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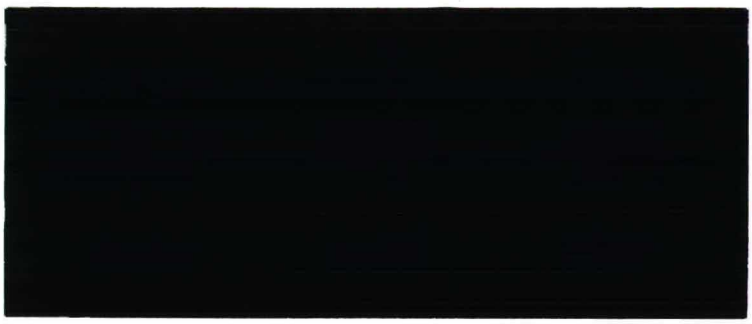

ANALYSIS AND COMPUTATION OF (n,N)-STRATEGIES FOR MAINTENANCE OF A TWO-COMPONENT SYSTEM

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Abstract

In this paper we investigate (n,N)-strategies for the maintenance of a two component series system. Vergin and Scriabin [9] provided some numerical evidence on the near-optimality of this type of policies. In a recent paper Özekici [6] gave a characterization of the structure of the optimal policy and its possible deviations from the (n,N)-structure for the discounted cost criterion. The same kind of structure is shown to hold for the average cost criterion. In view of the complicated form of the optimal policy and the near-optimality of the (n,N)-policies we present a fast computational method to compute the average costs under a given (n,N)-policy. This method is based on a well-know embedding technique. Moreover, a heuristic based on this computational method, is presented by which the optimal values of n and N can be determined.

Key words and phrases: Multicomponent reliability system; preventive and opportunistic replacement; Markov decision processes; control limit rules; computational aspects.

1. Introduction

In this paper we investigate the two component maintenance scheduling problem introduced by Vergin and Scriabin [9]. This problem concerns the maintenance of a series system, consisting of two (identical) components, which are subject to stochastic failures. The components are functioning independent of each other. When one of the components breaks down the system fails which involves break-down costs. To avoid a frequent occurence of this cost preventive replacement is allowed. Additionally we have the option of opportunistic replacement, which refers to the possibility to replace both components, when only one of them has failed or reached its preventive age. This makes sense not only because it usually is cheaper to replace two components simultaneously rather than separately but also due to the series structure of the system. Replacements can only take place at inspection epochs, which are assumed to be equidistant discrete time epochs.

To find an optimal maintenance schedule one might use standard policy-improvement or value iteration techniques from Markov decision theory. However, there are two drawbacks. First of all due to the two dimensional state space (denoting the ages of both components) these standard techniques are in general very time-consuming and even prohibitive for moderate size problems. In the second place the actual optimal policy, found by one of these procedures, may have a rather irregular structure, which makes it necessary to explicitly specify the optimal decision for all possible age-combinations. From a practical point of view one might be interested in policies which are easy to characterize and, on the other hand, close to optimality. Moreover, it is important that for the proposed policy the actual average expected costs per unit time can be computed rather easily. A class of policies satisfying these conditions are the so-called (n,N)-policies. An (n,N)-policy prescribes to replace a component when it has failed or when its age has reached the value N and, if one of the components is replaced, to replace the other simultaneously when its age is greater than or equal to n ($\leq$ N). In this sence an (n,N)-policy seems to be the natural two-dimensional generalization of the well-known one-dimensional control-limit rules.

Vergin and Scriabin [9] have shown by numerical comparisons that (n,N)-policies are close to optimality for a wide range of cost parameters. Özekici [6] provided some additional support for this conclusion based on the analysis of the optimal policy. These results are mentioned in more detail in section 3. The main result of this paper is dealt with in section 4. There we present an exact computational method for the average costs under a given (n,N)-policy. This method is based on a well-known embedding technique (see e.g. Tijms and Van der Duyn Schouten [8]). Finally we use this computational method in section 5 as a building block in a heuristic to determine the best policy within the class of (n,N)-policies.

Recently various authors studied models which are closely related to ours. Epstein and Wilamowsky [2] investigate a system consisting of two components one of which has an exponential lifetime, while the other has a fixed finite lifelength with failure rate equal to zero. A characterization of the optimal replacement policy is given. Opportunistic maintenance is also the subject of a paper by Bäckert and Rippin [1]. They consider a three component system. At failure epochs of one of these components opportunistic replacement of the other components is considered. The performance of three different solution techniques, among which the Markov decision approach, are compared. Finally we mention a paper by Haurie and l'Ecuyer [3]. They analyse the optimal preventive replacement rule over a finite horizon of a parallel system comprised of m identical independent components. There are fixed as well as linear replacement costs but no break-down costs. It is shown that the optimal policy may have some counterintuitive properties with respect to monotonicity in its state vector.

2. Mathematical model and its solution techniques

Consider a series system with two identical and independently operating components 1 and 2. Both components are subject to stochastic failures. The times until failure are stochastic variables with known probability distribution function. The system is inspected only at discrete equidistant time epochs. When upon inspection a component turns out to have failed during the last period it has to be replaced immediately and a breakdown cost is incurred. The time needed for replacement is negligible.

On the other hand when at an inspection epoch both components are still working there are four possible decisions to make: replace none component, replace only component 1, replace only component 2 or replace both components. A replacement in this situation is called a preventive replacement or an opportunistic replacement. The former refers to the opportunity to prevent breakdown cost by timely replacement, the latter refers to the opportunity to prevent cumulation of replacement costs by joint replacement instead of separate replacements. After replacement of a component a new identical component starts with age equal to zero. The lifetime distributions are supposed to have finite support i.e. when a component has reached its maximal lifelenght it certainly will fail during the next period.
The following non-negative costs are incurred:

b := system breakdown cost.
r_1 := replacement cost for a single component (either 1 or 2).
r_{12} := cost of joint replacement of both components 1 and 2.

Note that due to the series structure the same breakdown cost b is incurred when either one or both components fail.

Assumption 1. $r_1 \leq r_{12} \leq 2r_1$ □

Next we consider the aging process of the system. As stated before we assume that both components are independent. For a relaxation of this assumption we refer to the recent paper of Özekici [6]. (see also section 3).
Define

L_i := the lifetime of component i expressed in inspection periods, i=1,2.

Let

$\mu := E(L_i)$, i=1,2 and

$$q_n := P\,(L_i=n \mid L_i \geq n),\ n=0,\ldots,m\ ;\ i=1,2$$

where

$$m := \min\{n: P\,(L_i \geq n+1) = 0\}$$

and

$$p_n := 1 - q_n\ ,\ n=0,\ldots,m.$$

Note that q_n denotes to probability that a component will fail during the next period given that its age equals n at the beginning of this period. We say that a component at an inspection epoch is found in state n when it has not failed and its age equals n. When the component has failed we say that it is in state m+1.

To model this system as a Markov decision process we introduce as a state space

$$S = \{1, 2,\ldots,m+1\}^2$$

where for $(i,j) \in S$ we denote by i(j) the state of the first (second) component.
The set of possible actions A consists of:

i := replace only component i, i=1,2
12 := replace both components
0 := replace no component

The set of feasible actions in state $s \in S$ is denoted by A(s).
The one-step cost function c(s,a), $s \in S$, $a \in A$ is easily composed from the given cost parameters b, r_1 and r_{12}. For example

$$c\ ((m+1,j),12) = b + r_{12}\ ,\ 1 \leq j \leq m+1$$

$$c\ ((i,j),12) = r_{12}\ ,\ 1 \leq i,j \leq m$$

$$c((i,j),1) = r_1 \ , \ 1 \le i,j \le m$$

Finally the one-step transition probabilities $p_{st}(a)$, $s,t \in S$, $a \in A$ are completely determined by the survival probabilities p_n. For example

$$p_{(i,j),(i+1,j+1)}(0) = p_i p_j \ , \ 1 \le i,j \le m-1$$

$$p_{(i,j),(m+1,j+1)}(0) = q_i p_j \ , \ 1 \le i \le m, \ 1 \le j \le m-1$$

$$p_{(i,j),(1,1)}(12) = p_0^{\,2} \ , \ 1 \le i,j \le m+1.$$

Assumption 2. $0 = p_m < p_{m-1} \le \ldots \le p_1 \le p_0 < 1$. □

As far as the lifetime distribution of the components is concerned knowledge of the sequence $(p_k)_{k=0}^{m}$ suffices. It is irrelevant whether the lifetimes themselves have a discrete or continuous probability distribution. □

Denote by $X(t) = (X_1(t), X_2(t))$ the states of both components at the t-th inspection epoch. For every stationary policy R the process $\{X(t), t = 0,1,2,..\}$ constitutes a discrete-time Markov chain on S. It is easy to see that (m+1,m+1) is a positive recurrent state under every stationary policy R. Due to assumption 2 it follows that (m+1,m+1) can be reached from every starting state (i,j). This implies that there exists only one recurrent class under every policy R. Let C(R) denote this recurrent class and let D(R) be the set of transient states under policy R. Note that state $(i,j) \in D(R)$ iff it cannot be reached from (m+1,m+1). Finally we note that X(t) is aperiodic on C(R) for every policy R since we can return to (m+1,m+1) in one step. We are now interested in that stationary policy R that minimizes

$$g(R) := \lim_{n\to\infty} E_R \frac{1}{n+1} \sum_{t=0}^{n} c(X(t), R(X(t)))$$

The standard tools to solve this Markov decision problem numerically are the policy-iteration and the value-iteration algorithm (see e.g. Tijms [7]). From a numerical point of view the value-iteration algorithm is

preferred in particular for big state spaces. This algorithm recursively solves the finite horizon Bellman optimality equation

$$(2.1) \quad V_n(s) = \min_{a \in A(s)} \{c(s,a) + \sum_{t \in S} p_{st}(a)\, V_{n-1}(t)\}, \quad s \in S,\ n \geq 1.$$

On the other hand we know from Markov decision theory (see e.g. Howard [4] or Tijms [7]) that for any stationary policy R the average costs g(R) and the so-called relative values $v_R(s)$, $s \in S$ are the unique solution to the set of linear equations

$$(2.2) \quad \begin{cases} v_R(s) = c(s,R(s)) - g(R) + \sum_{t \in S} p_{st}(R(s))\, v_R(t), & s \in S \\ v_R(m+1,m+1) = 0. \end{cases}$$

A step in the general policy improvement algorithm proceeds as follows. First we define for a given policy R the policy improvement quantity

$$(2.3) \quad T_R(s,a) := c(s,a) - g(R) + \sum_{t \in S} p_{st}(a)\, v_R(t)$$

For any stationary policy $\bar{R}$ we have (see Tijms [7])

$$g(\bar{R}) \leq g(R)$$

whenever

$$T_R(s,\bar{R}(s)) \leq v_R(s) \quad \text{for all } s \in S.$$

In section 5 we use this result to derive a policy improvement procedure within the class of (n,N)-policies.

To conclude this section we note that the average optimal policy R^* can be found as the minimizing action in the average optimality equation (see e.g. Tijms [7]

$$(2.4)\qquad \begin{cases} v(s) = \min\limits_{a\in A(s)} \{c(s,a) - g + \sum\limits_{t\in S} p_{st}(a)\, v(t)\} \\ \\ v(m+1,m+1) = 0 \end{cases}$$

3. The structure of the optimal policy

Up to now the most complete characterization of the structure of the optimal policy has been given by Özekici [6]. Özekici's model is more general than ours in the followings aspects. The system considered by Özekici consists of n different components whose ages develop over time as an increasing Hunt process, which implies amongst others that the ages of the components are not necessarily independent and may increase at different rates as well as by jumps. Our cost structure is also a special case of the cost structure assumed by Özekici: a general maintenance/replacement cost function depending on the components to be replaced (but not on their ages) and an operating cost function of the age-vector. Under fairly general assumptions on both cost functions and aging process Özekici shows that for the optimal replacement policy the four regions of the state space where the four different actions 0, 1, 2 and 12 are optimal are connected subsets of the state space with certain regularity conditions for the boundaries. A typical example of the optimal policy for two components in Özekici's model is shown in figure 1

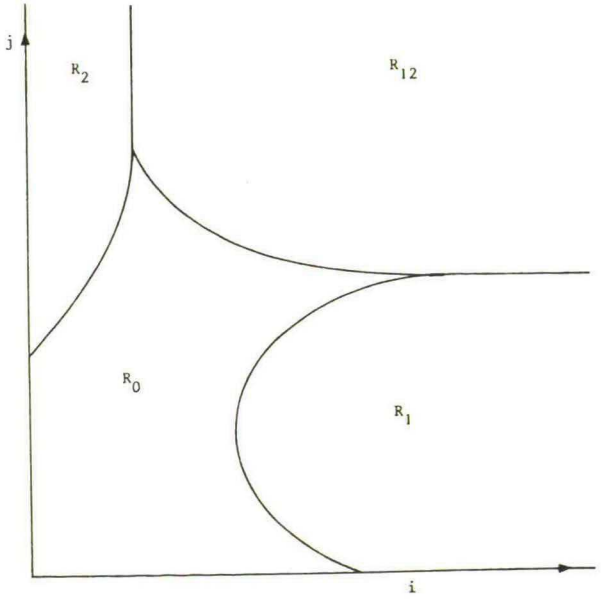

Figure 1 The general form of the optimal policy in Özekici's model.

Let us denote the region where the optimal policy prescribes action a by R_a, $a \in A$.
The special characterics of the optimal policy are:

(i) the boundaries between R_1 and R_{12} and between R_2 and R_{12} are straight lines.

(ii) the other boundaries do not necessarily have a convex or concave shape

(iii) for fixed age i of one component the replacement of the other component is controlled by a control limit rule i.e.

$(i,j) \in R_2 \cup R_{12}$ implies $(i,j+k) \in R_2 \cup R_{12}$ for $k \geq 0$.

(iv) If $(i,j) \in R_2$ than $(i,k) \in R_2$ for $m+1 \geq k \geq j$

(v) If $(i,j) \in R_{12}$ than $(k,\ell) \in R_{12}$ for all $m+1 \geq k \geq i$ and all $m+1 \geq \ell \geq j$

(vi) it may occur that $(i,j) \in R_2$ and $(i+k,j) \in R_0$ for some $k \geq 0$ (similarly for R_1 and R_0)

(vii) it may occur that $(i,j) \in R_0$ and $(i,j+k) \in R_1$ for some $k \geq 0$ (similarly for R_0 and R_2)

The characteristics (i) up to (v) are proven by Özekici for the discounted cost criterion and below we will show that they hold for the average cost criterion. From extensive numerical experiments we have found situations where (ii) and (vi) occur for the average cost criterion (see figure 2). The situation mentioned under (vii) did not occur. We conjecture that it cannot occur, although we were not able to prove it.

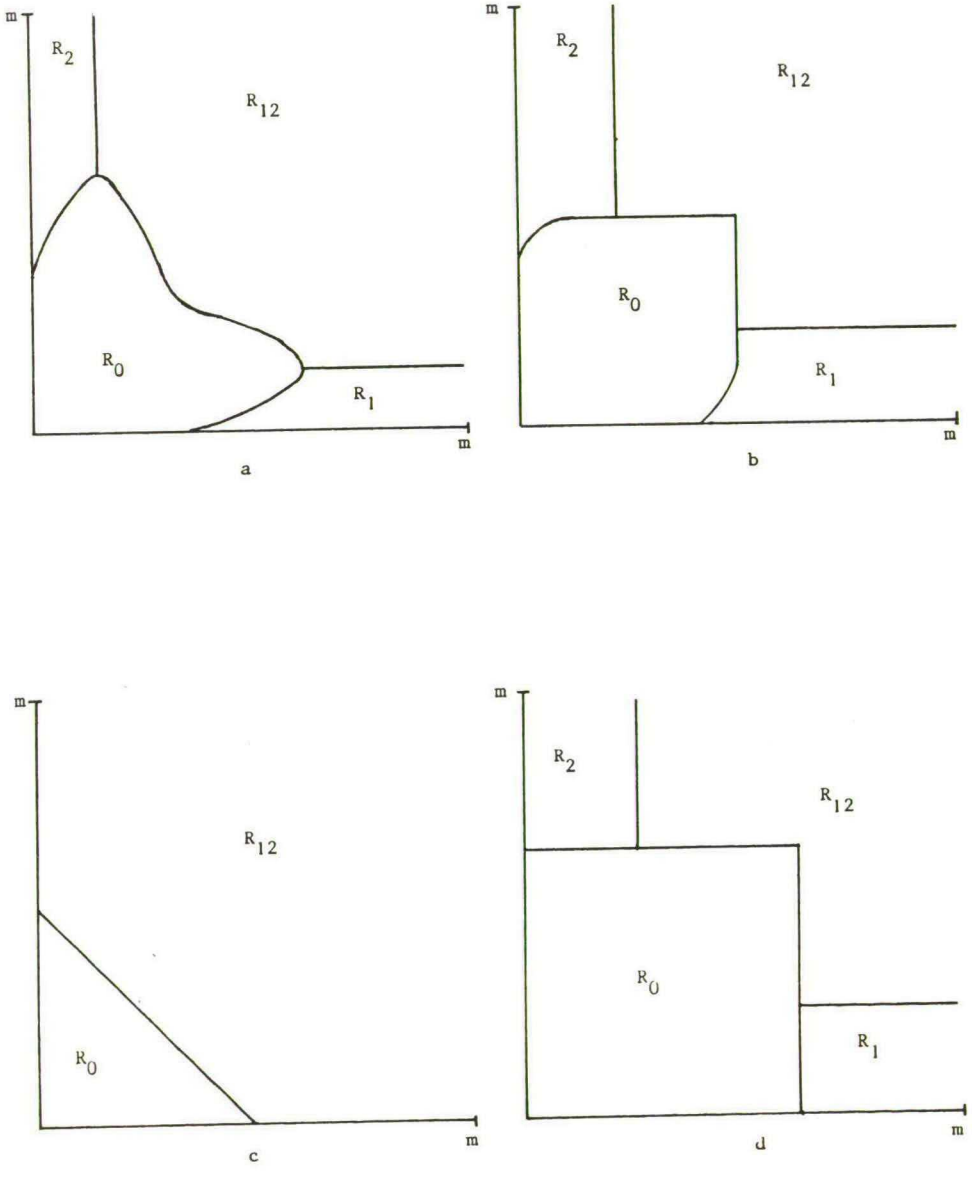

Figure 2 Some specific forms of the optimal maintenance policy.

Next we derive the properties mentioned above of the average optimal policy. Let us specify the Bellman optimality equation (2.1) for our model. Define for $0 \le i,j \le m$

$$(3.1)\quad W_{n+1}(i,j) := p_i p_j\, V_n(i+1,j+1) + (1-p_i)p_j\, V_n(m+1,j+1) + p_i(1-p_j)\, V_n(i+1,m+1) + (1-p_i)\,(1-p_j)\, V_n(m+1,m+1)$$

Now relation (2.1) can be rewritten as follows

$$(3.2)\quad V_{n+1}(i,j) = \min\{W_{n+1}(i,j),\ r_1 + W_{n+1}(0,j),\ r_1 + W_{n+1}(i,0),\ r_{12} + W_{n+1}(0,0)\}$$

$$(3.3)\quad V_{n+1}(i,m+1) = \min\{b + r_1 + W_{n+1}(i,0),\ b + r_{12} + W_{n+1}(0,0)\}$$

$$(3.4)\quad V_{n+1}(m+1,j) = \min\{b + r_1 + W_{n+1}(0,j),\ b + r_{12} + W_{n+1}(0,0)\}$$

$$(3.5)\quad V_{n+1}(m+1,m+1) = b + r_{12} + W_{n+1}(0,0).$$

Similarly the average optimality equation (2.4) can be rewritten into

$$(3.6)\quad v(i,j) = \min\{-g + w(i,j),\ r_1 - g + w(0,j),\ r_1 - g + w(i,0),\ r_{12} - g + w(0,0)\}$$

$$(3.7)\quad v(i,m+1) = \min\{b + r_1 - g + w(i,0),\ b + r_{12} - g + w(0,0)\}$$

$$(3.8)\quad v(m+1,j) = \min\{b + r_1 - g + w(0,j),\ b + r_{12} - g + w(0,0)\}$$

$$(3.9)\quad v(m+1,m+1) = b + r_{12} - g + w(0,0)$$

where for $0 \le i,j \le m$

$$(3.10)\quad w(i,j) = p_i p_j\, v(i+1,j+1) + (1-p_i)\, p_j\, v(m+1,j+1) +$$

$$+ p_i(1-p_j)\, v(i+1,m+1) + (1-p_i)(1-p_j)\, v(m+1,m+1)$$

Theorem 1 The average optimal policy R^* is symmetric.

Proof From (3.2) unto (3.5) it easily follows by induction on n that $V_n(.,.)$ is symmetric starting with $V_0(.,.) \equiv 0$. Since the solution $v(.,.)$ of the average optimality equations (3.6) unto (3.9) satisfy (see Tijms [7])

$$v(i,j) - v(k,\ell) = \lim_{n\to\infty} \{V_n(i,j) - V_n(k,\ell)\}$$

it follows that $v(.,.)$ is symmetric, which in turn implies that the minimizing action in the right hand side of (3.6) unto (3.9) is symmetric. The observation that the minimizing action in the right hand side of the average optimality equation consitutes an average optimal policy (see Tijms [7]) completes the proof. □

Theorem 2 Any solution of the equations (3.6) unto (3.9) satisfy

$$(3.11)\quad v(i,j) \le v(i,j+1)\ ,\ 1 \le i \le m+1\ ,\ 1 \le j \le m$$

and

$$(3.12)\quad v(i,j) \le v(i+1,j)\ ,\ 1 \le i \le m\ ,\ 1 \le j \le m+1$$

Proof By theorem 1 it is sufficient to show that (3.11) holds. From (3.6) unto (3.9) it follows that for $1 \le i,j \le m+1$

$$(3.13)\quad v(i,j) \le b + r_{12} - g + w(0,0) = v(m+1,m+1)$$

Rewrite (3.10) into

$$(3.14)\quad w(i,j) = v(m+1,m+1) - p_i p_j\,(v(m+1,m+1)-v(i+1,j+1)) -$$

$$- (1-p_i)p_j\,(v(m+1,m+1)-v(m+1,j+1)) - p_i(1-p_j)\,(v(m+1,m+1) -$$

$$- v(i+1,m+1))$$

From (3.6) unto (3.9) it follows that (3.11) trivially holds for j=m and all $1 \le i \le m+1$. Now suppose that (3.11) holds for all $k+1 \le j \le m$ and all $1 \le i \le m+1$.
To show that (3.11) also holds for j=k and all $1 \le i \le m+1$ it suffices to prove that for all $0 \le i \le m$

$$(3.15)\quad w(i,k) \le w(i,k+1)$$

From (3.14) it follows that

$$w(i,k) - w(i,k+1) = -\Delta_1 - \Delta_2 + \Delta_3$$

where

$$\Delta_1 := p_i p_k\,(v(m+1,m+1) - v(i+1,k+1)) - p_i p_{k+1}\,(v(m+1,m+1) -$$

$$- v(i+1,k+2))$$

$$\Delta_2 := (1-p_i)p_k\,(v(m+1,m+1) - v(m+1,k+1)) - (1-p_i)p_{k+1}\,(v(m+1,m+1) -$$

$$- v(m+1,k+2))$$

$$\Delta_3 := p_i(p_k-p_{k+1})\,(v(m+1,m+1) - v(i+1,m+1))$$

From assumption 2 and the induction hypothesis it follows that $\Delta_1 \ge 0$ and $\Delta_2 \ge 0$. Next $\Delta_3 \ge 0$ follows from assumption 2 and relation (3.13). Finally we note that by the induction hypothesis

$$\Delta_1 \geq p_i p_k \, (v(m+1,m+1) - v(i+1,k+2)) - p_i p_{k+1} \, (v(m+1,m+1) -$$

$$- v(i+1,k+2)) = p_i (p_k - p_{k+1}) \, (v(m+1,m+1) - v(i+1,k+2))$$

$$\geq p_i (p_k - p_{k+1}) \, (v(m+1,m+1) - v(i+1,m+1))$$

$$= \Delta_3$$

which yields (3.15) □

Corollary 1 The average optimal policy R^* with regions R_a^*, $a \in A$ has the following properties:

(i) If $j > i$ than the optimal policy can be chosen such that $R^*(i,j) \neq 1$

(ii) If $(i,j) \in R_2^*$ than $(i,j+1) \in R_2^*$

(iii) If $(i,j) \in R_{12}^*$ than $(k,\ell) \in R_{12}^*$ for all $k \geq i$ and all $\ell \geq j$.

Proof

(i) For $j > i$ we have

$$w(i,0) = w(0,i) \leq w(0,j).$$

Hence

$$r_1 - g + w(i,0) \leq r_1 - g + w\,(0,j)$$

which implies that action 2 is always as good as action 1 in the optimization of the right hand side of (3.6)

(ii) For $(i,j) \in R_2^*$ it follows that

$$r_1 + w(i,0) \leq \min \{w(i,j),\ r_1 + w(0,j),\ r_{12} + w(0,0)\}$$

$$\leq \min \{\ w(i,j+1),\ r_1 + w(0,j+1),\ r_{12} + w(0,0)\}$$

Hence $(i,j+1) \in R_2^*$.

(iii) The proof of (iii) proceeds similarly as (ii). □

Note that corollary 1 implies that the boundary between R_2^* and R_{12}^* is a straight line.

Besides the theoretical results containted in corollary 1 additional insight can be obtained by numerical experiments. By the value iteration algorithm we generated optimal policies under a wide variety of cost parameters and lifetime distributions. This general purpose algorithm works satisfactorily on a personal computer for a state space upto $m = 15$. The number of iterations required varied between 20 and 100.

In the appendix we present the numerical results. Apart from the optimal value of the average costs g^* we also give the average costs $g^*_{(n,N)}$ of the best policy of (n,N)-type. A policy R is called of (n,N)-type if there exist natural numbers $n \leq N$ such that $R(i,j) = 0$ if $0 \leq i,j \leq N$, $R(i,j) = 1$ if $i \geq N$, $j < n$; $R(i,j) = 2$ if $j \geq N$, $i < n$ and $R(i,j) = 12$ otherwise (see figure 2d).

A summary of the numerical results is presented in table 1 below where the frequency of occurrence of different values of $g^*_{(n,N)} - g^*$ as a percentage of g^* is tabulated.

%	0	0-0.1	0.1-0.2	0.2-0.3	0.3-0.4	0.4-0.5	0.5-0.6	>0.6
#	30	6	5	2	0	1	1	0

Table 1 Frequency of relative differences between $g^*_{(n,N)}$ and g^*.

In those cases where the difference between $g^*_{(n,N)}$ and g^* was equal to zero, the optimal policy was not necessarily of (n,N)-type. It sometimes occurs that the optimal policy itself (as it is generated by the value iteration algorithm) is not of (n,N)-type, but can be put into this form by modification of the actions on the set of transient states. It is not hard to see that such a modification only affects the value of the relative values on the transient states but not of the average costs itself. Moreover such a modification does not change the set of recurrent states.
We conclude this section with a remark concerning another experimental finding. We generated a number of surival distributions by a discretization of a given Weibull distribution. We investigated the effect of a decreasing grid size (increasing number of possible states) on the form of the optimal policy. It turned out that the value of $g^*_{(n,N)}$ approached the value of g^*, although we could not conclude that the form of the optimal policy converged to a (n,N)-form.

Both the theoretical and numerical results reported in this section justify a special attention to policies of (n,N)-type, in particular because these kind of policies are easy to handle in practice. Therefore we present in the next section an embedding technique by which the average costs under a given (n,N)-policy can be efficiently computed. This technique is used as a building block in a heuristical algorithm to compute the best (n,N)-policy. This algorithm will be presented in section 5.

4. Analysis of a (n,N)-policy via embedding

In this section we present an efficient numerical method to compute the average costs g(n,N) under a fixed policy of (n,N)-type. This method turns out to be faster than the application of the value iteration method for a fixed policy. Vergin and Scriabin [9] suggest simulation as an effective method to compute g(n,N).
In this section we consider a fixed (n,N)-policy with average costs g(n,N).
Let $X(t) = (X_1(t),X_2(t))$ denote the Markov chain on S describing the states of both components under the given (n,N)-policy and suppose that $X(0) = (m+1,m+1)$. Denote by T_n, $n \geq 1$ the epoch of the n-th replacement

(either a single replacement of one of both types or a joint replacement) and define

$$Z_0 := X(0)$$

and

$$Z_n := X(T_n),\ n \geq 1$$

Then $(Z_n)_{n=0}^{\infty}$ constitutes a Markov chain on state space

$$\tilde{S} := \{(i,N) : 1 \leq i \leq N\} \cup \{(N,j) : 1 \leq j \leq N\} \cup \{(i,m+1) : 1 \leq i \leq N\} \cup$$

$$\cup \{(m+1,j) : 1 \leq j \leq N\} \cup \{(m+1,m+1)\}$$

Note that state (i,N) with $i < N-n+1$ can only be entered by $(Z_n)_{n=0}^{\infty}$ from (N,N-i) or (m+1,N-i). However, since $N-i > n-1$ the policy prescribes action 12 in (N,N-i) and (m+1,N-i). Hence (i,N) is not accessible under the given policy, so that the state space can be reduced to

$$E := \{(i,N) : N-n+1 \leq i \leq N\} \cup \{(N,j) : N-n+1 \leq j \leq N\} \cup$$

$$\cup \{(i,m+1) : 1 \leq i \leq N\} \cup \{(m+1,j) : 1 \leq j \leq N\} \cup \{(m+1,m+1)\}.$$

Note that this state space is of dimension $2(n+N) \leq 4m$, while S is of dimension m^2.
$(Z_n)_{n=0}^{\infty}$ is an irreducible, aperiodic, positive recurrent Markov chain on E. Let p(.,.) denote the one-step transition probabilities of $(Z_n)_{n=0}^{\infty}$ and $\pi(.)$ its stationary distribution.
For $s \in E$ we define

$T(s)$:= the time until next replacement, given that the present replacement is carried out in state s

$$\tau(s) := E\ T(s)$$

$c(s)$:= the expected costs incurred until next replacement, given that the present replacement is carried out in state s.

Then we have by the theory of regenerative processes (see e.g. Tijms et al. [8])

$$(4.1) \qquad g(n,N) = \frac{\sum_{s\in E} c(s)\pi(s)}{\sum_{s\in E} \tau(s)\pi(s)}$$

Before we derive explicit expressions for $c(s)$ and $\tau(s)$ we note that because of symmetry it suffices to compute both functions in the states (i,N) for $i \leq N$ and $(i,m+1)$ of E.
For the function $c(s)$ we have

$$c(i,N) = r_1 \ , \ c(i,m+1) = b + r_1 \quad \text{for } i<n$$

$$c(i,N) = r_{12} \ , \ c(i,m+1) = b + r_{12} \quad \text{for } i\geq n$$

For $i<n$ we have

$$(4.2) \qquad \tau(i,N) = \tau(i,m+1) = \sum_{k=1}^{\infty} P(T(i,m+1)\geq k) = 1 + \sum_{k=2}^{N-i} r(k-1,i)\, r(k-1,0)$$

where

$$r(k,i) := \prod_{\ell=0}^{k-1} p_{i+\ell}$$

Note that $r(k,i)$ equals $P(L\geq i+k \mid L\geq i)$, where L denotes the lifetime of a single component.
Similarly we find for $i\geq n$

$$(4.3) \qquad \tau(i,N) = \tau(i,m+1) = \tau(m+1,m+1) = 1 + \sum_{k=2}^{N} r^2(k-1,0).$$

Finally the stationary distribution can be obtained by solution of the set of stationary equations $\pi=\pi P$ together with the normalizing equation

$$\sum_{s\in E} \pi(s) = 1$$

This set of equations can be solved by standard procedures (see Tijms [7]) once the one step transition probabilities p(s,t) are known.
In figure 3 below the possible transitions from (i,N) are shown.

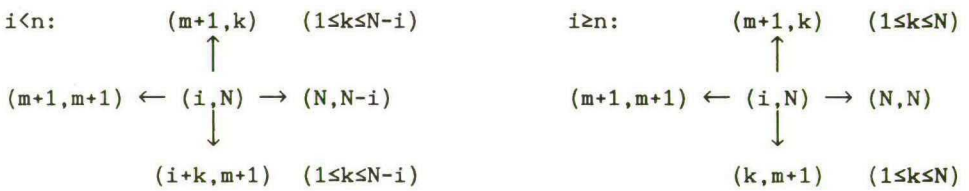

Figure 3 The possible one-step transitions from (i,N)

For i<n the transition probabilities are given by

$$p((i,N),\ (N,N-i)) = r(N-i,i)\ r(N-i,0)$$

$$p((i,N,\ (i+k,m+1)) = r(k,i)\ r(k-1,0)\ (1-p_{k-1})\quad (1\le k\le N-i)$$

$$p((i,N),\ (m+1,k)) = r(k-1,i)\ (1-p_{i+k-1})\ r(k,0)\quad (1\le k\le N-i)$$

$$p((i,N),\ (m+1,m+1)) = 1 - \sum_{s\neq(m+1,m+1)} p((i,N),s)$$

For p((i,N),s) with i≥n we get the same expressions with i=0 substituted in the right hand sides. Finally we note that

(4.4) $\quad p((i,m+1,s) = p((i,N),s)$ for all i.

Instead of the solution of the stationary equations $\pi=\pi P$ by a standard numerical procedure, like Gauss-Seidel, we may follow another approach to find the value of g(n,N), namely the solution of the optimality equation corresponding with the semi-Markov decision process in which decisions can be taken at epochs of entrance into the states N and m+1 and where the decisions are prescribed by the given (n,N)-rule.

This yields an "optimality" equation

$$(4.5)\qquad \begin{cases} v(s) = c(s) - g\,\tau(s) + \sum_{t\in E} p(s,t)\,v(t) \quad , s\in E. \\ \\ v(m+1,m+1) = 0. \end{cases}$$

This system can be solved by the value-iteration method for semi-Markov decision processes (see Tijms [7]). Again we emphasize that the solution of (20) and (21) is much easier than solution of (3) because of the reduction of the state space. In section 5 we will show that (4.5) can even be reduced to a system of only n equations.
The choice between both methods is rather arbitrary and may depend on the preference of the user. The solution of the steady state equations yields the stationary distribution, while solution of (4.5) by the value iteration method yields bounds on the value of g(n,N) during the computational procedure. In our calculations we used the value iteration method.

5. A Heuristic for the optimal (n,N)-policy

In this section we describe a heuristic to compute the optimal replacement policy within the class of (n,N)-policies. In this algorithm the method to compute g(n,N) for fixed (n,N)-policy as described in the previous section is used as a building block. The main idea of the algorithm is contained in the following three steps.

Step 1. Choose good starting values for n and N. Go to 2.

Step 2. Compute g(n,N) and the relative values $v_{(n,N)}(s)$, $s\in S$ and go to 3.

Step 3. Use a special form of the policy improvement procedure to establish whether improved values of n and N exist. If so go to 2 else stop.

Below we will elaborate on these steps in more detail.

Step 1. In step 1 we use as an initial choice of n and N the optimal control limit T^* for the one component replacement model with breakdown costs

b and replacement costs r_1. By standard arguments from renewal theory it can be shown that T^* is the minimizing argument of

$$g(T) := \frac{r_1 + b \sum_{n=0}^{T} a_k}{\sum_{k=0}^{T} \sum_{\ell=k}^{m} a_\ell}$$

where

$$a_k := P(L_i=k),\ k=0,1,2,\ldots;\ i=1,2.$$

Using assumption 2 it is easily verified that T^* is the smallest natural number k satisfying

$$(5.1) \quad q_k \sum_{i=0}^{k-1} \sum_{\ell=i}^{m} a_\ell - \sum_{i=0}^{k-1} a_i \geq r_1/b$$

A natural number k satisfying (5.1) exists if

$$(5.2) \quad \mu - 1 \geq r_1/b$$

If (5.2) is not satisfied we have $T^* = m+1$.

Step 2. In step 2 the g(n,N) as well as the relative values $v_{(n,N)}(s)$, $s \in S$ are computed for a given (n,N)-policy. First we note that apart from the set of equations (2.2) the g(n,N) and $v_{(n,N)}(s)$, $s \in E$ are also the unique solution to the embedded set (4.5). For a proof of this result we refer to Tijms [7] (pp. 229-230).
So we first solve (4.5). At this point it is worthwhile to note that (4.5) becomes a rather small set of only n equations with n unknowns due the special structure of our problem. To prove this we note that

$$v(i,m+1) = b + r_2 - g\,\tau(i,m+1) + \sum_{t \in E} p((i,m+1),t)\, v(t),\ 1 \leq i < n$$

and

$$v(i,N) = r_2 - g\,\tau(i,N) + \sum_{t\in E} p((i,N),t)\, v(t),\quad 1 \le i < n$$

while

$$v(i,m+1) = b + r_{12} - g\,\tau(i,m+1) + \sum_{t\in E} p((i,m+1),t)\, v(t),\quad n \le i \le N \tag{5.3}$$

and

$$v(i,N) = r_{12} - g\,\tau(i,N) + \sum_{t\in E} p((i,N),t)\, v(t),\quad n \le i \le N.$$

From (4.2), (4.3) and (4.4) we conclude that

$$v(i,m+1) = b + v(i,N) \quad \text{for all } 1 \le i \le N$$

Moreover we see from (4.3), (4.4) and the fact that $p((i,N), s)$ for $i \ge n$ is independent of i that the right hand side of (5.3) is also independent of i.

This, together with the symmetry of $v(.,.)$ (see proof of theorem 1) reduces (4.5) to a set of n equations.

Next the values of $v_{(n,N)}(s)$ for $s\in E^c$ can be computed by single pass calculations from $g(n,N)$ and $\{v_{(n,N)}(s) : s\in E\}$ as follows. Deleting in the notation the dependency on the (n,N)-policy we have

$$v(i,m+1) = b + r_1 - g + w(i,0) = b + v(i,j)\ ,\ 1 \le i < n;\ N \le j \le m$$

and

$$v(i,m+1) = b + r_{12} - g + w(0,0) = b + v(i,j)\ ,\ n \le i \le N;\ N \le j \le m$$

Hence we conclude that

$$v(i,j) = v(i,m+1) - b \quad \text{for all } 1 \le i \le N;\ N \le j \le m.$$

Similarly

$$v(i,j) = v(m+1,j) - b \quad \text{for} \quad N \leq i \leq m;\ 1 \leq j \leq N.$$

Next we note that

$$v(i,j) = v(m+1,m+1) - b \quad \text{for} \quad N < i \leq m;\ N < j \leq m$$

Finally for $1 \leq i,j < N$ the values $v(i,j)$ can be recursively computed starting at the boundaries $\{(N,j) : 1 \leq j \leq N\}$ and $\{(i,N) : 1 \leq i \leq N\}$ and proceeding downwards along diagonals (see figure 4).

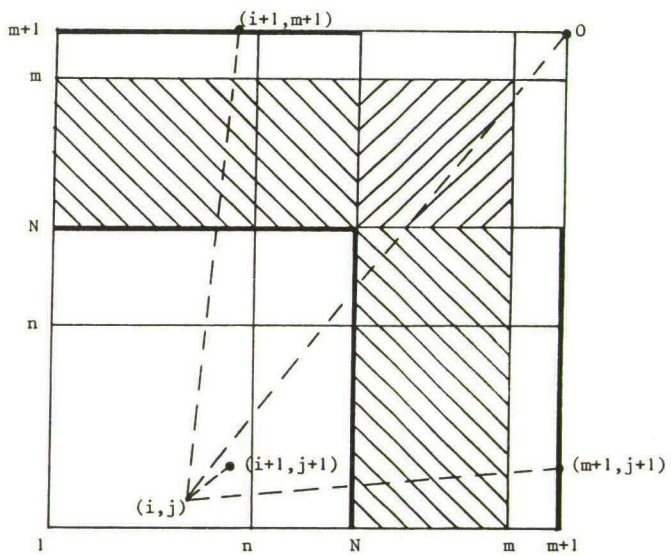

Figure 4 The computation of $v(i,j)$.

Step 3. With $g(n,N)$ and $v_{(n,N)}(s)$, $s \in S$ obtained in step 2 we apply the policy improvement procedure described at the end of section 2 in the following way. Since we want to stick to (n,N)-policies we modify the general improvement procedure in such a way that only (n,N)-policies will be encountered. First we investigate whether an improvement is obtained by decreasing n to n-1. This implies that we change the action in the states

$\{(n-1,j),\ N \le j \le m+1\}$ from 2 into 12 and in the states $\{(i,n-1),\ N \le i \le m+1\}$ from 1 into 12. In all other states no change of action occurs. Hence we compute the test value $T_{(n,N)}(s,12)$ (see (2.3)) for

$$s \in \{(n-1,j),\ N \le j \le m+1\} \cup \{(i,n-1),\ N \le i \le m+1\}$$

which yields:

$$T_{(n,N)}((n-1,j),12) = r_{12} - g + w(0,0)$$

while

$$v_{(n,N)}(n-1,j) = r_2 - g + w(n-1,0)$$

($w(.,.)$ is defined by (3.10))
Since neither $T_{(n,N)}((n-1,j),12)$ nor $v_{(n,N)}(n-1,j)$ depends on j we conclude that either

$$(5.4)\quad T_{(n,N)}((n-1,j),12) < v_{(n,N)}(n-1,j) \quad \text{for } \underline{\text{all}} \quad N \le j \le m+1$$

or

$$(5.5)\quad T_{(n,N)}((n-1,j),12) \ge v_{(n,N)}(n-1,j) \quad \text{for } \underline{\text{all}} \quad N \le j \le m+1.$$

If (5.4) holds (n-1,N) yields an improvement over the policy (n,N). If, on the contrary (5.5) is satisfied we conclude that the policy (n,N) is not improved by (n-1,N). So we can compare the policies (n,N) and (n-1,N) without computation of g(n-1,N). If (n-1,N) yields an improvement over (n,N) we continue by comparing the policy (n,N) with (n-2,N) and continue is this way until no improvement is obtained any more. On the other hand if (5.5) holds we compare (n,N) with (n+1,N) in a similar way.
Next we consider possible improvements of the value of N, with n fixed. This turns out to be more delicate than changing n. A modification of (n,N) into (n,N+1) yields a change of action on the set $\{(i,N) : 1 \le i \le N\} \cup \{(N,j) : 1 \le j \le N\}$.

For $1 \leq i \leq N$ we have

$$(5.6) \quad T_{(n,N)}((i,N),0) = -g + w(i,N)$$

while

$$(5.7) \quad v_{(n,N)}(i,N) = \begin{cases} r_1 - g + w(i,0) & \text{for } i < n \\ r_{12} - g + w(0,0) & \text{for } i \geq n \end{cases}$$

From (5.6) and (5.7) we conclude that in general no comparison between $T_{(n,N)}((i,N),0)$ and $v_{(n,N)}(i,N)$ is possible uniformly in $1 \leq i \leq N$. Hence no unambigions comparison between the policies (n,N) and (n,N+1) can be made based on this policy improvement step. Indeed this ambiguity did occur in our numerical examples. Therefore in our algorithm we compare (n,N) and (n,N+1) directly on base of g(n,N) and g(n,N+1).

The complete algorithm now reads as follows.

Algorithm

Step 0. Determine T^* and choose $n = N = T^*$. Go to step 1.

Step 1. Compute $g(n,N)$ and $v_{(n,N)}(s)$, $s \in E$ and go to step 2.

Step 2. Compute $v_{(n,N)}(s)$, $s \in S$ and go to step 3.

Step 3. Apply the improvement procedure to compare (n,N) and (n-1,N). If improvement is established compare (n,N) with (n-2,N) and continue in this way until no further improvement is obtained. Set n equal to the last value for which improvement was established and go to step 1. If (n-1,N) is not better than (n,N) apply the policy improvement procedure to compare (n,N) and (n+1,N) and proceed analogously.
If neither (n-1,N) nor (n+1,N) is better than (n,N) than go to step 4.

Step 4. Compute g(n,N+1). If g(n,N+1) < g(n,N) then put N := N+1 and go to step 1 else compute g(n,N-1). If g(n,N-1) < g(n,N) and n < N then put N := N-1 and go to step 1. If g(n,N-1) < g(n,N) and n=N then put N := N-1 and n := n-1 and go to step 1 else go to step 5.

Step 5. Perform an overall policy improvement step using (2.2) and (2.3) and stop.

The algorithm works quite satisfactorily. In table 2 below the computation times for several numerical examples are compared with those for the value-iteration algorithm. The numerical data used in this table are (5,4,5) and (5,7,10) for (b,r_1,r_{12}) while the lifetime distributions are given by $p^{(i)}$, i=1,...,5 as given in the appendix. We have made the comparison with the value-iteration algorithm because this algorithm usually works better on large state spaces than the policy iteration method.

	(5,4,5)	(5,7,10)
$p^{(1)}$	4:18 (18:34)	7:03 (17:41)
$p^{(2)}$	4:18 (47:45)	6:10 (36:45)
$p^{(3)}$	4:88 (39:28)	7:14 (41:25)
$p^{(4)}$	20:10 (14:01)	31:40 (24:01)
$p^{(5)}$	8:09 (3:07:62)	8:62 (1:22:56)

Table 2 Computation times of the algorithm in hundreds of seconds. In brackets are given the computation times according to the value iteration algorithm, yielding the overall optimal policy.

Although our algorithm theoretically can stick to a local minimum this did not occur in our numerical examples.
Note that our algorithm is a hybrid method containing elements from the value iteration method (steps 1 and 2), from the policy iteration method (step 3) as well as brute comparisons (step 4).

Step 1 is in general the most time consuming step, in particular when m is large. Therefore we emphasize that this step has to be carried out only once each passage through step 3.

Step 5 can be carried out as a final step in the algorithm in order to check the final (n,N)-policy upon overall optimality. In most of our numerical experiments we only needed just one step in the overall policy improvement procedure starting at the optimal (n,N)-policy to get the overall optimal policy.

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APPENDIX Numerical results

In the tables below we present numerical results for a number of costs parameters (b, r_1, r_{12}) and probability vectors $p^{(i)}$.
In the first column we see whether the optimal strategy has the (n,N)-form or not (y/n) or (n,N) on the recurrent class (y'). From the 4th and 5th column we can make a comparison between the optimal strategy and the strategy $(n,N)^*$, where the 5th column gives the percentage increases of $g(n,N)^*$ over g^*. In addition we mention in the last column the number of times the average cost of a policy is calculated by the heuristic.

$p^{(1)}$ = (0.80, 0.80, 0.75, 0.66, 0.55, 0.25, 0.15, 0.10, 0.05, 0.01) (m=10)

b	r_1	r_{12}		$(n,N)^*$	T^*	g^*	%	#
5	1	1.6	n	(2,3)	3	2.613	0.19	(4)
5	2	3	n	(2,4)	4	3.212	0.03	(4)
5	2	4	n	(4,4)	4	3.436	0.00	(3)
5	4	5	y'	(2,5)	5	4.078	0.00	(4)
5	4	7.5	n	(4,5)	5	4.645	0.06	(4)
5	7	8	y	(2,11)	5	5.347	0.00	(7)
5	7	10	y'	(3,11)	5	5.887	0.00	(6)
5	7	13	y'	(5,11)	5	6.399	0.00	(6)
5	12	18	y'	(4,11)	5	8.703	0.00	(4)

$p^{(2)}$ = (0.96, 0.92, 0.87, 0.77, 0.60, 0.40, 0.31, 0.15, 0.05, 0.05) (m=10)

b	r_1	r_{12}		(n,N)*	T*	g*	%	#
5	1	1.6	y	(2,3)	3	1.348	0.00	(5)
5	2	3	n	(2,3)	3	1.863	0.21	(5)
5	2	4	n	(3,4)	3	2.160	0.18	(3)
5	4	5	y'	(1,4)	4	2.506	0.00	(4)
5	4	7.5	n	(3,5)	4	3.189	0.00	(4)
5	7	8	y'	(1,5)	5	3.396	0.00	(5)
5	7	10	n	(2,5)	5	3.919	0.02	(5)
5	7	13	n	(4,7)	5	4.587	0.13	(7)
5	12	18	y'	(3,11)	7	5.981	0.00	(5)

$p^{(3)}$ = (0.90, 0.90, 0.88, 0.85, 0.65, 0.45, 0.25, 0.12, 0.10, 0.10) (m=10)

b	r_1	r_{12}		(n,N)*	T*	g*	%	#
5	1	1.6	y	(2,4)	4	1.583	0.00	(4)
5	2	3	y	(2,4)	4	2.045	0.00	(5)
5	2	4	y	(4,4)	4	2.254	0.00	(3)
5	4	5	n	(2,4)	4	2.724	0.51	(5)
5	4	7.5	n	(4,5)	4	3.300	0.27	(3)
5	7	8	n	(2,5)	5	3.655	0.03	(5)
5	7	10	y'	(3,6)	5	4.140	0.00	(5)
5	7	13	n	(4,8)	5	4.713	0.13	(6)
5	12	18	y'	(3,11)	6	6.234	0.00	(6)

$p^{(4)}$ = (0.806, 0.703, 0.648, 0.609, 0.578, 0.552, 0.530, 0.510, 0.493, 0.477, 0.463, 0.450, 0.438, 0.427) (m=14)

b	r_1	r_{12}		(n,N)*	T*	g*	%	#
5	1	1.6	y'	(1,3)	2	3.003	0.00	(4)
5	2	3	y'	(2,7)	5	3.631	0.00	(5)
5	2	4	n	(6,11)	5	3.837	0.00	(9)
5	4	5	y	(2,14)	12	4.573	0.00	(5)
5	4	7.5	n	(11,14)	12	5.029	0.00	(6)
5	7	8	y	(1,15)	14	5.957	0.00	(4)
5	7	10	y	(5,15)	14	6.521	0.00	(5)
5	7	13	n	(14,14)	14	6.833	0.00	(3)
5	12	18	y	(9,15)	15	9.441	0.00	(3)

$p^{(5)}$ = (0.995, 0.968, 0.916, 0.843, 0.754, 0.656, 0.555, 0.457, 0.366, 0.285, 0.216, 0.159, 0.114, 0.079) (m=14)

b	r_1	r_{12}		(n,N)*	T*	g*	%	#
5	1	1.6	y	(2,3)	3	0.928	0.00	(5)
5	2	3	n	(2,3)	4	1.407	0.07	(6)
5	2	4	n	(3,4)	4	1.678	0.48	(4)
5	4	5	y'	(1,4)	5	1.957	0.00	(5)
5	4	7.5	n	(3,5)	5	2.555	0.12	(4)
5	7	8	y'	(1,5)	6	2.689	0.00	(5)
5	7	10	y'	(2,6)	6	3.136	0.00	(5)
5	7	13	n	(4,8)	6	3.728	0.03	(5)
5	12	18	n	(3,10)	8	4.823	0.00	(5)

$p^{(4)}$ and $p^{(5)}$ are obtained from a discretization of the Weibull $(\alpha,1)$ distribution for $\alpha=1.4$ and $\alpha=3$ respectively.

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