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Engwerda, J.C.; Ran, A.C.M.; Rijkeboer, A.L.

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VERSITY
EKE
UNIVERSITEIT
BRABANT

POSTBOX 90153
5000 LE TILBURG
THE NETHERLANDS

[redacted]

DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A POSITIVE DEFINITE SOLUTION OF THE MATRIX EQUATION $X + A^{T}X^{-1}A = I$

Jacob C. Engwerda, André C.M. Ran
Arie L. Rijkeboer

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Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation $X + A^T X^{-1} A = I$.

by

Jacob C. Engwerda*
André C.M. Ran**
Arie L. Rijkeboer*

* Tilburg University
P.O. Box 90153
5000 LE Tilburg
The Netherlands

** Vrije Universiteit
Faculteit Wiskunde en Informatica
De Boelelaan 1081
1081 HV Amsterdam
The Netherlands

Abstract

In this paper we consider the problem under which conditions the matrix equation $X + A^T X^{-1} A = I$ (with A a real matrix) has a real symmetric positive definite solution X.
This equation plays a crucial role in solving a special case of the indefinite discrete-time Riccati equation.
We present both necessary and sufficient conditions for solvability of this equation. This result is obtained by using an analytic factorization approach. Moreover, we present algebraic recursive algorithms to compute the largest and smallest solution of the equation, respectively. Finally, we discuss the number of solutions.

I. Introduction

In (1990), Engwerda considered the problem under which conditions the matrix equation $X + A^T X^{-1} A = I$ has a real symmetric positive definite solution X, if A is a real matrix. Apart from being a natural extension of the familiar scalar quadratic equation he showed that this equation also results as a special case of the discrete-time algebraic Riccati equation. This equation plays a role in optimal control theory. In particular the smallest and largest solution, respectively, are important to solve a special case of the indefinite Linear Quadratic regulator problem.
Engwerda showed that the quadratic matrix equation has a solution in case matrix A is normal if and only if the spectral radius of A is smaller or equal than $\frac{1}{2}$. It turned out that this condition is necessary too to solve the general case. Furthermore a number of other necessary conditions were provided. A complete analytic solution for the general case remained, however, an open problem.
In this paper we present both analytical conditions and numerical algorithms to solve the equation. In particular we give algorithms to calculate the largest and smallest solution, respectively, if they exist. The analytic conditions are obtained by using a factorization approach of the problem. The paper is organized as follows. In section II we formalize the

problem and derive some technical results. These technical preliminaries are used in section III to obtain the main result of this paper, namely, both necessary and sufficient conditions for solvability of the equation. Section IV deals with two recurrence equations for calculating the largest and smallest solution, respectively. In section V we pay attention to the number of solutions. In particular we discuss conditions under which the equation has exact one solution, and give a complete characterization of the number of solutions in case matrix A is normal. We conclude the paper by indicating a number of interesting topics which remain to be solved, and by applying the basic result of this paper to an example of optimal control theory.

II. Problem statement and mathematical preliminaries

In this paper we study the problem under which conditions the matrix equations

$$Y + A^T Y^{-1} A = I \tag{1}$$

and

$$X + A X^{-1} A^T = I \tag{2}$$

have a real symmetric positive definite solution Y and X, respectively, if A is a real matrix and I is the identity matrix. To solve this problem we start with a number of elementary results on rational functions. Their proofs can be found in the appendix. But first we introduce some notation that will be used throughout this paper.

Basic notation

In the sequel the following notation is used

- $X > 0$ denotes that X is a symmetric positive definite matrix
- $X \geq 0$ denotes that X is a symmetric semi-positive definite matrix
- $j := \sqrt{-1}$ and $\bar{\alpha}$ is the conjugate of α
- $D = \{z \in \mathbb{C} \mid |z| < 1\}$; $D^c = \{z \in \mathbb{C} \mid |z| \geq 1\}$

- $\bar{D} = \{z \in \mathbb{C} \mid |z| \le 1\}$
- $\partial D = \{z \in \mathbb{C} \mid |z| = 1\}$
- $\mathbb{R}[z]$:= the set of all polynomials with real entries
- $\mathbb{C}[z]$:= the set of all polynomials with complex entries
- $\mathbb{C}(z)$:= the set of all rational functions
- Similar definitions hold w.r.t. $\mathbb{R}^{n\times n}[z]$, $\mathbb{C}^{n\times n}[z]$ and $\mathbb{C}^{n\times n}(z)$.
- If $V \in \mathbb{C}^{n\times n}(z)$ then we denote by $\bar{V}$ the element of $\mathbb{C}^{n\times n}(z)$ such that $\bar{V}_{ij}(z) = \overline{V_{ij}(z)}$ for all $z \in \mathbb{R}$ for which $V_{ij}(z)$ exists ($i = 1,..,n$; $j = 1,..,n$).
- If $V \in \mathbb{C}^{n\times n}(z)$ then V^* denotes the transpose of $\bar{V}$.
- $\sigma(A)$:= the set of all eigenvalues of A
- $r(A)$:= spectral radius of A, i.e. $\max\{|\lambda_i| \mid \lambda_i \in \sigma(A)\}$.

Lemma 1

Let $f(z)$ be a rational function different from zero, i.e. $f(z) \in \mathbb{C}(z)\backslash\{0\}$, with the property that $f(z) \ge 0$ for all $z \in \mathbb{R}$.
Then, there exist α_i in the open upper half plane of $\mathbb{C}$ and $a_i \in \mathbb{R}$ such that $f(z) = c \prod_{i=1}^{k} (z-\alpha_i)^{n_i}(z-\bar{\alpha}_i)^{n_i}(z-a_i)^{2m_i}$, where $c > 0$ and $n_i, m_i \in \mathbb{Z}$. □

Lemma 2

Let $f(z)$ be a rational function, different from zero, with the property that $f(z) \ge 0$ for all $z \in \partial D$.
Then $f(z)$ can be factorized as $q(z)q^*(\frac{1}{z})$ with $q(z)$ a rational function which has all its poles and zeros in D^c. □

The next theorem generalizes lemma 2 for matrix valued functions $F(z)$. This factorization of $F(z)$ into $Q(z)Q^*(\frac{1}{z})$ will be used in section III to derive a method to calculate the largest solution X_L of equation (2) exactly. The proof of this theorem is closely related to the one Rozanov gave in theorem 10.1 (1967).

Theorem 3

Let $f_{ij} \in \mathbb{C}(z)$ be such that $f_{ij}(z) = f^*_{ji}(\frac{1}{z})$, $i,j = 1,..,n$. Consider the rational matrix function $F(z)$ defined by $(f_{ij}(z))$, $i,j = 1,..,n$.

If $F(z) \geq 0$ $\forall z \in \partial D$ and det $F(z)$ is different from zero, then $F(z)$ can be factorized as $F(z) = Q(z)Q^*(\frac{1}{z})$ with $Q(z)$ a rational matrix function, which is both analytic and invertible in D.

<u>Proof</u>

We prove this theorem by constructing a matrix $Q(z)$ that has all the above stated requirements. The algorithm to obtain this factorization is split into three parts.

In the first part we use Gaussian Elimination to obtain an $L(z)D(z)L^*(\frac{1}{z})$ decomposition of $F(z)$ where $L(z)$ is some lower triangular matrix that is analytic in D and $D(z)$ is a diagonal matrix. The next step is to factorize $D(z)$ into $D_1(z)D_1^*(\frac{1}{z})$ such that $D_1(z)$ is analytic in D too. In the last part we show how from the factorization obtained in step 2, a factorization of $F(z)$ can be derived which is also invertible in D.

<u>Algorithm</u>

<u>Step 1</u>: ($L(z)D(z)L^*(\frac{1}{z})$ factorization of $F(z)$).

Let $m_1 := f_{11}$ be the first minor of $F(z)$.

Due to our assumptions on $F(z)$, $m_1(z) \not\equiv 0$ and, moreover, $m_1(z) \geq 0$ $\forall z \in \partial D$.

Consequently, with $L_1 := \begin{bmatrix} 1 & & & \\ -\frac{f_{21}}{m_{11}} & 1 & & \\ \vdots & & \ddots & \\ -\frac{f_{n1}}{m_{11}} & & & 1 \end{bmatrix}$ we have that

$$L_1(z)F(z)L^*(\frac{1}{z}) = \begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & * \dots & * \\ & * & & \\ & \vdots & & \\ 0 & * & \dots & * \end{bmatrix}.$$

Note that $L_1(z)$ is unimodular. Therefore the same argument yields now that $m_2(z) \not\equiv 0$ and, moreover, $m_2(z) \geq 0$ $\forall z \in \partial D$.

By induction it is now easily shown that there exist unimodular matrices $L_1,\dots,L_{n-1}$ such that

$$L_{n-1}(z)\cdot\ldots\cdot L_1(z)F(z)L_1^*(\tfrac{1}{z})\cdot\ldots\cdot L_{n-1}^*(\tfrac{1}{z}) = \operatorname{diag}(m_1(z),\dots,m_n(z)),$$

where $m_i(z) \geq 0 \ \forall z \in \partial D$, $i = 1,\dots,n$.
So, $F(z) = L(z)D(z)L^*(\frac{1}{z})$, where $L(z)$ is an unimodular rational matrix, and $D(z) = \operatorname{diag}(m_1(z),\dots,m_n(z))$.

Step 2: (analytic factorization of $F(z)$).
Let $n(z)$ be the product of all denominators of $L(z)$. Then $n(z)L(z)$ is a polynomial matrix, and $N(z) := n(z)n^*(\frac{1}{z})$ is a rational function different from zero which has the property that $N(z) \geq 0$ for all $z \in \partial D$. So, $N(z) = p(z)p^*(\frac{1}{z})$ with $p(z)$ a rational function which has all its poles and zeros in D^c (see lemma 2).
Since $m_i(z) \geq 0 \ \forall z \in \partial D$, using lemma 2 again, we have that $m_i(z)$ can be factorized too as $\mu_i(z)\mu_i^*(\frac{1}{z})$ with both $\mu_i(z)$ and its inverse analytic in D. This leads to the factorization of $\frac{D(z)}{N(z)} = D_1(z)D_1^*(\frac{1}{z})$ with $D_1(z) = \operatorname{diag}\left[\frac{\mu_1(z)}{p(z)},\dots,\frac{\mu_n(z)}{p(z)}\right]$.
So $F(z) = L(z)D(z)L^*(\frac{1}{z}) = n(z)L(z)\frac{D(z)}{N(z)} L^*(\frac{1}{z})n^*(\frac{1}{z}) = Q_1(z)Q_1^*(\frac{1}{z})$ where $Q_1(z) := n(z)L(z)D_1(z)$ is analytic in D.

Step 3: (zero cancelation).
Since $\det F(z)$ is different from zero, $\det Q_1(z)$ differs from zero too and has a finite number of zeros in D.
Let $\alpha \in D$ be a zero of $Q_1(z)$. Then, there is an unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $U^*Q_1(\alpha)U = \begin{bmatrix} 0 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \dots & * \end{bmatrix}$. So the components in the first column of $U^*Q_1(z)U$ have a zero at $z = \alpha$. Now, define the rational matrix

$$\Lambda(z) := \begin{bmatrix} \frac{1-\bar{\alpha}z}{z-\alpha} & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Then $\Lambda(z)\Lambda^*(\frac{1}{z}) = I$ and $U^*Q_1(z)U\Lambda(z)$ is analytic in D, hence also $Q_2(z) = Q_1(z)U\Lambda(z)$ is analytic in D, and it is easy to verify that $Q_2(z)Q_2^*(\frac{1}{z})$ is also a factorization of F(z).

Furthermore, $\det Q_2(z) = \det Q_1(z) \det U \det \Lambda(z) = \dfrac{\det Q_1(z)(1-\bar{\alpha}z)}{z-\alpha} \det U$ from which we see that every zero of $\det Q_2(z)$ in D is also a zero of $\det Q_1(z)$, but the total multiplicity of zeros in D is strictly smaller in case of $\det Q_2(z)$ than in case of $\det Q_1(z)$.
That's why, by repeating this procedure a finite number of times, we obtain a factorization $F(z) = Q(z)Q^*(\frac{1}{z})$ where Q(z) is analytic in D and $\det Q(z)$ has no zeros in D, that is to say, also the inverse of Q(z) is analytic in D.

□

If F(z) is a rational matrix function which is a linear combination of powers of z and $\frac{1}{z}$, we can be more specific about this factorization of F(z) into $Q(z)Q^*(\frac{1}{z})$.

Theorem 4

Let all assumptions of theorem 3 be satisfied and assume, moreover, that $F(z) = \sum_{k=-m}^{m} F_k z^k$.
According to theorem 3, there exists a factorization of F(z) into $Q(z)Q^*(\frac{1}{z})$ where Q(z) is both analytic and invertible in D.
Then $Q(z) = Q_0 + Q_1 z + \dots + Q_m z^m$.

Proof

Q(z) is analytic in D, so $Q(z) = \sum_{k=0}^{\infty} C_k z^k$ for all $z \in D$. Since $F_{ii}(z) = \sum_{j=1}^{n} Q_{ij}(z)Q_{ji}^*(\frac{1}{z}) = \sum_{j=1}^{n} Q_{ij}(z)\overline{Q_{ij}}(\frac{1}{z}) = \sum_{j=1}^{n} |Q_{ij}(z)|^2$ if $z \in \partial D$, and $F_{ii}(z) = \sum_{k=-m}^{m} F_{k,ii} z^k$ is integrable over ∂D, $|Q_{ij}(z)|^2$ must be integrable over ∂D too. This implies in particular that $Q_{ij}(z)$ has no poles on ∂D. Therefore, Q(z) is analytic on a disk around zero with radius $R > 1$. So for all z, $|z| > \frac{1}{R}$, we have $z^m Q^*(\frac{1}{z}) = z^m \sum_{k=0}^{\infty} C_k^*(\frac{1}{z})^k = C_0^* z^m + \dots + C_m^* + \sum_{k=1}^{\infty} C_{k+m}^*(\frac{1}{z})^k$, and $z^m Q^*(\frac{1}{z})$ is analytic for all $z \in D^c$. On the other hand, the inverse of Q(z) is analytic on D, and $z^m F(z) = \sum_{k=0}^{2m} F_k z^k$ is analytic on D, and

$z^mQ^*(\frac{1}{z}) = Q^{-1}(z)z^mF(z)$, from which it follows that $z^mQ^*(\frac{1}{z})$ is also analytic on D.

So we have $z^mQ^*(\frac{1}{z})$ is analytic on $\mathbb{C}$, and $\sum_{k=1}^{\infty} C^*_{k+m}(\frac{1}{z})^k = 0$. □

III. The solvability conditions

We are now able to present the necessary and sufficient conditions under which the matrix equation (2) has a solution. The result reads as follows

Theorem 5

Equation (2) has a solution $X > 0$ if and only if the following two conditions are satisfied:

i) $\psi(z) := I + zA + \frac{1}{z}A^\top \geq 0 \quad \forall z \in \partial D$

ii) det $\psi(z)$ is different from zero

Proof

"⇒" Suppose $X > 0$ is a solution. Then $(I + zAX^{-1})X(I + \frac{1}{z}X^{-1}A^\top) = X + AX^{-1}A^\top + zA + \frac{1}{z}A^\top = \psi(z)$.

From this relationship, the necessity of both conditions is obvious.

"⇐" From lemma 4 and theorem 3 we have that $\psi(z)$ can be factorized as $(Q_0 + Q_1z)(Q_0^* + Q_1^*\frac{1}{z})$, where $Q_0 + Q_1z$ is invertible $\forall z \in D$. So, in particular Q_0 is invertible.

Consequently $Q_0Q_0^* + Q_1Q_1^* = I$ and $Q_1Q_0^* = A$. Let $X := Q_0Q_0^*$. Then $X > 0$ and it is easily verifed that X satisfies equation (2). □

Next we show that the solution $Q_0Q_0^*$ mentioned in the proof above is the largest solution for the matrix equation (2). That is, for any other solution X of (2) we have that $X \leq Q_0Q_0^*$. A direct consequence of this observation is that the algorithm presented in theorem 3 can be used to calculate the largest solution of (2) exactly.

Theorem 6

Let $(Q_0 + Q_1z)(Q_0^* + Q_1^*\frac{1}{z})$ be the factorization of $\psi(z)$, obtained via the algorithm presented in the proof of theorem 3. Then $X_L := Q_0Q_0^*$ is the largest solution of the equation $X + AX^{-1}A^\top = I$ (2). Moreover, $X = X_L$ is the only positive definite solution of (2) for which $X + zA$ is invertible for all $z \in D$.

Proof

Let $X > 0$ be any solution of (2) and $X^{\frac{1}{2}} > 0$ be the square root of X. Then, $(X^{\frac{1}{2}} + zAX^{-\frac{1}{2}})(X^{\frac{1}{2}} + \frac{1}{z}X^{-\frac{1}{2}}A^{\top}) = \psi(z)$.

So, $U(z) := (Q_0 + Q_1 z)^{-1}(X^{\frac{1}{2}} + zA^{\top}X^{-\frac{1}{2}})$ is analytic in D. Since $U(z)U^*(\frac{1}{z}) = I$, and I is integrable over ∂D, the same argument as in the proof of Lemma 4 shows that $U_{ij}(z)$ has no poles on ∂D, so $U(z)$ is analytic on a disk around zero with a radius $R > 1$.

So we can write $U(z) = \sum_{i=0}^{\infty} U_i z^i$ for all $z \in \mathbb{C}$ with $|z| < R$ and consequently $U^*(\frac{1}{z}) = \sum_{i=0}^{\infty} U_i (\frac{1}{z})^i$ for all $z \in \mathbb{C}$ with $|z| > \frac{1}{R}$.

Substitution of the power series expansion for $\frac{1}{R} < |z| < R$ yields:

$$I = U(z)U^*(\tfrac{1}{z}) = \left[\sum_{i=0}^{\infty} U_i z^i\right]\left[\sum_{i=0}^{\infty} U_i^*(\tfrac{1}{z})^i\right] = \sum_{i=0}^{\infty} U_i U_i^*.$$

So, in particular we have $U_0 U_0^* \le I$.

It is easily verified that $Q_0 U_0 = X^{\frac{1}{2}}$. Therefore, $X = Q_0 U_0 U_0^* Q_0^* \le Q_0 Q_0^*$. So, $X_L = Q_0 Q_0^*$ is the largest solution of (2). By Lemma 4 we see that $U(z)$ is invertible for all $z \in D$ if and only if $U(z) = U_0$, that is to say $U_0 U_0^* = I$. Since $X + zA = (Q_0 + Q_1 z)U(z)X^{\frac{1}{2}}$ this implies the last statement of the theorem.

□

We shall also show how to calculate the smallest solution for the matrix equation (2) by the same algorithm. It will turn out that this is easy when A is invertible, whereas we have to transform the equation a little bit in case A is singular. The clue lies in the relationship between solutions of the equation $X + AX^{-1}A^{\top} = I$ (2) and the equation $Y + A^{\top}Y^{-1}A^* = I$ (1) in case A is invertible.

Theorem 7

Let A be invertible.

Then X solves (2) iff $Y = I - X = AX^{-1}A^{\top}$ solves (1). X_L is the largest solution of (2) iff $Y_S = I - X_L$ is the smallest solution of (1).

Moreover, $Y = Y_S$ is the only positive definite solution of (1) for which $Y + zA^T$ is invertible for all $z \in D^c\backslash\partial D$.

Proof

The first two statements follow by direct substitution.
The last statement is a direct consequence of the fact that

$$Y + zA^T = AX^{-1}A^T + zA^T = z(\frac{1}{z}A + X)X^{-1}A^T. \qquad \square$$

Remark 8

If A is singular, we can obtain the smallest solution of equation (2) by the following reduction (actually this reduction was basically treated by Engwerda in (1990)).

There is an orthogonal transformation S such that $SAS^* = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix}$ where A_{11} is a square matrix. Note that X solves (2) iff $SXS^* + SAS^*\, SX^{-1}S^*\, SA^TS^* = I$. From this it follows easily that

$$X \text{ solves (2) iff } SXS^* = \begin{pmatrix} X_{11} & 0 \\ 0 & I \end{pmatrix} \text{ where } X_{11} > 0 \text{ and } X_{11} \text{ solves}$$

$$X_{11} + A_{11}X_{11}^{-1}A_{11}^T = I - A_{12}A_{12}^T \tag{3}$$

Let $P := I - A_{12}A_{12}^T$. If X_{11} solves (3), then $P \geq X_{11} > 0$. In that case, let $Z := P^{-\frac{1}{2}}X_{11}P^{-\frac{1}{2}}$.
Then we have:
X solves (2) iff $P > 0$ and Z solves $Z + (P^{-\frac{1}{2}}A_{11}P^{-\frac{1}{2}})Z^{-1}(P^{-\frac{1}{2}}A_{11}^TP^{-\frac{1}{2}}) = I$ (4)
and it is also clear that X is the smallest solution of (2) iff $P > 0$ and Z is the smallest solution of (4).
If A_{11} is singular, then also $P^{-\frac{1}{2}}A_{11}P^{-\frac{1}{2}}$ is singular, and we can repeat the process of reduction. After repeating a finite number of times, we end up with $\tilde{A}_{11}$ invertible or $\tilde{A}_{11}$ the 1×1 zero-matrix. In the first case, theorem 7 tells us how to calculate the smallest solution, in the second case there is only one solution for the reduced equation.
Furthermore, we have that $Z + zP^{-\frac{1}{2}}A_{11}P^{-\frac{1}{2}}$ is invertible iff $X_{11} + zA_{11}$ is invertible iff $\begin{pmatrix} X_{11} & 0 \\ 0 & I \end{pmatrix} + z\begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix}$ is invertible iff $X + zA$ is invertible.

Therefore, we can generalize the last statement of Theorem 7, to be valid also in case A is singular. When X_S is the smallest solution of (2), then $X = X_S$ is the only solution of (2) for which $X + zA$ is invertible for all $z \in D^c \backslash \partial D$.

Theorem 9

Suppose that equation $X + AX^{-1}A^T = I$ (2) has a positive definite symmetric solution X.

Then this equation has a largest and a smallest solution, X_L and X_S respectively. Moreover, X_L and X_S are real and

- $X = X_L$ is the only solution for which $X + zA$ is invertible for all $z \in D$
- $X = X_S$ is the only solution for which $X = zA$ is invertible for all $z \in P^c \backslash \partial D$.

Proof

The only thing that remains to be proved is that X_L and X_S are real.

Note that: X solves (2) iff $\bar{X}$ solves (2); and $X + zA$ is invertible iff $\bar{X} + \bar{z}A$ is invertible.

Hence we conclude from the two last statements of the same theorem that the smallest and the largest solution are real. □

We conclude this section by showing that in general the set of solutions of equation (1) does not satisfy the partial ordering relationship "≥".

Example 10

Consider equation (1) with $a = \frac{\sqrt{3}}{4} I$. Then, $X_1 = \frac{3}{4} I$, $X_2 = \begin{bmatrix} 3/4 & 0 \\ 0 & 1/4 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1/4 & 0 \\ 0 & 3/4 \end{bmatrix}$ and $X_4 = \frac{1}{4} I$ are solutions for this equation. Moreover, it is obvious that neither $X_2 \geq X_3$ nor $X_3 \geq X_2$ holds. □

IV. Two recurrence equations

In the previous section we saw that whenever our matrix equation (1) has a solution, then it has automatically a largest and smallest solution, denoted by X_L and X_s, respectively. Moreover, we presented an algorithm to calculate these solutions X_L and X_s. In this section we show that these solutions can also be obtained via a recurrence equation. The advantage of these recurrence equations are that they are directly related to the original equation (1) and very simple to implement. Whether both solutions X_L and X_s are obtained from these equations in a numerically reliable way remains at this point an open question, and therefore a problem for future research.

We will see that the algorithm to calculate the largest (real) solution X_L is the most easiest one. To calculate X_s, we will in fact implement the dual algorithm for calculating X_L. However, since the dual algorithm only works if matrix A is invertible, in general we first have to apply some transformations, we already mentioned in remark 8, to equation (1) before we can use this dual algorithm.

The algorithm to calculate X_L is as follows.

<u>Algorithm 11</u>

Consider the recurrence equation

$$\begin{aligned} X_0 &= I \\ X_{n+1} &= I - A^T X_n^{-1} A \end{aligned} \tag{3}$$

If equation (1) has a solution $X > 0$, then $X_n \rightarrow X_L$.

<u>Proof</u>

We show that X_n is a monotonically decreasing sequence that is bounded from below, and thus converges. To that end we first show by induction that $X_k \geq X \; \forall k \in \mathbb{N}$. Note that as a consequence then $X_k > 0$ for any $k \in \mathbb{N}$, and since X is an arbitrarily chosen solution of (1), we have that $X_k \geq X_L$ $\forall k \in \mathbb{N}$.

For $k = 0$, the statement is trivially satisfied. So, assume that the statement holds for $k = n$. Then, $X_{k+1} - X = A^T(X^{-1} - X_k^{-1})A \geq 0$, since $X_k \geq X > 0$, which completes the first part of our argument.

Next we show that X_k is a monotonically decreasing sequence. The proof is quite similar to the previous argument. first, consider $X_0 - X_1$. From the definition of X_n we have that $X_0 - X_1 = I - (I - A^T X_0^{-1} A) = A^T A \geq 0$. So, the statement holds for $k = 0$. Next, assume that $X_k - X_{k+1} \geq 0$ for $k = n$. Then, using the induction argument and the fact that $X_k > 0$ for any k, $X_{n+1}-X_{n+2} = A^{\top}(X_{n+1}^{-1}-X_n^{-1})A \geq 0$. So, the induction argument is complete with this. Combination of both results yields that $X_n \to X_L$. □

To calculate X_s, the following algorithm can be used.

Algorithm 12

If equation (1) has a solution $X > 0$, then the next algorithm gives us the smallest (real) solution X_s of this equation

1.i) If A is invertible then go to part 2 of this algorithm

ii) Else apply an orthogonal transformation T such that
$$A = T^{\top}\begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix}T.$$

iii) If $A_{11} = 0$, then $X_S := T^{\top}\begin{bmatrix} I-A_{21}^{\top}A_{21} & 0 \\ 0 & I \end{bmatrix}T$ and the algorithm stops.

iv) Else $X_S := T^{\top}\begin{bmatrix} Y_S & 0 \\ 0 & I \end{bmatrix}T$, with $Y_S > 0$ the smallest solution of equation (1), where A is replaced by $(I-A_{21}^{\top}A_{21})^{-\frac{1}{2}}A_{11}(I-A_{21}^{\top}A_{21})^{-\frac{1}{2}}$. Now return to i).

2. Consider the recurrence equation
$$X_0 := AA^{\top}$$
$$X_{n+1} := A(I-X_n)^{-1}A^{\top}$$
Then $X_n \to X_s$.

Proof

Part 1 of the algorithm follows from remark 8. So what is left to be proved is that part 2 of the algorithm works under the assumption that A is invertible. Using theorem 7 and algorithm 11, this is however straightforward proved now, and therefore omitted. □

V. The number of solutions

In this section we discuss the number of solutions for equation (2). We start by a result which tells us under which conditions there is exactly one solution.

Theorem 13

Equation (1) has exact one solution if and only if the following three conditions are satisfied

i) det $\psi(z)$ is different from zero

ii) $\psi(z) \geq 0 \ \forall z \in \partial D$

iii) either all roots of det $\psi(z)$ are on ∂D or det $\psi(z)$ has no roots.

Proof

The first two conditions were already proved in theorem 5. So what is left to be shown is that condition iii) is both necessary and sufficient for the existence of exactly one solution.

Now, $\psi(z) = (X + zA)X^{-1}(X + \frac{1}{z} A^\top)$ whenever X is a solution of (2). Therefore det $\psi(z)$ = det X^{-1}.det$(X + zA)$.$\overline{\text{det}(X + \frac{1}{\bar{z}} A)}$. So, if det $\psi(z)$ has only roots on ∂D, the same holds for det$(X + zA)$, i.e. $X + zA$ is invertible $\forall z \in D$. According to theorem 9 equation (2) has thus only one solution.

On the other hand, if equation (2) has only one solution this solution is automatically both the smallest and largest solution. Thus, using theorem 9, det$(X + zA)$ is invertible $\forall z \notin \partial D$. Consequently, det $\psi(z)$ has no roots outside ∂D.

□

The next theorem gives an exact characterization of the number of solutions if matrix A in (1) is normal (i.e. $AA^\top = A^\top A$). To prove this result we first give two lemma's.

Lemma 14

Consider the equation $X + X^{-1} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$, with $A_{22} \in \mathbb{R}^{k\times k}$. Assume that the multiplicity of all eigenvalues of A and which appear in A_{22} is one.

Then, $X = \begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}$, where $X_{22} \in \mathbb{R}^{k\times k}$.

Proof

Let λ be an eigenvalue of A_{22} and x the corresponding eigenvector. Then, by definition, we have $Ax = \lambda x$.

On the other hand we have that

$$X^2x - AXx + x = 0$$

$$X^2x - \lambda Xx + x = 0.$$

From which we deduce $AXx = \lambda Xx$.
So, Xx is an eigenvector of A corresponding with the eigenvalue λ. But, since the multiplicity of all eigenvalues of A that appear in A_{22} is one, we consequently have $Xx = cx$ for some $c \in \mathbb{C}$. Or, stated differently, x is an eigenvector of X too. This yields the stated result. □

<u>Lemma 15</u>
Let $X := \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} > 0$, where $X_{11} \in \mathbb{R}^{m\times m}$ and $X_{22} \in \mathbb{R}^{k\times k}$.
Then, $X^{-1} = \begin{bmatrix} * & * \\ * & X_{22}^{-1} \end{bmatrix}$ iff $X_{12} = 0$. □

<u>Proof</u>
Since $X > 0$, we have that (see e.g. Kailath (pp. 656)) $X^{-1} = \begin{bmatrix} * & * \\ * & \Delta^{-1} \end{bmatrix}$, where $\Delta = X_{22} - X_{12}^T X_{11}^{-1} X_{12}$. So, $\Delta = X_{22}$ iff $X_{12}^T X_1^{-1} X_{12} = 0$. Since $X_1 > 0$, the stated result is obvious now. □

<u>Theorem 16</u>
Let A be normal and $r(A) \le \frac{1}{2}$.
Define S_A by: $S_A := \{\lambda_i \in \sigma(A) | \lambda_i = 0 \vee |\lambda_i| = \frac{1}{2}\}$, and C_A: by $C_A := \{\lambda_i \in \sigma(A) | \text{Im } \lambda_i \ge 0\} \backslash S_A$.
Let q be the number of elements in C_A.
Then, equation (1) has
1) 2^q solutions iff $C_A = \emptyset$ or the multiplicity of all eigenvalues of A that are in C_A equals one.
2) infinitely many solutions iff there exists an eigenvalue of A in C_A with multiplicity greater than one.

<u>Proof</u>
Engwerda showed in (1990) that if A is normal the condition $r(A) \le \frac{1}{2}$ is both necessary and sufficient for the existence of a solution for equation (1). So, we concentrate here on the number of solutions. Since A is normal

there exists an orthogonal matrix U such that $U^{\top}AU = \mathrm{diag}(\Lambda_i)$, where each Λ_i is either a scalar or a real 2×2 matrix of the form $\Lambda_i = \begin{bmatrix} \lambda_i & \mu_i \\ -\mu_i & \lambda_i \end{bmatrix}$ with $\mu_i \neq 0$.

Consequently X solves equation (1) iff $Z(= U^{\top}XU) > 0$ solves

$$Z + \Lambda^{\top}Z^{-1}\Lambda = I, \text{ where } \Lambda := \mathrm{diag}(\Lambda_i). \qquad (1')$$

So, to determine the number of solutions for equation (1) it suffices to calculate the number of solutions for this equation (1'). We show now by induction on k that the theorem holds for equation (1') if n = 2k-1 and n = 2k. First the case k = 1.

If n = 1, elementary calculation shows that equation (1') has exact one solution iff $\lambda \in S_A$, and that it has two solutions iff $0 < |\lambda| < \frac{1}{2}$.

To study the case n = 2, we discern three cases:

i) $\Lambda = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$, with $\mu \neq 0$

ii) $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, with $\lambda_1 \neq \lambda_2$

iii) $\Lambda = \lambda I$.

Case i) corresponds with the case that A has complex eigenvalues, case ii) with the case of two different eigenvalues and case iii) with the case that A has eigenvalues with a multiplicity larger than one.

Now, consider case i). Then, equation (1') reduces to

$Z + U\begin{bmatrix} \lambda-j\mu & 0 \\ 0 & \lambda+j\mu \end{bmatrix}U^{*}Z^{-1}U\begin{bmatrix} \lambda+j\mu & 0 \\ 0 & \lambda-j\mu \end{bmatrix}U^{*} = I$, where $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}$. So, whenever $Z > 0$ ($Z \in \mathbb{R}^{2\times 2}$) solves (1), then $\tilde{Z} := U^{*}ZU > 0$ ($\tilde{Z} \in \mathbb{C}^{2\times 2}$) solves $\tilde{Z} + \begin{bmatrix} \lambda-j\mu & 0 \\ 0 & \lambda+j\mu \end{bmatrix}\tilde{Z}^{-1}\begin{bmatrix} \lambda+j\mu & 0 \\ 0 & \lambda-j\mu \end{bmatrix} = I$. Therefore, we first study the number of solutions of this equation. Introducing $\tilde{Z} =: \begin{bmatrix} z_{11} & z_{12} \\ \bar{z}_{12} & z_{22} \end{bmatrix}$, we can rewrite this equation as

$$z_{11} + \frac{1}{\mathrm{Det}}(\lambda^2 + \mu^2)z_{22} = 1 \qquad (a)$$

$$z_{12} - \frac{1}{\mathrm{Det}}(\lambda - j\mu)^2\bar{z}_{12} = 0 \qquad (b)$$

$$z_{22} + \frac{1}{\text{Det}}(\lambda^2 + \mu^2)z_{11} = 1 \qquad \text{(c)},$$

where $\text{Det} = z_{11}z_{22} - z_{12}\bar{z}_{12} > 0$. Elementary calculation shows that equation (b) implies that $z_{12} = 0$ (since $\mu \neq 0$).

Now, introduce $r := \frac{1 + \sqrt{1-4(\lambda^2+\mu^2)}}{2}$ and $s := \frac{1 - \sqrt{1-4(\lambda^2+\mu^2)}}{2}$. Then it is obvious that whenever $\lambda^2+\mu^2 \neq \frac{1}{4}$, the equation has four solutions, namely,

$$\tilde{Z}_1 := rI;\ \tilde{Z}_2 := \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix};\ \tilde{Z}_3 := \begin{bmatrix} s & 0 \\ 0 & r \end{bmatrix} \text{ and } \tilde{Z}_4 := sI.$$

If $\lambda^2+\mu^2 = \frac{1}{4}$, there is only one solution: $\tilde{Z} = \frac{1}{z} I$. So, if $r(A) < \frac{1}{2}$ equation (1) has at most four solutions $Z_i = U\tilde{Z}_iU_i^*$, $i = 1,\ldots,4$. However, it is easily verified that only Z_1 and Z_4 are real solutions. So for this case we have the result that whenever $r(A) < \frac{1}{2}$ equation (1') has two solutions, and if $r(A) = \frac{1}{2}$ it has one solution.

Next, consider case ii). Then, it is shown in a similar way that whenever $\lambda_i \notin S_A$, $i = 1,2$, the equation has exact four solutions. Moreover, it is easily verified that if both λ_1 and λ_2 are in S_A the equation has only one solution, and if either λ_1 or λ_2 is in S_A (but not both), the equation has two solutions.

Finally, consider case iii). Elementary calculation shows that for all

$$a \in \left[\frac{1 - \sqrt{1-4\lambda^2}}{2}, \frac{1 + \sqrt{1-4\lambda^2}}{2}\right],\quad X := \begin{bmatrix} 1-a & \sqrt{-a^2+a-\lambda^2} \\ \sqrt{-a^2+a-\lambda^2} & a \end{bmatrix} \text{ satisfies}$$

equation (1'), whenever $\lambda \notin S_A$. If either $\lambda = 0$, $\frac{1}{2}$ or $-\frac{1}{2}$ it is again easily shown that the equation has only one solution. This completes the initialization step of the induction proof.

Next assume that the theorem holds for $n = \bar{k}$.

So, consider equation (1') with $\Lambda = \begin{bmatrix} \bar{\Lambda} & 0 \\ 0 & \Lambda_{\bar{k}+1} \end{bmatrix}$, where $\bar{\Lambda} \in \mathbb{R}^{2\bar{k}\times 2\bar{k}}$.

First assume that $\sigma(\Lambda_{\bar{k}+1}) \subset S_A$. Then simple calculations show that the number of solutions of this equation coincides with the number of solutions of equation (1') if we consider $\Lambda = \begin{bmatrix} \bar{\Lambda} & 0 \\ 0 & \Lambda'_{\bar{k}+1} \end{bmatrix}$, where $\Lambda'_{\bar{k}+1}$ is obtained from $\Lambda_{\bar{k}+1}$ by dropping all its zero eigenvalues. Therefore, without loss of generality, we can assume that $0 \notin \sigma(\Lambda_{\bar{k}+1})$. Next write $Z =: \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^{T} & Z_{22} \end{bmatrix}$, where $Z_{11} \in \mathbb{R}^{2\bar{k}\times 2\bar{k}}$. Then in particular the equation $Z_{22} + \Lambda_{\bar{k}+1}(Z_{22}-Z_{12}^{T}Z_{11}^{-1}Z_{12})^{-1}\Lambda_{\bar{k}+1} = I$, with $\sigma(\Lambda_{\bar{k}+1}) \subset \{\lambda \mid |\lambda| = \tfrac{1}{2}\}$, must be solvable. Using lemma 15 it is easily seen that this equation has a solution iff $Z_{12} = 0$, and that in that case the solution is uniquely determined.

Concluding, we have that if $\sigma(\Lambda_{\bar{k}+1}) \subset S_A$ the number of solutions of equation (1'), with $\Lambda := \begin{bmatrix} \bar{\Lambda} & 0 \\ 0 & \Lambda_{\bar{k}+1} \end{bmatrix}$ coincides with the number of solutions with $\Lambda := \bar{\Lambda}$.

If only one eigenvalue of $\sigma(\Lambda_{\bar{k}+1}) \in S_A$, then the same argument shows that the number of solutions coincides with the number of solutions with $\Lambda := \begin{bmatrix} \bar{\Lambda} & 0 \\ 0 & \lambda \end{bmatrix}$, where $\lambda \notin S_A$.

So, we can concentrate on the number of solutions for equation (1') if $\sigma(\Lambda_{n+1}) \cap S_A = \emptyset$.

First consider the case $\sigma(\Lambda_{\bar{k}+1}) \cap \sigma(\Lambda) \neq \emptyset$.

If this intersection contains a real element, then the same argument we used for $\bar{k} = 1$, case iii), shows that there are infinitely many solutions. In case the intersection contains a complex eigenvalue, straightforward extension of the next example shows that there are infinitely many solutions too. In this example, we take $\Lambda_1 = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$ with $\mu \neq 0$, and consider the equation

$$Z + \begin{bmatrix} \Lambda_1^{T} & 0 \\ 0 & \Lambda_1^{T} \end{bmatrix} Z^{-1} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_1 \end{bmatrix} = I.$$

Straightforward calculation shows that

$$\forall a \in \left[\frac{1 - \sqrt{1-4(\lambda^2+\mu^2)}}{2}, \frac{1 + \sqrt{1-4(\lambda^2+\mu^2)}}{2} \right], \text{ with } b := \sqrt{-a^2+a-4(\lambda^2+\mu^2)},$$

$$Z = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ b & 0 & 1-a & 0 \\ 0 & b & 0 & 1-a \end{bmatrix} \text{ satisfies the equation.}$$

So, the case that remains to be solved is if $\sigma(\Lambda) \cap \sigma(\Lambda_{\bar{k}+1}) = \emptyset$, where $\Lambda_{\bar{k}+1}$ contains no eigenvalues with a multiplicity two. Then lemma 14 shows that all solutions of equation (1') have the form $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$, where $X_1 \in \mathbb{R}^{2\bar{k}\times 2\bar{k}}$. Using the induction argument and the result for $\bar{k} = 1$ yields then the stated results. □

Corollary 17

Let A be normal.

Then, $\det \psi(z)$ either has no roots or all its roots are on ∂D iff $\sigma(A) \subset S_A$. □

That the condition $\sigma(A) \subset S_A$ is sufficient to conclude that there is exact one solution for equation (1) can also be obtained by a more direct proof. Note that whenever $\lambda \in S_A\backslash\{0\}$ is an eigenvalue of A, then $\det \psi(-2\bar{\lambda}) = 0$. In particular this shows that the condition is also sufficient to conclude that there is exact one solution if we drop the normality assumption on A.

VI. Conclusion

In this paper we considered the question under which conditions the quadratic matrix equation (1) has a real positive definite solution. In general, this equation will have more than one solution. In fact there may be infinitely many solutions. So questions arise like whether there exists a smallest and largest solution, respectively, and what will be the relationship between matrix A and the number of solutions. We derived algorithms for calculating the largest and smallest solution and stated conditions under which the solution of the equation is uniquely determined. Moreover, we gave an exact characterization of the number of solutions in case matrix A is normal. In fact, in this latter case, we presented all solutions for the equation (as long as there are only a finite number of solutions to it).

To analyze the general case, it seems to us that the factorizational approach still offers many possibilities to solve questions like the relationship between matrix A and the number of solutions, and the stability of the largest and smallest solution w.r.t. small perturbations of A (see Ran et al. (1989) for formal definitions on this subject). This remains a topic for future research, as well as the question whether it is possible to find methods which more quickly yield an appropriate factorization of $\psi(z)$ into $Q(z)Q^*(\frac{1}{z})$.

We conclude the paper by stating a refinement of corollary 16 in Engwerda (1990), which treats solvability conditions for the optimal control problem:

$$\min_{u[0,.]} \lim_{N\to\infty} J_N \text{ w.r.t. } x(k+1) = Ax(k) + Bu(k);\ x(.),u(.) \in \mathbb{R}^n;\ x(0) = x.$$

with the additional constraint that $\lim_{N\to\infty} x(N) = 0$ and B is invertible where

$$J_N = \sum_{k=0}^{N-1} \{x^T(k)Qx(k) + u^T(k)Ru(k)\}$$

and both Q and R are symmetric.

We have the next result:

Example 18

Let $M := RB^{-1}A'B$, and $N := B^TA'^TB^{-T}RB^{-1}A'B + R + B^TQB$. Then this problem has a solution if the following conditions are satisfied:

1) $N > 0$

2) i) $\det \psi(z) \neq 0$, where $\psi(z) = I + zA + \frac{1}{z} A^T$ and $A = N^{-\frac{1}{2}}MN^{-\frac{1}{2}}$.

   ii) $\psi(z) \geq 0\ \forall z \in \partial D$

3) $r(X_LN^{\frac{1}{2}}MN^{\frac{1}{2}}) < 1$, where X_L is the largest solution of equation (1).

Moreover an optimal control strategy is $u(k) = -(I - N^{\frac{1}{2}}X_LN^{\frac{1}{2}}RB^{-1})A'x(k)$. □

Appendix

Proof of lemma 1:

From the factorization theorem we have that f(z) can be written as

$$c\prod_{i=1}^{\ell}(z-\beta_i)^{p_i}, \text{ where } \beta_i \in \mathbb{C},\ p_i \in \mathbb{Z}$$

Since $f(z) = \overline{f(z)}$ for all $z \in \mathbb{R}$, we can write: $f(z) = c\prod_{i=1}^{k}(z-\alpha_i)^{n_i}(z-\bar{\alpha}_i)^{n_i}(z-a_i)^{m_i}$, where α_i in the open upper half plane of $\mathbb{C}$, $a_i \in \mathbb{R}$, $n_i, m_i \in \mathbb{Z}$, and it is also obvious that $c \in \mathbb{R}$. Note that $(z-\alpha_i)(z-\bar{\alpha}_i) \geq 0$ for all $z \in \mathbb{R}$. So, from the assumption that $f(z) \geq 0$ for all $z \in \mathbb{R}$ we obtain now immediately that the sign of $c\prod_{i=1}^{k}(z-a_i)^{m_i}$ must be positive on $\mathbb{R}$. Consequently c must be a positive number and the m_i, $i = 1,\dots,k$ must be even numbers. □

Proof of lemma 2:

Consider the transformation $T(u) := \frac{j-u}{j+u}$ which maps the open upper-half-plane of $\mathbb{C}$ onto the open unit disk and the real line into the unit circle. As a mapping from $\mathbb{C}\backslash\{-j\}$ to $\mathbb{C}\backslash\{-1\}$, T satisfies the rule: $T(-\bar{u}) = \overline{T(u)}$ $(u \neq -j)$.

Its inverse transformation is $T^{-1}(z) = j\,\frac{1-z}{1+z}$, and T^{-1} satisfies the rule: $T^{-1}(\frac{1}{z}) = -T^{-1}(z)$.

Next consider $g(u) := f(T(u))$. g is a rational function different from zero, and $g(u) \geq 0$ for all $u \in \mathbb{R}$. So according to Lemma 1 we have that $g(u) = c\prod_{i=1}^{k}(u-\alpha_i)^{n_i}(u-\bar{\alpha}_i)^{n_i}(u-a_i)^{2m_i}$ for some α_i in the open upper-half-plane of $\mathbb{C}$, $a_i \in \mathbb{R}$, $n_i, m_i \in \mathbb{Z}$ and $c > 0$.

Now $f(z) = g(T^{-1}(z))$. Using the rule: $T^{-1}(\frac{1}{z}) = -T^{-1}(z)$, we can write:

$$f(z) = c\prod_{i=1}^{k}(T^{-1}(z)-\alpha_i)^{n_i}(T^{-1}(z)-\bar{\alpha}_i)^{n_i}(T^{-1}(z)-a_i)^{2m_i} =$$

$$= c\prod_{i=1}^{k}(T^{-1}(z)-\alpha_i)^{n_i}(T^{-1}(z)-a_i)^{m_i}\prod_{i=1}^{k}(-T^{-1}(\tfrac{1}{z})-\bar{\alpha}_i)^{n_i}(-T^{-1}(\tfrac{1}{z})-a_i)^{m_i}$$

Now denote $\prod_{i=1}^{k} (T^{-1}(z) - \alpha_i)^{n_i}(T^{-1}(z) - a_i)^{m_i}$ by $t(z)$ and $\prod_{i=1}^{k} (-T^{-1}(\frac{1}{z}) - \bar{\alpha}_i)^{n_i}(-T^{-1}(\frac{1}{z}) - a_i)^{m_i}$ by $p(\frac{1}{z})$.

Let us take a look at the poles and zeros of $t(z)$ and $p(w)$.
The poles/zeros of $t(z)$ are respectively: -1 with multiplicity $-\sum_{i=1}^{k} (n_i+m_i)$ and the union of $T(\alpha_i) \in D$ with multiplicity n_i and $T(a_i) \in \partial D$ with multiplicity m_i. Whereas the poles/zeros of $p(w)$ are respectively: -1 with multiplicity $-\sum_{i=1}^{k} (n_i+m_i)$ and the union of $T(-\bar{\alpha}_i) = \overline{T(\alpha_i)} \in D$ with multiplicity n_i and $T(-a_i) = \overline{T(a_i)} \in \partial D$ with multiplicity m_i.

So, the poles/zeros of p are conjugate of poles/zeros of t and have the same multiplicity.
Therefore, $p(z) = \tilde{c}t^*(z)$, for certain constant $\tilde{c}$ and $f(z) = c\tilde{c}t(z)t^*(\frac{1}{z})$.
Because $f(z) \geq 0$ $\forall z \in \partial D$, $c\tilde{c}$ must be positive, so $f(z) = q(z)q^*(\frac{1}{z})$ where $q(z) = \sqrt{c\tilde{c}}t^*(\frac{1}{z})$.
Finally note, that all poles and zeros of q are in the complement of D. □

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