

Tilburg University

On the existence of a positive definite solution of the matrix equation $X+AXA=I$

Engwerda, J.C.

Publication date:
1989

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):

Engwerda, J. C. (1989). *On the existence of a positive definite solution of the matrix equation $X+AXA=I$* . (Research memorandum / Tilburg University, Department of Economics; Vol. FEW 397). Unknown Publisher.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

CBM

CBM
R




7626
1989
397

POSTBOX 90153
5000 LE TILBURG
THE NETHERLANDS



DEPARTMENT OF ECONOMICS
RESEARCH MEMORANDUM



 K.U.B.
BIBLIOTHEEK
TILBURG

ON THE EXISTENCE OF A POSITIVE
DEFINITE SOLUTION OF THE MATRIX
EQUATION $X + A^T X^{-1} A = I$

Jacob C. Engwerda

FEW 397

On the existence of a positive definite solution of the
matrix equation $X + A^T X^{-1} A = I$

by

Jacob C. Engwerda

Abstract

In this paper the question is raised under which conditions on the real (square) matrix A , the matrix equation $X + A^T X^{-1} A = I$ has a real symmetric positive definite solution X . Both necessary and sufficient solvability conditions on A are derived. Moreover, we give an algorithm to calculate the solution. For a number of special cases we also present an analytic solution.

I. Introduction

As already mentioned in the abstract the central issue in this paper is to find solvability conditions for the existence of a positive definite solution X of the matrix equation $X + A^T X^{-1} A = I$. This problem can be viewed as a natural extension of giving solvability conditions for the scalar problem $x + \frac{a^2}{x} = 1$. From calculus we know that the existence of the real square root $\sqrt{1 - 4a^2}$ plays here an important role. We will see that this condition generalizes straightforwardly to the matrix case, if A has the additional property that it is normal (i.e. $A^T A = A A^T$). However, if A has not this additional property things become more complicated.

We show that the general problem has a solution if and only if a related recursive algorithm converges to a positive definite solution. Moreover we use this algorithm to prove that, provided matrix A satisfies a certain condition, the matrix equation is solvable and to calculate a solution numerically.

Separately, we derive a number of necessary conditions and show by means of a counterexample that these are in general not sufficient.

The paper is organized as follows. First in section 2 we introduce some notation and study the general problem together with the recursive algorithm. Then, we derive a number of necessary conditions. Section 4 contains a number of special cases in which a solution exists. Before we discuss the main results in section 6, we give in section 5 an example of this equation in the field of optimal control theory.

II. The general problem

Mathematically, the problem analyzed in this paper is to find conditions under which:

$$\exists X > 0: X + A^T X^{-1} A = I, \quad (1)$$

where X and I are real square $\bar{n} \times \bar{n}$ matrices.

Here $X > 0$ means that X is symmetric positive definite, denotes A^T the transpose of A and is I the identity matrix. In the sequel also the notation \geq is used to indicate that a matrix is symmetric semi-positive definite and is $A > B$ used as a different notation for $A - B > 0$. Moreover, $\text{Ker } A$ denotes the kernel of A and $\text{Im } A$ its image.

Further on we show that this problem has a solution if and only if the next recursion problem is solvable.

$$\forall n \in \mathbb{N} \text{ is } X_n > AA^T \text{ in}$$

$$X_0 = I \tag{2}$$

$$X_{n+1} = I - A^T X_n^{-1} A$$

To prove this result we start with some intermediate results which are interesting in itself. The first thing we prove is that in fact it suffices to solve problem (1) for invertible matrices. We show that in case A is not invertible, problem (1) can be reduced to a similar problem with an invertible A matrix. How this can be accomplished is the contents of theorem 1. Its proof contains an algorithm which will be used later on again.

Theorem 1

If we can solve problem (1) whenever matrix A is invertible, then we can solve this problem without this invertibility restriction too.

Proof

We prove this theorem by reducing the problem to a similar problem of lower dimension.

The reduction is achieved via the next algorithm

- (i) If A is invertible then the algorithm is finished.
- (ii) Else, there exists an orthogonal transformation T such that

$$A = T^T \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix} T.$$

Consequently problem (1) has a solution if and only if (iff.) the next problem is solvable

$$\exists Y > 0: Y = \begin{pmatrix} A_{11}^T & A_{21}^T \\ 0 & 0 \end{pmatrix} Y^{-1} \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix} = I;$$

which is the case iff.

$$\exists Z > 0: Z + A_{11}^T Z^{-1} A_{11} = I - A_{21}^T A_{21}, \text{ where } G := I - A_{21}^T A_{21} > 0.$$

Now define $A := G^{-\frac{1}{2}} A_{11} G^{-\frac{1}{2}}$. Then this problem can be rewritten into the original form (1), unless $A_{11} = 0$. If $A_{11} \neq 0$ we return to (i), otherwise $Z := I - A_{21}^T A_{21}$ and the algorithm stops. \square

So, to solve problem (1) we could restrict us to invertible matrices. But from the algorithm it is clear that then the solvability conditions for non-invertible matrices become rather involved. For that reason we will not make this invertibility assumption w.r.t. A from the outset on. The next preparatory lemma gives a lowerbound for any solution to problem (1).

Lemma 2

If equation (1) has a solution X , then $X > AA^T$.

Proof

Rewriting (1) yields that

$$X = I - A^T X^{-1} A. \tag{i}$$

Since X is by assumption positive definite we immediately obtain from this equation that $X \leq I$. Consequently, $X^{-1} \geq I$ and thus $A^T A \leq A^T X^{-1} A = I - X \leq I$. From this inequality we conclude that

$$A^T A \leq I. \tag{ii}$$

From (i) we have, moreover, that X^{-1} equals $(I - A^T X^{-1} A)^{-1}$. Applying Schur's lemma (see e.g. Kailath (1980) pp. 656) yields that

$$X^{-1} = I - A^T (AA^T - X)^{-1} A. \quad (\text{iii})$$

As $X^{-1} > I$, we obtain the inequality

$$A^T (X - AA^T)^{-1} A \geq 0.$$

So, $x^T (X - AA^T)^{-1} x > 0$ whenever $x \notin \text{Ker } A$. Since $X - AA^T$ is a symmetric matrix, $x^T (X - AA^T) x$ will be positive too for any $x \notin \text{Ker } A$. So what is left to be shown is that $x^T (X - AA^T) x > 0$ for any $x \in \text{Ker } A$. This immediately results from (ii). For, let $x \in \text{Ker } A$. Then, $x^T X x = x^T (I - A^T X^{-1} A) x = x^T I x \geq x^T AA^T x$. As $X - AA^T$ is invertible, it is clear that $X - AA^T$ is positive definite.

Corollary 3

If problem (1) has a solution X , then $I - AA^T - A^T A > 0$.

Proof

Since $X + A^T X^{-1} A = I$, we obtain by substitution of (iii) from lemma 2 that

$$X + A^T A - A^T (AA^T - X)^{-1} A^2 = I.$$

So $X - AA^T = I - AA^T - A^T A - A^T (X - AA^T)^{-1} A^2$.

Application of lemma 2 yields that

$$I - AA^T - A^T A = (X - AA^T) + A^T (X - AA^T)^{-1} A^2 > 0. \quad \square$$

A similar result as lemma 1 holds w.r.t. problem (2).

Lemma 4

If problem (2) has a solution, then there exists a positive constant α such that $X_n > \alpha I \forall n \in \mathbb{N}$.

Proof

The proof is similar to the one of theorem 1.

(i) In case matrix A is invertible the above statement is trivial.

(ii) In case A is not invertible we decompose A again into $A =:$

$T^T \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix} T$ and note that the algorithm of theorem 1 implies that the next algorithm has a solution $X' > 0$.

$$X'_0 = I$$

$$X'_{n+1} = I - A_{21}^T A_{21} - A_{11}^T X'_n{}^{-1} A_{11} \text{ where } X'_n = T^T \begin{pmatrix} X'_n & 0 \\ 0 & I \end{pmatrix} T.$$

Rewriting this equation yields

$$\exists Y > 0 \text{ in}$$

$$Y_0 = I$$

$$Y_{n+1} = I - A'' Y_n^{-1} A'', \text{ where } A'' := (I - A_{21}^T A_{21})^{-\frac{1}{2}} A_{11} (I - A_{21}^T A_{21})^{-\frac{1}{2}}.$$

If A_{11} is non-zero then we return to (i). In case $A_{11} = 0$ then it is clear that $X'_n = I - A_{21}^T A_{21} > 0 \forall n$, and thus $X'_n > \alpha I \forall n$ for some α too. Finally notice that this algorithm stops after at most $\bar{n}-1$ iterations, and that

the nested solution $X = T_1^T \begin{pmatrix} T_2^T (\text{etc.}) T_2 & 0 \\ 0 & I \end{pmatrix} T_1$ can always be estimated by αI for some α , which completes the proof. \square

Using these two lemmas we can prove that problem (1) has a solution whenever problem (2) has one, and vice versa.

Theorem 5

Problem (1) has a solution iff. problem (2) is solvable.

Proof

" \Leftarrow " First we prove that X'_n is a monotonically non-increasing sequence. This is proved by induction.

Note that the initialization step is trivially satisfied, for,

$$X_0 - X_1 = I - (I - A^T A) = A^T A \geq 0.$$

Now, let $X_n \leq X_{n-1}$. Then, since $X_n > 0$, we have that $X_n^{-1} \geq X_{n-1}^{-1}$. So $X_n - X_{n+1} = A^T (X_{n-1}^{-1} - X_n^{-1}) A \geq 0$, which completes the induction step. Therefore X_n is a monotonically non-increasing sequence which is, according to lemma 4, bounded from below by some positive definite matrix. Consequently X_n converges to a positive definite limit which satisfies equation (1).

" \Rightarrow " We prove this part by induction.

According to corollary 3 we have that whenever problem (1) has a solution then $I - A^T A - AA^T$ is positive definite. Since $X_1 - AA^T = I - A^T A - AA^T$, this completes the first part of the proof. Now assume that $X_i - AA^T > 0 \forall i \leq n$.

Then it is easily seen by induction that $X_i - X = A^T (X^{-1} - X_{i-1}^{-1}) A \geq 0 \forall i \leq n$. So in particular $X - AA^T \leq X_n - AA^T$. Application of this inequality yields that

$$\begin{aligned} X_{n+1} - AA^T &= I - AA^T - A^T X_n^{-1} A \\ &= I - A^T (X_n - AA^T)^{-1} A - AA^T \\ &= (I + A^T (X_n - AA^T)^{-1} A)^{-1} - AA^T \\ &\geq (I + A^T (X - AA^T)^{-1} A)^{-1} - AA^T \\ &= I - A^T X^{-1} A - AA^T \\ &= X - AA^T \end{aligned}$$

> 0 , which completes the proof. \square

III. Necessary conditions

In this section we discuss a number of conditions on A that must be satisfied in order to solve the matrix equation. Moreover we show by means of a counterexample that these conditions are in general not sufficient to solve the problem.

We start this section again with a preliminary lemma. In this lemma, as well as in the rest of the paper, we use the notation $r(A)$ to denote the spectral radius of matrix A (i.e. $\max_{\lambda_i} |\lambda_i|$, where λ_i are the eigenvalues of matrix A).

Lemma 6

Let P and Q be two arbitrary compatible matrices. Then, $r(P^T Q - Q^T P) \leq r(P^T P + Q^T Q)$.

Proof

By elementary calculus we have that

$$r(P^T Q - Q^T P) = r\left[(P^T \ Q^T) \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix}\right].$$

Since $r(AB) = r(BA)$ for any two compatible matrices, we have that

$$r\left[(P^T \ Q^T) \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix}\right] = r\left[\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} (P^T \ Q^T)\right].$$

Now, $r(A) \leq \|A\|_2$, where $\|\cdot\|_2$ denotes the operator norm (i.e. the largest singular value of matrix A).

So

$$\begin{aligned} r\left[\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} (P^T \ Q^T)\right] &\leq \left\| \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} (P^T \ Q^T) \right\|_2 \leq \\ &\left\| \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} P \\ Q \end{bmatrix} (P^T \ Q^T) \right\|_2. \end{aligned}$$

As $\begin{bmatrix} P \\ Q \end{bmatrix} (P^T \ Q^T)$ is a normal matrix, and $\|A\|_2 = r(A)$ for any matrix A of this type, we can rewrite the above expression as follows:

$$\begin{aligned}
r(P^T Q - Q^T P) &\leq \left\| \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} P \\ Q \end{bmatrix} (P^T Q^T) \right\|_2 \\
&= 1 \, r \left[\begin{bmatrix} P \\ Q \end{bmatrix} (P^T Q^T) \right] \\
&= r \left[(P^T Q^T) \begin{bmatrix} P \\ Q \end{bmatrix} \right] \\
&= r(P^T P + Q^T Q), \text{ which completes the proof.} \quad \square
\end{aligned}$$

Theorem 7

Assume that problem (1) is solvable.

Then matrix A satisfies the following inequalities.

- (i) $r(A) \leq \frac{1}{2}$
- (ii) $r(A + A^T) \leq 1$
- (iii) $r(A - A^T) \leq 1$.

Proof

i) Let x be an eigenvector corresponding to an eigenvalue λ of A . Then rewriting the equality

$$x^T X x + x^T A^T X^{-1} A x = x^T x \text{ yields } x^T X x + |\lambda|^2 x^T X^{-1} x = x^T x.$$

From which we deduce that

$$|\lambda|^2 = \frac{x^T (I - X) x}{x^T X^{-1} x} \quad (*)$$

Since X is a symmetric positive definite matrix we can make a singular value decomposition of X into $U^T \Sigma U$, where U is an orthogonal matrix and $\Sigma = \text{diag}(\sigma_i^2)$ (see e.g. Kailath 1980 pp. 667).

Now, introduce the variable $y = Ux$. Then we have from (*) that

$$|\lambda|^2 = \frac{y^T (I - \Sigma) y}{y^T \Sigma^{-1} y}.$$

So, it suffices to prove that $\frac{y^T (I - \Sigma) y}{y^T \Sigma^{-1} y} \leq \frac{1}{4}$, or equivalently, that $y^T (I - \Sigma - \frac{1}{4} \Sigma^{-1}) y \leq 0$.

$$\text{As } y^T(I - \Sigma - \frac{1}{4} \Sigma^{-1})y = \sum_{i=1}^{\bar{n}} y_i^2 \left[1 - \sigma_i^2 - \frac{1}{4 \sigma_i^2} \right] = \sum_{i=1}^{\bar{n}} y_i^2 \left(\sigma_i^2 - \frac{1}{2} \right)^2 \frac{1}{\sigma_i^2},$$

which is clearly smaller than zero this proves the first claim.

ii) To prove the other two claims we introduce the following notation

$$P := X^{\frac{1}{2}} - X^{-\frac{1}{2}}A$$

$$Q := X^{\frac{1}{2}} + X^{-\frac{1}{2}}A.$$

With this notation equation (1) can be rewritten as either $P^T P = I - A - A^T$ or $Q^T Q = I + A + A^T$.

Since both $P^T P$ and $Q^T Q$ are semi-positive definite this proves claim ii).

iii) Using the above notation we have, moreover, that

$$A - A^T = \frac{1}{2} (P^T Q - Q^T P).$$

Application of lemma 6 yields then that

$$\begin{aligned} r(A - A^T) &= \frac{1}{2} r(P^T Q - Q^T P) \leq \frac{1}{2} r(P^T P + Q^T Q) \\ &= \frac{1}{2} r(I - A - A^T + I + A + A^T) \\ &= \frac{1}{2} r(2I) \\ &= 1. \end{aligned}$$

The proof of this part is completed by noting that

$$r((A-A^T)(A-A^T)^T) = r(-(A-A^T)^2) = r((A-A^T)^2) = (r(A-A^T))^2. \quad \square$$

Other necessary conditions can be formulated too, like e.g. $r(AA^T + A^T A) < 1$ (see corollary 3) or $r(A^2 + A^{T^2}) \leq 1/2$. These additional conditions do, however, not give much extra information about matrix A. Moreover, they are together with the conditions posed in theorem 6 not yet sufficient too to conclude solvability of the matrix equation as will be shown in example 8. For that reason we will not go into any further details on this subject here.

Example 8

$$\text{Let } A = \begin{bmatrix} 0.5 & -0.45 \\ 0.45 & 0 \end{bmatrix}.$$

Then all necessary conditions mentioned before are satisfied. However, from the simulation results performed with algorithm (2) (see appendix 1) it is clear that X is not positive definite. so according to theorem 5, problem (1) does not have a solution. \square

We conclude this section with two examples on the 2×2 matrix case in which the above mentioned conditions are sufficient. They might be useful in future research to obtain a general analytic expression for a solution of the equation. That the stated solutions indeed satisfy the equation can be verified by elementary calculation.

Example 9

$$\text{Let } A = \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix}. \text{ Then, with } x_{22} = \frac{1 - a_{12}^2 + \sqrt{(1 - a_{12}^2)^2 - 4a_{22}^2}}{2},$$

$$X = \begin{bmatrix} 1 & 0 \\ 0 & x_{22} \end{bmatrix} \text{ satisfies equation (1).}$$

Moreover, X can be rewritten as follows

$$X = \frac{1}{2} (I - G + \sqrt{(I + G)^2 - 4A^T A}), \quad (*)$$

$$\text{where } G = (A - A^T)(A - A^T)^T. \quad \square$$

Example 10

$$\text{Let } A = \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix}. \text{ Then, with } x_{11} = 1 + a_{12}^2 - a_{21}^2 + \sqrt{(1 - a_{21}^2 + a_{12}^2) - 4a_{12}^2}$$

$$\text{and } x_{22} = 1 + a_{21}^2 - a_{12}^2 + \sqrt{(1 - a_{12}^2 + a_{21}^2)^2 - 4a_{21}^2}, X = \frac{1}{2} \begin{bmatrix} x_{11} & 0 \\ 0 & x_{22} \end{bmatrix} \text{ satisfies equation (1).}$$

Moreover X can be rewritten like (*) in example 9 with G replaced by $G = AA^T - A^T A$. \square

Note that the two analytic solutions stated here do not coincide. Now, one might hope that this is due to the fact that in example 9 matrix A is

generically invertible, whereas in example 8 this is not the case. But unfortunately this is not the case. The analytic solution presented in example 9 does not solve the equation for every invertible matrix A. Take e.g. $A = \begin{bmatrix} 0.2 & 0.4 \\ 0.05 & 0.25 \end{bmatrix}$, then $X := \begin{bmatrix} 0.95 & -0.117 \\ -0.117 & 0.701 \end{bmatrix}$, and simple calculations show that this is a counterexample for this conjecture. However, there is a class of matrices for which this formula does make sense. These are the normal matrices. In the next section we will see that if matrix A is normal, condition i) of theorem 7 is already sufficient to conclude solvability of equation (1), and that a solution is given by (*) where G is as in example 10.

IV Some special cases

Using the developed theory of the previous section, we derive in the present section a sufficient condition for existence of a solution. The claim is that whenever the operator norm of A is smaller than $\frac{1}{2}$, then there exists a solution. In particular if matrix A is normal this implies that the equation has a solution iff. the spectral radius of A is smaller than $\frac{1}{2}$. We first prove this lastmentioned result. This, since in that case a geometric approach is possible which facilitates a constructive proof.

Theorem 11

Let A be normal.

Then problem (1) has a solution iff. $r(A) \leq \frac{1}{2}$.

Proof

That the spectral condition is necessary was already proved in theorem 7. To prove the sufficiency of the condition we recall from elementary matrix theory (see e.g. Horn et al. pp. 105) the result that matrix A is normal iff. there is a real orthogonal matrix U such that

$$U^T A U = \text{diag}(D_i)$$

where each D_i is either a real 1×1 matrix or is a real 2×2 matrix of the form

$$D_i = \begin{bmatrix} \lambda_i & \mu_i \\ -\mu_i & \lambda_i \end{bmatrix}.$$

An immediate consequence of this result is that problem (1) is solvable iff. $\exists Z > 0: Z + D^T Z^{-1} D = I$, where $D := \text{diag}(D_i)$ and $r(D_i) \leq \frac{1}{2}$. By construction we show now that this problem always has a diagonal solution $Z := \text{diag}(z_i)$.

To that end we first consider the case that D_i is a 1×1 matrix. Then we have to solve the equation $z_i + \frac{D_i}{z_i} = 1$. Since $D_i \leq \frac{1}{2}$ it is easily seen that this quadratic equation always has a positive solution.

In case D is a 2×2 matrix we note that the assumption $r(D_i) \leq \frac{1}{2}$ in particular implies that $1 - 4(\lambda_i^2 + \mu_i^2)$ is semi-positive.

Now take $Z = \text{diag}(1 + \sqrt{1 - 4(\lambda_i^2 + \mu_i^2)})$. Straightforward calculation shows then that Z indeed satisfies the equation $Z + D^T Z^{-1} D = I$ and that, moreover, Z is positive definite. This completes the proof. \square

Remark 12

By some matrix manipulation it can be shown that always $X_1 = \frac{1}{2}(I + (I - 4A^T A)^{\frac{1}{2}})$ and $X_2 = \frac{1}{2}(I - (I - 4A^T A)^{\frac{1}{2}})$ satisfy the equation. Expressions which clearly generalize the scalar case.

A question which now immediately arises in this context is whether the set of all solutions satisfying equation (1) has a smallest (X') c.q. largest (X'') element in the sense that any other solution X satisfies the inequality $X' \leq X \leq X''$.

In the particular case of theorem 11 a natural guess of X' and X'' would then be X_2 and X_1 respectively. This remains, however, a topic for future research.

In the next theorem we show that in general the condition that the largest singular value of matrix A is smaller than $\frac{1}{2}$ is sufficient to conclude that problem (1) has a solution. The proof is given by showing that problem (2) has a solution under this assumption. The disadvantage of this approach is that the connection with analytic solution is lost.

Theorem 13

Let σ^2 denote the largest singular value of A.

Then problem (1) has a solution if $\sigma^2 \leq \frac{1}{2}$.

Proof

Consider the "equivalent" problem (2).

We show by induction that under the above mentioned assumption

$$X_n > AA^T + \frac{1}{4} I.$$

The first step is rather trivial.

For, since $\sigma^2 \leq \frac{1}{2}$, $AA^T \leq \frac{1}{4} I$. Consequently, $X_0 = I > \frac{1}{2} I \geq AA^T + \frac{1}{4} I$.

Now assume that $X_n \geq AA^T + \frac{1}{4} I$.

$$\begin{aligned} \text{Then, } X_{n+1} &= I - A^T (X_n - AA^T + AA^T)^{-1} A \\ &= (I + A^T (X_n - AA^T)^{-1} A)^{-1} \\ &\geq (I + 4A^T A)^{-1} \\ &\geq \frac{1}{2} I \\ &\geq AA^T + \frac{1}{4} I. \end{aligned}$$

So, $X_n > AA^T \forall n \in \mathbb{N}$. Therefore, according to theorem 3 problem (1) has a solution. □

V. An example from control theory

In this section we give an example in the field of control theory, where the solvability of equation (1) plays an important role.

Consider the next optimal control problem

$$\min_{u[0, \dots]} \lim_{N \rightarrow \infty} J_N \text{ w.r.t. } x(k+1) = A^T x(k) + B u(k); x(\cdot) \in \mathbb{R}^n; u(\cdot) \in \mathbb{R}^m, \\ x(0) = x.$$

$$\text{with the additional constraint that } \lim_{N \rightarrow \infty} x(N) = 0; \quad (3)$$

where

$$J_N = \sum_{k=0}^{N-1} \{x^T(k) Q x(k) + u^T(k) R u(k)\} \text{ and both } Q \text{ and } R \text{ are symmetric.}$$

It is well known that it is difficult to find explicit solvability conditions for this so-called indefinite Linear Quadratic (LQ) control problem (see e.g. Jonckheere et al (1978) and Molinari (1973)). We will show that

in case matrix B is invertible the solvability of an appropriate equation of the type (1) plays a crucial role. But first we state sufficient general solvability conditions for problem (3).

Theorem 14

Problem (3) has a solution if there exists a real solution K' of the next Algebraic Riccati Equation

$$(ARE) \quad K = A'^T \{K - KB(R + B^T KB)^{-1} B^T K\} A' + Q$$

which additionally satisfies the requirements

i) $R + B^T K' B > 0$

ii) $r(A' + BF) < 1$, where $F = -(R + B^T K' B)^{-1} B^T K' A'$.

Proof

It is well known that by introducing the variable $v(k) = u(k) - Fx(k)$ the cost functional can be rewritten as

$$\begin{aligned} \min_{u[0, \dots]} \lim_{N \rightarrow \infty} J_N &= \min_{u[0, \dots]} \lim_{N \rightarrow \infty} (J_N + x^T(N) K' x(N) - x^T(N) K' x(N)) \\ &= \min_{u[0, \dots]} \lim_{N \rightarrow \infty} \left\{ \sum_{k=0}^{N-1} v^T(k) (R + B^T K' B) v(k) + x^T(0) K' x(0) - \right. \\ &\quad \left. x^T(N) K' x(N) \right\} \end{aligned} \quad (*)$$

$$\text{where } x(k+1) = (A' + BF)x(k) + Bv(k).$$

Now take $v(\cdot) = 0$. Then, due to our assumption on $r(A' + BF)$, $x(N)$ converges to zero. Consequently, the minimum value of problem (3) is always equal or smaller than $x^T K' x$.

Moreover since the control sequence must be such that $x(N)$ converges to zero and $R + B^T K' B > 0$, we have from (*) that always $\min \lim_{N \rightarrow \infty} J_N \geq x^T K' x$. So $v(k) = 0$ solves the problem, which completes the proof. \square

Thus the problem left to be solved is to give conditions under which there exists a real symmetric matrix K' to (ARE) which additionally satisfies 13.i) and 13.ii).

Theorem 15

Let $M := RB^{-1}A'B$ and $N := B^T A'^T B^{-T} R B^{-1} A'B + R + B^T Q B$.

There exists a real symmetric matrix K' to (ARE) satisfying 13.i) iff.

- 1) $N > 0$.
- 2) Problem (1) has a solution with $A := N^{-\frac{1}{2}} M N^{-\frac{1}{2}}$.

Proof

Consider (ARE). Some elementary matrix manipulation shows that (ARE) has a real symmetric solution satisfying 13.i) iff. the following equation has this property

$$R + B^T K B = -B^T A'^T K B (R + B^T K B)^{-1} B^T K A' B + B^T A'^T K A' B + R + B^T Q B.$$

This equation can be rewritten as

$$R + B^T K B = -B^T A'^T B^{-T} R (R + B^T K B)^{-1} R B^{-1} A' B + B^T A'^T B^{-T} R B^{-1} A' B + R + B^T Q B.$$

So, introducing $Y := R + B^T K B$, we see that there is a solution iff. there exists a real positive definite solution Y to

$$Y = -M^T Y^{-1} M + N.$$

The stated conditions 1) and 2) now immediately result from this equation. □

Combining the main results of this section and the previous one we have the following corollary.

Corollary 16

With the notation of theorem 15 the indefinite LQ problem (3) has a solution if the following conditions are satisfied

- 1) $N > 0$

- 2) $\|N^{-\frac{1}{2}} MN^{-\frac{1}{2}}\|_2 \leq \frac{1}{2}$
 3) $r(X^{-1}M) < 1$. □

Note that in the definite LQ problems condition 3) is always satisfied.

VI. Concluding remarks

In this paper we introduced a nonlinear equation which directly extends the well known scalar quadratic equation. It turned out that it is rather difficult to find necessary and sufficient conditions for the existence of a real symmetric positive definite solution. For that reason we formulated a recursive algorithm from which always numerically a solution can be calculated whenever the equation is solvable.

Of course, the equation has in general more than one solution. Therefore the question arises whether all solutions can be ordered in some way, and in particular, whether there exists a smallest and largest element.

Drawing the parallel with the properties of the solutions satisfying the Algebraic Riccati Equation (see e.g. Willems (1971) and Trentelman (1987)), we believe that this minimal and maximal element exist, and that our recursive algorithm converges to the maximal one. But this remains a topic for future research.

Here, we concentrated on finding solvability conditions which can be easily verified, and the derivation of an analytic solution. We showed that whenever the operator norm of matrix A is smaller than $\frac{1}{2}$ the equation is always solvable. In case matrix A is normal this condition is both necessary and sufficient, and we gave an analytic solution. To find an explicit solution in other cases was rather difficult. Only in the 2×2 case for some particular situations general formulas were derived, which unfortunately do not solve the equation in general.

Since we were not able to solve the general problem we also derived a number of simple non-trivial necessary conditions that are expressed in terms of spectral radii.

In this context it is interesting to note that Lerer studied in a recent paper (1989) quadratic matrix equations too. He treats these problems from a factorization point of view. Maybe that this different approach will

give rise to additional explicit solvability conditions. But this is again a matter of future research.

We concluded the paper with an example from optimal control theory. It concerns the indefinite linear quadratic optimization problem. We showed that in case the input matrix B is invertible the optimal control problem can in essence be reduced to the question whether a special quadratic matrix equation of the type we studied is solvable. Using the developed theory we gave sufficient solvability conditions.

References

- Horn R.A. and Johnson C.A., 1985, *Matrix Analysis* (London, Cambridge University Press).
- Jonckheere E.A. and Silvermann L.M., 1978, Spectral theory of the linear quadratic optimal control problem; discrete-time single-input case, I.E.E.E. CAS-25.
- Kailath T., 1980, *Linear Systems* (Englewood Cliffs, N.J., Prentice Hall).
- Lerer L., 1989, The algebraic Riccati equation and inverse spectral problems for rational matrix functions, paper presented at the MTNS-89 conference in Amsterdam.
- Molinari B.P., 1973, The stabilizing solution of the algebraic Riccati equation, *Siam Journal Control and Optimization* Vol. 11, pp. 262-271.
- Trentelman H.L., 1987, Families of linear-quadratic problems, I.E.E.E. Trans. Automatic Control Vol. AC-32, pp. 323-329.
- Willems J.C., 1971, Least squares stationary optimal control and the algebraic Riccati equation, I.E.E.E. Trans. Automatic Control Vol. AC-16, pp. 621-634.

Appendix 1

The first 4 values for X_n in $X_{n+1} = I - A^T X_n^{-1} A$; $X_0 = I$,
with

$$A = \begin{pmatrix} 0.500 & -0.450 \\ 0.450 & 0.000 \end{pmatrix}$$

are:

$$X_0 = \begin{pmatrix} 1.000 & 0.000 \\ 0.000 & 1.000 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 0.548 & 0.225 \\ 0.225 & 0.798 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 0.459 & 0.347 \\ 0.347 & 0.582 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 0.439 & 0.414 \\ 0.414 & 0.196 \end{pmatrix}$$

$$X_4 = \begin{pmatrix} 0.433 & 0.465 \\ 0.465 & 1.463 \end{pmatrix}$$

IN 1988 REEDS VERSCHENEN

- 297 Bert Bettonvil
Factor screening by sequential bifurcation
- 298 Robert P. Gilles
On perfect competition in an economy with a coalitional structure
- 299 Willem Selen, Ruud M. Heuts
Capacitated Lot-Size Production Planning in Process Industry
- 300 J. Kriens, J.Th. van Lieshout
Notes on the Markowitz portfolio selection method
- 301 Bert Bettonvil, Jack P.C. Kleijnen
Measurement scales and resolution IV designs: a note
- 302 Theo Nijman, Marno Verbeek
Estimation of time dependent parameters in linear models
using cross sections, panels or both
- 303 Raymond H.J.M. Gradus
A differential game between government and firms: a non-cooperative
approach
- 304 Leo W.G. Strijbosch, Ronald J.M.M. Does
Comparison of bias-reducing methods for estimating the parameter in
dilution series
- 305 Drs. W.J. Reijnders, Drs. W.F. Verstappen
Strategische bespiegelingen betreffende het Nederlandse kwaliteits-
concept
- 306 J.P.C. Kleijnen, J. Kriens, H. Timmermans and H. Van den Wildenberg
Regression sampling in statistical auditing
- 307 Isolde Woittiez, Arie Kapteyn
A Model of Job Choice, Labour Supply and Wages
- 308 Jack P.C. Kleijnen
Simulation and optimization in production planning: A case study
- 309 Robert P. Gilles and Pieter H.M. Ruys
Relational constraints in coalition formation
- 310 Drs. H. Leo Theuns
Determinanten van de vraag naar vakantiereizen: een verkenning van
materiële en immateriële factoren
- 311 Peter M. Kort
Dynamic Firm Behaviour within an Uncertain Environment
- 312 J.P.C. Blanc
A numerical approach to cyclic-service queueing models

- 313 Drs. N.J. de Beer, Drs. A.M. van Nunen, Drs. M.O. Nijkamp
Does Morkmon Matter?
- 314 Th. van de Klundert
Wage differentials and employment in a two-sector model with a dual labour market
- 315 Aart de Zeeuw, Fons Groot, Cees Withagen
On Credible Optimal Tax Rate Policies
- 316 Christian B. Mulder
Wage moderating effects of corporatism
Decentralized versus centralized wage setting in a union, firm, government context
- 317 Jörg Glombowski, Michael Krüger
A short-period Goodwin growth cycle
- 318 Theo Nijman, Marno Verbeek, Arthur van Soest
The optimal design of rotating panels in a simple analysis of variance model
- 319 Drs. S.V. Hannema, Drs. P.A.M. Versteijne
De toepassing en toekomst van public private partnership's bij de grote en middelgrote Nederlandse gemeenten
- 320 Th. van de Klundert
Wage Rigidity, Capital Accumulation and Unemployment in a Small Open Economy
- 321 M.H.C. Paardekooper
An upper and a lower bound for the distance of a manifold to a nearby point
- 322 Th. ten Raa, F. van der Ploeg
A statistical approach to the problem of negatives in input-output analysis
- 323 P. Kooreman
Household Labor Force Participation as a Cooperative Game; an Empirical Model
- 324 A.B.T.M. van Schaik
Persistent Unemployment and Long Run Growth
- 325 Dr. F.W.M. Boekema, Drs. L.A.G. Oerlemans
De lokale produktiestructuur doorgelicht.
Bedrijfstakverkenningen ten behoeve van regionaal-economisch onderzoek
- 326 J.P.C. Kleijnen, J. Kriens, M.C.H.M. Lafleur, J.H.F. Pardoel
Sampling for quality inspection and correction: AOQL performance criteria

- 327 Theo E. Nijman, Mark F.J. Steel
Exclusion restrictions in instrumental variables equations
- 328 B.B. van der Genugten
Estimation in linear regression under the presence of heteroskedasticity of a completely unknown form
- 329 Raymond H.J.M. Gradus
The employment policy of government: to create jobs or to let them create?
- 330 Hans Kremers, Dolf Talman
Solving the nonlinear complementarity problem with lower and upper bounds
- 331 Antoon van den Elzen
Interpretation and generalization of the Lemke-Howson algorithm
- 332 Jack P.C. Kleijnen
Analyzing simulation experiments with common random numbers, part II: Rao's approach
- 333 Jacek Osiewalski
Posterior and Predictive Densities for Nonlinear Regression. A Partly Linear Model Case
- 334 A.H. van den Elzen, A.J.J. Talman
A procedure for finding Nash equilibria in bi-matrix games
- 335 Arthur van Soest
Minimum wage rates and unemployment in The Netherlands
- 336 Arthur van Soest, Peter Kooreman, Arie Kapteyn
Coherent specification of demand systems with corner solutions and endogenous regimes
- 337 Dr. F.W.M. Boekema, Drs. L.A.G. Oerlemans
De lokale produktiestructuur doorgelicht II. Bedrijfstakverkenningen ten behoeve van regionaal-economisch onderzoek. De zeescheepsnieuwbouwindustrie
- 338 Gerard J. van den Berg
Search behaviour, transitions to nonparticipation and the duration of unemployment
- 339 W.J.H. Groenendaal and J.W.A. Vingerhoets
The new cocoa-agreement analysed
- 340 Drs. F.G. van den Heuvel, Drs. M.P.H. de Vor
Kwantificering van ombuigen en bezuinigen op collectieve uitgaven 1977-1990
- 341 Pieter J.F.G. Meulendijks
An exercise in welfare economics (III)

- 342 W.J. Selen and R.M. Heuts
A modified priority index for Günther's lot-sizing heuristic under capacitated single stage production
- 343 Linda J. Mittermaier, Willem J. Selen, Jeri B. Waggoner, Wallace R. Wood
Accounting estimates as cost inputs to logistics models
- 344 Remy L. de Jong, Rashid I. Al Layla, Willem J. Selen
Alternative water management scenarios for Saudi Arabia
- 345 W.J. Selen and R.M. Heuts
Capacitated Single Stage Production Planning with Storage Constraints and Sequence-Dependent Setup Times
- 346 Peter Kort
The Flexible Accelerator Mechanism in a Financial Adjustment Cost Model
- 347 W.J. Reijnders en W.F. Verstappen
De toenemende importantie van het verticale marketing systeem
- 348 P.C. van Batenburg en J. Kriens
E.O.Q.L. - A revised and improved version of A.O.Q.L.
- 349 Drs. W.P.C. van den Nieuwenhof
Multinationalisatie en coördinatie
De internationale strategie van Nederlandse ondernemingen nader beschouwd
- 350 K.A. Bubshait, W.J. Selen
Estimation of the relationship between project attributes and the implementation of engineering management tools
- 351 M.P. Tummers, I. Woittiez
A simultaneous wage and labour supply model with hours restrictions
- 352 Marco Versteijne
Measuring the effectiveness of advertising in a positioning context with multi dimensional scaling techniques
- 353 Dr. F. Boekema, Drs. L. Oerlemans
Innovatie en stedelijke economische ontwikkeling
- 354 J.M. Schumacher
Discrete events: perspectives from system theory
- 355 F.C. Bussemaker, W.H. Haemers, R. Mathon and H.A. Wilbrink
A (49,16,3,6) strongly regular graph does not exist
- 356 Drs. J.C. Caanen
Tien jaar inflatieneutrale belastingheffing door middel van vermogensaftrek en voorraadaf trek: een kwantitatieve benadering

- 357 R.M. Heuts, M. Bronckers
A modified coordinated reorder procedure under aggregate investment
and service constraints using optimal policy surfaces
- 358 B.B. van der Genugten
Linear time-invariant filters of infinite order for non-stationary
processes
- 359 J.C. Engwerda
LQ-problem: the discrete-time time-varying case
- 360 Shan-Hwei Nienhuys-Cheng
Constraints in binary semantical networks
- 361 A.B.T.M. van Schaik
Interregional Propagation of Inflationary Shocks
- 362 F.C. Drost
How to define UMVU
- 363 Rommert J. Casimir
Infogame users manual
Rev 1.2 December 1988
- 364 M.H.C. Paardekooper
A quadratically convergent parallel Jacobi-process for diagonal
dominant matrices with nondistinct eigenvalues
- 365 Robert P. Gilles, Pieter H.M. Ruys
Characterization of Economic Agents in Arbitrary Communication
Structures
- 366 Harry H. Tigelaar
Informative sampling in a multivariate linear system disturbed by
moving average noise
- 367 Jörg Glombowski
Cyclical interactions of politics and economics in an abstract
capitalist economy

IN 1989 REEDS VERSCHENEN

- 368 Ed Nijssen, Will Reijnders
"Macht als strategisch en tactisch marketinginstrument binnen de distributieketen"
- 369 Raymond Gradus
Optimal dynamic taxation with respect to firms
- 370 Theo Nijman
The optimal choice of controls and pre-experimental observations
- 371 Robert P. Gilles, Pieter H.M. Ruys
Relational constraints in coalition formation
- 372 F.A. van der Duyn Schouten, S.G. Vanneste
Analysis and computation of (n,N) -strategies for maintenance of a two-component system
- 373 Drs. R. Hamers, Drs. P. Verstappen
Het company ranking model: a means for evaluating the competition
- 374 Rommert J. Casimir
Infogame Final Report
- 375 Christian B. Mulder
Efficient and inefficient institutional arrangements between governments and trade unions; an explanation of high unemployment, corporatism and union bashing
- 376 Marno Verbeek
On the estimation of a fixed effects model with selective non-response
- 377 J. Engwerda
Admissible target paths in economic models
- 378 Jack P.C. Kleijnen and Nabil Adams
Pseudorandom number generation on supercomputers
- 379 J.P.C. Blanc
The power-series algorithm applied to the shortest-queue model
- 380 Prof. Dr. Robert Bannink
Management's information needs and the definition of costs, with special regard to the cost of interest
- 381 Bert Bettonvil
Sequential bifurcation: the design of a factor screening method
- 382 Bert Bettonvil
Sequential bifurcation for observations with random errors

- 383 Harold Houba and Hans Kremers
Correction of the material balance equation in dynamic input-output models
- 384 T.M. Doup, A.H. van den Elzen, A.J.J. Talman
Homotopy interpretation of price adjustment processes
- 385 Drs. R.T. Frambach, Prof. Dr. W.H.J. de Freytas
Technologische ontwikkeling en marketing. Een oriënterende beschouwing
- 386 A.L.P.M. Hendrikx, R.M.J. Heuts, L.G. Hoving
Comparison of automatic monitoring systems in automatic forecasting
- 387 Drs. J.G.L.M. Willems
Enkele opmerkingen over het inversificerend gedrag van multinationale ondernemingen
- 388 Jack P.C. Kleijnen and Ben Annink
Pseudorandom number generators revisited
- 389 Dr. G.W.J. Hendrikse
Speltheorie en strategisch management
- 390 Dr. A.W.A. Boot en Dr. M.F.C.M. Wijn
Liquiditeit, insolventie en vermogensstructuur
- 391 Antoon van den Elzen, Gerard van der Laan
Price adjustment in a two-country model
- 392 Martin F.C.M. Wijn, Emanuel J. Bijnen
Prediction of failure in industry
An analysis of income statements
- 393 Dr. S.C.W. Eijffinger and Drs. A.P.D. Gruijters
On the short term objectives of daily intervention by the Deutsche Bundesbank and the Federal Reserve System in the U.S. Dollar - Deutsche Mark exchange market
- 394 Dr. S.C.W. Eijffinger and Drs. A.P.D. Gruijters
On the effectiveness of daily interventions by the Deutsche Bundesbank and the Federal Reserve System in the U.S. Dollar - Deutsche Mark exchange market
- 395 A.E.M. Meijer and J.W.A. Vingerhoets
Structural adjustment and diversification in mineral exporting developing countries
- 396 R. Gradus
About Tobin's marginal and average q
A Note

Bibliotheek K. U. Brabant



17 000 01086034 5