On the existence of a positive definite solution of the matrix equation X+AXA=I

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RESEARCH MEMORANDUM

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ON THE EXISTENCE OF A POSITIVE DEFINITE SOLUTION OF THE MATRIX EQUATION $X + A^{T}X^{-1}A = I$

Jacob C. Engwerda

FEW 397

On the existence of a positive definite solution of the matrix equation $X + A^T X^{-1} A = I$

by

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Abstract

In this paper the question is raised under which conditions on the real (square) matrix A, the matrix equation $X + A^T X^{-1} A = I$ has a real symmetric positive definite solution X. Both necessary and sufficient solvability conditions on A are derived. Moreover, we give an algorithm to calculate the solution. For a number of special cases we also present an analytic solution.

I. Introduction

As already mentioned in the abstract the central issue in this paper is to find solvability conditions for the existence of a positive definite solution X of the matrix equation $X + A^T X^{-1} A = I$. This problem can be viewed as a natural extension of giving solvability conditions for the scalar problem $x + \frac{a^2}{x} = 1$. From calculus we know that the existence of the real square root $\sqrt{1 - 4a^2}$ plays here an important role. We will see that this condition generalizes straightforwardly to the matrix case, if A has the additional property that it is normal (i.e. $A^T A = AA^T$). However, if A has not this additional property things become more complicated.
We show that the general problem has a solution if and only if a related recursive algorithm converges to a positive definite solution. Moreover we use this algorithm to prove that, provided matrix A satisfies a certain condition, the matrix equation is solvable and to calculate a solution numerically.
Seperately, we derive a number of necessary conditions and show by means of a counterexample that these are in general not sufficient.
The paper is organized as follows. First in section 2 we introduce some notation and study the general problem together with the recursive algorithm. Then, we derive a number of necessary conditions. Section 4 contains a number of special cases in which a solution exists. Before we discuss the main results in section 6, we give in section 5 an example of this equation in the field of optimal control theory.

II. The general problem

Mathematically, the problem analyzed in this paper is to find conditions under which:

$$\exists\, X > 0:\; X + A^T X^{-1} A = I, \qquad (1)$$

where X and I are real square $\bar{n} \times \bar{n}$ matrices.

Here $X > 0$ means that X is symmetric positive definite, denotes A^T the transpose of A and is I the identity matrix. In the sequal also the notation $\geq$ is used to indicate that a matrix is symmetric semi-positive definite and is $A > B$ used as a different notation for $A - B > 0$. Moreover, Ker A denotes the kernel of A and Im A its image.
Further on we show that this problem has a solution if and only if the next recursion problem is solvable.

$$\forall n \in \mathbb{N} \text{ is } X_n > AA^T \text{ in}$$

$$X_0 = I \tag{2}$$

$$X_{n+1} = I - A^T X_n^{-1} A$$

To prove this result we start with some intermediate results which are interesting in itself. The first thing we prove is that in fact it suffices to solve problem (1) for invertible matrices. We show that in case A is not invertible, problem (1) can be reduced to a similar problem with an invertible A matrix. How this can be accomplished is the contents of theorem 1. Its proof contains an algorithm which will be used later on again.

Theorem 1
If we can solve problem (1) whenever matrix A is invertible, then we can solve this problem without this invertibility restriction too.

Proof
We prove this theorem by reducing the problem to a similar problem of lower dimension.
The reduction is achieved via the next algorithm
(i) If A is invertible then the algorithm is finished.
(ii) Else, there exists an orthogonal transformation T such that

$$A = T^T \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix} T.$$

Consequently problem (1) has a solution if and only if (iff.) the next problem is solvable

$$\exists\, Y > 0:\ Y = \begin{bmatrix} A_{11}^{T} & A_{21}^{T} \\ 0 & 0 \end{bmatrix} Y^{-1} \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix} = I;$$

which is the case iff.

$$\exists\, Z > 0:\ Z + A_{11}^{T} Z^{-1} A_{11} = I - A_{21}^{T} A_{21},\ \text{where } G := I - A_{21}^{T} A_{21} > 0.$$

Now define $A := G^{-\frac{1}{2}} A_{11} G^{-\frac{1}{2}}$. Then this problem can be rewritten into the original form (1), unless $A_{11} = 0$. If $A_{11} \neq 0$ we return to (i), otherwise $Z := I - A_{21}^{T} A_{21}$ and the algorithm stops. □

So, to solve problem (1) we could restrict us to invertible matrices. But from the algorithm it is clear that then the solvability conditions for non-invertible matrices become rather involved. For that reason we will not make this invertibility assumption w.r.t. A from the outset on. The next preparatory lemma gives a lowerbound for any solution to problem (1).

Lemma 2

If equation (1) has a solution X, then $X > AA^{T}$.

Proof

Rewriting (1) yields that

$$X = I - A^{T} X^{-1} A. \tag{i}$$

Since X is by assumption positive definite we immediately obtain from this equation that $X \leq I$. Consequently, $X^{-1} \geq I$ and thus $A^{T}A \leq A^{T}X^{-1}A = I - X \leq I$. From this inequality we conclude that

$$A^{T}A \leq I. \tag{ii}$$

From (i) we have, moreover, that X^{-1} equals $(I - A^T X^{-1} A)^{-1}$. Applying Schur's lemma (see e.g. Kailath (1980) pp. 656) yields that

$$X^{-1} = I - A^T(AA^T - X)^{-1}A. \tag{iii}$$

As $X^{-1} > I$, we obtain the inequality

$$A^T(X - AA^T)^{-1}A \geq 0.$$

So, $x^T(X - AA^T)^{-1} x > 0$ whenever $x \notin \mathrm{Ker}\, A$. Since $X - AA^T$ is a symmetric matrix, $x^T(X - AA^T) x$ will be positive too for any $x \notin \mathrm{Ker}\, A$. So what is left to be shown is that $x^T(X - AA^T)x > 0$ for any $x \in \mathrm{Ker}\, A$. This immediately results from (ii). For, let $x \in \mathrm{Ker}\, A$. Then, $x^T X x = x^T(I - A^T X^{-1} A)x = x^T I x \geq x^T AA^T x$. As $X - AA^T$ is invertible, it is clear that $X - AA^T$ is positive definite.

Corollary 3

If problem (1) has a solution X, then $I - AA^T - A^TA > 0$.

Proof

Since $X + A^T X^{-1} A = I$, we obtain by substitution of (iii) from lemma 2 that

$$X + A^TA - A^{T^2} (AA^T - X)^{-1}A^2 = I.$$

So $X - AA^T = I - AA^T - A^TA - A^{T^2}(X - AA^T)^{-1}A^2$.

Application of lemma 2 yields that

$$I - AA^T - A^TA = (X - AA^T) + A^{T^2}(X - AA^T)^{-1}A^2 > 0. \qquad \square$$

A similar result as lemma 1 holds w.r.t. problem (2).

Lemma 4

If problem (2) has a solution, then there exists a positive constant α such that $X_n > \alpha I \ \forall n \in \mathbb{N}$.

Proof

The proof is similar to the one of theorem 1.

(i) In case matrix A is invertible the above statement is trivial.

(ii) In case A is not invertible we decompose A again into $A =: T^{\top}\begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix} T$ and note that the algorithm of theorem 1 implies that the next algorithm has a solution $X' > 0$.

$$X'_0 = I$$

$$X'_{n+1} = I - A_{21}^{\top}A_{21} - A_{11}^{\top}X_n'^{-1}A_{11} \text{ where } X_n = T^{\top}\begin{bmatrix} X'_n & 0 \\ 0 & I \end{bmatrix}T.$$

Rewriting this equation yields

$$\exists\, Y > 0 \text{ in}$$

$$Y_0 = I$$

$$Y_{n+1} = I - A'' Y_n^{-1} A'', \text{ where } A'' := (I - A_{21}^{\top}A_{21})^{-\frac{1}{2}} A_{11}(I - A_{21}^{\top}A_{21})^{-\frac{1}{2}}.$$

If A_{11} is non-zero then we return to (i). In case $A_{11} = 0$ then it is clear that $X'_n = I - A_{21}^{\top}A_{21} > 0 \;\forall\, n$, and thus $X_n > \alpha I \;\forall n$ for some α too. Finally notice that this algorithm stops after at most $\bar{n}-1$ iterations, and that the nested solution $X = T_1^{\top}\begin{bmatrix} T_2^{\top}\,(\text{etc.})T_2 & 0 \\ 0 & I \end{bmatrix} T_1$ can always be estimated by αI for some α, which completes the proof. □

Using these two lemmas we can prove that problem (1) has a solution whenever problem (2) has one, and vice versa.

Theorem 5

Problem (1) has a solution iff. problem (2) is solvable.

Proof

"⇐" First we prove that X_n is a monotonically non-increasing sequence. This is proved by induction.

Note that the initialization step is trivially satisfied, for,

$$X_0 - X_1 = I - (I - A^T A) = A^T A \geq 0.$$

Now, let $X_n \leq X_{n-1}$. Then, since $X_n > 0$, we have that $X_n^{-1} \geq X_{n-1}^{-1}$. So $X_n - X_{n+1} = A^T(X_{n-1}^{-1} - X_n^{-1})A \geq 0$, which completes the induction step. Therefore X_n is a monotonically non-increasing sequence which is, according to lemma 4, bounded from below by some positive definite matrix. Consequently X_n converges to a positive definite limit which satisfies equation (1).

"⇒" We prove this part by induction.

According to corollary 3 we have that whenever problem (1) has a solution then $I - A^T A - AA^T$ is positive definite. Since $X_1 - AA^T = I - A^T A - AA^T$, this completes the first part of the proof. Now assume that $X_i - AA^T > 0 \ \forall i \leq n$.

Then it is easily seen by induction that $X_i - X = A^T(X^{-1} - X_{i-1}^{-1})A \geq 0 \ \forall i \leq n$. So in particular $X - AA^T \leq X_n - AA^T$. Application of this inequality yields that

$$\begin{aligned}
X_{n+1} - AA^T &= I - AA^T - A^T X_n^{-1} A \\
&= I - A^T(X_n - AA^T)^{-1}A - AA^T \\
&= (I + A^T(X_n - AA^T)^{-1}A)^{-1} - AA^T \\
&\geq (I + A^T(X - AA^T)^{-1}A)^{-1} - AA^T \\
&= I - A^T X^{-1} A - AA^T \\
&= X - AA^T \\
&> 0, \text{ which completes the proof.} \qquad \square
\end{aligned}$$

III. Necessary conditions

In this section we discuss a number of conditions on A that must be satisfied in order to solve the matrix equation. Moreover we show by means of a counterexample that these conditions are in general not sufficient to solve the problem.

We start this section again with a preliminary lemma. In this lemma, as well as in the rest of the paper, we use the notation r(A) to denote the spectral radius of matrix A (i.e. $\max_{\lambda_i}|\lambda_i|$, where λ_i are the eigenvalues of matrix A).

Lemma 6

Let P and Q be two arbitrary compatible matrices. Then, $r(P^TQ - Q^TP) \le r(P^TP + Q^TQ)$.

Proof

By elementary calculus we have that

$$r(P^TQ - Q^TP) = r\left[(P^T\ Q^T)\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} P \\ Q \end{bmatrix}\right].$$

Since r(AB) = r(BA) for any two compatible matrices, we have that

$$r\left[(P^T\ Q^T)\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} P \\ Q \end{bmatrix}\right] = r\left[\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} P \\ Q \end{bmatrix}(P^T\ Q^T)\right].$$

Now, $r(A) \le \|A\|_2$, where $\|.\|_2$ denotes the operator norm (i.e. the largest singular value of matrix A).

So

$$r\left[\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} P \\ Q \end{bmatrix}(P^T\ Q^T)\right] \le \left\|\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} P \\ Q \end{bmatrix}(P^T\ Q^T)\right\|_2 \le$$

$$\left\|\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\right\|_2\left\|\begin{bmatrix} P \\ Q \end{bmatrix}(p^T\ Q^T)\right\|_2.$$

As $\begin{bmatrix} P \\ Q \end{bmatrix}(P^T\ Q^T)$ is a normal matrix, and $\|A\|_2 = r(A)$ for any matrix A of this type, we can rewrite the above expression as follows:

$$r(P^T Q - Q^T P) \le \left\| \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} P \\ Q \end{bmatrix} (P^T \; Q^T) \right\|_2$$

$$= 1 \; r\left[\begin{bmatrix} P \\ Q \end{bmatrix} (P^T \; Q^T) \right]$$

$$= r\left[(P^T \; Q^T) \begin{bmatrix} P \\ Q \end{bmatrix} \right]$$

$$= r(P^T P + Q^T Q)\text{, which completes the proof.} \qquad \square$$

Theorem 7

Assume that problem (1) is solvable.

Then matrix A satisfies the following inequalities.

(i) $r(A) \le \frac{1}{2}$

(ii) $r(A + A^T) \le 1$

(iii) $r(A - A^T) \le 1$.

Proof

i) Let x be an eigenvector corresponding to an eigenvalue λ of A. Then rewriting the equality

$$x^T X x + x^T A^T X^{-1} A x = x^T x \text{ yields } x^T X x + |\lambda|^2 x^T X^{-1} x = x^T x.$$

From which we deduce that

$$|\lambda|^2 = \frac{x^T (I - X) x}{x^T X^{-1} x} \qquad (*)$$

Since X is a symmetric positive definite matrix we can make a singular value decomposition of X into $U^T \Sigma U$, where U is an orthogonal matrix and $\Sigma = \operatorname{diag}(\sigma_i^2)$ (see e.g. Kailath 1980 pp. 667).

Now, introduce the variable $y = Ux$. Then we have from (*) that

$$|\lambda|^2 = \frac{y^T (I - \Sigma) y}{y^T \Sigma^{-1} y} .$$

So, it suffices to prove that $\frac{y^T (I - \Sigma) y}{y^T \Sigma^{-1} y} \le \frac{1}{4}$, or equivalently, that $y^T (I - \Sigma - \frac{1}{4} \Sigma^{-1}) y \le 0$.

As $y^T(I - \Sigma - \frac{1}{4}\Sigma^{-1})y = \sum_{i=1}^{\bar{n}} y_i^2\left[1 - \sigma_i^2 - \frac{1}{4\sigma_i^2}\right] = \sum_{i=1}^{\bar{n}} - y_i^2(\sigma_i^2 - \frac{1}{2})^2 \frac{1}{\sigma_i^2}$, which is clearly smaller than zero this proves the first claim.

ii) To prove the other two claims we introduce the following notation

$$P := X^{\frac{1}{2}} - X^{-\frac{1}{2}}A$$

$$Q := X^{\frac{1}{2}} + X^{-\frac{1}{2}}A.$$

With this notation equation (1) can be rewritten as either $P^TP = I - A - A^T$ or $Q^TQ = I + A + A^T$.

Since both P^TP and Q^TQ are semi-positive definite this proves claim ii).

iii) Using the above notation we have, moreover, that

$$A - A^T = \frac{1}{2}(P^TQ - Q^TP).$$

Application of lemma 6 yields then that

$$r(A - A^T) = \frac{1}{2} r(P^TQ - Q^TP) \le \frac{1}{2} r(P^TP + Q^TQ)$$

$$= \frac{1}{2} r(I - A - A^T + I + A + A^T)$$

$$= \frac{1}{2} r(2I)$$

$$= 1.$$

The proof of this part is completed by noting that

$$r((A-A^T)(A-A^T)^T) = r(-(A-A^T)^2) = r((A-A^T)^2) = (r(A-A^T))^2. \qquad \square$$

Other necessary conditions can be formulated too, like e.g. $r(AA^T + A^TA) < 1$ (see corollary 3) or $r(A^2 + A^{T^2}) \le 1/2$. These additional conditions do, however, not give much extra information about matrix A. Moreover, they are together with the conditions posed in theorem 6 not yet sufficient too to conclude solvability of the matrix equation as will be shown in example 8. For that reason we will not go into any further details on this subject here.

Example 8

Let $A = \begin{bmatrix} 0.5 & -0.45 \\ 0.45 & 0 \end{bmatrix}$.

Then all necessary conditions mentioned before are satisfied. However, from the simulation results performed with algorithm (2) (see appendix 1) it is clear that X is not positive definite. so according to theorem 5, problem (1) does not have a solution. □

We conclude this section with two examples on the 2×2 matrix case in which the above mentioned conditions are sufficient. They might be useful in future research to obtain a general analytic expression for a solution of the equation. That the stated solutions indeed satisfy the equation can be verified by elementary calculation.

Example 9

Let $A = \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix}$. Then, with $x_{22} = \dfrac{1 - a_{12}^2 + \sqrt{(1 - a_{12}^2)^2 - 4a_{22}^2}}{2}$,

$X = \begin{bmatrix} 1 & 0 \\ 0 & x_{22} \end{bmatrix}$ satisfies equation (1).

Moreover, X can be rewritten as follows

$$X = \frac{1}{2} \left(I - G + \sqrt{(I + G)^2 - 4\, A^T A}\,\right), \qquad (*)$$

where $G = (A - A^T)(A - A^T)^T$. □

Example 10

Let $A = \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix}$. Then, with $x_{11} = 1 + a_{12}^2 - a_{21}^2 + \sqrt{(1-a_{21}+a_{12}^2) - 4a_{12}^2}$

and $x_{22} = 1 + a_{21}^2 - a_{12}^2 + \sqrt{(1-a_{12}^2 + a_{21}^2)^2 - 4a_{21}^2}$, $X = \frac{1}{2}\begin{bmatrix} x_{11} & 0 \\ 0 & x_{22} \end{bmatrix}$ satisfies equation (1).

Moreover X can be rewritten like (*) in example 9 with G replaced by $G = AA^T - A^TA$. □

Note that the two analytic solutions stated here do not coincide. Now, one might hope that this is due to the fact that in example 9 matrix A is generically invertible, whereas in example 8 this is not the case. But unfortunately this is not the case. The analytic solution presented in example 9 does not solve the equation for every invertible matrix A. Take e.g. $A = \begin{bmatrix} 0.2 & 0.4 \\ 0.05 & 0.25 \end{bmatrix}$, then $X := \begin{bmatrix} 0.95 & -0.117 \\ -0.117 & 0.701 \end{bmatrix}$, and simple calculations show that this is a counterexample for this conjecture. However, there is a class of matrices for which this formula does make sense. This are the normal matrices. In the next section we will see that if matrix A is normal, condition i) of theorem 7 is already sufficient to conclude solvability of equation (1), and that a solution is given by (*) where G is as in example 10.

IV Some special cases

Using the developed theory of the previous section, we derive in the present section a sufficient condition for existence of a solution. The claim is that whenever the operator norm of A is smaller than ½, then there exists a solution. In particular if matrix A is normal this implies that the equation has a solution iff. the spectral radius of A is smaller than ½. We first prove this lastmentioned result. This, since in that case a geometric approach is possible which facilitates a constructive proof.

Theorem 11
Let A be normal.
Then problem (1) has a solution iff. $r(A) \le \frac{1}{2}$.

Proof
That the spectral condition is necessary was already proved in theorem 7. To prove the sufficiency of the condition we recall from elementary matrix theory (see e.g. Horn et al. pp. 105) the result that matrix A is normal iff. there is a real orthogonal matrix U such that

$$U^T A\ U = \mathrm{diag}(D_i)$$

where each D_i is either a real 1×1 matrix or is a real 2×2 matrix of the form

$$D_i = \begin{bmatrix} \lambda_i & \mu_i \\ -\mu_i & \lambda_i \end{bmatrix}.$$

An immediate consequence of this result is that problem (1) is solvable iff. $\exists\ Z > 0$: $Z + D^T Z^{-1} D = I$, where $D := \text{diag}(D_i)$ and $r(D_i) \le \frac{1}{2}$. By construction we show now that this problem always has a diagonal solution $Z := \text{diag}(z_i)$.

To that end we first consider the case that D_i is a 1×1 matrix. Then we have to solve the equation $z_i + \frac{D_i}{z_i} = 1$. Since $D_i \le \frac{1}{2}$ it is easily seen that this quadratic equation always has a positive solution.

In case D is a 2×2 matrix we note that the assumption $r(D_i) \le \frac{1}{2}$ in particular implies that $1 - 4(\lambda_i^2 + \mu_i^2)$ is semi-positive.

Now take $Z = \text{diag}(1 + \sqrt{1-4(\lambda_i^2 + \mu_i^2)})$. Straightforward calculation shows then that Z indeed satisfies the equation $Z + D^T Z^{-1} D = I$ and that, moreover, Z is positive definite. This completes the proof. □

Remark 12

By some matrix manipulation it can be shown that always $X_1 = \frac{1}{2}(I + (I - 4A^TA)^{\frac{1}{2}})$ and $X_2 = \frac{1}{2}(I - (I - 4A^TA)^{\frac{1}{2}})$ satisfy the equation. Expressions which clearly generalize the scalar case.

A question which now immediately arises in this context is whether the set of all solutions satisfying equation (1) has a smallest (X') c.q. largest (X") element in the sense that any other solution X satisfies the inequality $X' \le X \le X''$.

In the particular case of theorem 11 a natural guess of X' and X" would then be X_2 and X_1 respectively. This remains, however, a topic for future research.

In the next theorem we show that in general the condition that the largest singular value of matrix A is smaller than ½ is sufficient to conclude that problem (1) has a solution. The proof is given by showing that problem (2) has a solution under this assumption. The disadvantage of this approach is that the connection with analytic solution is lost.

Theorem 13

Let σ^2 denote the largest singular value of A.

Then problem (1) has a solution if $\sigma^2 \le \frac{1}{2}$.

Proof

Consider the "equivalent" problem (2).

We show by induction that under the above mentioned assumption $X_n > AA^T + \frac{1}{4} I$.

The first step is rather trivial.

For, since $\sigma^2 \le \frac{1}{2}$, $AA^T \le \frac{1}{4} I$. Consequently, $X_0 = I > \frac{1}{2} I \ge AA^T + \frac{1}{4} I$.

Now assume that $X_n \ge AA^T + \frac{1}{4} I$.

$$\begin{aligned}\text{Then, } X_{n+1} &= I - A^T(X_n - AA^T + AA^T)^{-1}A \\ &= (I + A^T(X_n - AA^T)^{-1}A)^{-1} \\ &\ge (I + 4A^TA)^{-1} \\ &\ge \frac{1}{2} I \\ &\ge AA^T + \frac{1}{4} I.\end{aligned}$$

So, $X_n > AA^T$ $\forall n \in \mathbb{N}$. Therefore, according to theorem 3 problem (1) has a solution. □

V. An example from control theory

In this section we give an example in the field of control theory, where the solvability of equation (1) plays an important role.

Consider the next optimal control problem

$$\min_{u[0,.]} \lim_{N\to\infty} J_N \text{ w.r.t. } x(k+1) = A'x(k) + B\,u(k);\ x(.) \in \mathbb{R}^n;\ u(.) \in \mathbb{R}^m,$$
$$x(0) = x.$$

with the additional constraint that $\lim_{N\to\infty} x(N) = 0$; (3)

where

$J_N = \sum_{k=0}^{N-1} \{x^T(k)Qx(k) + u^T(k)Ru(k)\}$ and both Q and R are symmetric.

It is well known that it is difficult to find explicit solvability conditions for this so-called indefinite Lineair Quadratic (LQ) control problem (see e.g. Jonckheere et al (1978) and Molinari (1973)). We will show that in case matrix B is invertible the solvability of an appropriate equation of the type (1) plays a crucial role. But first we state sufficient general solvability conditions for problem (3).

Theorem 14

Problem (3) has a solution if there exists a real solution K' of the next Algebraic Riccati Equation

$$\text{(ARE)} \quad K = A'^T\{K - KB(R + B^TKB)^{-1}B^TK\}A' + Q$$

which additionally satisfies the requirements

i) $R + B^TK'B > 0$

ii) $r(A' + BF) < 1$, where $F = -(R + B^TK'B)^{-1} B^TK'A'$.

Proof

It is well known that by introducing the variable $v(k) = u(k) - Fx(k)$ the cost functional can be rewritten as

$$\min_{u[0,.]} \lim_{N\to\infty} J_N = \min_{u[0,.]} \lim_{N\to\infty} (J_N + x^T(N)K'x(N) - x^T(N)K'x(N))$$

$$= \min_{u[0,.]} \lim_{N\to\infty} \{ \sum_{k=0}^{N-1} v^T(k)(R + B^TK'B)v(k) + x^T(0)K'x(0) - x^T(N)K'x(N)\} \qquad (*)$$

where $x(k+1) = (A' + BF)x(k) + Bv(k)$.

Now take $v(.) = 0$. Then, due to our assumption on $r(A'+BF)$, $x(N)$ converges to zero. Consequently, the minimum value of problem (3) is always equal or smaller than $x^TK'x$.

Moreover since the control sequence must be such that $x(N)$ converges to zero and $R+B^T K'B > 0$, we have from (*) that always $\min \lim J_N \geq x^TK'x$. So $v(k) = 0$ solves the problem, which completes the proof. □

Thus the problem left to be solved is to give conditions under which there exists a real symmetric matrix K' to (ARE) which additionally satisfies 13.i) and 13.ii).

Theorem 15

Let $M := RB^{-1} A'B$ and $N := B^{T} A'^{T}B^{-T}RB^{-1}A'B + R + B^{T}QB$.

There exists a real symmetric matrix K' to (ARE) satisfying 13.i) iff.

1) $N > 0$.

2) Problem (1) has a solution with $A := N^{-\frac{1}{2}}MN^{-\frac{1}{2}}$.

Proof

Consider (ARE). Some elementary matrix manipulation shows that (ARE) has a real symmetric solution satisfying 13.i) iff. the following equation has this property

$$R + B^{T}KB = -B^{T}A'^{T} KB(R+B^{T} KB)^{-1} B^{T}KA'B + B^{T}A'^{T} KA'B + R + B^{T} QB.$$

This equation can be rewritten as

$$R+B^{T}KB = -B^{T}A'^{T}B^{-T}R(R+B^{T}KB)^{-1}RB^{-1}A'B + B^{T}A'^{T}B^{-T}RB^{-1}A'B + R + B^{T}QB.$$

So, introducing $Y := R + B^{T}KB$, we see that there is a solution iff. there exists a real positive definite solution Y to

$$Y = -M^{T} Y^{-1}M + N.$$

The stated conditions 1) and 2) now immediately result from this equation.

□

Combining the main results of this section and the previous one we have the following corollary.

Corollary 16

With the notation of theorem 15 the indefinite LQ problem (3) has a solution if the following conditions are satisfied

1) $N > 0$

2) $\|N^{-\frac{1}{2}} MN^{-\frac{1}{2}}\|_2 \le \frac{1}{2}$

3) $r(X^{-1}M) < 1.$ □

Note that in the definite LQ problems condition 3) is always satisfied.

VI. Concluding remarks

In this paper we introduced a nonlinear equation which directly extends the well known scalar quadratic equation. It turned out that it is rather difficult to find necessary and sufficient conditions for the existence of a real symmetric positive definite solution. For that reason we formulated a recursive algorithm from which always numerically a solution can be calculated whenever the equation is solvable.

Of course, the equation has in general more than one solution. Therefore the question arises whether all solutions can be ordered in some way, and in particular, whether there exists a smallest and largest element.

Drawing the parallel with the properties of the solutions satisfying the Algebraic Riccati Equation (see e.g. Willems (1971) and Trentelman (1987)), we believe that this minimal and maximal element exist, and that our recursive algorithm converges to the maximal one. But this remains a topic for future research.

Here, we concentrated on finding solvability conditions which can be easily verified, and the derivation of an analytic solution. We showed that whenever the operator norm of matrix A is smaller than $\frac{1}{2}$ the equation is always solvable. In case matrix A is normal this condition is both necessary and sufficient, and we gave an analytic solution. To find an explicit solution in other cases was rather difficult. Only in the 2×2 case for some particular situations general formulas were derived, which unfortunately do not solve the equation in general.

Since we were not able to solve the general problem we also derived a number of simple non-trivial necessary conditions that are expressed in terms of spectral radii.

In this context it is interesting to note that Lerer studied in a recent paper (1989) quadratic matrix equations too. He treats these problems from a factorization point of view. Maybe that this different approach will

give rise to additional explicit solvability conditions. But this is again a matter of future research.
We concluded the paper with an example from optimal control theory. It concerns the indefinite linear quadratic optimization problem. We showed that in case the input matrix B is invertible the optimal control problem can in essence be reduced to the question whether a special quadratic matrix equation of the type we studied is solvable. Using the developed theory we gave sufficient solvability conditions.

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Appendix 1

The first 4 values for X_n in $X_{n+1} = I - A^T X_n^{-1} A$; $X_0 = I$, with

$$A = \begin{pmatrix} 0.500 & -0.450 \\ 0.450 & 0.000 \end{pmatrix}$$

are:

$$X_0 = \begin{pmatrix} 1.000 & 0.000 \\ 0.000 & 1.000 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 0.548 & 0.225 \\ 0.225 & 0.798 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 0.459 & 0.347 \\ 0.347 & 0.582 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 0.439 & 0.414 \\ 0.414 & 0.196 \end{pmatrix}$$

$$X_4 = \begin{pmatrix} 0.433 & 0.465 \\ 0.465 & 1.463 \end{pmatrix}$$

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