finite buffers and with polling according to a table or with random polling, see also Blanc [3].

It holds for both service disciplines that the distribution of the total amount of work in the system is equal to that of an M/G/1 system with arrival rate Λ and with β_1 and β_2 as first two moments of the service time distribution:

$$\Lambda := \sum_{j=1}^{s} \lambda_j, \qquad \beta_1 := \sum_{j=1}^{s} \frac{\lambda_j}{\Lambda}\frac{1}{\mu_j}, \qquad \beta_2 := 2\sum_{j=1}^{s} \frac{\lambda_j}{\Lambda}\left(\frac{1}{\mu_j}\right)^2. \tag{2.1}$$

Therefore, the traffic intensities ρ of the polling systems are defined as

$$\rho := \Lambda\,\beta_1 = \Lambda \sum_{j=1}^{s} \frac{\lambda_j}{\Lambda}\frac{1}{\mu_j} = \sum_{j=1}^{s} \frac{\lambda_j}{\mu_j}, \tag{2.2}$$

and a necessary and sufficient condition for ergodicity of the systems is $\rho < 1$. It will be assumed throughout that the systems are in steady state. Finally, we introduce the following load-independent quantities:

$$a_j := \lambda_j/\rho, \qquad \eta_j := a_j/\mu_j, \qquad j = 1,\ldots,s. \tag{2.3}$$

Note that definitions (2.1), (2.2) and (2.3) imply the following relations

$$\sum_{j=1}^{s} \eta_j = 1, \qquad \sum_{j=1}^{s} a_j = \frac{\Lambda}{\rho} = \frac{1}{\beta_1}, \qquad \beta_2 = 2\,\beta_1 \sum_{j=1}^{s} \frac{\eta_j}{\mu_j}. \tag{2.4}$$

3. Balance equations for limited service models

The power-series algorithm has been described in Blanc [3] for systems with Bernoulli schedules. Below we will only present the balance equations for the state probabilities of systems with limited service. The recursive scheme of the power-series algorithm can be derived from these equations in the same way as in Blanc [3]. Let N_j denote the number of jobs in queue j (waiting or being served), j=1,...,s. In order to transform the queue length process of limited service systems into a Markov process we introduce a polling table and a supplementary variable H, indicating the actual position in the table. The polling table is described as follows. Let $L := \sum_{j=1}^{s} K_j$ be the length of the table. The mapping $\ell(h)$ from table entry to queue number is defined by

$$\ell(h) = j, \quad \text{if} \quad \sum_{i=1}^{j-1} K_i < h \leq \sum_{i=1}^{j} K_i, \quad \text{for} \quad j = 1,\dots,s, \quad h = 1,\dots,L, \tag{3.1}$$

and it is continued as a periodic function by the convention

$$\ell(h + k\,L) = \ell(h), \qquad h \in \{1,\dots,L\}, \qquad k \in \mathbb{Z}. \tag{3.2}$$

The value of the variable H is increased by one whenever a service has been completed or when queue $\ell(H)$ is empty, unless the whole system has become empty; in the latter case the value of H is set and kept equal to 1 until a new arrival occurs. The value of $\ell(H)$ determines the queue to which the server is attending. Let $\bar{n} = (n_1,\dots,n_s)$ be a vector with non-negative integer entries. The state probabilities are defined as follows:

$$p(\bar{n},h) := \Pr\{N_j=n_j,\ j=1,\dots,s;\ H=h\}, \quad \bar{n} \in \mathbb{N}^s, \quad h = 1,\dots,L. \tag{3.3}$$

Let I{E} stand for the indicator function of the event E, and let $\bar{e}_j$ be a vector with zero entries except an entry of one at the j^{th} position (j=1,...,s). The balance equations for the state probabilities (3.3) in models with a limited service discipline are readily verified to be, for $h = 1,\dots,L,\ \bar{n} \in \mathbb{N}^s$,

$$[\rho \sum_{j=1}^{s} a_j + \mu_{\ell(h)}]\, p(\bar{n},h) = \rho \sum_{j=1}^{s} a_j\, p(\bar{n}-\bar{e}_j,h)\; I\{n_j>0;\; n_j>1 \text{ if } j=\ell(h)\}$$

$$+ \sum_{i=1}^{L} \mu_{\ell(h-i)}\, p(\bar{n}+\bar{e}_{\ell(h-i)};h-i)\; I\{n_{\ell(\nu)}=0,\; \nu=h-i+1,\dots,h-1\}$$

$$+ a_{\ell(h)}\rho\, p(\bar{0},1)\; I\{\bar{n}=\bar{e}_{\ell(h)} \wedge \ell(i)\neq\ell(h), i=1,\dots,h-1\},\quad n_{\ell(h)}>0; \qquad (3.4)$$

$$\rho \sum_{j=1}^{s} a_j\, p(\bar{0},1) = \sum_{h=1}^{L} \mu_{\ell(h)}\, p(\bar{e}_{\ell(h)},h). \qquad (3.5)$$

Note that a state $(\bar{n},h)$ with $n_{\ell(h)}=1$ can be entered through an arrival at queue h only if $\bar{n} = \bar{e}_{\ell(h)}$ and if h is the first entry on the polling table with the value $\ell(h)$, $h=1,\dots,L$. Further, it should be noted that the balance equations (3.4) and (3.5) are valid for models with arbitrary polling tables, i.e. for arbitrary surjective mappings $\ell : \{1,\dots,L\} \to \{1,\dots,s\}$. The examples in this paper, however, are restricted to models with limited service, cf. (3.1). The reader is referred to Blanc [3] for details concerning the derivation of a recursive scheme from the balance equations, the computation of moments of the joint queue length distribution, and the application of the ε-algorithm.

4. Waiting times

This section is concerned with a discussion of the stationary distributions of the waiting times W_j (excluding service times) of jobs arriving at queue j $(j=1,\dots,s)$, for both limited service and Bernoulli schedules. Firstly, some general relations will be reviewed. Then, two terms of the power-series expansions of the mean waiting times in light traffic will be derived from the recursive scheme of the power-series algorithm. Finally, several properties concerning the heavy traffic behaviour of the mean waiting times, which have been found on the basis of numerical data, will be discussed.

The power-series algorithm computes the joint queue-length distribution. The waiting time distributions are related to this distribution through (Blanc [3]):

$$E\{z^{N_j}\} = \frac{1}{1 + (1-z)\eta_j\rho}\, E\{e^{-a_j\rho(1-z)W_j}\},\quad |z| \leq 1,\quad j = 1,\dots,s. \qquad (4.1)$$

Let W be the waiting time of an arbitrary job. Then, with (2.2) and (2.3),

$$E\{W\} = \sum_{j=1}^{s} \frac{\lambda_j}{\Lambda} E\{W_j\} = \beta_1 \sum_{j=1}^{s} a_j \, E\{W_j\}. \tag{4.2}$$

We recall that the mean waiting times at the various queues of a polling system satisfy the following conservation law, cf. Kleinrock [11], Boxma [7],

$$\sum_{j=1}^{s} \eta_j \, E\{W_j\} = \frac{\rho}{1-\rho} \sum_{j=1}^{s} \frac{\eta_j}{\mu_j} = \frac{\rho}{1-\rho} \frac{\beta_2}{2\beta_1}. \tag{4.3}$$

Next, we discuss the light traffic behaviour of the mean waiting times. For this purpose we introduce for $j=1,\dots,s$, $\bar{q} = (q_1,\dots,q_s)$, the quantities:

$$\Xi(j) := 2 \sum_{i=1}^{s-1} \eta_{j+i} \sum_{\nu=0}^{i-1} \frac{\eta_{j+\nu}}{\mu_{j+\nu}}, \qquad \Psi(j,\bar{q}) := 2 \sum_{i=1}^{s} q_i \frac{\eta_i^2}{\mu_i} - 2\, q_j \frac{\eta_j}{\mu_j};$$

here, indices exceeding s should be read modulo s.

Theorem 1a. *For cyclic polling systems with Bernoulli schedules* $\bar{q}$ *and with exponential service time distributions it holds for* $j = 1,\dots,s$, *that,*

$$E\{W_j\} = \rho \frac{\beta_2}{2\beta_1} + \rho^2 \left[\eta_j \frac{\beta_2}{2\beta_1} + \Xi(j) + \Psi(j,\bar{q})\right] + O(\rho^3), \quad \textit{as } \rho \downarrow 0. \tag{4.4}$$

Theorem 1b. *For cyclic polling systems with limited service discipline* $\bar{K}$ *and with exponential service time distributions the power-series expansions* (4.4) *hold with* q_j *replaced by* $I\{K_j \geq 2\}$, $j = 1,\dots,s$.

Proof. In order to derive these light traffic asymptotes we determine first the coefficients of the power-series expansions of the state probabilities and the mean queue lengths up to the 3^{rd} power of ρ according to the recurrence relations of the power-series algorithm (see Blanc [3]). This leads for Bernoulli schedules to: for $j=1,\dots,s$, as $\rho \downarrow 0$,

$$E\{N_j\} = \eta_j \rho + a_j \rho^2 \frac{\beta_2}{2\beta_1} + a_j \rho^3 \left[\eta_j \frac{\beta_2}{2\beta_1} + \Xi(j) + \Psi(j,\bar{q})\right] + O(\rho^4).$$

Then, the coefficients of the power-series expansions of the mean waiting times follow with the aid of Little's formula, cf. (4.1). □

The first term of the coefficient of ρ^2 in the power-series expansions of the mean waiting times is independent of the order in which the queues are placed and of the service disciplines at the queues. The term $\Xi(.)$ reflects the influence of the order in which the server visits the queues, and the term $\Psi(.,.)$ depends on the service disciplines at the queues. Note that the coefficients of the power-series expansions of the mean waiting times up to the m^{th} power of ρ, $m=1,2,\ldots$, are the same for all $K_i \geq m$, $i=1,\ldots,s$, in systems with limited service. It is possible, but increasingly tedious, to determine more coefficients of the power-series expansions of the mean waiting times in a similar way. The appendix contains a discussion of the above mentioned functions and properties for some special cases.

For the description of heavy traffic properties we introduce the limits:

$$\omega_j := \lim_{\rho\uparrow 1} (1-\rho)\, E\{W_j\}, \quad j = 1,\ldots,s, \qquad \omega_0 := \lim_{\rho\uparrow 1} (1-\rho)\, E\{W\}, \tag{4.5}$$

$$x_j := \lim_{\rho\uparrow 1} \left[E\{W_j\} - \frac{\omega_j}{(1-\rho)}\right], \quad j = 1,\ldots,s, \qquad x_0 := \lim_{\rho\uparrow 1} \left[E\{W\} - \frac{\omega_0}{(1-\rho)}\right]. \tag{4.6}$$

The conservation law (4.3) and relation (4.2) imply that

$$\sum_{j=1}^{s} \eta_j \omega_j = \frac{\beta_2}{2\beta_1}, \qquad \omega_0 = \beta_1 \sum_{j=1}^{s} a_j \omega_j,$$

$$\sum_{j=1}^{s} \eta_j x_j = -\frac{\beta_2}{2\beta_1}, \qquad x_0 = \beta_1 \sum_{j=1}^{s} a_j x_j. \tag{4.7}$$

Theorem 2. *If the service discipline is exhaustive at each queue, i.e.* $q_j = 1$, $j=1,\dots,s$, *then the limits defined in* (4.5) *are given by:*

$$\omega_j = \frac{1-\eta_j}{\sum_{i=1}^{s} \eta_i(1-\eta_i)} \frac{\beta_2}{2\beta_1}, \qquad j = 1,\dots,s. \tag{4.8}$$

Proof. From the results of Cooper [8] it can be deduced that when the service discipline is exhaustive at each queue, it holds that $\omega_j = \delta_{j-1}(1)$, $j=1,\dots,s$, where the quantities $\delta_j(k)$, $j=1,\dots,s$, $k=1,\dots,s-1$, satisfy the following set of $s\times(s-1)$ linear equations (read $\delta_{j+s}(k)$ for $\delta_j(k)$ and η_{j+s} for η_j whenever $j<1$):

$$(1-\eta_{j+1})\ \delta_j(k) = \sum_{h=0}^{s-1-k} \frac{\eta_{j-h}^2}{1-\eta_{j-h}}\ \delta_{j-1-h}(1) + \sum_{h=0}^{s-2-k} \eta_{j-h}\ \delta_{j-1-h}(k+1+h)$$
$$+ \sum_{h=0}^{s-1-k} \eta_{j-h}\ \delta_{j-1-h}(2+h)\ I\{h<s-2\}.$$

It is rather tedious, but straightforward, to verify with the aid of (2.4) that solutions of this set of equations are of the form

$$\delta_j(k) = C \sum_{i=0}^{s-1-k} \eta_{j-i}, \qquad j = 1,\dots,s, \quad k = 1,\dots,s-1.$$

This implies that the quotient $\omega_j/(1-\eta_j)$ has the same value for each j, $j=1,\dots,s$. The constant C can be determined with the aid of the conservation law (4.3). □

The proof and the assertion of theorem 2 remain valid for general service time distributions if β_2 is read as the second moment of the job-averaged service time distribution, cf. (2.1), because the set of equations for $\delta_j(k)$, $j=1,\dots,s$, $k=1,\dots,s-1$, only depends on the mean service times, and because the conservation law (4.3) holds for general service time distributions.

Property 1. *The limit* ω_j, $j=1,\dots,s$, *is positive if and only if (in the cases of limited service and Bernoulli schedules, respectively)*

$$a_j/K_j = \max_{i=1,..,s} \{a_i/K_i\}, \qquad a_j(1-q_j) = \max_{i=1,..,s} \{a_i(1-q_i)\}. \qquad (4.9)$$

Remark. When the lefthand sides of (4.9) are close to, but not equal to, the maximum at the righthand sides of these relations for some j, j=1,...,s, the limit ω_j is zero, but $E\{W_j\}$ has a large finite limit as $\rho \uparrow 1$. That only the arrival rates, and not the service rates, play a role in property 1, can be explained by the fact that a certain (integer) number of jobs is served during each cycle of the server along the queues according to the limited service discipline as well as to the Bernoulli schedules. If service disciplines would be considered in which the server spends a certain amount of time at each queue during a cycle and in which service of a job can be interrupted and resumed in a later cycle, then it might be expected that a similar property as property 1 holds, but with the arrival rates replaced by the relative loads at the queues. Note that the relations (4.9) hold for each queue in systems with exhaustive service at each queue ($K_j = \infty$ or $q_j = 1$, j=1,...,s). □

As a consequence of property 1 the ε-algorithm which is being used to accelerate the convergence of the power-series occuring in the algorithm, should not be modified as described in Blanc [3] for moments of the marginal queue length distributions at queues where (4.9) does not hold. Another implication is that approximations for mean waiting times in polling systems, which do not possess property 1 will behave poorly under heavy traffic circumstances. For instance, the approximations for the mean waiting times in Boxma and Meister [4] have the right heavy traffic limits in case of exhaustive service (formula (17) in [4]) according to theorem 2, but they do not have the proper heavy traffic behaviour in case of 1-limited service (formula (20) in [4]) according to property 1.

Corollary 1. *If there exists a queue* j *such that for each* i, i=1,...,s, i≠j,

$$a_i/K_i < a_j/K_j, \quad \textit{respectively} \quad a_i(1-q_i) < a_j(1-q_j), \qquad (4.10)$$

then property 1 implies that $\omega_i = 0$ *for each* i, i=1,...,s, i≠j, *and that*

$$\omega_j = \frac{1}{\eta_j} \frac{\beta_2}{2\beta_1}, \qquad \omega_0 = \frac{1}{2} \mu_j \beta_2. \tag{4.11}$$

Remark. The above corollary is a consequence of property 1 and of (4.7). Watson [19] derived a similar result for systems with 1-limited service and non-negligible switching times. □

Property 2. *If the system consists of two queues, if the service discipline is Bernoulli at each queue, and if* $a_1(1-q_1) > a_2(1-q_2)$, *then:*

$$\omega_2 = 0, \qquad x_2 = \frac{\mu_1\mu_2 \frac{\beta_2}{2\beta_1} + \eta_2[\mu_2(1-q_2)-\mu_1 q_1]}{a_1(1-q_1) - a_2(1-q_2)} \frac{1}{\mu_2}; \tag{4.12}$$

and the quantities ω_1, ω_0, x_1, *and* x_0 *follow from* (4.7).

Remark. Property 2 has been found on the basis of numerical experiments with several values of the parameters a_1, a_2, μ_1, μ_2, q_1 and q_2. Further, this property agrees with the following observations. Corollary 1 implies that $\omega_2=0$. The denominator of x_2 vanishes when $a_1(1-q_1) = a_2(1-q_2)$, which is in agreement with property 1. In the case $q_1=0$, $q_2=1$, jobs at queue 2 have non-preemptive priority over jobs at queue 1. For this model it is known (see Jaiswal [10], §V.3) that for $0 < \rho < 1$,

$$E\{W_2\} = (1-\rho)\, E\{W_1\} = \frac{\rho}{1-\eta_2\rho} \frac{\beta_2}{2\beta_1},$$

from which it is readily seen that property 2 holds in this case. □

We did not find a general result similar to property 2 for systems with two queues and limited service, except that ω_2 vanishes if $a_1/K_1 > a_2/K_2$, by corollary 1. Only in the case that $K_2 = \infty$ we have found that

$$x_2 = \frac{1}{\eta_1} \frac{\beta_2}{2\beta_1} + \frac{1}{2\mu_1} (K_1-1). \tag{4.13}$$

Note that in this case ($K_2 = \infty$ or $q_2 = 1$) the limits x_2 in (4.12) and (4.13) agree when $K_1 = (1+q_1)/(1-q_1)$.

5. Numerical examples

This section contains descriptions of cyclic polling systems for which data have been generated with the aid of the power-series algorithm together with the ε-algorithm (cf. Wynn [20], Blanc [3]).

In table 1 the standard deviation $\sigma\{W\}$ of the waiting times has been listed for symmetrical systems with limited service ($K_j = K$, $j=1,\dots,s$). Table 2 contains the standard deviation of the waiting times for symmetrical systems with Bernoulli schedules ($q_j = q$, $j=1,\dots,s$). The mean waiting time follows directly from (4.3) for symmetrical systems, and does not depend on the service discipline. In both tables, $\mu_j = 1$ for $j=1,\dots,s$.

Tables 3a and 4a show in which way the waiting time distributions depend on the arrival rates and the service rates, respectively. Both tables are concerned with models with three queues, with traffic intensity $\rho=0.90$ and with relative loads in the proportion of $\eta_1 : \eta_2 : \eta_3 = 1 : 2 : 3$. In table 3a, the service rates at the three queues are equal ($\mu_j=1$, $j=1,2,3$), and the arrival rates are $\lambda_1 = 0.15$, $\lambda_2 = 0.30$, and $\lambda_3 = 0.45$. In table 4a, the arrival rates at the three queues are equal ($\lambda_j = 0.30$, $j=1,2,3$), and the service rates are $\mu_1 = 2$, $\mu_2 = 1$, $\mu_3 = 2/3$. Because the mean waiting time $E\{W\}$ is equal to 9.000 for the models in table 3a, independently of the service discipline, cf. (4.2), (4.3), this quantity has been omitted. The value of the righthand side of the conservation law (4.3) is equal to 10.500 for the models in table 4a. In both tables the data on the lefthand side concern systems with a limited service discipline (K_1,K_2,K_3) and those on the righthand side concern systems with Bernoulli schedules (q_1,q_2,q_3). In these and the following tables, "dcp" stands for service discipline. The server visits the queues in the usual order (1,2,3,1,..) in the models in the tables 3a and 4a. Tables 3b and 4b contain data for the same models as tables 3a and 4a respectively, but the order in which the server inspects the queues has been reversed (1,3,2,1,..). The routing of the server does not seem to have a major influence on the performance measures, compared with the influence of the parameters of the service disciplines. Still, altering the polling order may lead to differences in mean waiting times of more than 10%.

Tables 5a and 5b show the influence of a relatively heavily loaded queue on the mean waiting times at queues which are four times less heavily loaded. The parameters of the system are in the case of s queues, $s=2,\dots,6$: $\mu_1 = 1$, $\mu_j = 2$, $j=2,\dots,s$; $a_1 = 2a_j = \frac{4}{s+3}$, $j=2,\dots,s$ (hence,

$\beta_2/\beta_1 = \frac{s+7}{s+3}$); and $\rho = 0.75$. Table 6 contains data for the same models as table 5a, but for a traffic intensity of $\rho = 0.95$. The examples concern only service disciplines with $K_j = K_2$, $j=2,\dots,s$, or $q_j = q_2$, $j=2,\dots,s$; therefore, the service discipline is indicated either by (K_1,K_2) or by (q_1,q_2). Note that the differences in mean waiting times of the lightly loaded queues are not negligible, although their arrival and service rates and their service disciplines are the same. In the cases of 4, 5 and 6 queues the values of $E\{W_3\},\dots,E\{W_{s-1}\}$ lie, in this order, in between those of $E\{W_2\}$ and $E\{W_s\}$ for all considered service disciplines.

Although we do not have the disposal of bounds on errors for data generated by the power-series algorithm together with the ε-algorithm, we estimate on the basis of differences between successive terms that relative errors are below 1% for almost all quantities listed in the tables, and even below 0.01% for most quantities. In general, errors increase with increasing traffic intensity, with increasing number of queues, with increasing length of the polling table (L), with increasing differences in the arrival rates and the service rates, and when equality in (4.9) is approximated (i.e. when the mean waiting time at one or more queues possesses a large but finite heavy traffic limit). Also, errors for standard deviations are usually larger than those for averages.

6. Conclusions

When comparing data for the waiting time distributions in systems with a limited service discipline with those with Bernoulli schedules we arrive at the following conclusions. Tables 1 and 2 show that Bernoulli schedules lead to higher variances of the waiting times than limited service disciplines in symmetrical systems. In fact, the standard deviation of the waiting times seems to be a convex function of K for models with limited service disciplines and a concave function of q for models with Bernoulli schedules. It is interesting to note that while the standard deviation of the waiting times in symmetrical systems with exhaustive service (q=1) is larger than that in symmetrical systems with 1-limited service (q=0) when the number of queues is small ($s \leq 4$), this property does no longer hold when the number of queues increases. The limited service disciplines and the Bernoulli schedules agree in that the mean waiting times pass globally through similar trajectories (though continuously in case of Bernoulli schedules and with jumps in case of limited service), when we consider them as functions of one parameter, e.g., $K_j = K$, $q_j = q$ $(j=1,\dots,s)$ or as function of K_j and q_j for some j, $j=1,\dots,s$, while the other queues have exhaustive or 1-limited disciplines. But there seems to be no general relationship between the parameters of the Bernoulli and of the limited service schedules, which give approximately the same position on these trajectories. See tables 3, 4,5,6, and compare (4.10), which suggests a relation between K_j and $1/(1-q_j)$, with the observation below (4.13). It should be noted that mean waiting times are not in every case monotonous functions of the parameters of the service discipline; see for instance table 3a where $E\{W_1\}$ is larger for $q_1 = 0.90$ then for both $q_1 = 0.00$ and $q_1 = 1.00$ when q_j is related to q_1 by $a_j(1-q_j)=a_1(1-q_1)$, $j=2,3$. A general property suggested by the examples, and supported by theorems 1a and 1b, is that $E\{W_j\}$ is minimal over all disciplines $\bar{K}$ and $\bar{q}$, for fixed arrival and service rates, when $q_j=0$ $(K_j=1)$ and $q_i=1$ $(K_i=\infty)$, $i\neq j$, $i=1,\dots,s$, and maximal in the reversed case $q_j=1$ and $q_i=0$, $i\neq j$, $i=1,\dots,s$ $(j=1,\dots,s)$. Finally, we note that examples show that the ordering of the mean waiting times in certain models is not the same for all values of ρ (compare, e.g., the ordering for $q_1=q_2=0.90$ in the tables 5a and 6; see also table A.3). This is supported by theorem 1a (1b) and property 1.

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Table 1. Standard deviation of the waiting time for symmetrical systems with limited service disciplines.

s	K	ρ=.5	ρ=.75	ρ=.9	ρ=.95	ρ=.98	s	K	ρ=.5	ρ=.75	ρ=.9	ρ=.95	ρ=.98
2	1	1.8350	4.246	11.187	22.67	57.09	3	1	1.8805	4.434	11.87	24.2	61.2
2	2	1.8301	4.237	11.174	22.66	57.08	3	2	1.8739	4.410	11.82	24.1	61.1
2	3	1.8320	4.231	11.159	22.64	57.05	3	3	1.8788	4.396	11.77	24.1	61.0
2	4	1.8388	4.230	11.145	22.62	57.03	3	4	1.8888	4.394	11.73	24.0	60.9
2	5	1.8479	4.234	11.133	22.60	57.00	3	5	1.8991	4.401	11.69	23.9	60.8
2	6	1.8575	4.243	11.124	22.58	56.97	3	6	1.9076	4.415	11.67	23.9	60.7
2	7	1.8664	4.254	11.118	22.56	56.95	3	7	1.9138	4.432	11.64	23.8	60.4
2	8	1.8741	4.269	11.114	22.54	56.92	3	8	1.9180	4.452	11.63	23.7	60.3
2	9	1.8805	4.285	11.113	22.53	56.89	3	∞	1.9254	4.610	12.41	25.3	64.1
2	10	1.8857	4.302	11.115	22.51	56.85							
2	12	1.8927	4.337	11.126	22.49	56.78	4	1	1.9057	4.549	12.31	25.2	64.
2	14	1.8968	4.371	11.146	22.47	56.74	4	2	1.8973	4.507	12.22	25.1	64.
2	16	1.8990	4.403	11.173	22.46	56.68	4	3	1.9033	4.483	12.11	24.9	63.
2	18	1.9002	4.432	11.207	22.46	56.62	4	4	1.9131	4.478	12.00	24.6	63.
2	20	1.9008	4.456	11.245	22.46	56.59	4	5	1.9214	4.485	11.90	24.3	62.
2	∞	1.9014	4.569	12.362	25.29	64.03	4	∞	1.9345	4.624	12.42	25.3	64.

Table 2. Standard deviation of the waiting time for symmetrical systems with Bernoulli schedules.

q	s	ρ=.5	ρ=.75	ρ=.9	ρ=.95	ρ=.98	s	ρ=.5	ρ=.75	ρ=.9	ρ=.95	ρ=.98
.00	2	1.8350	4.246	11.187	22.67	57.09	5	1.9213	4.624	12.62	25.9	65.9
.25	2	1.8448	4.289	11.340	23.01	57.99	5	1.9283	4.665	12.80	26.4	67.2
.50	2	1.8575	4.347	11.547	23.47	59.22	5	1.9352	4.706	13.01	26.9	68.8
.75	2	1.8748	4.430	11.849	24.15	61.01	5	1.9407	4.734	13.22	27.6	70.9
.90	2	1.8891	4.502	12.116	24.74	62.59	5	1.9415	4.716	13.19	27.7	72.0
.95	2	1.8949	4.533	12.230	25.00	63.26	5	1.9409	4.689	13.02	27.4	71.5
.98	2	1.8987	4.554	12.307	25.17	63.70	5	1.9401	4.659	12.78	26.7	70.0
.99	2	1.9000	4.562	12.334	25.23	63.86	5	1.9398	4.646	12.63	26.2	68.4
1.0	2	1.9014	4.569	12.362	25.29	64.03	5	1.9394	4.630	12.43	25.4	64.1
.00	3	1.8805	4.434	11.865	24.20	61.17	6	1.9316	4.677	12.84	26.5	67.4
.25	3	1.8904	4.483	12.055	24.63	62.35	6	1.9373	4.712	13.02	26.9	68.7
.50	3	1.9019	4.542	12.296	25.20	63.91	6	1.9425	4.744	13.20	27.4	70.2
.75	3	1.9149	4.612	12.604	25.95	66.05	6	1.9457	4.756	13.34	27.9	72.0
.90	3	1.9224	4.646	12.782	26.46	67.70	6	1.9452	4.724	13.24	27.9	72.5
.95	3	1.9243	4.644	12.773	26.51	68.12	6	1.9441	4.692	13.03	27.4	71.8
.98	3	1.9251	4.630	12.665	26.28	67.83	6	1.9431	4.662	12.77	26.7	70.1
.99	3	1.9252	4.622	12.574	26.02	67.16	6	1.9428	4.649	12.63	26.2	68.4
1.0	3	1.9254	4.610	12.408	25.33	64.07	6	1.9423	4.634	12.43	25.4	64.1
.00	4	1.9057	4.549	12.307	25.21	63.93	7	1.9388	4.713	13.0	26.8	68.
.25	4	1.9142	4.594	12.501	25.68	65.22	7	1.9434	4.742	13.1	27.2	69.
.50	4	1.9233	4.645	12.734	26.25	66.86	7	1.9472	4.763	13.3	27.6	71.
.75	4	1.9319	4.693	12.993	26.95	68.99	7	1.9489	4.765	13.4	27.9	72.
.90	4	1.9352	4.696	13.059	27.28	70.39	7	1.9474	4.726	13.2	27.8	72.
.95	4	1.9353	4.677	12.958	27.13	70.43	7	1.9462	4.693	13.0	27.4	71.
.98	4	1.9350	4.652	12.757	26.63	69.46	7	1.9452	4.663	12.8	26.6	67.
.99	4	1.9348	4.639	12.627	26.21	68.25	7	1.9448	4.650	12.6	26.2	65.
1.0	4	1.9345	4.624	12.422	25.35	64.09	7	1.9444	4.637	12.4	25.4	64.

Table 3a. The waiting time distributions for systems with three queues, unequal arrival rates and equal service rates.

dcp	$E\{W_1\}$	$E\{W_2\}$	$E\{W_3\}$	$\sigma\{W\}$	dcp	$E\{W_1\}$	$E\{W_2\}$	$E\{W_3\}$	$\sigma\{W\}$
111	2.445	5.048	13.820	13.746	.25 .25 .25	2.856	5.430	13.428	13.555
222	2.790	5.141	13.643	13.646	.50 .50 .50	3.570	6.066	12.766	13.261
333	3.216	5.264	13.419	13.534	.75 .75 .75	5.146	7.306	11.414	12.794
444	3.671	5.406	13.173	13.406	.90 .90 .90	7.578	8.733	9.652	12.526
666	4.583	5.727	12.666	13.253	.95 .95 .95	9.186	9.361	8.697	12.575
888	5.439	6.077	12.136	12.926	.99 .99 .99	11.148	9.791	7.756	12.732
123	11.828	9.158	7.952	12.005	.00 .50 .67	10.936	9.476	8.037	12.326
246	11.892	9.251	7.868	11.938	.50 .75 .83	11.457	9.597	7.783	12.740
369	11.939	9.345	7.790	11.899	.80 .90 .93	11.905	9.713	7.556	13.059
4812	11.969	9.424	7.723	11.785	.90 .95 .97	12.022	9.769	7.480	13.116
∞∞∞	11.776	9.836	7.517	12.727	.96 .98 .99	11.990	9.810	7.463	13.032
∞22	2.075	5.279	13.789	13.760	1.0 .50 .50	2.276	6.299	13.042	13.462
∞11	1.373	5.258	14.037	13.915	1.0 .00 .33	1.696	11.173	9.986	12.596
∞23	2.524	11.043	9.796	12.347	1.0 .50 .67	2.793	11.285	9.546	12.811
∞46	4.097	10.848	9.402	12.092	1.0 .75 .83	4.342	11.284	9.030	12.909
∞69	5.397	10.695	9.066	11.926	1.0 .97 .98	9.565	10.450	7.845	12.737
1∞1	3.547	1.530	15.798	15.233	.00 1.0 .67	15.106	2.681	11.177	14.143
1∞3	15.059	2.356	11.410	13.996	.80 1.0 .93	14.749	6.100	9.017	13.875
2∞6	14.665	3.613	10.703	13.562	.90 1.0 .97	13.829	7.495	8.394	13.488
3∞9	14.301	4.662	10.123	13.220	.97 1.0 .99	12.550	8.989	7.824	12.985
11∞	7.371	20.504	1.873	17.795	.00 .50 1.0	17.108	14.827	2.413	16.212
12∞	17.194	14.970	2.289	16.151	.80 .90 1.0	15.525	12.011	4.817	15.117
24∞	16.382	14.062	3.165	15.556	.90 .95 1.0	14.354	11.111	5.808	14.341
48∞	15.069	12.717	4.498	14.616	.96 .98 1.0	13.084	10.405	6.702	13.520
1∞∞	32.977	4.859	3.768	22.649	.50 1.0 1.0	28.947	5.770	4.505	20.897
2∞∞	29.244	5.739	4.426	21.083	.75 1.0 1.0	23.877	6.929	5.421	18.503
4∞∞	23.622	7.061	5.418	18.582	.90 1.0 1.0	17.715	8.370	6.515	15.362
8∞∞	17.229	8.563	6.549	15.506	.95 1.0 1.0	14.806	9.072	7.017	13.912
∞1∞	3.562	21.997	2.356	18.684	1.0 .50 1.0	4.647	20.467	2.806	17.875
∞2∞	4.322	20.861	2.652	18.098	1.0 .75 1.0	6.208	18.247	3.766	16.672
∞4∞	5.635	18.907	3.517	17.072	1.0 .90 1.0	8.574	14.817	5.264	14.817
∞8∞	7.602	15.990	4.808	15.535	1.0 .95 1.0	9.998	12.687	6.209	13.754
∞∞1	1.952	1.732	16.195	15.513	1.0 1.0 .50	2.789	2.533	15.381	15.010
∞∞2	2.584	2.237	15.648	15.180	1.0 1.0 .75	4.124	3.787	14.101	14.255
∞∞4	3.725	3.156	14.654	14.590	1.0 1.0 .90	6.563	5.981	11.825	13.118
∞∞8	5.596	4.680	13.014	13.676	1.0 1.0 .95	8.433	7.543	10.160	12.580

Table 3b. The model as in table 3a, with the polling order reversed.

dcp	$E\{W_1\}$	$E\{W_2\}$	$E\{W_3\}$	$\sigma\{W\}$	dcp	$E\{W_1\}$	$E\{W_2\}$	$E\{W_3\}$	$\sigma\{W\}$
111	2.484	5.017	13.827	13.749	1.0 1.0 1.0	11.969	9.677	7.559	12.733
444	3.742	5.340	13.192	13.417	.75 .75 .75	5.270	7.198	11.444	12.799
123	11.841	9.151	7.952	12.006	.00 .50 .67	10.961	9.450	8.046	12.326
369	12.019	9.272	7.814	11.886	.90 .95 .97	12.158	9.636	7.524	13.118
12∞	17.228	14.957	2.286	16.154	.00 .50 1.0	17.084	14.827	2.421	16.211
1∞3	15.068	2.358	11.405	13.995	.00 1.0 .67	15.204	2.662	11.158	14.150
∞23	2.526	11.018	9.812	12.346	1.0 .00 .33	1.702	11.148	10.001	12.595
1∞∞	33.010	4.931	3.710	22.661	1.0 .00 .00	1.408	5.223	14.048	13.919
∞1∞	3.367	21.989	2.218	18.674	.00 1.0 .00	3.671	1.481	15.789	15.231
∞∞1	2.133	1.640	16.196	15.516	.00 .00 1.0	7.265	20.513	1.903	17.795

Table 4a. The waiting time distributions for systems with three queues, equal arrival rates and unequal service rates.

dcp	$E\{W_1\}$	$E\{W_2\}$	$E\{W_3\}$	$E\{W\}$	dcp	$E\{W_1\}$	$E\{W_2\}$	$E\{W_3\}$	$E\{W\}$
111	9.095	10.147	11.203	10.149	.25 .25 .25	9.683	10.316	10.895	10.298
222	9.316	10.225	11.078	10.206	.50 .50 .50	10.521	10.560	10.453	10.511
333	9.566	10.320	10.932	10.272	.75 .75 .75	11.860	10.952	9.745	10.852
444	9.819	10.421	10.780	10.340	.90 .90 .90	13.152	11.343	9.054	11.183
666	10.304	10.616	10.488	10.470	.95 .95 .95	13.666	11.530	8.758	11.318
888	10.739	10.803	10.208	10.584	.99 .99 .99	13.878	11.728	8.555	11.387
123	32.432	8.794	4.326	15.184	.00 .50 .67	31.144	8.673	4.837	14.884
246	31.837	8.825	4.465	15.055	.50 .75 .83	28.719	9.022	5.412	14.384
369	30.905	9.044	4.671	14.869	.80 .90 .93	24.232	9.856	6.354	13.480
4812	29.961	9.159	4.907	14.676	.90 .95 .97	20.654	10.550	7.082	12.762
∞∞∞	13.755	11.786	8.558	11.366	.96 .98 .99	17.130	11.212	7.815	12.052
211	2.525	11.406	12.554	8.828	.50 .00 .00	3.060	11.304	12.444	8.936
811	1.664	11.574	12.729	8.656	.90 .00 .00	1.843	11.539	12.693	8.692
∞11	1.660	11.575	12.730	8.655	1.0 .50 .50	2.835	12.143	11.959	8.979
∞22	2.581	11.527	12.455	8.854	1.0 .90 .90	7.403	12.805	9.996	10.068
∞44	4.208	11.522	11.916	9.215	1.0 .95 .95	9.526	12.639	9.398	10.521
121	12.262	3.084	14.857	10.067	.00 .50 .00	12.075	3.532	14.620	10.076
181	12.862	1.800	15.513	10.058	.00 .90 .00	12.772	2.002	15.408	10.061
1∞1	12.877	1.768	15.529	10.058	.50 1.0 .50	14.443	2.797	14.321	10.520
2∞2	12.824	2.519	15.046	10.130	.90 1.0 .90	16.472	6.809	10.970	11.417
4∞4	12.856	3.864	14.139	10.286	.95 1.0 .95	15.959	8.588	9.955	11.501
112	16.044	17.466	4.008	12.506	.00 .00 .50	15.744	17.132	4.331	12.402
118	17.786	19.254	2.235	13.092	.00 .00 .90	17.609	19.067	2.419	13.032
11∞	17.900	19.369	2.121	13.130	.50 .50 1.0	18.251	18.208	2.778	13.079
22∞	17.516	18.809	2.622	12.982	.90 .90 1.0	17.770	14.665	5.300	12.578
44∞	16.864	17.818	3.500	12.727	.95 .95 1.0	16.741	13.480	6.433	12.218
12∞	33.779	11.002	2.406	15.729	.00 .50 1.0	33.300	11.068	2.522	15.630
1∞∞	40.320	5.445	3.930	16.565	.50 1.0 1.0	37.775	6.025	4.392	16.064
2∞∞	38.112	5.971	4.315	16.133	.75 1.0 1.0	33.782	6.943	5.111	15.278
4∞∞	34.240	6.896	4.990	15.375	.90 1.0 1.0	26.636	8.611	6.381	13.876
8∞∞	28.176	8.330	6.033	14.189	.95 1.0 1.0	21.403	9.860	7.293	12.852
∞1∞	4.277	25.762	2.400	10.813	1.0 .50 1.0	5.374	24.221	3.061	10.885
∞2∞	5.032	24.639	2.897	10.856	1.0 .75 1.0	7.006	21.911	4.057	10.991
∞4∞	6.352	22.680	3.763	10.932	1.0 .90 1.0	9.658	18.095	5.718	11.157
∞8∞	8.403	19.639	5.103	11.050	1.0 .95 1.0	11.397	15.520	6.854	11.257
∞∞1	2.327	2.068	18.845	7.747	1.0 1.0 .50	3.524	3.227	17.674	8.142
∞∞2	3.243	2.814	18.043	8.033	1.0 1.0 .75	5.340	4.955	15.917	8.737
∞∞4	4.853	4.145	16.619	8.539	1.0 1.0 .90	8.370	7.712	13.069	9.717
∞∞8	7.357	6.252	14.380	9.330	1.0 1.0 .95	10.457	9.467	11.203	10.376

Table 4b. The model as in table 4a, with the polling order reversed.

dcp	$E\{W_1\}$	$E\{W_2\}$	$E\{W_3\}$	$E\{W\}$	dcp	$E\{W_1\}$	$E\{W_2\}$	$E\{W_3\}$	$E\{W\}$
111	9.130	10.108	11.218	10.152	1.0 1.0 1.0	14.397	11.257	8.696	11.450
444	10.006	10.218	10.852	10.359	.75 .75 .75	11.997	10.817	9.790	10.868
123	32.491	8.792	4.312	15.196	.00 .50 .67	31.196	8.656	4.831	14.894
369	31.172	8.891	4.685	14.911	.90 .95 .97	20.984	10.355	7.102	12.814
1∞∞	40.365	5.465	3.902	16.577	1.0 .00 .00	1.674	11.515	12.765	8.651
∞1∞	4.102	25.727	2.481	10.770	.00 1.0 .00	12.943	1.748	15.520	10.070
∞∞1	2.606	1.913	18.856	7.792	.00 .00 1.0	17.951	19.329	2.130	13.137

Table 5a. The influence of one relatively heavily loaded queue, ρ = 0.75.

	s = 2, limited					s = 2, Bernoulli			
dcp	E{W$_1$}	E{W$_2$}	E{W}	σ{W}	dcp	E{W$_1$}	E{W$_2$}	E{W}	σ{W}
1 1	3.037	1.351	2.475	3.588	.25 .00	2.909	1.865	2.561	3.595
2 1	2.661	2.858	2.726	3.808	.50 .00	2.639	2.942	2.740	3.909
3 1	2.313	4.250	2.958	4.678	.67 .00	2.336	4.157	2.943	4.735
4 1	2.113	5.047	3.091	5.368	.75 .00	2.165	4.841	3.057	5.342
8 1	1.836	6.156	3.276	6.449	.90 .00	1.867	6.030	3.255	6.496
16 1	1.727	6.590	3.348	6.942	.95 .00	1.775	6.399	3.316	6.861
∞ 1	1.688	6.750	3.375	7.210	.99 .00	1.705	6.681	3.364	7.142
∞ 4	1.852	6.091	3.265	6.791	1.0 .75	1.868	6.027	3.254	6.695
∞ 8	1.963	5.649	3.192	6.437	1.0 .90	1.974	5.605	3.184	6.338
∞ 16	2.052	5.293	3.132	6.089	1.0 .99	2.078	5.187	3.114	5.943
4 2	2.606	3.075	2.763	3.852	.75 .50	2.519	3.423	2.821	4.231
8 4	2.495	3.519	2.836	4.012	.90 .80	2.363	4.048	2.925	4.742
12 6	2.403	3.889	2.898	4.223	.95 .90	2.266	4.436	2.989	5.112
16 8	2.330	4.180	2.947	4.447	.98 .96	2.178	4.787	3.048	5.483
20 10	2.274	4.404	2.984	4.661	.99 .98	2.140	4.942	3.074	5.659
2 2	2.996	1.516	2.503	3.576	.25 .25	2.984	1.565	2.511	3.590
4 4	2.895	1.919	2.570	3.577	.50 .50	2.896	1.918	2.570	3.626
8 8	2.704	2.683	2.697	3.700	.75 .75	2.720	2.620	2.687	3.831
12 12	2.554	3.286	2.798	3.927	.90 .90	2.482	3.572	2.845	4.385
16 16	2.440	3.742	2.874	4.190	.95 .95	2.335	4.158	2.943	4.868
∞ ∞	2.093	5.128	3.105	5.885	.99 .99	2.155	4.881	3.063	5.599
1 2	3.144	0.922	2.404	3.643	.00 .50	3.119	1.025	2.421	3.630
1 4	3.172	0.810	2.385	3.666	.00 .75	3.150	0.900	2.400	3.651
1 ∞	3.176	0.794	2.382	3.670	.00 .90	3.166	0.835	2.389	3.663
8 ∞	2.722	2.610	2.685	3.695	.75 1.0	2.874	2.003	2.584	3.675
16 ∞	2.442	3.731	2.872	4.187	.99 1.0	2.171	4.818	3.053	5.536

	s = 3, limited						s = 3, Bernoulli				
dcp	E{W$_1$}	E{W$_2$}	E{W$_3$}	E{W}	σ{W}	dcp	E{W$_1$}	E{W$_2$}	E{W$_3$}	E{W}	σ{W}
1 1	3.040	1.404	1.434	2.230	3.401	.25 .00	2.818	1.847	1.880	2.341	3.430
2 1	2.419	2.644	2.679	2.540	3.717	.50 .00	2.416	2.649	2.687	2.542	3.798
3 1	1.963	3.551	3.597	2.769	4.508	.67 .00	2.033	3.411	3.456	2.733	4.492
4 1	1.728	4.017	4.071	2.886	5.021	.75 .00	1.835	3.805	3.854	2.832	4.924
8 1	1.431	4.607	4.667	3.034	5.729	.90 .00	1.508	4.456	4.513	2.996	5.670
16 1	1.337	4.794	4.856	3.081	6.005	.95 .00	1.410	4.651	4.710	3.045	5.892
∞ 1	1.317	4.834	4.896	3.091	6.099	.99 .00	1.335	4.799	4.860	3.082	6.059
∞ 2	1.452	4.526	4.664	3.024	5.884	1.0 .50	1.450	4.548	4.653	3.025	5.927
∞ 4	1.627	4.119	4.372	2.936	5.563	1.0 .75	1.583	4.253	4.414	2.958	5.708
4 2	2.333	2.791	2.879	2.584	3.771	.75 .50	2.256	2.951	3.024	2.622	4.114
6 3	2.250	2.929	3.071	2.625	3.855	.90 .80	2.081	3.268	3.409	2.710	4.511
8 4	2.179	3.047	3.236	2.660	3.957	.95 .90	1.993	3.414	3.616	2.754	4.727
10 5	2.121	3.144	3.372	2.690	4.066	.99 .98	1.908	3.532	3.836	2.796	4.929
2 2	2.955	1.562	1.618	2.273	3.386	.50 .50	2.795	1.881	1.938	2.352	3.475
4 4	2.761	1.927	2.031	2.370	3.406	.75 .75	2.526	2.397	2.498	2.487	3.728
6 6	2.584	2.259	2.404	2.458	3.495	.90 .90	2.224	2.962	3.140	2.638	4.220
8 8	2.439	2.533	2.713	2.531	3.625	.95 .95	2.075	3.232	3.469	2.713	4.535
∞ ∞	1.888	3.550	3.900	2.806	4.964	.99 .99	1.928	3.485	3.804	2.786	4.872
1 2	3.243	0.986	1.041	2.128	3.471	.00 .50	3.191	1.098	1.140	2.155	3.453
1 4	3.296	0.874	0.943	2.102	3.504	.00 .75	3.250	0.973	1.026	2.125	3.482
1 ∞	3.302	0.861	0.931	2.099	3.510	.00 .90	3.282	0.905	0.967	2.109	3.499
4 ∞	2.841	1.766	1.870	2.330	3.414	.50 1.0	3.066	1.310	1.426	2.217	3.448
8 ∞	2.459	2.495	2.667	2.520	3.619	.90 1.0	2.378	2.617	2.870	2.561	3.946

Table 5b. The influence of one relatively heavily loaded queue, ρ = 0.75.

dcp	s = 4, limited E{W_1}	E{W_2}	E{W_4}	E{W}	σ{W}	dcp	s = 4, Bernoulli E{W_1}	E{W_2}	E{W_4}	E{W}	σ{W}
1 1	3.022	1.447	1.495	2.091	3.273	.25 .00	2.734	1.828	1.882	2.206	3.319
2 1	2.240	2.485	2.542	2.404	3.627	.50 .00	2.266	2.447	2.510	2.394	3.677
3 1	1.754	3.123	3.199	2.598	4.286	.67 .00	1.868	2.973	3.046	2.553	4.222
4 1	1.521	3.429	3.515	2.691	4.668	.75 .00	1.672	3.232	3.311	2.631	4.536
6 1	1.325	3.687	3.782	2.770	5.011	.90 .00	1.353	3.651	3.742	2.759	5.056
∞ 1	1.170	3.891	3.990	2.832	5.348	.99 .00	1.187	3.868	3.967	2.825	5.321
∞ 2	1.355	3.591	3.799	2.758	5.112	1.0 .50	1.328	3.650	3.810	2.769	5.212
∞ 4	1.573	3.235	3.580	2.671	4.795	1.0 .75	1.479	3.418	3.642	2.708	5.027
4 2	2.138	2.582	2.721	2.445	3.676	.75 .50	2.097	2.648	2.761	2.461	3.944
6 3	2.048	2.664	2.877	2.481	3.740	.95 .90	1.862	2.888	3.153	2.555	4.346
2 2	2.894	1.598	1.687	2.142	3.257	.50 .50	2.703	1.852	1.941	2.219	3.368
3 3	2.754	1.767	1.894	2.198	3.265	.75 .75	2.388	2.243	2.393	2.345	3.617
4 4	2.621	1.930	2.090	2.252	3.299	.90 .90	2.079	2.612	2.853	2.469	3.998
∞ ∞	1.793	2.926	3.312	2.583	4.412	.95 .95	1.944	2.765	3.063	2.523	4.196
1 2	3.308	1.049	1.133	1.977	3.344	.00 .50	3.229	1.163	1.229	2.009	3.324
1 ∞	3.389	0.933	1.036	1.945	3.389	.00 .75	3.314	1.043	1.124	1.975	3.356
2 ∞	3.119	1.289	1.401	2.052	3.306	.50 1.0	3.056	1.347	1.511	2.077	3.322
4 ∞	2.713	1.810	1.964	2.215	3.300	.75 1.0	2.674	1.824	2.059	2.231	3.411
6 ∞	2.434	2.158	2.361	2.326	3.411	.90 1.0	2.241	2.365	2.676	2.404	3.761

dcp	s = 5, limited E{W_1}	E{W_2}	E{W_5}	E{W}	σ{W}	dcp	s = 5, Bernoulli E{W_1}	E{W_2}	E{W_5}	E{W}	σ{W}
1 1	2.991	1.481	1.540	2.003	3.18	.25 .00	2.657	1.811	1.877	2.114	3.24
2 1	2.103	2.361	2.433	2.299	3.55	.50 .00	2.159	2.302	2.380	2.280	3.57
3 1	1.622	2.833	2.935	2.461	4.09	.67 .00	1.768	2.688	2.778	2.411	4.00
4 1	1.400	3.057	3.160	2.532	4.37	.75 .00	1.579	2.873	2.971	2.474	4.24
5 1	1.288	3.160	3.250	2.574	4.50	.90 .00	1.277	3.168	3.280	2.574	4.63
∞ 1	1.104	3.336	3.457	2.632	4.85	.95 .00	1.188	3.255	3.371	2.604	4.74
∞ 2	1.325	3.065	3.296	2.559	4.60	1.0 .50	1.277	3.133	3.317	2.574	4.73
4 2	1.994	2.423	2.599	2.335	3.56	.75 .50	1.993	2.442	2.576	2.336	3.79
2 2	2.814	1.629	1.736	2.062	3.17	.50 .50	2.621	1.828	1.934	2.126	3.29
∞ ∞	1.737	2.594	2.962	2.421	4.06	.75 .75	2.285	2.134	2.305	2.238	3.51
1 2	3.347	1.106	1.206	1.884	3.25	.00 .50	3.243	1.220	1.299	1.919	3.23
1 ∞	3.445	1.002	1.119	1.852	3.29	.00 .75	3.349	1.106	1.201	1.884	3.26
2 ∞	3.085	1.357	1.486	1.972	3.20	.50 1.0	3.031	1.386	1.568	1.990	3.23
4 ∞	2.582	1.838	2.018	2.139	3.22	.75 1.0	2.591	1.796	2.045	2.136	3.31

dcp	s = 6, limited E{W_1}	E{W_2}	E{W_6}	E{W}	σ{W}	dcp	s = 6, Bernoulli E{W_1}	E{W_2}	E{W_6}	E{W}	σ{W}
1 1	2.955	1.507	1.573	1.942	3.10	.25 .00	2.589	1.796	1.871	2.046	3.18
2 1	1.996	2.263	2.345	2.215	3.47	.50 .00	2.080	2.194	2.281	2.191	3.47
4 1	1.327	2.751	2.919	2.398	4.11	.75 .00	1.525	2.628	2.735	2.350	4.01
∞ 1	1.074	2.975	3.108	2.478	4.49	.95 .00	1.153	2.915	3.043	2.456	4.41
∞ 2	1.323	2.732	2.971	2.410	4.25	1.0 .50	1.256	2.802	2.996	2.427	4.38
2 2	2.746	1.651	1.764	2.001	3.07	.50 .50	2.549	1.808	1.924	2.058	3.22
∞ ∞	1.700	2.387	2.729	2.300	3.80	.75 .75	2.205	2.054	2.232	2.156	3.42
4 2	1.888	2.301	2.489	2.246	3.47	.75 .50	1.921	2.294	2.438	2.237	3.65
1 2	3.365	1.158	1.265	1.824	3.17	.00 .50	3.240	1.268	1.354	1.860	3.15
1 ∞	3.479	1.063	1.187	1.792	3.21	.00 .75	3.365	1.162	1.264	1.824	3.18
2 ∞	3.033	1.413	1.550	1.919	3.12	.50 1.0	2.996	1.422	1.608	1.930	3.15
4 ∞	2.452	2.515	2.844	2.085	3.15	.75 1.0	2.518	1.777	2.023	2.066	3.23

Table 6. The influence of one relatively heavily loaded queue, $\rho = 0.95$.

dcp	s = 2, limited				dcp	s = 2, Bernoulli			
	$E\{W_1\}$	$E\{W_2\}$	$E\{W\}$	$\sigma\{W\}$		$E\{W_1\}$	$E\{W_2\}$	$E\{W\}$	$\sigma\{W\}$
1 1	20.78	2.38	14.65	20.2	.25 .00	20.32	4.22	14.95	19.8
2 1	16.89	17.96	17.24	20.1	.50 .00	16.56	19.24	17.46	20.9
3 1	8.24	52.56	23.01	46.3	.67 .00	8.42	51.83	22.89	46.0
4 1	6.25	60.49	24.33	52.2	.75 .00	6.45	59.69	24.20	51.9
8 1	4.48	67.59	25.51	57.0	.90 .00	4.40	67.89	25.57	57.4
16 1	3.90	69.91	25.90	58.9	.95 .00	3.95	69.71	25.87	58.5
∞ 1	3.56	71.25	26.13	59.5	.99 .00	3.63	70.96	26.08	59.3
∞ 4	4.10	69.09	25.76	58.5	1.0 .75	4.50	67.50	25.50	57.6
∞ 8	4.74	66.52	25.34	57.6	1.0 .90	5.84	62.15	24.61	54.7
∞ 12	5.32	63.62	24.94	56.5	1.0 .99	10.05	45.32	21.80	42.1
4 2	16.81	18.26	17.29	20.1	.75 .50	15.60	23.11	18.10	23.2
8 4	16.62	19.02	17.42	20.2	.90 .80	14.21	28.67	19.03	27.2
12 6	16.41	19.85	17.56	20.3	.95 .90	13.24	32.53	19.67	30.3
16 8	16.21	20.65	17.69	20.4	.98 .96	12.29	36.34	20.31	33.6
20 10	16.01	21.48	17.82	20.4	.99 .98	11.84	38.15	20.61	35.3
2 2	20.73	2.59	14.68	20.2	.25 .25	20.65	2.90	14.73	20.1
4 4	20.58	3.19	14.78	20.0	.50 .50	20.40	3.92	14.90	19.9
8 8	20.22	4.61	15.02	19.7	.75 .75	19.72	6.63	15.36	19.5
12 12	19.85	6.10	15.27	19.4	.90 .90	18.17	12.83	16.39	19.6
16 16	19.48	7.58	15.51	19.2	.95 .95	16.52	19.42	17.49	21.6
∞ ∞	11.26	40.46	20.99	37.5	.99 .99	12.99	33.55	19.84	31.1
1 2	21.04	1.33	14.47	20.5	.00 .25	20.91	1.87	14.56	20.4
1 4	21.10	1.10	14.43	20.6	.00 .50	21.00	1.52	14.50	20.5
1 ∞	21.11	1.06	14.43	20.6	.00 .90	21.09	1.13	14.44	20.6
8 ∞	20.33	4.19	14.95	19.8	.75 1.0	20.48	3.58	14.85	20.0
16 ∞	19.52	7.42	15.49	19.2	.99 1.0	14.03	29.39	19.15	27.7

dcp	s = 3, limited					dcp	s = 3, Bernoulli				
	$E\{W_1\}$	$E\{W_2\}$	$E\{W_3\}$	$E\{W\}$	$\sigma\{W\}$		$E\{W_1\}$	$E\{W_2\}$	$E\{W_3\}$	$E\{W\}$	$\sigma\{W\}$
1 1	22.39	2.70	2.75	12.56	20.1	.25 .00	21.41	4.64	4.70	13.04	19.4
2 1	15.33	16.83	16.86	16.09	20.0	.50 .00	15.05	17.37	17.42	16.22	20.6
3 1	6.01	35.45	35.51	20.74	36.5	.67 .00	6.43	34.62	34.68	20.54	36.0
4 1	4.30	38.86	38.92	21.60	39.3	.75 .00	4.70	38.07	38.13	21.42	38.8
8 1	2.90	41.67	41.73	22.30	41.2	.90 .00	3.01	41.45	41.52	22.25	41.2
12 1	2.61	42.24	42.31	22.44	41.7	.95 .00	2.65	42.17	42.24	22.43	41.7
∞ 1	2.34	42.78	42.84	22.58	42.0	.99 .00	2.40	42.66	42.73	22.55	42.0
∞ 2	2.72	41.96	42.15	22.39	41.6	1.0 .50	2.85	41.74	41.86	22.32	42.2
∞ 4	3.41	40.44	40.88	22.04	40.9	1.0 .75	3.68	40.07	40.23	21.91	42.1
4 2	15.19	17.07	17.16	16.15	20.0	.75 .50	13.76	19.93	20.03	16.87	22.9
6 3	15.04	17.35	17.50	16.23	20.0	.90 .80	12.20	22.99	23.21	17.65	26.2
8 4	14.88	17.65	17.83	16.31	20.0	.95 .90	11.36	24.60	24.97	18.07	28.1
10 5	14.70	18.02	18.20	16.40	19.9	.99 .98	10.64	25.83	26.59	18.43	29.6
2 2	22.26	2.94	3.03	12.62	20.0	.50 .50	21.58	4.29	4.39	12.96	19.5
4 4	21.90	3.63	3.77	12.80	19.7	.75 .75	20.22	6.96	7.16	13.64	18.9
6 6	21.49	4.42	4.59	13.01	19.4	.90 .90	17.48	12.33	12.76	15.01	19.2
8 8	21.08	5.24	5.44	13.21	19.3	.95 .95	15.10	16.96	17.62	16.20	21.3
∞ ∞	10.80	25.43	26.36	18.35	28.1	.99 .99	11.72	23.53	24.60	17.89	26.9
1 2	22.97	1.52	1.62	12.27	20.5	.00 .50	22.86	1.74	1.82	12.32	20.4
1 4	23.10	1.25	1.37	12.20	20.6	.00 .75	23.00	1.44	1.54	12.25	20.6
1 ∞	23.12	1.20	1.34	12.19	20.7	.00 .90	23.07	1.29	1.41	12.21	20.6
4 ∞	22.29	2.85	2.99	12.60	20.0	.50 1.0	22.64	2.10	2.48	12.43	20.3
8 ∞	21.28	4.85	5.03	13.11	19.3	.90 1.0	19.75	7.54	8.48	13.88	18.9

Appendix

The examples described in section 5 will be further elaborated in this appendix. The heavy traffic behaviour of the moments of the waiting time distributions for the model of table 6 is shown in table A.1 as function of q_1 for q_2=0.00. Note the strong sensitivity, especially of $E\{W_2\}$, with respect to the parameter q_1 in the neighborhood of q_1=0.50. For q_1=0.50 the queues are balanced in the sense that the length of both queues tends to infinity in heavy traffic. Table A.2 contains estimations of heavy traffic limits for the same model, but for Bernoulli schedules with $a_1(1-q_1) = a_2(1-q_2)$, the case which has not been covered by the conjecture of property 2. Although limited service and Bernoulli schedules agree in the sense of property 1, the actual values of the limits ω_j, $j=0,1,..,s$, behave quite differently for the two service disciplines. For limited service with $K_1 = 2K_2$ we find the following estimates (these limits seem to vary hardly with K_2, as far as they can be determined with sufficient accuracy):

Table A.1. Model of tables 5,6, Bernoulli schedules.

dcp	s = 2, ρ = 0.98				s = 2, ρ = 0.99			
	$E\{W_1\}$	$E\{W_2\}$	$E\{W\}$	$\sigma\{W\}$	$E\{W_1\}$	$E\{W_2\}$	$E\{W\}$	$\sigma\{W\}$
.00 .00	54.48	2.59	37.18	52.0	110.71	2.67	74.69	105.0
.25 .00	53.91	4.87	37.56	51.3	110.10	5.11	75.10	104.3
.45 .00	50.61	18.05	39.76	48.5	105.98	21.58	77.83	100.1
.49 .00	45.59	38.14	43.11	48.8	96.11	60.85	84.43	95.7
.50 .00	42.65	49.91	45.07	52.5	86.11	101.06	91.09	105.3
.51 .00	38.82	65.21	47.62	61.3	70.35	164.14	101.55	147.1
.55 .00	25.51	118.43	56.49	104.0	34.99	305.20	125.16	259.2
.67 .00	11.73	173.59	65.68	143.6	13.38	391.98	139.58	311.0
.75 .00	8.25	187.49	68.00	151.7	9.05	409.28	142.46	319.8
.90 .00	5.20	199.69	70.03	158.1	5.52	423.41	144.82	326.0
1.0 .00	4.08	204.17	70.78	160.3	4.28	428.37	145.64	328.1

Table A.2. Model of tables 5,6: estimations of heavy traffic limits.

dcp	s = 2, Bernoulli				s = 3, Bernoulli				
	ω_1	ω_2	ω_0	ζ_0	ω_1	ω_2	ω_3	ω_0	ζ_0
.50 .00	0.869	1.023	0.921	1.06	0.788	0.925	0.924	0.856	1.05
.75 .50	0.815	1.239	0.957	1.18	0.715	1.070	1.070	0.893	1.18
.90 .80	0.736	1.558	1.010	1.40	0.624	1.253	1.252	0.938	1.36
.95 .90	0.679	1.783	1.047	1.57	0.571	1.359	1.357	0.965	1.48
.98 .96	0.623	2.008	1.085	1.75	0.528	1.444	1.442	0.986	1.58
.99 .98	0.596	2.115	1.102	1.83	0.518	1.463	1.465	0.991	1.61
1.0 1.0	0.563	2.250	1.125	1.95	0.556	1.389	1.389	0.972	1.43

s=2: $\omega_1 = 0.889$, $\omega_2 = 0.946$, $\omega_0 = 0.908$, $\zeta_0 = 1.02$;
s=3: $\omega_1 = 0.806$, $\omega_2 = 0.889$, $\omega_3 = 0.889$, $\omega_0 = 0.847$, $\zeta_0 = 1.02$.

The limits in table A.2 have been determined by means of extrapolation from values of the performance measures for values of ρ close to 1. It is not clear whether differences between ω_2 and ω_3 (case s=3) exist or stem from inaccuracies of the computations. In the case of exhaustive service these limits are equal by theorem 2. Above we have used, in addition to (4.5), the following notation:

$$\zeta_0 := \lim_{\rho \uparrow 1} (1-\rho)\, \sigma\{W\}. \tag{A.1}$$

Next, we will consider the result of theorem 1a in more detail. Firstly, it is readily verified with the aid of (2.4) that, for all vectors $\bar{q}$,

$$\sum_{j=1}^{s} \eta_j \Psi(j,\bar{q}) = 0; \tag{A.2}$$

hence, by the conservation law, cf. (4.3), it must hold that

$$\frac{\beta_2}{2\beta_1} \sum_{j=1}^{s} \eta_j^2 + \sum_{j=1}^{s} \eta_j\, \Xi(j) = \frac{\beta_2}{2\beta_1}. \tag{A.3}$$

For the job weighted mean waiting time, cf. (4.2), it is found that

$$E\{W\} = \rho \frac{\beta_2}{2\beta_1} + \rho^2\Big[\frac{1}{2}\beta_2 \sum_{j=1}^{s} a_j\eta_j + \beta_1 \sum_{j=1}^{s} a_j\, \Xi(j) + 2\sum_{j=1}^{s} q_j\eta_j^2(\frac{1}{\mu_j} - \beta_1)\Big] + O(\rho^3), \quad \text{as } \rho \downarrow 0. \tag{A.4}$$

This expansion indicates that $E\{W\}$ is minimal in light traffic when $q_j = 1$ if $1/\mu_j < \beta_1$, and $q_j = 0$ if $1/\mu_j > \beta_1$, for $j=1,\dots,s$ (cf. the end of section 6). If all service rates are equal (i.e. $\mu_j = \mu$, $j=1,\dots,s$), then the quantities appearing in the power-series expansion (4.4) become, for $j=1,\dots,s$,

$$\Xi(j) = \frac{2}{\mu}\sum_{i=1}^{s-1} \eta_{j+i} \sum_{\nu=0}^{i-1} \eta_{j+\nu} = \frac{2}{\mu}\sum_{i=2}^{s} \eta_i \sum_{\nu=1}^{i-1} \eta_\nu, \qquad \text{(independent of } j\text{)},$$

$$\Psi(j,\bar{q}) = \frac{2}{\mu}\left[\sum_{i=1}^{s} q_i\, \eta_i^2 - q_j\, \eta_j\right], \qquad \sum_{j=1}^{s} a_j\, \Psi(j,\bar{q}) = 0. \qquad (A.5)$$

If, moreover, the arrival rates are equal (i.e. $\eta_j = 1/s$, $j=1,\dots,s$), then

$$\Xi(j) = \frac{s-1}{s}\,\frac{1}{\mu}, \qquad \Psi(j,\bar{q}) = \frac{2}{s\mu}\left[\frac{1}{s}\sum_{i=1}^{s} q_i - q_j\right], \qquad j = 1,\dots,s;$$

note that $\Psi(j,\bar{q})$ vanishes when also $q_j = q$ for all j, $j=1,\dots,s$.
For models with $a_j = a_2$, $\mu_j = \mu_2$, $j=2,\dots,s$, (cf. tables 5 and 6) we have

$$\frac{\beta_2}{2\beta_1} = \frac{\eta_1}{\mu_1} + (s-1)\,\frac{\eta_2}{\mu_2}; \qquad \Xi(1) = 2(s-1)\eta_2\left[\frac{\eta_1}{\mu_1} + \frac{s-2}{2}\,\frac{\eta_2}{\mu_2}\right];$$

$$\Xi(j) = 2\eta_1(s+1-j)\,\frac{\eta_2}{\mu_2} + 2\eta_2(j-2)\,\frac{\eta_1}{\mu_1} + \eta_2(s-1)(s-2)\,\frac{\eta_2}{\mu_2}, \qquad j=2,\dots,s;$$

$$\Psi(j,\bar{q}) = 2\left[q_1\,\frac{\eta_1^2}{\mu_1} + \frac{\eta_2^2}{\mu_2}\sum_{i=2}^{s} q_i - q_j\,\frac{\eta_j}{\mu_j}\right], \qquad j = 1,\dots,s.$$

Note that $\Xi(j)$ is increasing with j, $j=2,\dots,s$, if and only if $1/\mu_1 > 1/\mu_2$. For the specific models considered in the tables 5 and 6 these quantities become:

$$\beta_1 = \frac{s+3}{2(s+1)}, \qquad \frac{\beta_2}{2\beta_1} = \frac{s+7}{2(s+3)}, \qquad \eta_1 = \frac{4}{s+3}, \qquad \eta_j = \frac{1}{s+3}, \qquad j=2,\dots,s;$$

$$\Xi(1) = \frac{(s-1)(s+14)}{2(s+3)^2}, \qquad \Xi(j) = \frac{(s-1)(s-2)}{2(s+3)^2} + \frac{4(s+j-3)}{(s+3)^2}, \qquad j=2,\dots,s;$$

$$\Psi(1,\bar{q}) = \frac{1}{(s+3)^2}\left[\sum_{i=2}^{s} q_i - 8(s-1)q_1\right] = \frac{s-1}{(s+3)^2}\,(q_2-8q_1);$$

$$\Psi(j,\bar{q}) = \frac{1}{(s+3)^2}\left[32q_1 + \sum_{i=2}^{s} q_i - (s+3)q_j\right] = \frac{4}{(s+3)^2}(8q_1-q_2), \quad j=2,\dots,s.$$

For general models with two queues these quantities become:

$$\Xi(j) = 2\eta_1\eta_2/\mu_j, \quad j = 1,2;$$

$$\Psi(1,\bar{q}) = 2\eta_2 \left[q_2 \frac{\eta_2}{\mu_2} - q_1 \frac{\eta_1}{\mu_1}\right], \qquad \Psi(2,\bar{q}) = 2\eta_1 \left[q_1 \frac{\eta_1}{\mu_1} - q_2 \frac{\eta_2}{\mu_2}\right].$$

From these relations it readily follows that $E\{W_1\} > E\{W_2\}$ for small values of ρ if $(1-2q_1)\,\eta_1/\mu_1 > (1-2q_2)\,\eta_2/\mu_2$. On the other hand, property 1 implies that $E\{W_1\} > E\{W_2\}$ for values of ρ close to 1 if $a_1(1-q_1) > a_2(1-q_2)$. For the model of tables 5 and 6, with 2 queues, the above implies that there is a different ordering of $E\{W_1\}$ and $E\{W_2\}$ in light and heavy traffic if $7+2q_2 < 16\, q_1 < 8+8q_2$. This feature is illustrated in table A.3 where the mean waiting times are shown for several values of the load ρ for service disciplines with $q_1=q_2=q$ (respectively $K_1=K_2=K$). Note in this table also the close resemblence of the mean waiting times in the cases $K=16$ and $q=1.0$ ($K=\infty$) for $\rho \leq 0.5$ (cf. the observation below theorem 1), and the important differences for larger values of ρ. This rather abrupt change in the behaviour of the performance measures causes slow convergence of the power-series for larger values of ρ for such models. At this point it is also important to note the discontinuity in the heavy traffic limits of the mean waiting times, for instance when $q_1 = q_2 = q$ and q approaches 1: for $q < 1$ these limits are determined by corollary 1, while for $q = 1$ they are given by theorem 2 (see also table A.3).
The quantities appearing in theorem 1 become for the model of table 3:

$$\beta_1 = 1, \quad \frac{\beta_2}{2\beta_1} = 1, \quad \eta_1 = \frac{1}{6}, \quad \eta_2 = \frac{1}{3}, \quad \eta_3 = \frac{1}{2};$$

$$\Psi(1,\bar{q}) = \frac{2}{9}q_2 + \frac{1}{2}q_3 - \frac{5}{18}q_1, \qquad \Psi(2,\bar{q}) = \frac{1}{18}q_1 + \frac{1}{2}q_3 - \frac{4}{9}q_2,$$

$$\Psi(3,\bar{q}) = \frac{1}{18}q_1 + \frac{2}{9}q_2 - \frac{1}{2}q_3; \qquad \Xi(j) = \frac{11}{18}, \quad j = 1,2,3.$$

The values of $\Xi(j)$ are independent of j and of the polling order for models with equal service rates, cf. (A.5). However, higher order terms of the power-series expansions of the mean waiting times do depend on the polling order; see the differences in the tables 3a and 3b.
The quantities appearing in theorem 1 become for the model of table 4:

$$\beta_1 = 1, \quad \frac{\beta_2}{2\beta_1} = \frac{7}{6}, \quad \eta_1 = \frac{1}{6}, \quad \eta_2 = \frac{1}{3}, \quad \eta_3 = \frac{1}{2};$$

$$\Xi(1) = \frac{17}{36}, \quad \Xi(2) = \frac{25}{36}, \quad \Xi(3) = \frac{29}{36}, \quad \sum_{j=1}^{3} a_j\ \Xi(j) = \frac{71}{108};$$

$$\Psi(1,\bar{q}) = \frac{2}{9} q_2 + \frac{3}{4} q_3 - \frac{5}{36} q_1, \qquad \Psi(2,\bar{q}) = \frac{1}{36} q_1 + \frac{3}{4} q_3 - \frac{4}{9} q_2,$$

$$\Psi(3,\bar{q}) = \frac{1}{36} q_1 + \frac{2}{9} q_2 - \frac{3}{4} q_3, \qquad \sum_{j=1}^{3} a_j\ \Psi(j,\bar{q}) = \frac{1}{4} q_3 - \frac{1}{36} q_1.$$

When the polling order is reversed in this model then we have:

$$\Xi(1) = \frac{23}{36}, \quad \Xi(2) = \frac{19}{36}, \quad \Xi(3) = \frac{31}{36}, \quad \sum_{j=1}^{3} a_j\ \Xi(j) = \frac{73}{108}.$$

Tables A.4 and A.5 contain estimations for heavy traffic limits of waiting time characteristics for the models of table 3 and table 4 respectively. These tables should be read as follows. An entry without brackets stands for an estimation for ω_j, j=1,2,3; an entry between brackets indicates that $\omega_j = 0$, cf. property 1, and stands for an estimation for x_j, j=1,2,3, cf. (4.6). For the model of table 3 we have $\omega_0 = 1.00$, independent of the service discipline.

Table A.3. Model of tables 5,6; 2 queues, $q_1=q_2=q$ respectively $K_1=K_2=K$.

q=.00	ρ=0.1	ρ=0.5	ρ=0.7	ρ=0.8	ρ=0.9	ρ=0.92	ρ=0.94	ρ=0.96	ρ=0.98
$E\{W_1\}$	.1014	0.9606	2.3313	4.1119	9.6107	12.39	17.05	27.95	54.48
$E\{W_2\}$	.0944	0.6576	1.1747	1.5525	2.0574	2.18	2.31	2.45	2.59
q=.50	ρ=0.1	ρ=0.5	ρ=0.7	ρ=0.8	ρ=0.9	ρ=0.92	ρ=0.94	ρ=0.96	ρ=0.98
$E\{W_1\}$	.1001	0.9192	2.2201	3.9309	9.3125	12.06	16.68	25.98	54.03
$E\{W_2\}$	.0998	0.8232	1.6197	2.2764	3.2499	3.50	3.77	4.07	4.39
q=.75	ρ=0.1	ρ=0.5	ρ=0.7	ρ=0.8	ρ=0.9	ρ=0.92	ρ=0.94	ρ=0.96	ρ=0.98
$E\{W_1\}$	.0993	0.8796	2.0902	3.6906	8.8413	11.52	16.05	25.25	53.17
$E\{W_2\}$	.1030	0.9818	2.1391	3.2375	5.1348	5.68	6.29	7.00	7.81
q=.90	ρ=0.1	ρ=0.5	ρ=0.7	ρ=0.8	ρ=0.9	ρ=0.92	ρ=0.94	ρ=0.96	ρ=0.98
$E\{W_1\}$	.0987	0.8388	1.9256	3.3377	7.9577	10.42	14.68	23.50	50.90
$E\{W_2\}$	.1052	1.1448	2.7974	4.6490	8.6693	10.05	11.79	14.01	16.91
q=.95	ρ=0.1	ρ=0.5	ρ=0.7	ρ=0.8	ρ=0.9	ρ=0.92	ρ=0.94	ρ=0.96	ρ=0.98
$E\{W_1\}$	.0985	0.8190	1.8298	3.1053	7.2217	9.43	13.30	21.50	47.84
$E\{W_2\}$	.1059	1.2240	3.1810	5.5789	11.6131	14.01	17.29	22.00	29.14
q=.99	ρ=0.1	ρ=0.5	ρ=0.7	ρ=0.8	ρ=0.9	ρ=0.92	ρ=0.94	ρ=0.96	ρ=0.98
$E\{W_1\}$	.0984	0.7989	1.7168	2.8031	6.0571	7.73	10.61	16.69	37.15
$E\{W_2\}$	.1066	1.3042	3.6328	6.7875	16.2721	20.83	28.07	41.25	71.89
q=1.0	ρ=0.1	ρ=0.5	ρ=0.7	ρ=0.8	ρ=0.9	ρ=0.92	ρ=0.94	ρ=0.96	ρ=0.98
$E\{W_1\}$	.0983	0.7931	1.6795	2.6957	5.5924	7.02	9.38	14.08	28.16
$E\{W_2\}$	.1067	1.3276	3.7820	7.2174	18.1303	23.69	32.99	51.67	107.85
K=16	ρ=0.1	ρ=0.5	ρ=0.7	ρ=0.8	ρ=0.9	ρ=0.92	ρ=0.94	ρ=0.96	ρ=0.98
$E\{W_1\}$	.0983	0.8001	1.8558	3.3712	8.5187	11.22	15.79	25.04	53.04
$E\{W_2\}$	.1067	1.2996	3.0766	4.5150	6.4253	6.87	7.34	7.83	8.35
K=8	ρ=0.1	ρ=0.5	ρ=0.7	ρ=0.8	ρ=0.9	ρ=0.92	ρ=0.94	ρ=0.96	ρ=0.98
$E\{W_1\}$	.0983	0.8329	2.0468	3.7268	9.1128	11.87	16.50	25.82	53.88
$E\{W_2\}$	.1067	1.1684	2.3127	3.0928	4.0488	4.26	4.49	4.73	4.97
K=2	ρ=0.1	ρ=0.5	ρ=0.7	ρ=0.8	ρ=0.9	ρ=0.92	ρ=0.94	ρ=0.96	ρ=0.98
$E\{W_1\}$	.0993	0.9354	2.2931	4.0672	9.5596	12.34	16.99	26.33	54.42
$E\{W_2\}$	.1030	0.7584	1.3274	1.7311	2.2614	2.39	2.52	2.66	2.82

Table A.4. The model as in table 3: estimations of heavy traffic limits.

dcp	$\omega_1(x_1)$	$\omega_2(x_2)$	$\omega_3(x_3)$	ζ_0	dcp	$\omega_1(x_1)$	$\omega_2(x_2)$	$\omega_3(x_3)$	ζ_0
111	(3.20)	(9.27)	2.00	1.74	1.0 1.0 1.0	1.36	1.09	0.82	1.33
444	(4.92)	(10.04)	2.00	1.78	.75 .75 .75	(9.17)	(19.57)	2.00	1.73
123	1.35	1.01	0.88	1.26	.00 .50 .67	1.26	1.06	0.88	1.30
369	1.34	1.01	0.88	1.25	.96 .98 .99	1.51	1.08	0.78	1.47
∞11	(1.60)	(9.57)	2.00	1.74	1.0 .00 .33	(2.09)	1.29	1.14	1.34
∞23	(3.19)	1.28	1.15	1.32	1.0 .75 .83	(7.60)	1.38	1.08	1.43
1∞1	(5.03)	(1.82)	2.00	1.73	.00 1.0 .67	1.92	(3.73)	1.36	1.60
1∞3	1.82	(2.99)	1.39	1.55	.80 1.0 .93	2.41	(15.66)	1.17	1.81
11∞	(13.96)	3.00	(2.33)	2.24	.00 .50 1.0	2.22	1.89	(3.32)	1.88
12∞	2.18	1.91	(2.99)	1.86	.80 .90 1.0	2.54	1.70	(11.93)	2.05
1∞∞	6.00	(8.48)	(6.32)	3.32	.50 1.0 1.0	6.00	(10.96)	(8.31)	3.34
2∞∞	6.00	(10.47)	(7.81)	3.32	.75 1.0 1.0	6.00	(15.89)	(12.25)	3.37
∞1∞	(5.08)	3.00	(2.81)	2.24	1.0 .50 1.0	(7.16)	3.00	(3.94)	2.24
∞2∞	(6.33)	3.00	(3.55)	2.24	1.0 .75 1.0	(11.31)	3.00	(6.21)	2.24
∞∞1	(2.42)	(2.12)	2.00	1.73	1.0 1.0 .50	(3.67)	(3.33)	2.00	1.73
∞∞2	(3.26)	(2.79)	2.00	1.73	1.0 1.0 .75	(6.15)	(5.75)	2.00	1.73

Table A.5. The model as in table 4: estimations of heavy traffic limits.

dcp	$\omega_1(x_1)$	$\omega_2(x_2)$	$\omega_3(x_3)$	ω_0	dcp	$\omega_1(x_1)$	$\omega_2(x_2)$	$\omega_3(x_3)$	ω_0
111	1.01	1.13	1.25	1.13	1.0 1.0 1.0	1.59	1.27	0.95	1.27
444	1.01	1.12	1.25	1.13	.75 .75 .75	1.36	1.21	1.07	1.21
123	7.00	(24.09)	(6.91)	2.33	.00 .50 .67	7.00	(23.98)	(8.22)	2.33
∞11	(2.00)	1.32	1.45	0.92	1.0 .50 .50	(3.87)	1.41	1.39	0.93
∞22	(3.24)	1.32	1.45	0.92	1.0 .90 .90	(17.74)	1.61	1.25	0.96
1∞1	1.52	(2.12)	1.83	1.12	.50 1.0 .50	1.78	(3.73)	1.74	1.17
2∞2	1.52	(3.12)	1.83	1.12	.90 1.0 .90	2.55	(15.87)	1.46	1.35
11∞	2.22	2.39	(2.66)	1.54	.50 .50 1.0	2.35	2.32	(3.82)	1.56
22∞	2.22	2.39	(3.41)	1.54	.90 .90 1.0	2.81	2.06	(12.52)	1.63
1∞∞	7.00	(9.23)	(6.48)	2.33	.50 1.0 1.0	7.00	(10.47)	(7.48)	2.33
2∞∞	7.00	(10.22)	(7.23)	2.33	.75 1.0 1.0	7.00	(12.96)	(9.47)	2.33
∞1∞	(6.08)	3.50	(3.14)	1.17	1.0 .50 1.0	(8.16)	3.50	(4.28)	1.17
∞2∞	(7.33)	3.50	(3.89)	1.17	1.0 .75 1.0	(12.31)	3.50	(6.55)	1.17
∞∞1	(2.91)	(2.55)	2.33	0.78	1.0 1.0 .50	(4.77)	(4.36)	2.33	0.78
∞∞2	(4.16)	(3.55)	2.33	0.78	1.0 1.0 .75	(8.49)	(7.99)	2.33	0.78

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