Parameter estimation in the exponential distribution, confidence intervals and a monte carlo study for some goodness of fit tests

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PARAMETER ESTIMATION IN THE EXPONENTIAL DISTRIBUTION, CONFIDENCE INTERVALS AND A MONTE CARLO STUDY FOR SOME GOODNESS OF FIT TESTS

BY

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Section 1

In this paper we first consider the two-parameter exponential density function

$$f(x;\ \alpha,\beta) = \frac{1}{\beta}\, e^{-(\frac{x-\alpha}{\beta})} \qquad (x > \alpha). \qquad .....(1.1)$$

from which n independent observations are drawn.
We shall derive unbiased estimators for α and β which have minimum variance.
In doing this we make use of the following theorem [1]:

For variables coming from an exponential density function, we can prove the following:

1. $\sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)})$ has a $\Gamma(0,\ \beta,\ n-1)$ density;

2. $\underline{x}_{(1)}$ and $\sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)})$ are statistically independent;

3. $\underline{x}_{(1)}$ has a $\Gamma(\alpha,\ \frac{\beta}{n},\ 1)$ density.

Here $\Gamma(\alpha,\ \beta,\ \gamma) = \frac{1}{\beta^{\gamma}\Gamma(\gamma)}\, e^{-(\frac{x-\alpha}{\beta})}\, (x-\alpha)^{\gamma-1} \qquad (x > \alpha,\ \beta > 0,\ \gamma > 0)$

and $\underline{x}_1 \dots \underline{x}_n$ (random variables) are drawings from (1.1),
$\underline{x}_{(1)} \overset{\text{def}}{=} \min(\underline{x}_1, \dots, \underline{x}_n)$.

The expectation of a $\Gamma(\alpha,\beta,\gamma)$ density function is $\alpha + \beta\gamma$.
Now let's say we have $\underline{x}_1 \dots \underline{x}_n$ independent drawings from a density function given by formula (1.1), then by making use of the quoted theorem, we have $\sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)})$ following a $\Gamma(0,\beta,n-1)$ density function, so with an expectation

$$\mathcal{E}\{\sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)})\} = (n-1)\beta \quad . \qquad .....(1.2)$$

An unbiased estimator $\hat{\underline{\beta}}$ for β is

$$\underline{\hat{b}} = \sum_{i=1}^{n} \frac{(\underline{x}_i - \underline{x}_{(1)})}{n-1} \quad . \qquad .....(1.3)$$

Above all $\underline{\hat{b}}$ has minimum variance [2] , [3] .
For α an unbiased estimator is

$$\underline{\hat{a}} = \underline{x}_{(1)} - \frac{\sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)})}{n(n-1)} \quad . \qquad .....(1.4)$$

This is easy to see

$$\mathcal{E}(\underline{\hat{a}}) = \mathcal{E}(\underline{x}_{(1)}) - \frac{1}{n(n-1)} \, \mathcal{E}\{\sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)})\} = \alpha + \frac{\beta}{n} - \frac{\beta}{n} = \alpha.$$

B. Epstein [3] has proved that $\underline{\hat{a}}$ has also minimum variance.
To test the hypothesis H_o: $\beta = \beta_o$ we can use as a test statistic

$$\underline{T}_1 = \frac{2(n-1)\underline{\hat{b}}}{\beta_o} = \frac{2 \sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)})}{\beta_o} \qquad .....(1.5)$$

which is distributed as $\Gamma(0,2,n-1)$ under the null hypothesis, or as a a $\chi^2_{2(n-1)}$.
It is now possible to construct a two-sided confidence interval for β with an unreliability of 2ε,

$$\left(\frac{2 \sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)})}{\chi^2_{2(n-1)}(1-\varepsilon)} \; ; \; \frac{2 \sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)})}{\chi^2_{2(n-1)}(\varepsilon)} \right) \qquad .....(1.6),$$

in which $\chi^2_{2(n-1)}(1-\varepsilon)$ and $\chi^2_{2(n-1)}(\varepsilon)$ are respectively the upper and lower tail percentage points of the $\chi^2_{2(n-1)}$-distribution.
We may construct a <u>shortest confidence interval</u> for β.
The term shortest confidence interval needs clarification.
Let $\underline{x}_1 \ldots \underline{x}_n$ be a random drawing from a distribution with density function $f(x;\theta)$. In using the standard method for obtaining a confidence interval for θ, one seeks a random variable $\underline{T}(\underline{x}_1, \ldots \underline{x}_n; \theta) = \underline{T}(\theta)$ whose distribution is independent of θ. Then the probability statement

$$P(a < \underline{T}(\theta) < b) = 1 - \gamma$$

is converted to

$$P(\underline{w}_1 < \theta < \underline{w}_2) = 1-\gamma$$

and, after observing $x_1 \ldots x_n$ the specific numbers w_1, w_2 are calculated and form the endpoints of the confidence interval.
For every $\underline{T}(\theta)$, a and b can be chosen in different ways, one of which is to make $\underline{w}_2 - \underline{w}_1$ a minimum.
Such an interval based up on $\underline{T}(\theta)$ is called the shortest interval.
It may, however, be possible to find another random variable $\underline{T}'(\theta)$ which yields an even shorter interval.
We know that

$$\sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)}) \text{ is } \Gamma(0,\beta,n-1) \text{ distributed,}$$

and making the transformation

$$\underline{z} = \frac{2\underline{y}}{\beta}, \quad \text{where } \underline{y} = \sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)}), \text{ we find}$$

$$f(z) = \frac{1}{2^{n-1}\Gamma(n-1)} \; e^{-\frac{z}{2}} . \; z^{n-2} \Rightarrow \Gamma(0,2,n-1) \text{ or } \chi^2_{2(n-1)}.$$

So the stochastic variable

$$\underline{z} = \frac{2 \sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)})}{\beta} \text{ has a } \chi^2_{2(n-1)} \text{ density.}$$

Then the probability statement is

$$P(a < \frac{2 \sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)})}{\beta} < b) = 1-\gamma \qquad \ldots\ldots(1.7)$$

or

$$P\left(\frac{2 \sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)})}{b} < \beta > \frac{2 \sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)})}{a} \right) = 1-\gamma \qquad .....(1.8)$$

The length of the interval is

$$\underline{L} = 2 \sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)}) \{\frac{1}{a} - \frac{1}{b}\} . \qquad .....(1.9)$$

The Lagrange function can be written as

$$\Phi (a,b,\lambda) = 2 \sum_{i=1}^{n} (\underline{x}_i - \underline{x}_{(1)}) \{\frac{1}{a} - \frac{1}{b}\} + \lambda \{ \int_a^b f(z)\, dz - (1-\gamma)\} ...(1.10)$$

The resulting conditions for a and b are

$$\begin{cases} a^2 f(a) = b^2 f(b) \\ \int_a^b f(z)\, dz = 1-\gamma . \end{cases} \qquad .....(1.11)$$

The numerical solution of (1.11) for a and b has been obtained to four significant figures by Tate and Klett [4], for

$\nu = 2\ (1)\ 29$; ν degrees of freedom

$1-\gamma = .90, .95, .99, .995, .999.$

To construct a confidence interval for α and a test for the hypothesis $H_o : \alpha = \alpha_o$, we make use of the following theorems.

First theorem [1]:

If $\underline{x}_1$ and $\underline{x}_2$ are statistically independent with density function $\Gamma(0,\beta, \gamma_1)$ and $\Gamma(0,\beta, \gamma_2)$, then the stochastic variable $\underline{y} = \frac{\underline{x}_1}{\underline{x}_1+\underline{x}_2}$ has a β-density function

$$\beta(y; \gamma_1, \gamma_2) = \frac{\Gamma(\gamma_1+\gamma_2)}{\Gamma(\gamma_1)\, \Gamma(\gamma_2)}\, y^{\gamma_1-1} (1-y)^{\gamma_2-1} \quad (0 \le y \le 1) .$$

Second theorem [5]:

The density of $\underline{u} = \frac{\underline{x}_1}{\underline{x}_2} = \frac{\underline{y}}{1-\underline{y}}$ can be derived from the β-density-function.

Third theorem [5]:

If $\underline{x}_i$ $(i = 1,\ldots n)$ are independent stochastic variables with density functions $\Gamma(\alpha_i, \beta, \gamma_i)$, then the stochastic variable $\sum_{i=1}^{n} \underline{x}_i$ has a $\Gamma(\sum_i \alpha_i, \beta, \sum_i \gamma_i)$ density.
So the test statistic

$$\underline{T}_2 = \frac{n(\underline{x}_{(1)} - \alpha_o)}{\sum_{i=1}^{n}(\underline{x}_i - \underline{x}_{(1)}) + n(\underline{x}_{(1)} - \alpha_o)} = \frac{n(\underline{x}_{(1)} - \alpha_o)}{\sum_{i=1}^{n}(\underline{x}_i - \alpha_o)} \qquad \ldots\ldots(1.12)$$

has a $\beta(1,n-1)$ density function under H_o, namely

$$\beta(T_2;\ 1,n-1) = (n-1)\ (1-T_2)^{n-2} \qquad (0 \leq T_2 \leq 1)\ . \qquad \ldots\ldots(1.13)$$

The significance levels for a T_2 value are

$$k_l = \int_o^{T_2} (n-1)\ (1-u)^{n-2}\ du = 1-(1-T_2)^{n-1}\ , \qquad \ldots\ldots(1.14)$$

$$k_r = (1-T_2)^{n-1}\ .$$

When $\underline{T}_2$ is a test statistic and $f(T_2)$ its density function under H_o, then we mean with $T_2(\varepsilon)$ that value, for which

$$\int_o^{T_2(\varepsilon)} f(T_2)\ d\,T_2 = \varepsilon\ .$$

For the critical value $T_2(\varepsilon)$ we find

$$\varepsilon = \int_0^{T_2(\varepsilon)} (n-1)(1-u)^{n-2}\, du = 1-(1-T_2(\varepsilon))^{n-1}, \qquad \text{.....(1.15)}$$

or

$$T_2(\varepsilon) = 1-(1-\varepsilon)^{\frac{1}{n-1}} \quad \text{and analogous } T_2(1-\varepsilon) = 1-\varepsilon^{\frac{1}{n-1}}.$$

The upper tail of the confidence interval can be found as follows

$$P(\underline{T}_2 \leq T_2(\varepsilon)) = \varepsilon \rightarrow$$

$$P\left(\frac{n(\underline{x}_{(1)}-\alpha_o)}{\sum_{i=1}^{n}(\underline{x}_i-\alpha_i)} \leq T_2(\varepsilon)\right) = \varepsilon \rightarrow$$

$$P\left(\alpha_o \geq \frac{\underline{x}_{(1)}-\bar{\underline{x}}\, T_2(\varepsilon)}{1-T_2(\varepsilon)}\right) = \varepsilon$$

Moreover α is anyhow smaller than the smallest drawing from the density (1.1)

$$P(\alpha \leq \underline{x}_{(1)}) = 1$$

so

$$\left(\frac{\underline{x}_{(1)}-\bar{\underline{x}}\, T_2(\varepsilon)}{1-T_2(\varepsilon)};\ \underline{x}_{(1)}\right) \qquad \text{.....(1.16)}$$

is a confidence interval for α with unreliability ε.
The question is if this is the best interval in the sense that the expected length of the interval is minimal in revue to other possible intervals.
First of all we shall calculate the expected length of (1.16) and then look for a better interval.
The expected length of (1.16) is

$$\frac{(\alpha+\frac{\beta}{n})(1-T_2(\varepsilon)) - \alpha - \frac{\beta}{n} + T_2(\varepsilon)(\alpha+\beta)}{1-T_2(\varepsilon)} =$$

$$\frac{T_2\ (\varepsilon)}{1-T_2\ (\varepsilon)}\ \beta(1-\frac{1}{n})\ .$$

If we deal equivalent with both sides of the interval, namely

$$\left(\frac{\underline{x}_{(1)} - \bar{\underline{x}}\ T_2\ (\frac{\varepsilon}{2})}{1-T_2\ (\frac{\varepsilon}{2})};\quad \frac{\underline{x}_{(1)} - \bar{\underline{x}}\ T_2(1-\frac{\varepsilon}{2})}{1-T_2(1-\frac{\varepsilon}{2})}\right) \qquad .....(1.17)$$

then we can prove that the expected length of this last interval is shorter than (1.16).

We shall now calculate the expected length of the new interval (1.17)

$$\frac{\alpha + \frac{\beta}{n}-(\alpha+\beta)\ T_2\ (1-\frac{\varepsilon}{2})}{1 - T_2(1-\frac{\varepsilon}{2})} - \frac{\alpha + \frac{\beta}{n} - T_2\ (\frac{\varepsilon}{2})(\alpha+\beta)}{1 - T_2\ (\frac{\varepsilon}{2})}$$

Suppose $\delta = \varepsilon/2$

$$= \frac{\alpha + \frac{\beta}{n} - \{1-\delta^{\frac{1}{n-1}}\}\ (\alpha+\beta)}{\delta^{\frac{1}{n-1}}} - \frac{\alpha + \frac{\beta}{n} - \{1-(1-\delta)^{\frac{1}{n-1}}\}\ (\alpha+\beta)}{(1-\delta)^{\frac{1}{n-1}}}$$

$$= (1 - \frac{1}{n})\ \beta\ \{\frac{1}{(1-\delta)^{\frac{1}{n-1}}} - \frac{1}{\delta^{\frac{1}{n-1}}}\}$$

$$= (1 - \frac{1}{n})\ \beta\ \{\frac{1}{(1-\frac{\varepsilon}{2})^{\frac{1}{n-1}}} - \frac{1}{(\frac{\varepsilon}{2})^{\frac{1}{n-1}}}\}$$

If interval (1.17) is better than (1.16), we must have

$$\frac{T_2(\varepsilon)}{1-T_2(\varepsilon)} > \frac{1}{(1-\frac{\varepsilon}{2})^{\frac{1}{n-1}}} - \frac{1}{(\frac{\varepsilon}{2})^{\frac{1}{n-1}}}$$

or

$$\frac{1-(1-\varepsilon)^{\frac{1}{n-1}}}{(1-\varepsilon)^{\frac{1}{n-1}}} > \frac{1}{(1-\frac{\varepsilon}{2})^{\frac{1}{n-1}}} - \frac{1}{(\frac{\varepsilon}{2})^{\frac{1}{n-1}}} .$$

We know that

$$-1 + \frac{1}{(\frac{\varepsilon}{2})^{\frac{1}{n-1}}} > 0$$

$$\frac{1}{(1-\varepsilon)^{\frac{1}{n-1}}} - \frac{1}{(1-\frac{\varepsilon}{2})^{\frac{1}{n-1}}} > 0 .$$

We have found now that interval (1.17) is better. It was not possible for us to construct a <u>shortest interval</u> for α with the test statistic

$$\underline{T}_2 = \frac{n(\underline{x}_{(1)}-\alpha)}{\sum_{i=1}^{n}(\underline{x}_i-\alpha)} .$$

<u>Section 2</u>

We now consider the one-parameter exponential density function

$$f(x;\beta) = \frac{1}{\beta} e^{-\frac{x}{\beta}} \qquad .....(2.1)$$

The likelihoodfunction for a sample of n independent observations is

$$L = (\frac{1}{\beta})^n \; e^{-\frac{\sum_{i=1}^{n} x_i}{\beta}} \qquad .....(2.2)$$

$$\ln L = n \ln \frac{1}{\beta} - \frac{1}{\beta} \sum_{i=1}^{n} x_i .$$

The likelihood estimator for β for n independent drawings $\underline{x}_1 \dots \underline{x}_n$ is

$$\left.\frac{d \ln L}{d\beta}\right|_{\beta=\hat{b}} = 0 \rightarrow \underline{\hat{b}} = \frac{\sum_{i=1}^{n} \underline{x}_i}{n} \qquad \text{.....(2.3)}$$

The density function for $\underline{z}$, where $\underline{z} = \sum_{i=1}^{n} \underline{x}_i$, is

$f(z;\beta,n) = \frac{1}{\beta^n \Gamma(n)} z^{n-1} e^{-\frac{z}{\beta}}$ and for the density function for $\underline{\hat{b}}$ we get

$$g(\hat{b};\beta, n) = \frac{n}{\beta^n \Gamma(n)} (\hat{b}n)^{n-1} e^{-\frac{\hat{b}n}{\beta}} .$$

A confidence interval is constructed as follows

$$\int_0^{h_1(\beta)} g(\hat{b};\beta,n) d\hat{b} = \varepsilon . \qquad \text{.....(2.4)}$$

Suppose $\frac{x}{2} = \frac{\hat{b}\, n}{\beta} \rightarrow x = \frac{2\hat{b}n}{\beta} \rightarrow \frac{d\hat{b}}{dx} = \frac{\beta}{2\,n}$

The integral in (2.4) becomes then

$$\int_0^{\frac{2nh_1(\beta)}{\beta}} \frac{1}{2^n\, \Gamma(n)} x^{n-1} e^{-\frac{x}{2}} dx = \varepsilon ,$$

from which $h_1(\beta)$ can be written as a function of β

$$h_1(\beta) = \frac{\beta \chi^2_{2n}(\varepsilon)}{2n} . \qquad \text{.....(2.5)}$$

Analogous we find the other limit

$$h_2(\beta) = \frac{\beta \chi^2_{2n}\,(1-\varepsilon)}{2n} . \qquad \text{.....(2.6)}$$

The confidence interval for β is now

$$\left(\frac{2 \sum_{i=1}^{n} \underline{x}_i}{\chi^2_{2n}(1-\varepsilon)} \quad ; \quad \frac{2 \sum_{i=1}^{n} \underline{x}_i}{\chi^2_{2n}(\varepsilon)} \right) \quad . \qquad .....(2.7)$$

It is again possible to construct a shortest confidence interval for β, via the test statistic $\underline{T} = \frac{2n\hat{\underline{b}}}{\beta}$ which follows a χ^2_{2n} distribution. We have as a probability statement

$$P(a < \frac{2n\hat{\underline{b}}}{\beta} < b) = 1 - \gamma$$

or

$$P(\frac{2n\hat{\underline{b}}}{b} < \beta < \frac{2n\hat{\underline{b}}}{a}) = 1 - \gamma \quad .$$

The length of the interval is

$\underline{L} = \frac{2n\hat{\underline{b}}}{a} - \frac{2n\hat{\underline{b}}}{b}$ and the Lagrange function

$$\Phi\ (a,b,\lambda) = \frac{2n\hat{\underline{b}}}{a} - \frac{2n\hat{\underline{b}}}{b} + \lambda\ \{\int_a^b f(T)dT - (1-\gamma)\} \quad . \qquad .....(2.8)$$

The resulting conditions for a and b are $a^2 f(a) = b^2 f(b)$ together with the integral $\int_a^b f_{2n}(t)dt = 1-\gamma$, where $f_{2n}(t)$ is the chi-square density with 2n degrees of freedom, which will give a solution for a and b which has been tabulated by Tate and Klett.

Section 3

Some goodness of fit tests for exponential distributions

3.1 The Cramèr - Von Mises - Smirnov statistic

The following statistical problem is treated: n independent drawings $\underline{x}_1 \ldots \underline{x}_n$ from a continuous function F(.) are given and we want to test the hypothesis H_o

$$H_o: F(x) = 1-e^{-\frac{x-\alpha}{\beta}} \qquad |\alpha| < \infty \quad \beta > 0 \quad . \qquad .....(3.1)$$

Thus, we want to test whether or not the observations are coming from an exponential distribution with location parameter α and scale parameter β.

The criterion for the test statistic is an integrated squared error between the empirical distribution $\underline{F}_n(.)$ of the data (i.e. $F_n(x) = \frac{k}{n}$ if k observations are $\leq x$) and the exponential distribution $\hat{\underline{F}}(.)$ obtained by estimating the unknown parameters in F(.) assuming H_o is true. The test function is then

$$\underline{C}_n = n \int\{\underline{F}_n(x) - \hat{\underline{F}}(x)\}^2 \, d\hat{\underline{F}}(x) =$$

$$n \int\underline{F}_n^2(x) \, d\,\hat{\underline{F}}(x) + n \int\hat{\underline{F}}^2(x) \, d\,\hat{\underline{F}}(x) - 2\, n \int\underline{F}_n(x)\,\hat{\underline{F}}(x) \, d\,\hat{\underline{F}}(x) \qquad .....(3.2)$$

Integration along the real line gives:

<u>first term</u>

$$n \int\hat{\underline{F}}^2(x) \, d\,\hat{\underline{F}}(x) = \frac{n}{3} \;;$$

<u>second term</u>

$$n \int\underline{F}_n^2(x) \, d\,\hat{\underline{F}}(x) = n \sum_{i=2}^{n} \int_{\underline{x}_{(i-1)}}^{\underline{x}_{(i)}} \underline{F}_n^2(x) \, d\,\hat{\underline{F}}(x) + n - n\,\hat{\underline{F}}(\underline{x}_{(n)}) =$$

$$n \sum_{i=2}^{n} \frac{(i-1)}{n^2} [\hat{\underline{F}}(\underline{x}_{(i)}) - \hat{\underline{F}}(\underline{x}_{(i-1)})] + n - n\,\hat{\underline{F}}(\underline{x}_{(n)}) = \quad *$$

$$- n \sum_{i=1}^{n} \frac{2\,i - 1}{n^2} \, \hat{\underline{F}}(\underline{x}_{(i)}) + n =$$

$$- \sum_{i=1}^{n} \frac{(2\,i - 1)}{n} \, \hat{\underline{F}}(\underline{x}_{(i)}) + n \;;$$

---

* Here we have underlined $\hat{\underline{F}}(\underline{x}_{(i)})$, because $\hat{F}$ is a function of the stochastic drawings, and the argument of the function is a stochastic <u>ordered</u> drawing.

third term

$$2\, n \int \underline{F}_n(x)\ \hat{\underline{F}}(x)\ d\ \hat{\underline{F}}(x) = 2\, n \sum_{i=2}^{n} \int_{\underline{x}_{(i-1)}}^{\underline{x}_{(i)}} \frac{(i-1)}{n}\ \hat{\underline{F}}(x)\ d\ \hat{\underline{F}}(x)\ +$$

$$+\ n \int_{\underline{x}_{(n)}}^{\infty} \hat{\underline{F}}(x)\ d\ \hat{\underline{F}}(x) = \sum_{i=2}^{n} (i-1)\ [\hat{\underline{F}}^2(\underline{x}_{(i)}) - \hat{\underline{F}}^2(\underline{x}_{(i-1)})]\ +$$

$$+\ n\ [1 - \hat{\underline{F}}^2(\underline{x}_{(n)})] = - \sum_{i=1}^{n} \hat{\underline{F}}^2(\underline{x}_{(i)}) + n\ ;$$

so that

$\underline{C}_n = \sum_{i=1}^{n} \hat{\underline{F}}^2(\underline{x}_{(i)}) - \frac{1}{n} \sum_{i=1}^{n} (2i-1)\ \hat{\underline{F}}(\underline{x}_{(i)}) + \frac{n}{3}$, where $\underline{x}_{(1)} \cdots \underline{x}_{(n)}$ denote stochastic ordered drawings.

The hypothesis H_o must be rejected if $\underline{C}_n$ is suitable large.
The distribution of $\underline{C}_n$ for n = 10, 20, ∞ is approximated by J.v. Soest [6] for the cases: a . α and β both unknown b . α known and β unknown.
A useful property for a Monte Carlo study is that $\underline{C}_n$ is invariant for the transformation $\frac{x-\alpha}{\beta}$, so that the statistic is independent for special values of α and β. This can be seen as follows

$$\int [\underline{F}_n\ (\tfrac{x-\alpha}{\beta}) - \hat{\underline{F}}\ (\tfrac{x-\alpha}{\beta})]^2\ d\ \hat{\underline{F}}\ (\tfrac{x-\alpha}{\beta})\ .$$

Suppose $\frac{x-\alpha}{\beta} = y \rightarrow dx = \beta\, dy$, so that $\underline{C}_n$ becomes $\int [\underline{F}_n(y) - \hat{\underline{F}}(y)]^2\ d\hat{\underline{F}}(y)$.
J. v. Soest has also calculated the power of the test, but unfortunely only for a sample size of n = 20.

3.2 The Kuiper statistic for goodness of fit

Kuiper [7] has proposed $\underline{V}_n$, an adaptation of the Kolmogorov statistic, to test the null hypothesis that a random sample of size n, comes from a populati n with **given** continuous **distribution** function.
The Kuiper test statistic is defined as

$$\underline{V}_n = \sup_{-\infty<x<\infty} \{\underline{F}_n(x) - \hat{\underline{F}}(x)\} - \inf_{-\infty<x<\infty} \{\underline{F}_n(x) - \hat{\underline{F}}(x)\}.$$

Kuiper has derived the asymptotic distribution function of $\underline{V}_n$.
It is independent of the form of $F(x)$, and the convergence of the cumulative distribution function of the test statistic to its asymptotic form is quite rapid.
M. Stephens [8] has given exact significance points of $\underline{V}_n$ for a <u>completely specified</u> hypothesis.
When the null hypothesis is not completely specified and some parameters must be estimated from the sample, the distribution of $\underline{V}_n$ is no longer independent of the particular form of $F(x)$, which implies that a table of significance points must be made for every form of $F(x)$.
Making use of [9] it may be shown that the distribution of the test statistic is independent of the <u>true</u> parameters of scale and location. (This is an important property for the Monte Carlo study, which will follow here after.)
The empirical distribution is the same as in section 3.1 and the exponential distribution $\hat{\underline{F}}(.)$ is as follows

$$\hat{\underline{F}}(x) = \int_0^x \frac{1}{\hat{\underline{\beta}}} e^{-\frac{(x-\hat{\underline{\alpha}})}{\underline{\beta}}} dx = e^{\frac{\hat{\underline{\alpha}}}{\hat{\underline{\beta}}}} \{1-e^{\frac{x}{\hat{\underline{\beta}}}}\} .$$

The procedure runs as follows.
Given a set of observations $x_1 \ldots x_n$ which are arranged in increasing order, the Kuiper test statistic can be calculated.
To derive the distribution of $\underline{V}_n$ we drew a random sample of size n from the two parameter exponential distribution with location α and scale β. We did the same also for the one parameter exponential distribution. Then β and α were estimated, or β alone [see (1.3) and (1.4)] and the statistic V_n was computed. For every sample size n we repeated this procedure until all the calculated critical points of $\underline{V}_n$ were accurate to at least 0.001.
The significance levels used in the computation are 1%, 2,5%, 5%, 10%, 15%, 20%, 25%, 30%, 35%, 40%, 45%, 50%.
In table 1 and 2 we give only the levels from 1% to 20%.

Table 1. Estimated critical points for the Kuiper test statistic for testing exponentiality with scale β and location α estimated

Sample size n	Number of replications N	1%	2.5%	5%	10%	15%	20%
5	30.000	0.470	0.428	0.388	0.352	0.328	0.308
6	24.000	0.461	0.423	0.392	0.349	0.324	0.306
7	39.000	0.459	0.418	0.386	0.350	0.325	0.304
8	37.000	0.450	0.412	0.379	0.343	0.320	0.302
9	27.000	0.439	0.401	0.368	0.334	0.312	0.295
10	22.000	0.425	0.393	0.362	0.329	0.307	0.290
11	25.000	0.419	0.383	0.353	0.321	0.300	0.284
12	40.000	0.412	0.376	0.346	0.313	0.293	0.278
13	20.000	0.402	0.369	0.340	0.308	0.288	0.273
14	28.000	0.393	0.358	0.331	0.301	0.282	0.267
15	20.000	0.382	0.350	0.324	0.295	0.276	0.261
16	17.000	0.378	0.342	0.317	0.289	0.270	0.256
17	16.000	0.366	0.336	0.312	0.284	0.267	0.252
18	20.000	0.359	0.329	0.306	0.279	0.261	0.248
19	19.000	0.354	0.322	0.300	0.273	0.256	0.243
20	14.000	0.347	0.318	0.296	0.269	0.253	0.240
21	29.000	0.340	0.312	0.289	0.265	0.248	0.236
22	25.000	0.333	0.305	0.283	0.260	0.244	0.232
23	24.000	0.330	0.301	0.280	0.256	0.240	0.228
24	26.000	0.322	0.297	0.276	0.253	0.238	0.226
25	20.000	0.318	0.293	0.272	0.248	0.233	0.222
26	12.000	0.312	0.287	0.266	0.246	0.230	0.219
27	19.000	0.310	0.284	0.264	0.241	0.227	0.216
28	24.000	0.303	0.279	0.259	0.238	0.224	0.214

		1%	2.5%	5%	10%	15%	20%
29	22.000	0.302	0.278	0.258	0.236	0.222	0.212
30	15.000	0.296	0.272	0.254	0.232	0.219	0.209
40	22.000	0.264	0.244	0.227	0.207	0.196	0.186
50	15.000	0.238	0.221	0.206	0.189	0.178	0.170
60	16.000	0.222	0.205	0.191	0.176	0.166	0.158
70	16.000	0.208	0.192	0.178	0.163	0.154	0.148
80	11.000	0.194	0.180	0.168	0.155	0.146	0.140
90	18.000	0.186	0.170	0.159	0.146	0.138	0.132
100	12.000	0.179	0.164	0.153	0.140	0.132	0.127
250	9.000	0.114	0.107	0.100	0.092	0.087	0.084

Table 2. Estimated critical points for the Kuiper test statistic for testing exponentiality with scale β estimated

Sample size n	Number of replications N	1%	2.5%	5%	10%	15%	20%
		significance levels					
5	47500	0.716	0.667	0.616	0.549	0.510	0.477
6	72500	0.701	0.647	0.592	0.531	0.490	0.459
7	47500	0.680	0.622	0.572	0.512	0.474	0.443
8	92500	0.656	0.598	0.549	0.491	0.455	0.426
9	52500	0.634	0.578	0.529	0.475	0.438	0.409
10	55000	0.618	0.560	0.511	0.457	0.422	0.397
11	31000	0.601	0.546	0.500	0.446	0.412	0.387
12	30000	0.581	0.527	0.479	0.430	0.399	0.373
13	31000	0.564	0.513	0.467	0.420	0.387	0.363
14	27000	0.550	0.500	0.457	0.407	0.378	0.355
15	25000	0.537	0.485	0.444	0.397	0.367	0.346
16	30000	0.533	0.480	0.438	0.391	0.362	0.339
17	29000	0.516	0.464	0.423	0.380	0.351	0.330
18	23000	0.502	0.455	0.415	0.373	0.345	0.324
19	42000	0.493	0.445	0.406	0.363	0.337	0.317
20	28000	0.485	0.438	0.400	0.357	0.329	0.310
21	31000	0.476	0.429	0.392	0.351	0.325	0.305
22	20000	0.460	0.421	0.384	0.345	0.320	0.300
23	24000	0.457	0.410	0.376	0.337	0.312	0.293
24	21000	0.452	0.407	0.372	0.332	0.308	0.290
25	26000	0.449	0.403	0.366	0.328	0.304	0.285
26	36000	0.436	0.392	0.357	0.321	0.297	0.279
27	23000	0.432	0.387	0.353	0.316	0.294	0.277
28	21000	0.424	0.382	0.346	0.311	0.288	0.270
29	19000	0.418	0.376	0.344	0.308	0.286	0.269
30	22000	0.407	0.370	0.336	0.301	0.279	0.263
40	22000	0.360	0.325	0.297	0.268	0.249	0.234
50	31000	0.326	0.294	0.269	0.242	0.225	0.212
60	21000	0.300	0.272	0.249	0.224	0.208	0.196
70	15000	0.280	0.255	0.235	0.211	0.195	0.185
80	15000	0.265	0.241	0.221	0.198	0.185	0.174
90	17000	0.249	0.224	0.206	0.187	0.174	0.164
100	18000	0.237	0.215	0.196	0.177	0.165	0.156

Because Monte Carlo simulation involves random values, the results are subject to statistical fluctuations. Thus any estimate will not be exact but will have an associated error band.
The larger the number of trials in the simulation, the more precise will be the final answer, and we can obtain as small an error as desired by conducting sufficient trials.
The number of replications in table 1 and 2 are found as follows:
Given the significance levels 1%, 2.5%, ----, 50%, we calculated the corresponding critical points to a given numerical accuracy.
For a certain sample size we can do the reverse procedure to determine the number of replications:
Given some critical points, we first specify ε, the maximum allowable error in estimating the percentage p, and $1 - \alpha$ the desired probability or confidence level that the estimated proportion $\hat{p}$ does not differ from p by more than $\pm\ \varepsilon$; and p^1 is an initial estimate of p.
When $\varepsilon = 0.10$ and $1 - \alpha = 0.95$, the following expression, based on the normal distribution approximation to the binomial distribution, may be used to estimate the number of trials in a more statistical way:

$$N = (19{,}6)^2 \frac{1-p^1}{p^1} \ .$$

But we can't use this procedure, because we don't have the critical points to estimate the significance levels, but the reverse.

Next we have estimated the power of the test statistic. A large number of samples of size n is drawn from an alternative distribution with specified parameter(s) and for each sample we test the null hypothesis that this sample has been drawn from a two-or one- parameter exponential distribution. The fraction of the number of times that the null hypothesis is rejected gives an estimate of the power. The procedure is done for several alternatives.
Table 3 is for the two-parameter case, table 4 for the one-parameter case.

Table 3. Empirical power for the two-parameter case on a 5% and 10% level of significance and for different sample-sizes.

Alternative distribution	n = 20		n = 30		n = 40		n = 50		n = 60		n = 70	
	α=0.05	α=0.10	α=0.05	α=0.10	α=0.05	α=0.10	α=0.05	α=0.10	α=0.05	α=0.10	α=0.05	α=0.10
$\chi^2_{d.f.1}$	0.283	0.396	0.465	0.577	0.611	0.723	0.746	0.828	0.814	0.880	0.870	0.925
Γ-distr. $\beta=\frac{1}{3}$ $\delta=3$	0.327	0.450	0.503	0.618	0.687	0.791	0.810	0.885	0.882	0.937	0.929	0.962
Γ-distr. $\beta=\frac{1}{4}$ $\delta=3$	0.315	0.458	0.497	0.623	0.675	0.781	0.804	0.886	0.903	0.940	0.983	0.967
Γ-distr. $\beta=\frac{1}{2}$ $\delta=3$	0.316	0.441	0.502	0.646	0.659	0.775	0.788	0.868	0.905	0.938	0.945	0.973
Γ-distr. $\beta=1$ $\delta=2$	0.187	0.283	0.294	0.413	0.375	0.501	0.504	0.622	0.594	0.717	0.700	0.813
Γ-distr. $\beta=1$ $\delta=3$	0.323	0.445	0.501	0.629	0.698	0.806	0.788	0.875	0.877	0.929	0.939	0.967
Γ-distr. $\beta=\frac{1}{5}$ $\delta=3$	0.304	0.431	0.528	0.658	0.658	0.784	0.796	0.869	0.896	0.948	0.944	0.972

Here we mean with $\chi^2_{d.f.1}$:

$$f(x^2) = \frac{e^{-x^2/2}}{2\sqrt{\frac{\pi x^2}{2}}}$$ and with Γ-distribution the following density:

$$f(x;\beta,\delta) = \frac{1}{\beta^\delta \Gamma(\delta)} e^{-\frac{x}{\beta}} x^{\delta-1} .$$

This table gives the empirical power results against a Chi-Square alternative and some Gamma alternatives, which are derived from 1.000 random samples.

Table 4.

Empirical power for the one-parameter case on a 5% and 10% level of significance and for different sample-sizes.

Alternative distribution	n = 20		n = 30		n = 40		n = 50		n = 60		n = 70	
	α=0.05	α=0.10	α=0.05	α=0.10	α=0.05	α=0.10	α=0.05	α=0.10	α=0.05	α=0.10	α=0.05	α=0.10
$\chi^2_{d.f.1}$	0.216	0.358	0.338	0.501	0.420	0.584	0.522	0.692	0.624	0.784	0.686	0.838
Γ-distr. $\beta=\frac{1}{3}$ $\delta=3$	0.345	0.579	0.655	0.843	0.829	0.936	0.934	0.983	0.979	0.995	0.995	0.999
Γ-distr. $\beta=\frac{1}{4}$ $\delta=3$	0.680	0.832	0.903	0.977	0.985	0.995	0.997	0.999	1.0	1.0	1.0	1.0
Γ-distr. $\beta=\frac{1}{2}$ $\delta=3$	0.879	0.958	0.985	0.997	0.998	1.0	1.0	1.0	1.0	1.0	1.0	1.0
Γ-distr. $\beta=1$ $\delta=2$	0.971	0.984	0.994	0.998	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
Γ-distr. $\beta=1$ $\delta=3$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
Γ-distr. $\beta=\frac{1}{5}$ $\delta=3$	0.951	0.987	0.996	0.999	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Conclusion

For the two parameter exponential distribution the Kuiper statistic seems not to be so good in power as the Cramèr - Von Mises - Smirnov and the Shapiro-Wilk statistic. But we see a very quick rise in power for the Kuiper statistic when rising the sample size.
The results concerning the one parameter exponential distribution seem to indicate that the Kuiper statistic is in general better in power than the Cramèr - Von Mises - Smirnov and the Shapiro - Wilk statistic.
See for a comparison our tables and the table mentioned by J. v. Soest [6].

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