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Publication date:
1993

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R41
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Jan van der Leeuw

FEW 611

Communicated by Prof.dr. B.B. van der Genugten

FIRST ORDER CONDITIONS FOR THE MAXIMUM LIKELIHOOD ESTIMATION OF AN EXACT ARMA MODEL

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Abstract

Using the exact covariance matrix of ARMA(p,q) errors first order conditions for the parameters are derived and solved. This is done for the pure MA case, the pure AR case and the general ARMA model. Our approach applies both to maximum likelihood and minimum distance estimation. The exact covariance is written in the form of lag matrices, which can simply be differentiated. The resulting first order conditions have at least one solution.

The difference between maximum likelihood and minimum distance estimation amounts to a function of the elements of the covariance matrix. This function is simple in case of the pure MA or AR case, but more complicated in the general ARMA case. Of course, the solutions for the AR and MA parameters are in general conditional. Only in the pure MA and AR case of a time series model without explanatory variables direct solutions are found.

[1]I am indebted to H.H. Tigelaar for his suggestions and comments on an earlier draft.

1. Introduction

In a well known article C.M. Beach and J. MacKinnon (1978) presented a maximum likelihood procedure for estimating the parameters of a linear regression model with first-order autocorrelation. For a fixed value of the AR(1) parameter they estimate the regression parameter, next they calculate the AR(1) parameter conditional on this estimate. J.Magnus (1978) showed in a more general way that such a procedure converges and that it is possible to derive simultaneously the maximum likelihood estimates of the regression parameters and the parameters of the covariance matrix.

More recently several authors gave procedures to estimate the covariance parameters, be it for a pure time series model (Kohn and Ansley, 1985) or for a regression model (Zinde-Walsh and Galbraith, 1991). However, without a closed form of the general ARMA covariance matrix, the resulting formulas and algorithms become very complicated. In this paper we give a generalization of the Beach/MacKinnon procedure, using an expression for the exact ARMA covariance matrix in closed form. This is possible as this form of the covariance matrix is simple enough to be differentiated analytically. Conditional on the Aitken estimator and the corresponding residuals we can derive first order conditions for the likelihood function of the ARMA-parameters and solve them.

The results hold also for the pure time series model, without a matrix of regressors. At the same time we are able to show the differences between maximum likelihood and minimum distance estimators. In the pure AR and MA case the difference amounts to a sum of the elements of the off-diagonals of the covariance matrix. The general ARMA model has the same property, be it that the function of the covariance is more complicated.

2. The linear model

Consider the linear model with ARMA-errors:

$$y=X\beta+\varepsilon,$$

where y has dimensions (Tx1), X (Txk), β (kx1) and ε (Tx1). The general form of ARMA distributed errors is given by

$$\varepsilon_t=-\sum_{i=0}^{p}\vartheta_i\varepsilon_{t-i}+v_t+\sum_{i=0}^{q}\alpha_i v_{t-i},\ \ t=1,..,T \tag{1}$$

where v_t is a sequence of independently and identically distributed random variables. ϑ denotes the vector $(\vartheta_1,\vartheta_2,..,\vartheta_p)'$ of AR-parameters, α is the vector $(\alpha_1,\alpha_2,..,\alpha_q)'$ of MA-parameters. Use σ^2V to denote the covariance matrix of ε: $\sigma^2V=E\varepsilon\varepsilon'$.

One way to estimate the unknown parameters β, ϑ and α is by minimizing the weighted sum of squares $\varepsilon'V^{-1}\varepsilon$. Supposing normally distributed errors, we may prefer to maximize the (concentrated) likelihood function, which is equivalent to minimizing $S=|V|^{1/T}e'V^{-1}e$ (Judge *et al.*, p.284). Here $e=y-Xb$, b being an estimator of β. It is clear, that this model reduces to a pure 'time series' model in case X is zero: e is identical to ε and y (see, *e.g.*, Anderson and Mentz, 1982). If X is non-zero than we have to estimate ε as $e=y-Xb$, with $b=(X'V^{-1}X)^{-1}X'V^{-1}y$, the Aitken estimator. In any case, S is a function of the parameter vectors α and ϑ.

Minimizing $S=|V|^{1/T}e'V^{-1}e$ is equivalent to solving the first order conditions, or $\partial S/\partial\vartheta=0$ and $\partial S/\partial\alpha=0$. We should realize that the differential $\mathbf{d}(e'V^{-1}e)$ is equal to $e'\mathbf{d}(V^{-1})e$ because $e'V^{-1}\mathbf{d}e=(y'-b'X')V^{-1}X\mathbf{d}b=0$. For the differential of the determinant part we get $\mathbf{d}(|V|^{1/T})=1/T|V|^{1/T}\mathbf{tr}V^{-1}\mathbf{d}V$. Otherwise stated, the differential of S is

$$\mathbf{d}S=|V|^{1/T}\{s^2\mathbf{tr}V^{-1}\mathbf{d}V+e'\mathbf{d}(V^{-1})e\} \tag{2}$$

with $s^2=e'V^{-1}e/T$. In the sequel we will show how ϑ and α can expressed as a function of e, be it computed or identical to y. First we will give some remarks about the exact covariance matrix before we present first order conditions in subsequent sections.

2. Covariance matrix

Following Pagan (1974), we introduce two matrices for both the AR parameters and the MA parameters. These are special types of Toeplitz matrices. We define a (square) lower band matrix P of dimensions TxT, and a Txp matrix Q as follows:

$$Q=\begin{bmatrix} \vartheta_p & \vartheta_{p-1} & \cdot & \vartheta_1 \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \\ & & & \cdot \\ 0 & & & \vartheta_p \\ 0 & \cdot & \cdot & 0 \\ \cdot & & & \cdot \\ 0 & \cdot & \cdot & 0 \end{bmatrix} \qquad P=\begin{bmatrix} 1 & & & & & \\ \vartheta_1 & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & & & & & \\ \vartheta_p & & & & & \\ & \cdot & & \cdot & \cdot & \\ & & \vartheta_p \cdot & \cdot & \vartheta_1 & 1 \end{bmatrix}$$

The upper triangular part of a lower band matrix consists of zeros and the lower part has off-diagonals with the same elements. As is well-known its inverse can be obtained by a simple algorithm. An other important characteristic of these matrices is that they commute and that their product is a matrix of the same type. Q consists of an upper pxp part with an upper band matrix and a lower (T-p)xp part, which consists of only zeros. Like P and Q will be used to describe the AR part of the error vector, so are M and N defined for the MA part, replacing ϑ by α and p by q.

Next define the auxiliary vectors $\bar{\varepsilon}$ and $\bar{v}$:

$$\bar{\varepsilon}= (\varepsilon_{-p+1},\varepsilon_{-p+2},\ldots,\varepsilon_{-1},\varepsilon_0)'$$

$$\bar{v}= (v_{-q+1},v_{-q+2},\ldots,v_{-1},v_0)'$$

Then we can write (1) in matrix form:

$$\begin{bmatrix} Q & P \end{bmatrix} \begin{bmatrix} \bar{\varepsilon} \\ \varepsilon \end{bmatrix} = \begin{bmatrix} N & M \end{bmatrix} \begin{bmatrix} \bar{v} \\ v \end{bmatrix}$$

As is proven elsewhere (Van der Leeuw, 1992) the exact covariance matrix for ARMA errors is equal to

$$V=[N\ \ M][\bar{P}'\bar{P}-\bar{Q}\bar{Q}']^{-1}[N\ \ M]' \tag{3}$$

where $\bar{P}$ is like P, but of order (T+p)x(T+p) and $\bar{Q}$ like Q, but of order (T+p)xp, if the usual invertibility conditions hold.

Next we will consider the first order conditions for the pure MA case, the AR case, and at last the ARMA case.

3. MA first order conditions

From (3) we see, that the covariance matrix in the MA case is

$$V=[N\ \ M][N\ \ M]' \tag{4}$$

To be able to take differentials we rewrite (4) in the form of lag matrices. The definition of a lag matrix and some of its properties are found in the Appendix.

Lemma

Using lag matrices we can write the MA-covariance matrix as

$$V=\sum_{i=0}^{q}\Big(\sum_{j=0}^{i}L_{j-i}'\alpha_j+\sum_{j=i+1}^{q}L_{i-j}\alpha_j\Big)\alpha_i \tag{5}$$

where $\alpha_0=1$.

The differential of V is

$$\mathbf{d}V=\sum_{i=0}^{q}\Big(\sum_{j=0}^{i}(L_{j-i}+L_{j-i}')\alpha_j+\sum_{j=i+1}^{q}(L_{i-j}+L_{i-j}')\alpha_j\Big)\mathbf{d}\alpha_i \tag{6}$$

Proof

Observe, that V can be written as $V=[O\ \ I]\bar{M}\bar{M}'[O\ \ I]'$, where $\bar{M}$ has the same structure as M, but is of order (T+q)x(T+q). O is a Txq zero matrix, I is the TxT identity matrix. Using lag matrices, we can write $\bar{M}$ as $\sum_{i=0}^{q}L_i(i,i)\alpha_i$ and for V we get

$$V=[O\ \ I]\Big(\sum_{i=0}^{q}L_i(i,i)\alpha_i\Big)\Big(\sum_{j=0}^{q}L_j(j,j)\alpha_j\Big)'[O\ \ I]'$$

Transposing the j-sum part and multiplying:

$$=[O\ \ I](\sum_{i=0}^{q}\sum_{j=0}^{q}L_{i-j}(i,\max(i-j,0))\alpha_i\alpha_j)[O\ \ I]'.$$

Denoting vectors of length T+q with a bar, we have $[O\ \ I]=\sum_{h=1}^{T}\mathbf{i}_h\bar{\mathbf{i}}'_{h+q}$ and thus

$$V=\sum_{h=1}^{T}\mathbf{i}_h\bar{\mathbf{i}}'_{h+q}\sum_{i=0}^{q}\sum_{j=0}^{q}L_{i-j}(i,\max(i-j,0))\alpha_i\alpha_j)(\sum_{h=1}^{T}\mathbf{i}_h\bar{\mathbf{i}}'_{h+q})',$$

which gives after some manipulations

$$V=\sum_{i=0}^{q}\sum_{j=0}^{q}\sum_{h=1+\max(0,i-j)}^{T-\max(0,i-j)+i-j}\mathbf{i}_h\mathbf{i}'_{h-i+j}\alpha_i\alpha_j.$$

This is equal to

$$=\sum_{i=0}^{q}\sum_{j=0}^{q}L_{i-j}(\max(0,i-j),\ \max(0,i-j)\alpha_i\alpha_j$$

or

$$=\sum_{i=0}^{q}(\sum_{j=0}^{i}L'_{j-i}\alpha_j+\sum_{j=i+1}^{q}L_{i-j}\alpha_j)\alpha_i,$$

the first part of the lemma. □

For the differential of V we get

$$\mathbf{d}V=\sum_{i=0}^{q}\sum_{j=0}^{q}L_{i-j}(\max(0,i-j),\max(0,i-j)(\alpha_i\mathbf{d}\alpha_j+\alpha_j\mathbf{d}\alpha_i)$$

Write out, interchange the indices in the second part:

$$=\sum_{i=0}^{q}(\sum_{j=0}^{q}L_{i-j}(\max(0,i-j),\max(0,i-j)\alpha_j+\sum_{j=0}^{q}L_{j-i}(\max(0,j-i),\max(0,j-i)\alpha_j)\mathbf{d}\alpha_i$$

Splitting the sum over the j-index and take the transpose of the first and last part to get the result as stated in the lemma. □

Now we can give the (conditional) solution of the MA-parameter. Substituting the expressions of the lemma in (2) gives the result stated in the next theorem.

Theorem

Let $\phi = V^{-1}e$ and $f_k = \sum_{i=1}^{T-k} \phi_i \phi_{i+k}$, where ϕ_i is the i^{th} element of ϕ and d_k the sum of the elements of the k^{th} off-diagonal of V^{-1} times s^2.

The MA-parameter vector α satisfies $H\alpha = -h$, where the $(i,j)^{th}$ element of H is $f_{|i-j|} - d_{|i-j|}$ and $h_i = h_{i,0}$.

Proof

A solution for the MA-parameter α is found by solving the first order conditions: $s^2 \mathbf{tr} V^{-1} \partial V / \partial\alpha + e'(\partial(V^{-1})/\partial\alpha)e = 0$. The determinantal part has as differential:

$$\mathbf{tr} V^{-1}\mathbf{d}V = \mathbf{tr} V^{-1} \sum_{i=0}^{q} \Big(\sum_{j=0}^{i} (L_{j-i} + L'_{j-i})\alpha_j + \sum_{j=i+1}^{q} (L_{i-j} + L'_{i-j})\alpha_j \Big) \mathbf{d}\alpha_i$$

or as $\mathbf{tr} V^{-1} L'_{j-i} = \mathbf{tr} L'_{j-i} V^{-1} = (\mathbf{tr} V^{-1} L_{j-i})' = \mathbf{tr} V^{-1} L_{j-i}$:

$$= 2 \sum_{i=0}^{q} \Big(\sum_{j=0}^{i} \mathbf{tr} V^{-1} L_{j-i}\alpha_j + \sum_{j=i+1}^{q} \mathbf{tr} V^{-1} L_{i-j}\alpha_j \Big) \mathbf{d}\alpha_i$$

Let $k = |i-j|$. Then the lag index for both parts is $-k$ and we have

$$\mathbf{tr} V^{-1} L_{-k} = \mathbf{tr} V^{-1} \sum_{h=1}^{T-k} i_h i'_{h+k} = \sum_{h=1}^{T-k} \mathbf{tr}(i'_{h+k} V^{-1} i_h) = \sum_{h=1}^{T-k} v^{\bullet}_{h+k,h}\,,$$ where $v^{\bullet}_{h+k,h}$ is the $(h+k,h)^{th}$ element of V^{-1}. Thus $\mathbf{tr} V^{-1} L_{-k}$ is the sum of the elements of the k^{th} off-diagonal of V^{-1}. Hence the derivative becomes

$$s^2 \mathbf{tr} V^{-1} \partial V / \partial\alpha_i = 2\Big(\sum_{j=0}^{i} d_{i-j}\alpha_j + \sum_{j=i+1}^{q} d_{j-i}\alpha_j \Big), \quad i = 1,..,q.$$

To find the derivatives of the second part, we need $e'\mathbf{d}(V^{-1})e$, which is equal to $-e'V^{-1}\mathbf{d}VV^{-1}e$, or $-\phi'\mathbf{d}V\phi$.

$$\phi'\mathbf{d}V\phi = \sum_{i=0}^{q} \Big(\sum_{j=0}^{i} \phi'(L_{j-i} + L'_{j-i})\phi\alpha_j + \sum_{j=i+1}^{q} \phi'(L_{i-j} + L'_{i-j})\phi\alpha_j \Big) \mathbf{d}\alpha_i$$

$$=\sum_{i=0}^{q}\Big(\sum_{j=0}^{i} 2\phi' L_{j-i}\phi\alpha_j+\sum_{j=i+1}^{q} 2\phi' L_{i-j}\phi\alpha_j\Big)\mathbf{d}\alpha_i$$

Now, $\phi' L_k\phi=\sum_{i=1}^{T+k}\phi'\mathbf{i}_i\mathbf{i}'_{i-k}\phi=\sum_{i=1}^{T+k}\phi_i\phi_{i-k}=f_{-k}$ and the derivative is

$$\phi'\partial V/\partial\alpha_i\phi=2\Big(\sum_{j=0}^{i} f_{i-j}\alpha_j+\sum_{j=i+1}^{q} f_{j-i}\alpha_j\Big),\ i=1,..,q.$$

Combining we get for the first order condition for $\partial S/\partial\alpha_i$:

$$\sum_{j=0}^{i} d_{i-j}\alpha_j+\sum_{j=i+1}^{q} d_{j-i}\alpha_j-\sum_{j=0}^{i} f_{i-j}\alpha_j-\sum_{j=i+1}^{q} f_{j-i}\alpha_j=0,\ i=1,..,q.$$

or because $\alpha_0=1$,

$$\sum_{j=1}^{i}(f_{i-j}-d_{i-j})\alpha_j+\sum_{j=i+1}^{q}(f_{j-i}-d_{j-i})\alpha_j=-(f_i-d_i),\ i=1,..,q.\ \square$$

4. AR first order conditions

From (3) it follows that the inverse of the covariance matrix in the pure AR-case is

$$V^{-1}=P'P-QQ' \tag{7}$$

P has dimensions TxT, Q Txp. Because of the definition we can write $P=\sum_{i=0}^{q}L_i(i,i)\vartheta_i$. To be able to apply lag-matrices to Q we define the (TxT) matrix $\bar{Q}$ as $\sum_{i=0}^{q}L_{i-q}\vartheta_i$. Then we have $Q=\bar{Q}[I_q\ O]'$, where $[I_q\ O]'$ is of order Txq, I_q being the qxq identity matrix and O a qx(T-q) zero matrix. The inverse of the covariance matrix can now be expressed in the form of lag-matrices and ϑ.

Lemma

Using lag matrices we can write the inverse of the AR-covariance matrix as

$$V^{-1}=\sum_{i=0}^{q}\sum_{j=0}^{q}L_{j-i}(j,j)\vartheta_i\vartheta_j \tag{8}$$

with $\vartheta_0=1$.

Its differential is

$$\mathbf{d}V^{-1}=\sum_{i=0}^{q}\sum_{j=0}^{q}(L_{j-i}(j,j)+L'_{j-i}(j,j))\vartheta_j\mathbf{d}\vartheta_i \tag{9}$$

Proof

From the definitions we have

$$V^{-1}=(\sum_{i=0}^{q}L_i(i,i)\vartheta_i)'(\sum_{j=0}^{q}L_j(j,j)\vartheta_j)-(\sum_{i=0}^{q}L_{i-q}\vartheta_i)[I_q\ \ O]'[I_q\ \ O](\sum_{j=0}^{q}L_{j-q}\vartheta_j)'$$

The part before the minus sign is $\sum_{i=0}^{q}\sum_{j=0}^{q}L_{j-i}(\max(0,i-j),j)\vartheta_i\vartheta_j$. The second part is more complicated. First observe, that $[I_q\ \ O]'[I_q\ \ O]=\sum_{h=1}^{q}\mathbf{i}_h\mathbf{i}'_h$, where $\mathbf{i}_h$ is a vector of length T, of which all elements are zero, except the h^{th}. Now,

$$(\sum_{i=0}^{q}L_{i-q}\vartheta_i)[I_q\ \ O]'[I_q\ \ O](\sum_{j=0}^{q}L_{j-q}\vartheta_j)'=\sum_{i=0}^{q}\sum_{j=0}^{q}\sum_{h=1}^{q}(L_{i-q}\mathbf{i}_h)(L_{j-q}\mathbf{i}_h)'\vartheta_i\vartheta_j.$$

Here $L_k\mathbf{i}_h$ (with $k\le 0$) is a lagged zero-one vector: $L_k\mathbf{i}_h=\mathbf{i}_{h+k}$. Therefore we have $L_{i-q}\mathbf{i}_h=\mathbf{i}_{i-q+h}$ and $L_{j-q}\mathbf{i}_h=\mathbf{i}_{j-q+h}$, with $1\le h\le q$, $1\le i-q+h\le T$ and $1\le j-q+h\le T$, which means $1+q-i+\max(0,i-j)\le h\le q$.

$$=\sum_{i=0}^{q}\sum_{j=0}^{q}\sum_{\substack{h=1+q-i+\\ \max(0,i-j)}}^{q}\mathbf{i}_{i-q+h}\mathbf{i}'_{j-q+h}\vartheta_i\vartheta_j.$$

Changing the index from h to h-i+q and interchange the i and j index:

$$=\sum_{i=0}^{q}\sum_{j=0}^{q}L_{j-i}(\max(0,j-i),T-i)\vartheta_i\vartheta_j.$$

The lag matrices of the first and second part are $L_{j-i}(\max(0,i-j),j)$ and $L_{j-i}(\max(0,j-i),T-i)$. The difference is $L_{j-i}(j,j)$. □

The differential is

$$\mathbf{d}V^{-1}=\sum_{i=0}^{q}\sum_{j=0}^{q}L_{j-i}(j,j)(\vartheta_j\mathbf{d}\vartheta_i+\vartheta_i\mathbf{d}\vartheta_j)$$

Writing out and interchanging the indices in the second part gives

$$=\sum_{i=0}^{q}\sum_{j=0}^{q}L_{j-i}(j,j)\vartheta_j\mathbf{d}\vartheta_i+\sum_{i=0}^{q}\sum_{j=0}^{q}L_{i-j}(i,i)\vartheta_j\mathbf{d}\vartheta_i$$

or

$$=\sum_{i=0}^{q}\sum_{j=0}^{q}(L_{j-i}(j,j)+L'_{j-i}(j,j))\vartheta_j\mathbf{d}\vartheta_i. \;\square$$

Before stating a theorem concerning the first order conditions of the AR-parameters, we give some properties of the determinant of the AR-covariance matrix. The determinant of the full TxT AR-matrix is equal to the determinant of its qxq (upper-left) submatrix. Moreover, this submatrix can be written in the same form as in (8), while the lag matrix is now of order qxq.

Lemma

Let $V_1^{-1}=P_1'P_1-Q_1Q_1'$, where P_1 is the qxq upper-left part of P and Q_1 is the upper qxq part of Q.

Then

1. $P_1'P_1-Q_1Q_1'$ is positive definite if the invertibility condition holds
2. $|V|=|V_1|$
3. $V_1^{-1}=\sum_{i=0}^{q}\left(\sum_{j=0}^{q-i-1}L_{j-i}(j,j)\vartheta_j-\sum_{j=q-i+1}^{q}L_{j-i}(q-i,q-i)\vartheta_j\right)\vartheta_i$

Proof

The first part of the lemma is proven in Van der Leeuw (1992). For the second part observe that P and Q can be partitioned in the following way:

$$P=\begin{bmatrix} P_1 & 0 \\ \hline P_2 & P_3 \end{bmatrix},\ Q=\begin{bmatrix} Q_1 \\ \hline 0 \end{bmatrix},\ \text{and thus } P'P-QQ'=\begin{bmatrix} P_1'P_1+P_2'P_2-Q_1'Q_1 & P_2'P_3 \\ \hline P_3'P_2 & P_3'P_3 \end{bmatrix}.$$

The upper-left element is P_1P_1', because $P_2'P_2=\begin{bmatrix} Q_1' & 0 \end{bmatrix}\begin{bmatrix} Q_1 \\ \hline 0 \end{bmatrix}=Q_1'Q_1$ and

$P_1'P_1+Q_1'Q_1=P_1P_1'+Q_1Q_1'$ (see Van der Leeuw, 1992). Apply the rules for the determinant of a partitioned inverse:

$$|V^{-1}|=|P_1P_1'|\,|P_1P_1'-P_2'P_3(P_3'P_3)^{-1}P_3'P_2|=|P_1'P_1-Q_1Q_1'|,\ \text{as } |P_1|=1.$$

For the third part we write the matrices of which V_1^{-1} consists as lag matrices. Of course L is now of order qxq. Using lag-matrices we get $P_1=\sum_{j=0}^{q-1} L_j(j,j)\vartheta_j$ and

$Q_1=\sum_{j=1}^{q} L_{j-q}\vartheta_j$, but as $L_q(q,q)\vartheta_q$ and $L_{-q}\vartheta_0=L_q'(q,q)$ are zero, we write

$$V_1^{-1}=\Big(\sum_{i=0}^{q} L_i(i,i)\vartheta_i\Big)'\sum_{j=0}^{q} L_j(j,j)\vartheta_j)-\Big(\sum_{i=0}^{q} L_{i-q}\vartheta_i\Big)\Big(\sum_{j=0}^{q} L_{j-q}\vartheta_j\Big)'$$

Rewriting the transposed parts, multiplying and interchanging the indices in the second part gives:

$$=\sum_{i=0}^{q}\sum_{j=0}^{q}(L_{j-i}(\max(0,j-i),j)-L_{j-i}(\max(0,j-i),q-i))\vartheta_i\vartheta_j$$

Conforming the definitions we get for the lag matrices:

$$\sum_{h=1+\max(0,j-i)}^{q-i} \mathbf{i}_h\mathbf{i}_{h+i-j}'-\sum_{h=1+\max(0,j-i)}^{j} \mathbf{i}_h\mathbf{i}_{h+i-j}'=$$

$$0 \text{ if } i+j=q,$$

$$=\sum_{h=j+1}^{q-i} \mathbf{i}_h\mathbf{i}_{h+i-j}'=L_{j-i}(j,j) \text{ if } j+i<q$$

$$= -\sum_{h=q-i+1}^{j} \mathbf{i}_h \mathbf{i}'_{h+i-j} = L_{j-1}(q-i,q-i) \text{ if } j+i>q.$$

Hence

$$V_1^{-1} = \sum_{i=0}^{q} \Big(\sum_{j=0}^{q-i-1} L_{j-i}(j,j)\vartheta_j - \sum_{j=q-i+1}^{q} L_{j-i}(q-i,q-i)\vartheta_j \Big)\vartheta_i$$

or, as V_1 is symmetric:

$$= \sum_{j=0}^{q} \Big(\sum_{i=0}^{q-j-1} L_{j-i}(j,j)\vartheta_i - \sum_{i=q-j+1}^{q} L_{j-i}(q-i,q-i)\vartheta_i \Big)\vartheta_j. \ \square$$

Theorem

Define $f_{i,j}$ as $\sum_{h=1}^{T-i-j} e_{h+j}e_{h+i}$ and $d_{i,j}$ as s^2 times the sum of the elements of the $|i-j|^{th}$ off-diagonal of V_1 without the first and last $\min(i,j)$ elements if $i+j<q$ and minus s^2 times the sum without the first and last $\min(q-i,q-j)$ elements if $i+j>q$.

The AR-parameter vector ϑ is satisfies $H\vartheta=-h$, where the $(i,j)^{th}$ element of H is $f_{i,j}-d_{i,j}$ and $h_i=h_{i,0}$.

Proof

A solution for the AR-parameter ϑ is found by solving the first order conditions $s^2\mathbf{tr}V^{-1}\partial V/\partial\vartheta + e'(\partial(V^{-1})/\partial\vartheta)e=0$. The determinantal part has as differential: $\mathbf{tr}V_1^{-1}\mathbf{d}V_1 = -\mathbf{tr}V_1\mathbf{d}V_1^{-1}$.

For the differential we get

$$\mathbf{d}V_1^{-1} = \sum_{i=0}^{q} \Big(\sum_{j=0}^{q-i-1} L_{j-i}(j,j)\vartheta_j - \sum_{j=q-i+1}^{q} L_{j-i}(q-i,q-i)\vartheta_j \Big)\mathbf{d}\vartheta_i +$$

$$\sum_{j=0}^{q} \Big(\sum_{i=0}^{q-j-1} L_{j-i}(j,j)\vartheta_i - \sum_{i=q-j+1}^{q} L_{j-i}(q-i,q-i)\vartheta_i \Big)\mathbf{d}\vartheta_j$$

Interchanging i and j in the second part and using $\mathbf{tr}V_1L_{i-j}(i,i) = \mathbf{tr}V_1L_{j-i}(j,j)$ as

$V_1=V_1'$, the result is

$$\mathbf{tr}V_1\mathbf{d}V_1^{-1}=2\sum_{i=0}^{q}\Big(\sum_{j=0}^{q-i-1}\mathbf{tr}V_1L_{j-i}(j,j)\vartheta_j-\sum_{j=q-i+1}^{q}\mathbf{tr}V_1L_{j-i}(q-i,q-i)\vartheta_j\Big)\mathbf{d}\vartheta_i.$$

The derivative is

$$s^2\mathbf{tr}V_1^{-1}\partial V_1/\partial\vartheta_i=-2s^2\Big(\sum_{j=0}^{q-i-1}\mathbf{tr}V_1L_{j-i}(j,j)\vartheta_j-\sum_{j=q-i+1}^{q}\mathbf{tr}V_1L_{j-i}(q-i,q-i)\vartheta_j\Big).$$

Here $\mathbf{tr}V_1L_{j-i}(j,j)=\sum_{h=j+1}^{q-i}\mathbf{tr}(\mathbf{i}_{h-j+i}'V_1\mathbf{i}_h)=v^*_{i+1,j+1}+..+v^*_{q-j,q-i}$, where $v^*_{i,j}$ is the $(i,j)^{th}$ element of V_1. In the same way we get for the second part $\mathbf{tr}V_1L_{j-i}(q-i,q-i)=v^*_{q-j+1,q-i+1}+..+v^*_{i,j}$.

Thus for the derivative to ϑ_i the coefficient of ϑ_j is the sum of the $(i-j)^{th}$ off-diagonal of V_1 without the first and last min(i,j) elements times $-2s^2$ if $i+j<q$. If $i+j>q$ we have a similar sum without the first and last min(q-i,q-j) elements times $2s^2$.

For the second part $e'(\partial(V^{-1})/\partial\vartheta)e$ we have

$$e'\mathbf{d}V^{-1}e=e'\Big(\sum_{i=0}^{q}\sum_{j=0}^{q}(L_{j-i}(j,j)+L_{j-i}'(j,j))\vartheta_j\mathbf{d}\vartheta_i\Big)e=2\sum_{i=0}^{q}\sum_{j=0}^{q}e'L_{j-i}(j,j)e\vartheta_j\mathbf{d}\vartheta_i.$$

But as $e'L_{j-i}(j,j)e=\sum_{h=1}^{T-i-j}e_{h+j}e_{h+i}=f_{i,j}$, we get for the differential $2\sum_{i=0}^{q}\sum_{j=0}^{q}f_{i,j}\vartheta_j\mathbf{d}\vartheta_i$ and for the derivative $e'\partial V^{-1}/\partial\vartheta_i e=2\sum_{j=0}^{q}f_{i,j}\vartheta_j$, $i=1,..,q$.

The first order condition becomes:

$$\sum_{j=0}^{q}f_{i,j}\vartheta_j-\sum_{j=0}^{q}d_{i,j}\vartheta_j=0,\quad i=1,..q$$

or as $\vartheta_0=1$:

$$\sum_{j=1}^{q}(f_{i,j}-d_{i,j})\vartheta_j=-(f_{i,0}-d_{i,0}),\quad i=1,..q.\ \square$$

5. ARMA first order conditions

In the ARMA case the covariance matrix is more complicated than in the MA or AR case. Nevertheless it is possible to find a (conditional) ML-solution for both the MA and AR parameters. First we will study the MA part, next the AR part. This is possible because the covariance matrix - if not inverted - is simple enough to isolate the two parameter vectors. We state the results in the following theorem, that is proved in the next sections. In this section we use q for the number of MA and AR parameters. This gives no loss of generality as it is always possible to fill up the shorter one with zeros.

Theorem

The first order conditions for the ARMA model can be split in a MA part and an AR part.

The MA parameter satisfies $H\alpha=-h$, where the $(i,j)^{th}$ element of H is

$$h_{i,j}=\phi'(i)\Delta^{-1}\phi(j)-\sum_{k=1}^{T}\sum_{l=1}^{T}\delta(i,j,k,l)\gamma(k,l) \text{ and } h_i=h_{i,0}.$$

Here

Δ is the inverse of the enlarged AR covariance matrix,

$\phi(i)=(\underbrace{0\ldots0}_{\leftarrow q-i\rightarrow}\ \underbrace{\phi_1\ldots\phi_T}_{\leftarrow T\rightarrow}\ \underbrace{0\ldots0}_{\leftarrow i\rightarrow})'$, ϕ_i is element i of $V^{-1}e$,

$\delta(i,j,k,l)$ is the $(k+q-i,l+q-j)^{th}$ element of Δ^{-1},

$\gamma(k,l)$ is the $(k,l)^{th}$ element of V^{-1}.

The AR-parameter vector satisfies $G\vartheta=-g$, where the $(i,j)^{th}$ element of G is

$$g_{i,j}=\sum_{k=1}^{T+q-i-j}\zeta_{k+i}\zeta_{k+j}-\psi(i,j) \text{ and } g_i=g_{i,0}.$$

Here

ζ_i is the i^{th} element of $\zeta=Ze$, $Z=\Delta^{-1}[N\ M]'V^{-1}$,

$\psi(i,j)$ is the sum of the elements of $(i-j)^{th}$ diagonal of

ZVZ', without the first and last $\min(i,j)$ elements.

5.1. MA-part conditions

To find the solution to the MA-part we proceed as follows. First define

$\bar{M}=\sum_{i=0}^{q}L_i(i,i)\alpha_i$ and $\phi=V^{-1}e$, as we did before, and $\Delta=\bar{P}'\bar{P}-\bar{Q}\bar{Q}'$. $\bar{M}$ and $\bar{P}$ have dimensions $(T+q)x(T+q)$, $\bar{Q}$ $(T+q)xq$.

For the inverse of the covariance matrix we can write

$V^{-1}=V^{-1}VV^{-1}=V^{-1}[0\ \ I]\bar{M}\Delta^{-1}\bar{M}'[0\ \ I]'V^{-1}$.

Now, $\bar{M}'[0\ \ I]'V^{-1}e=\bar{M}\begin{bmatrix}0\\ \phi\end{bmatrix}=\sum_{i=0}^{q}L_i'(i,i)\begin{bmatrix}0\\ \phi\end{bmatrix}\alpha_i=\sum_{i=0}^{q}L_{-i}(0,0)\begin{bmatrix}0\\ \phi\end{bmatrix}\alpha_i=\sum_{i=0}^{q}\phi(i)\alpha_i$,

where $\phi(i)$ is defined above.

The quadratic part becomes $e'V^{-1}e=\phi'V\phi=\sum_{i=0}^{q}\sum_{j=0}^{q}\phi'(i)\Delta^{-1}\phi(j)\alpha_i\alpha_j$ and its differential $\mathbf{d}(e'V^{-1}e)=e'\mathbf{d}V^{-1}e=2\sum_{i=0}^{q}\sum_{j=0}^{q}\phi'(i)\Delta^{-1}\phi(j)\alpha_j\mathbf{d}\alpha_i$.

For the determinantal part we need $\mathbf{tr}V^{-1}\mathbf{d}V$. As before we use $V=[0\ \ I]\bar{M}\Delta^{-1}\bar{M}'[0\ \ I]'$. Observe,

$$[0\ \ I]\bar{M}=(\sum_{h=1}^{T}\mathbf{i}_h\bar{\mathbf{i}}_{h+q}')(\sum_{i=0}^{q}\sum_{h=1+i}^{T+q}\bar{\mathbf{i}}_h\bar{\mathbf{i}}_{h-i}'\alpha_i).$$

Interchanging summations and replacing in the second part h by k=h-q, we get

$$=\sum_{i=0}^{q}\sum_{h=1}^{T}\mathbf{i}_h\bar{\mathbf{i}}'_{h+q-i}\alpha_i.$$

Thus, V becomes

$$V=(\sum_{i=0}^{q}\sum_{h=1}^{T}\mathbf{i}_h\bar{\mathbf{i}}'_{h+q-i}\alpha_i)\Delta^{-1}(\sum_{j=0}^{q}\sum_{k=1}^{T}\mathbf{i}_k\bar{\mathbf{i}}'_{k+q-j}\alpha_j)'$$

Interchange the summations and define the scalar $\delta(i,j,h,k)$ as $\bar{\mathbf{i}}'_{h+q-i}\Delta^{-1}\mathbf{i}_{k+q-j}$, the $(h+q-i,k+q-j)^{th}$ element of Δ^{-1}, the enlarged AR-covariance matrix. Now we have

$$V=\sum_{i=0}^{q}\sum_{j=0}^{q}\sum_{h=1}^{T}\sum_{k=1}^{T}\delta(i,j,h,k)\mathbf{i}_h\bar{\mathbf{i}}'_k\alpha_i\alpha_j,\text{ which gives}$$

$$\mathbf{d}V=\sum_{i=0}^{q}\sum_{j=0}^{q}\sum_{h=1}^{T}\sum_{k=1}^{T}\delta(i,j,h,k)\mathbf{i}_h\bar{\mathbf{i}}'_k(\alpha_j\mathbf{d}\alpha_i+\alpha_i\mathbf{d}\alpha_j)$$

$$=2\sum_{i=0}^{q}\sum_{j=0}^{q}\sum_{h=1}^{T}\sum_{k=1}^{T}\delta(i,j,h,k)\mathbf{i}_h\bar{\mathbf{i}}'_k\alpha_j\mathbf{d}\alpha_i$$

because $\delta(i,j,h,k)=\delta(i,j,h,k)'=\delta(j,i,k,h)=\delta(j,i,h,k)$.

For $\mathbf{tr}V^{-1}\mathbf{d}V$ we get, writing $\gamma(h,k)=\mathbf{i}'_hV^{-1}\mathbf{i}_k$, the $(h,k)^{th}$ element of V^{-1}, the inverse of the complete covariance matrix,

$$\mathbf{tr}V^{-1}\mathbf{d}V=\mathbf{tr}V^{-1}(2\sum_{i=0}^{q}\sum_{j=0}^{q}\sum_{h=1}^{T}\sum_{k=1}^{T}\delta(i,j,h,k)\mathbf{i}_h\bar{\mathbf{i}}'_k\alpha_j\mathbf{d}\alpha_i)$$

$$=2\sum_{i=0}^{q}\sum_{j=0}^{q}\sum_{h=1}^{T}\sum_{k=1}^{T}\delta(i,j,h,k)\gamma(h,k)\alpha_j\mathbf{d}\alpha_i.\ \square$$

5.2. AR-part conditions

Again we have to evaluate $\mathbf{tr}V^{-1}\mathbf{d}V$ and $e'\mathbf{d}V^{-1}e$. The expression for V^{-1} is $([N\ M][\bar{P}'\bar{P}-\bar{Q}\bar{Q}']^{-1}[N\ M]')^{-1}$, while we only can isolate the AR-parameters in the expression $\Delta=\bar{P}'\bar{P}-\bar{Q}\bar{Q}'$ (cf. (9)). Therefore we define $Z=\Delta^{-1}[N\ M]'V^{-1}$. Then $V^{-1}=Z'\Delta Z$, as is easily verified. The differential in the ϑ-direction is

$\mathbf{d}V^{-1}=-V^{-1}\mathbf{d}VV^{-1}=-V^{-1}[N\ M]\mathbf{d}\Delta^{-1}[N\ M]'V^{-1}=Z'\mathbf{d}\Delta Z$.

The quadratic form $e'\mathbf{d}V^{-1}e$ is, using $\zeta=Ze$ and (9),

$$e'\mathbf{d}V^{-1}e=-\zeta'\mathbf{d}\Delta\zeta$$

$$=-\zeta'\Big(\sum_{i=0}^{q}\sum_{j=0}^{q}(L_{j-i}(j,j)+L'_{j-i}(j,j))\vartheta_j\mathbf{d}\vartheta_i\Big)\zeta$$

$$=-2\sum_{h=1}^{T+q-i-j}\zeta_{h+i}\zeta_{h+j}\vartheta_j\mathbf{d}\vartheta_i.$$

The determinantal part becomes, using lag matrices and some basic properties of the trace operator:

$$\mathbf{tr}V^{-1}\mathbf{d}V=-\mathbf{tr}V\mathbf{d}V^{-1}=\mathbf{tr}VZ'\mathbf{d}\Delta Z=\mathbf{tr}ZVZ'\Big(\sum_{i=0}^{q}\sum_{j=0}^{q}(L_{j-i}(j,j)+L'_{j-i}(j,j))\vartheta_j\mathbf{d}\vartheta_i\Big)$$

$$=2\sum_{i=0}^{q}\sum_{j=0}^{q}\sum_{h=1}^{T+q-i-j}\mathbf{i}'_{h+j}ZVZ'\mathbf{i}_{h+i}\vartheta_j\mathbf{d}\vartheta_i.$$

Here $\psi(i,j)=\sum_{h=1}^{T+q-i-j}\mathbf{i}'_{h+j}ZVZ'\mathbf{i}_{h+i}$ is the sum of the elements of $(i-j)^{th}$ diagonal of ZVZ' without the first min(i,j) elements. □

Appendix

Lag matrix

1. Definition

Define $\mathbf{i}_i$ as the Nx1 vector of which all elements are zero, except element i, which is 1. Next define

$$L_k(n,m)=\sum_{i=n+1}^{N-m+k} \mathbf{i}_i\mathbf{i}'_{i-k},\quad |k|\le N-1,\ n\ge \max(0,k),\ m\ge \max(0,k).$$ If both n and m are zero we will write L_k.

This definition of L implies that L has one (off-)diagonal consisting of one's, all other elements being zero. If n=m=0, then every element of this diagonal is equal to one. We allow, however, the first or last elements of this diagonal to be zero. To define which elements are zero we have the choice between the numbers of first rows and the last columns on the one hand, or first columns and last rows on the other. We take, arbitrarily, the first way. This means that n and m are positive in case k is positive. If n=m=k=0 we get the identity matrix. If k is positive L can be regarded as a lag matrix, with the same property as the usual lag operator. Let a be an arbitrary vector of length N. Then, for $k\ge 0$:

$$\begin{aligned} L_k(k,k) &= \sum_{i=1+k}^{N} \mathbf{i}_i\mathbf{i}'_{i-k}a \\ &= \sum_{i=1+k}^{N} \mathbf{i}_i a'_{i-k} \\ &= (0..0\ a_1..a_{N-k})'. \end{aligned}$$

If n or m is greater than k the n-k first elements or m-k last elements disappear. We do, however, not exclude negative values for k. In this sense k is not an ordinary lag matrix. In the next sections we give some properties of L.

2. Properties

Transpose:

$$L'_k(n,m)=L_{-k}(n-k,m-k)$$

Proof

$$L'_k(n,m)=\Big(\sum_{i=1+n}^{N-m+k} \mathbf{i}_i\mathbf{i}'_{i-k}\Big)'$$

$$=\sum_{i=1+n}^{N-m+k} \mathbf{i}_{i-k}\mathbf{i}'_i$$

Changing the index i to j=i-k we get

$$=\sum_{j=1+n-k}^{N-(m-k)-k} \mathbf{i}_j\mathbf{i}'_{j+k}$$

$$=L_{-k}(n-k,m-k). \ \square$$

Multiplication:

$$L_{k_1}(n_1,m_1)\cdot L_{k_2}(n_2,m_2)=L_{k_1+k_2}(\max(n_1,n_2+k_1),\max(m_1+k_2,m_2))$$

Proof

$$L_{k_1}(n_1,m_1)\cdot L_{k_2}(n_2,m_2)=\sum_{i=1+n_1}^{N-m_1+k_1} \mathbf{i}_i\mathbf{i}'_{i-k_1}\sum_{i=1+n_2}^{N-m_2+k_2} \mathbf{i}_j\mathbf{i}'_{j-k_2}$$

$$=\sum_{i=1+n_1}^{N-m_1+k_1}\ \sum_{i=1+n_2}^{N-m_2+k_2} \mathbf{i}_i\mathbf{i}'_{i-k_1}\mathbf{i}_j\mathbf{i}'_{j-k_2}$$

This expression is only non-zero if $j=i-k_1$, or $i=j+k_1=h$.

$$=\sum_{h=1+n_1}^{N-m_1+k_1}\ \sum_{h=1+n_2+k_1}^{N-m_2+k_2+k_1} \mathbf{i}_h\mathbf{i}'_{h-k_1}\mathbf{i}_{h-k_1}\mathbf{i}'_{h-k_1-k_2}$$

Here h runs from max $(1+n_1,1+n_2+k_1)$ to min $(N-m_1+k_1,N-m_2+k_2+k_1)$, which is the same as from $1+\max(n_1,n_2+k_1)$ to $N-\max(m_1+k_2,m_2)+k_1+k_2$, while the lag is equal to k_1+k_2.

References

Anderson, T.W. and R.P. Mentz (1982), Maximum Likelihood Estimation in Autoregressive and Moving Average Models, *Time Series Analysis: Theory and Practice*, Amsterdam.

Beach, C.M. and J.G. MacKinnon (1978), A Maximum Likelihood Procedure for Regression with Autocorrelated Errors, *Econometrica*, 46.

Judge, G.C. *et al.*(1985), *The Theory and Practice of Econometrics*, New York.

Kohn, R. and C.F. Ansley (1985), Computing the Likelihood and its Derivatives for a Gaussian ARMA Model, *J. Statistical Computation and Simulation*, 22.

Van der Leeuw, J.L. (1992), The Covariance Matrix of ARMA-errors in Closed Form, *Journal of Econometrics*, forthcoming.

Magnus, J.R. (1978), Maximum Likelihood Estimation of the GLS Model with Unknown Parameters in the Disturbance Covariance Matrix, *J. of Econometrics*, 7.

Pagan, A. (1974), A Generalised Approach to the Treatment of Autocorrelation, *Australian Economic Papers*, 13.

Zinde-Walsh, V. and J.W. Galbraith (1991), Estimation of a Linear Regression Model with Stationary ARMA(p,q) Errors, *J. of Econometrics*, 47.

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