Jacobi-type algorithms for eigenvalues on vector- and parallel computers

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JACOBI-TYPE ALGORITHMS FOR EIGENVALUES ON VECTOR- AND PARALLEL COMPUTERS

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ABSTRACT

After a short introduction to Jacobi-like algorithms a review is given of a vector and a parallel implementation of the Jacobi method for symmetric matrices. In the last section a modification of Sameh's parallel eigen-algorithm is presented based on a problem formulation with so-called Euclidean parameters of non-orthogonal shears.

1. INTRODUCTION

The eigenvalue algorithm proposed by Jacobi, long before its time was ripe, was the favorite of the numerical analists of yesterday because of its simplicity and reliability. The method of Jacobi ([9], 1846) has been rediscovered in 1950. The rise of the QR algorithm made an end of the revival of the Jacobi algorithm. But the growing importance of vectorized and parallel computing restored the interest in Jacobi-like algorithms ([1], [12], [13], [19], [21]) and gave rise to its second revival.
An historical overview is in figure 1. The centre column describes the highlights in the history of the Jacobi algorithm for symmetric matrices. The other columns give the overview for the skew symmetric case, $A = -A^T$, the normal case, $A A^* = A^* A$, the nonnormal case and the singular value decomposition. The recent story of parallel eigenvalue computations starts in 1971 with Sameh's paper "Jacobi- and Jacobi-like Algorithms for a parallel computer" [19].
It is the purpose of the present contribution to review some recent developments both theoretical as practical.

In Jacobi-like methods for the computation of the eigenvalues $\lambda_1,\ldots,\lambda_n$ in the spectrum $\sigma(A)$ of matrix A $\mathbb{R}^{n\times n}$ a sequence $\{A_k\}$, $A_0 := A$, is constructed in which the matrices $A_k = (a_{ij}^{(k)})$, $k = 0,1,\ldots$, are recursively defined by the relation

$$A_{k+1} = V_k^{-1} A_k V_k, \quad k = 0,1,\ldots \; . \tag{1.1}$$

The matrix V_k differs from the unit matrix in the (ℓ_k,m_k)-plane. The nontrivial elements of V_k are the <u>Jacobi parameters</u>; they occur in the 2 × 2-matrix

$$\begin{pmatrix} v^{(k)}_{\ell_k,\ell_k} & v^{(k)}_{\ell_k,m_k} \\ v^{(k)}_{m_k,\ell_k} & v^{(k)}_{m_k,m_k} \end{pmatrix} = \begin{pmatrix} p_k & q_k \\ r_k & s_k \end{pmatrix} = \hat{V}_k \; . \tag{1.2}$$

This 2 × 2 matrix $\hat{V}_k$ will be called the (ℓ_k,m_k)-<u>restriction</u> of V_k. The choice of the successive pivot-pairs (ℓ_k,m_k) is called the <u>pivot-strategy</u>. In several Jacobi-like processes the pivot-pairs are selected in some cyclic order.

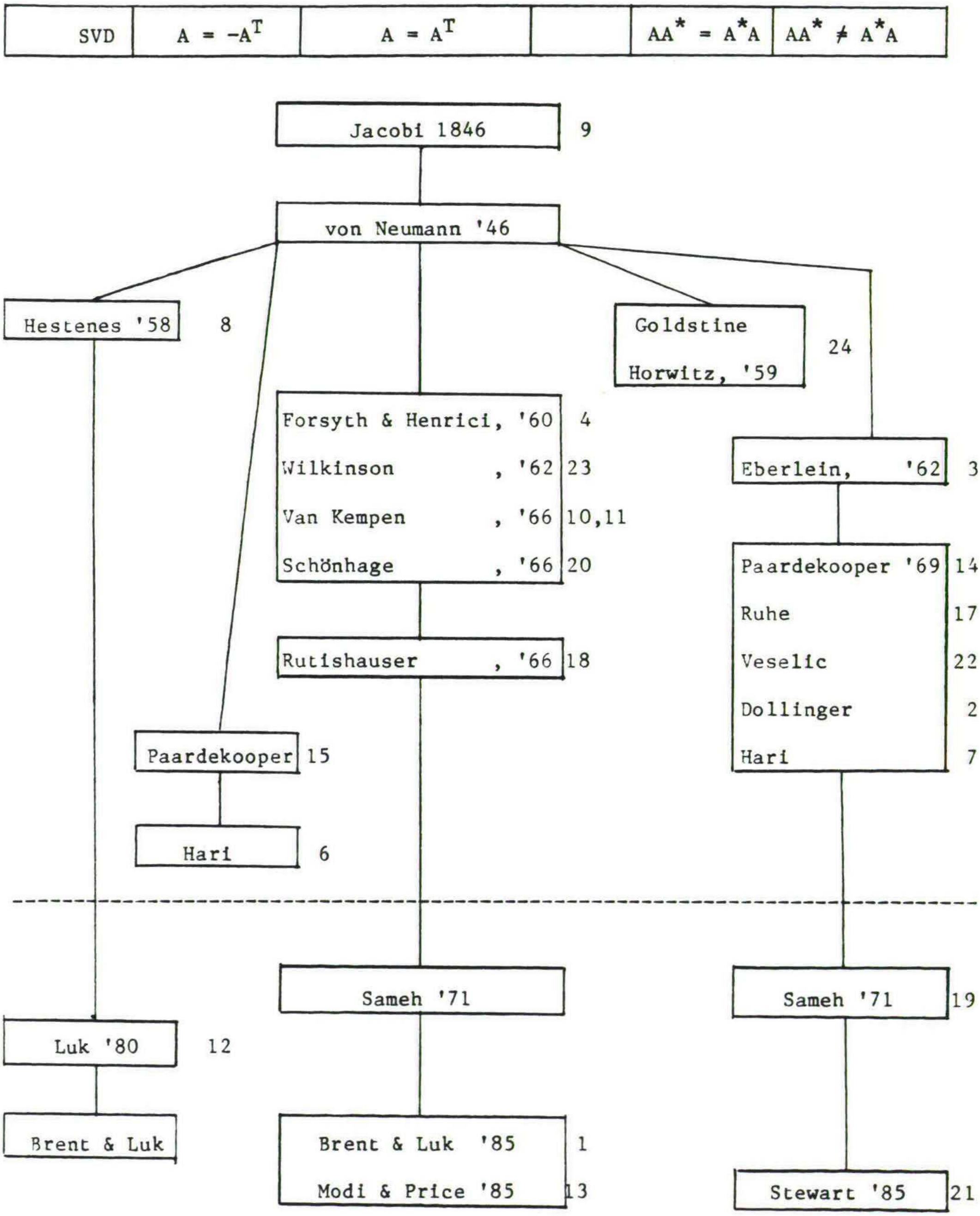

Figure 1. Historical Overview Jacobi-like Methods

In the Jacobi algorithm for the symmetric eigenvalue problem the Jacobi-parameters are

$$p_k = q_k = c_k = \cos \varphi_k , \; r_k = -q_k = s_k = \sin \varphi_k, \quad \varphi_k \in (-\tfrac{\pi}{4},\tfrac{\pi}{4}]. \qquad (1.3)$$

The rotation angle φ_k is chosen to minimize the sum of the squares of the non-diagonal elements:

$$\min_{-\frac{\pi}{4} < \varphi_k \leqq \frac{\pi}{4}} \sum_{i \neq j} (V_k^{-1} A_k V_k)_{ij}^2 (\varphi_k) .$$

In the non-normal case [2,3,14,17,22] non-unitary shears are used in order to diminish, even to minimize the Euclidean norms of the matrices in the sequence thus obtained. The minimization in each step of this process is more difficult than that in the symmetric case. After some easy but tedious calculations one observes that $\|V_k^{-1} A_k V_k\|_E$ is a rather simple function of the following variables:

$$x_k := p_k^2 + q_k^2, \; y := r_k^2 + s_k^2, \; z_k := p_k \bar{r}_k + q_k \bar{s}_k \qquad (1.4)$$

to be called the <u>Euclidean parameters</u> of V_k.

Since the Euclidean norm of A is invariant under a unitary similarity transformation the optimal normreducing shear V_k is determined except for a unitary factor, shear Q_k. Matrices S, P $\in C^{n\times n}$ will be called <u>row congruent</u> if S = PU for some unitary matrix U. It is easy to see that S and P are row congruent iff $SS^* = PP^*$. Hence the shears V_k and W_k on the same pivot pair are row congruent iff they share their Euclidean parameters, for

$$\hat{V}_k \hat{V}_k^* = \begin{pmatrix} x_k & z_k \\ \bar{z}_k & y_k \end{pmatrix} . \qquad (1.5)$$

This property of row contingent shears explains that $\|V_k^{-1} A_k V_k\|_E$ is a function of x_k, y_k and z_k and this quality will be used intensively in section 3.

Section 2 is concerned with the Jacobi process for symmetric matrices. Sequential as well as parallel algorithms are described. We emphasize the nice parallel Jacobi-algorithm of Brent-Luk [1] for a systolic array computer. The complexity result of that algorithm is impressive.

Section 3 reviews some Jacobi-like eigenalgorithms for non-normal matrices. The description of Eberlein's algorithm [3,14] with Euclidean parameters introduces our modification of Sameh's parallel algorithm [19].

2. JACOBI METHODS FOR THE SYMMETRIC EIGENVALUE PROBLEM

2.1. In each step of the Jacobi method for the symmetric eigenproblem the norm of the non-diagonal part of the new matrix $A^{(k+1)} = V_k^T A^{(k)} V_k$ is minimized with respect to the rotation parameter $\varphi_k \in (-\frac{\pi}{4}, \frac{\pi}{4}]$ of the Jacobi parameters of V_k. The minimum of

$$f(A^{(k+1)}) = \sum_{i \neq j} (V_k^T A^{(k)} V_k)_{ij}^2$$

is obtained iff $A^{(k+1)}_{\ell_k, m_k} = 0$. This annihilation of the (ℓ_k, m_k)-th element occurs iff

$$\tan 2\varphi_k = 2\, \mu_k / \nu_k, \qquad (2.1)$$

where

$$\mu_k = a^{(k)}_{\ell_k, m_k}\ , \quad \nu_k = a^{(k)}_{\ell_k, \ell_k} - a^{(k)}_{m_k, m_k}\ .$$

As an easy consequence of the orthogonality of V_k one finds

$$f(A^{(k+1)}) - f(A^{(k)}) = 2\, \mu_k^2\ . \qquad (2.2)$$

In the classical Jacobi iteration μ_k is the largest off-diagonal element in modulus. This optimal pivot-strategy is a special case of that described in

THEOREM 2.1. If $\mu_k^2 \geqq \frac{1}{n(n-1)} f(A^{(k)})$ for each $k \in \mathbb{N}$, then $\lim_{k \to \infty} A^{(k)} = \mathrm{diag}(\lambda_j)$

PROOF. With (2.2) we have

$$f(A^{(k+1)}) = f(A^{k)}) - 2\mu_k^2 \leqq (1 - \frac{2}{n(n-1)}) f(A^{(k)}),\ k \in \mathbb{N}\ .$$

Hence $f(A^{(k)}) \to 0(, k \to \infty)$. Let be $\delta = \min \{|\lambda_i - \lambda_j|; \lambda_i, \lambda_j \in \sigma(A),\ \lambda_i \neq \lambda_j\}$. Then $f(A^{(k)}) \leqq \frac{1}{16}\delta^2$ for each k larger than some N. For k > N there exists a permutation π of $\{1,\ldots,n\}$ such that $(\lambda_{\pi(i)} - a_{ii}^{(k)})^2 \leqq f(A^{(k)}) \leqq \frac{1}{16}\delta^2$ for each $i \in \{1,\ldots,n\}$, as follows from the Hoffmann-Wieland theorem. Since $|\varphi_k| \leqq \frac{\pi}{4}$ and $|a^{(k+1)}_{\ell_k,\ell_k} - a^{(k)}_{\ell_k,\ell_k}| = |a^{(k+1)}_{m_k,m_k} - a^{(k)}_{m_k}| = |\mu_k \tan\varphi_k| \leqq \frac{1}{4}\delta$,

$a^{(k+1)}_{\ell_k,\ell_k} \in [\lambda_{\pi(\ell_k)} - \delta/4, \lambda_{\pi(\ell_k)} + \delta/4]$. Similarly $a^{k+1}_{m_k,m_k}$ remains in the corresponding interval around $\lambda_{\pi(m_k)}$. The stationary matching of diagonal elements and eigenvalues together with $(\lambda_{\pi(i)} - a^{(k)}_{ii})^2 \leqq f(A^{(k)}) + O(,k \to \infty)$ implies the theorem.□

As an alternative for the pivot strategy mentioned in theorem 2.1 a cyclic method can be used, especially the row serial method. Then the elements are annihilated in the cyclic order (1,2),(1,3),...,(1,n),(2,3),...,(n-1,n), (1,2),... . Forsyth and Henrici [4] proved the convergence of the serial method. The ultimate convergence is quadratic, i.e. $f(A^{k+M}) < \text{constant } f^2(A^k)$, k large enough, where $M = \frac{1}{2} n(n-1)$; this has been investigated in [10,11,20,23].

2.2. The transformation $A^{(k+1)} = V_k^T A^{(k)} V_k$ requires 4n multiplications plus 2n additions. On a supercomputer like CYBER 205 it is recommendable to speed up this timeconsuming process with linked triads [16]. Let $A^{(k)} = D_k H^{(k)} D_k$, where $D_k = \text{diag}(d^{(k)},\ldots,d^{(k)}_n)$, with $D_0 = I$. Since

$$A^{(k+1)} = D_{k+1} H^{(k+1)} D_{k+1} = V_k^T D_k H^{(k)} D_k V_k , \tag{2.3}$$

where $\hat{V}_k = \begin{vmatrix} c_k & s_k \\ -s_k & c_k \end{vmatrix}$,

the obvious updating for D_k:

$$d_i^{(k+1)} = \begin{cases} d_i^{(k)} & , i \neq \ell_k, m_k \\ c_k d_i^{(k)} & , i \in \{\ell_k, m_k\} \end{cases} \tag{2.4}$$

brings about linked triads in the updating of $H^{(k)}$, viz.

$$H^{(k+1)} = J_k^T H^{(k)} J_k , \tag{2.5}$$

The (ℓ_k,m_k)-restriction of J_k is

$$\begin{vmatrix} 1 & \beta_k \\ -\alpha_k & 1 \end{vmatrix}, \begin{cases} \beta_k = t_k\, d^{(k)}_{m_k}/d^{(k)}_{\ell_k} \\ \alpha_k = t_k\, d^{(k)}_{\ell_k}/d^{(k)}_{m_k} \end{cases} , t_k = \tan \varphi_k .$$

Let be v_k the ℓ_k-th column of $H^{(k)}$ and w_k the m_k-th column of $H^{(k)}$. The corresponding columns of $H^{(k)}J_k$ are $\tilde{v}_k = v_k - \alpha_k w_k$, $w_k' = w_k + \beta_K v_k = (1+\alpha_k\beta_k)\ (w_k+\tilde{\beta}_k\tilde{v}_k)$ with $\tilde{\beta}_k = \beta_k/(1+\alpha_k\beta_k)$. With the updating scheme

$$\begin{cases} \tilde{v}_k = v_k - \alpha_k\ w_k \\ w_k' = w_k + \tilde{\beta}_k\ \tilde{v}_k \end{cases} , \begin{cases} d_{\ell_k}^{(k+1)} = c_k\ d_{\ell_k}^{(k)} \\ d_{m_k}^{(k+1)} = c_k^{-1} d_{m_k}^{(k)} \end{cases} , \qquad (2.6)$$

one avoids the necessity to copy v_k. In a similar way the new rows are computed. The multiplication $D_{k+1}H^{(k+1)}D_{k+1}$ is postponed until the end of the process. We conclude that variant (2.6) of a serial Jacobi-method is appropriate for a CYBER-like supercomputer [16].

2.3. The Jacobi methods described so far all were sequential. In essence the Jacobi-method with its nested loop for which computations are almost identical over the entire index set $\{(i,j)\}$ is pre-eminently suited for processing on an array processor with regular dataflow [1,12,13]. The systolic implementation of the Jacobimethod in [1] has a high degree of modularity, absence of long data paths, near-by connectivity and a simple synchronizing mechanism.

Assume n to be even. Consider the parallel Jacobi-like updating of the column pairs $2\ell - 1$, $2\ell(,\ell=1,\ldots,n/2)$ and there after the corresponding updating of the same pairs of rows. This achieves the annihilation of the elements $A_{2\ell-1,2\ell}$. In order to achieve an analogous updating of an other set of n/2 column-row pairs consider the permutation π of $\{1,\ldots,n\}$ such that

$$\pi(i) = \begin{cases} i & , \quad i = 1 \\ i + 2 & , \quad i = 2,4,\ldots,n-2 \\ i - 2 & , \quad i = 5,7,\ldots,n-1 \\ i - 1 & , \quad i = 3, \quad i = n . \end{cases}$$

The repeated exeration of this caterpillar permutation is illustrated in figure 2.

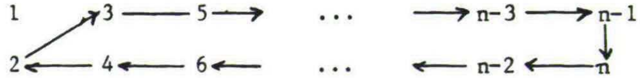

fig. 2. Caterpillar permutation $\pi(1,\ldots,n) = (1,4,2,6,\ldots,n-3,n-1)$.

The repeated annihilation of codiagonal elements can be performed on a systolic array processor, see figure 3. The square n/2 by n/2 array consists of processors $P_{ij}, i,j = 1,\ldots, n/2$, each containing the corresponding 2 × 2 submatrix, viz.

$$\tilde{A}_{ij} = \begin{vmatrix} \alpha_{ij} & \beta_{ij} \\ \gamma_{ij} & \delta_{ij} \end{vmatrix} = \begin{vmatrix} a_{2i-1,2j} & a_{2i-1,2j} \\ a_{2i,2j-1} & a_{2i,2j} \end{vmatrix} , \quad i,j = 1,\ldots,n/2.$$

In the first time step the processors P_{ii} compute $\begin{vmatrix} c_i & -s_i \\ s_i & c_i \end{vmatrix}$ and annihilate the elements $\beta_{ii}(=\gamma_{ii})$. Horizontal and vertical output lines (1) transport the rotation parameters $t_i = s_i/c_i$ away from the diagonal. In the second time step the transformation

$$\begin{vmatrix} c_i & -s_i \\ s_i & c_i \end{vmatrix} \begin{vmatrix} \alpha_{ij} & \beta_{ij} \\ \gamma_{ij} & \delta_{ij} \end{vmatrix} \begin{vmatrix} c_j & s_j \\ -s_j & c_j \end{vmatrix} \tag{2.7}$$

are executed in the codiagonal processor $P_{i,i\pm 1}$, the rotationparameters t_i are further transmitted along horizontal and vertical lines (2) to $P_{i,j}$ with $|i-j| = 2$ and the elements in the codiagonal registers are interchanged along lines (2), see figure 3. In the third time step the transformations (2.7) are performed in $P_{i,i\pm 2}$, the t_i are transfered to $P_{i,i\pm 3}$ and the processors P_{ij} are provided with the appropriate elements along the lines (3), ready for timestep 4 = 1(mod 3). Then the diagonal processors annihilate the elements $\beta_{ii} = a_{\pi(2i-1),\pi(2i)}$ and perform (2.7) in $P_{i,j}$, $|i - j| = 3$.

This systolic system pumps the data around the network. One sweep corresponds with 3(n-1) time steps. Hence the conclusion that a two-dimensional systolic array of $\frac{n}{2} \times \frac{n}{2}$ procession computes the eigenvalue of a n × n symmetric matrix in O(n log n) time units. Each time approximately $\frac{1}{3} n^2/4$ processors perform the transformation (2.7).

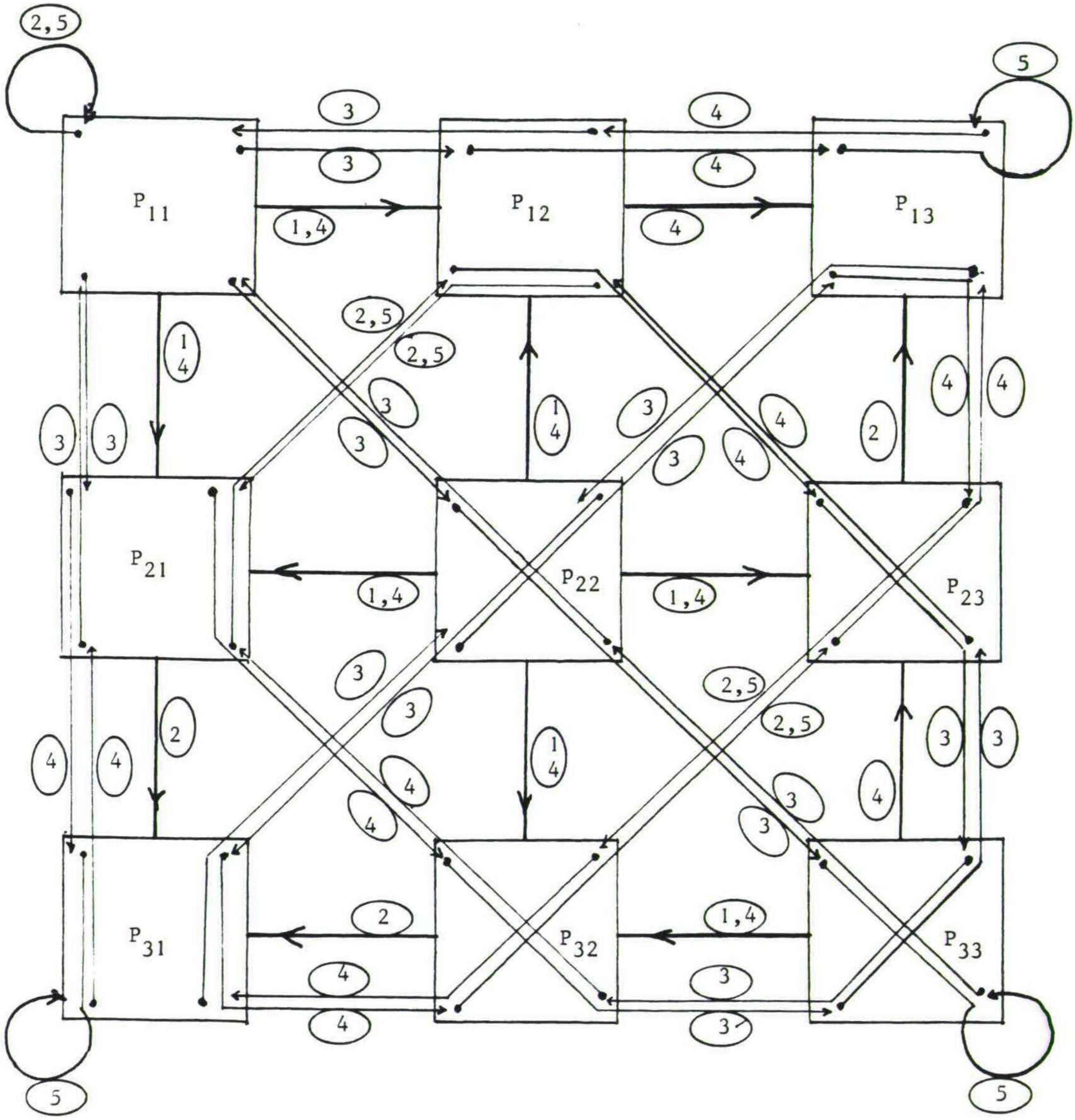

FIG. 3. SYSTOLIC ARRAY. The number of the lines indicate the time they are active.

3. A PARALLEL NORM-REDUCING ALGORITHM FOR THE NONSYMMETRIC EIGENPROBLEM

3.1. In 1962 Eberlein [3] proposed a Jacobi-like norm-reducing method for the non-normal eigenproblem. In each iteration $A_{k+1} := V_k^{-1} A_k V_k$ a non-orthogonal shear effectuates a norm reduction $\|A_k\|_E^2 - \|A_{k+1}\|_E^2$ in that k-th step. The pivot strategy and the non-trivial elements p_k, q_k, r_k, s_k of the successive unimodular V_k can be chosen such that [2,3,14,22] $\lim_{k\to\infty} \|A_k\|_E^2 = \sum_{j=1}^{n} |\lambda_j|^2, \lambda_j \in \sigma(A)$, eigenvalue of $A = A_0$. This means that sequence $\{A_k\}$ <u>converges to normality</u> [5,14]. The unimodularity of V_k implies that the Euclidean parameters (x_k, y_k, z_k) of the real shear V_k satisfy the conditions

$$x_k, y_k > 0, \quad x_k y_k - z_k^2 = 1. \tag{3.1}$$

Easy calculations give the result already announced in the introduction.

THEOREM 3.1 [14]. If V is a real unimodular shear with pivotpair (ℓ, m) and Euclidean parameters $(x,y,z) \in \mathbb{R}^3$, then

$$\|V^{-1}AV\|_E^2 = f(x,y,z) + \sigma + e , \tag{3.2}$$

with $\sigma = \sum_{i,j \notin \{1,m\}} a_{ij}^2$, $e = (a_{\ell\ell}+a_{mm})^2 - 2(a_{\ell\ell}a_{mm}-a_{\ell m}a_{m\ell})$ and

$$f(x,y,z) = \alpha x + \beta y + 2\gamma z + (-\lambda x+\mu y+\upsilon z)^2 \tag{3.3}$$

where $\alpha = \sum_{i\neq\ell,m} (a_{i\ell}^2+a_{mi}^2)$, $\beta = \sum_{i\neq\ell,m} (a_{\ell i}^2+a_{im}^2)$, $\gamma = \sum_{i\neq\ell,m} (a_{i\ell}a_{im} - a_{\ell i}\, a_{mi})$,

and $\lambda = a_{m\ell}$, $\mu = a_{\ell m}$, $\gamma = a_{\ell\ell} - a_{mm}$.□

So the minimization of f on $\mathcal{H} := \{(x,y,z) \mid x,y > 0,\ xy - z^2 = 1\}$ provides the optimal norm-reducing unimodular shears. An accurate analysis of that minimization problem leads to

THEOREM 3.2 [14] Let be $D = \alpha\mu - \beta\lambda - \gamma\nu$, $E = \nu^2+4\lambda\mu$, $F = \alpha\beta - \gamma^2$. If D and F are not 'oth equal to zero then f is minimal on H in the point

$$x = \frac{2\mu D - \beta(\rho-E)}{\rho(\rho-E)}, \quad y = \frac{-2\lambda D - \alpha(\rho-E)}{\rho(\rho-E)}, \quad z = \frac{-\nu D + \gamma(\rho-E)}{\rho(\rho-E)}, \tag{3.4}$$

where ρ is the unique root of the quartic equation

$$(\rho-E)^2(\rho-F) + D^2(2\rho-E) = 0$$

for which holds $\rho < \min\{0,E\}$. The infimum of f on $\mathcal{H}$ is not assumed when $D = 0 \wedge F = 0 \wedge (\alpha+\beta\neq 0 \vee (E=0\ \lambda\neq\mu))$. Then the intersection of the planes $\alpha x + \beta y + 2\gamma z = 0$ and $-\lambda x + \mu y + \nu z = 0$ is in the tangent cone $xy - z^2 = 0$ of $\mathcal{H}$. □

With the new variables

$$w := (x-y)/2, \quad t := t(w,z) = (x+y)/2 = (1+w^2+z^2)^{\frac{1}{2}} \tag{3.5}$$

we get $\|V^{-1}AV\|_E^2 = g(w,z) + \sigma + e$, where

$$g(w,z) = (\alpha+\beta)t + (\alpha-\beta)w + 2\gamma z + ((\mu-\lambda)\ t - (\mu+\lambda)w + \nu z)^2. \tag{3.6}$$

Now let be $C = C(A) = A^TA - AA^T$, the <u>commutator</u> of A, being a measure of non-normality. One easily finds

THEOREM 3.3[14]. Grad $g(0,0) = (c_{mm}-c_{\ell\ell}, 2c_{\ell m})^T$. Moreover $\|\tilde{V}^{-1}A\tilde{V}\|_E = \min\{\|V^{-1}AV\|_E \mid V \text{ unimodular } (\ell,m)\text{-shear}\}$ iff $\tilde{c}_{mm} - \tilde{c}_{\ell\ell} = \tilde{c}_{\ell m} = 0$, where $(\tilde{c}_{ij}) = C(\tilde{V}^{-1}A\tilde{V})$, the commutator of $\tilde{V}^{-1}A\tilde{V}$. □

This results gives an indication for an appropriate pivot strategy: choose in each step (ℓ,m) such that $\|\text{grad } g(0,0)\|^2 = (c_{mm}-c_{\ell\ell})^2 + 4c_{\ell m}^2$ is maximal. These choises of (ℓ_k,m_k) together with optimal normreducing shears V_k garantee that $C(A_k) \to 0(,k\to\infty)$[2,3,14,22].

3.1. The purpose of this subsection is to present an improved version of Sameh's parallel norm-reducing process [19]. Therefore we assume A to be of even order $n = 2k$ and partitioned as follows

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1_k} \\ A_{21} & A_{22} & \cdots & A_{2_k} \\ \vdots & \vdots & & \\ A_{k_1} & A_{k_2} & \cdots & A_{k_k} \end{pmatrix}, \tag{3.7}$$

where each submatrix $A_{\ell m}$ is given by

$$A_{\ell m} = \begin{vmatrix} a_{2\ell-1,2m-1} & a_{2\ell-1,2m} \\ a_{2\ell,2m-1} & a_{2\ell,2m} \end{vmatrix} \quad , \ \ell,\ m = 1,\ldots,k \ . \tag{3.8}$$

For convenience we define

$$\lambda_{\ell m} := a_{2\ell,m-1},\ \mu_{\ell m} := a_{2\ell-1,2m},\ \nu_{\ell m} := a_{2\ell-1,2\ell-1} - a_{2\ell,2\ell}.$$

Let be

$$\tilde{A} = V^{-1}AV \tag{3.9}$$

where $V = \mathrm{diag}(S_1,S_2,\ldots,S_k)$ with

$$S_i = S = \begin{vmatrix} p & q \\ r & s \end{vmatrix} \ , \ ps - qr = 1,\ i = 1,\ldots,\ k. \tag{3.10}$$

The computation of $V^{-1}AV$ is readily adapted to parallel computation: firstly simultaneous updating of the k column pairs, next the simulataneous updating of the k rows: $V^{-1}(AV)$. With Euclidean parameters (x,y,z) of S in (3.10)one obtains, analogous to theorem 3.1,

THEOREM 3.4. If V is $\mathrm{diag}(S_1,\ldots,S_k)$ where S_i as given in (3.10) then $\|V^{-1}AV\|_E^2 = h(x,y,z) + K$, where

$$h(x,y,z) = \sum_{\ell,m=1}^{k} (-\lambda_{\ell m}x + \mu_{\ell m}y + \nu_{\ell m}z)^2 \tag{3.11}$$

and $K = \sum_{\ell,m=1}^{k} (\mathrm{tr}^2 A_{\ell m} - 2\det(A_{\ell m}))$. □

The minimization of h on $\mathcal{H}$ leads to a generalized eigenproblem: $\det(B^TB-\rho H) = 0$, where

$$B = \left| b, b_2, b_3 \right| \in \mathbb{R}^{\frac{1}{2}n^2 \times 3}, \quad H = \begin{vmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \text{ with}$$

$$b_1 = -(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1k}, \lambda_{21}, \ldots, \lambda_{kk})^T$$

$$b_2 = (\mu_{11}, \mu_{12}, \ldots, \mu_{1k}, \mu_{21}, \ldots, \mu_{kk})^T \qquad (3.12)$$

$$b_3 = (\nu_{11}, \nu_{12}, \ldots, \nu_{1k}, \nu_{21}, \ldots, \nu_{kk})^T \quad .$$

With usual compactness, continuity and convexity arguments one reaches THEOREM 3.5. If rank (B) = 3, B = QR with R uppertriangular then the vector of optimal Euclidean parameters of S in V is eigenvector corresponding with the unique positive eigenvalue of $R^{-T}H\,R^{-1}$. □

In case of collinearity in the matrix B it may occur that the function h is not minimal on $\mathcal{H}$:

THEOREM 3.6. If rank (B) = 1 and $\mathrm{im}(B^T) = \{\tau(p_1, p_2, p_3)^T \mid \tau \in \mathbb{R}\}$ then

(i) $\min\{h(x,y,z) \mid x,y > 0,\ xy - z^2 = 1\} > 0$ if $p_3^2 - 4p_1p_2 < 0$.

(ii) $\inf\{h(x,y,z) \mid x,y > 0,\ xy - z^2 = 1\} = 0$ if $p_3^2 - 4p_1p_2 \geqq 0$; this infimum is assumed iff $p_3^2 - 4p_1p_2 > 0$. (3.13)

If rank (B) = 2 and $\ker(B) = \{\tau(p_1, p_2, p_3)^T \mid \tau \in \mathbb{R}\}$, then

(i) $\min\{h(x,y,z) \mid x,y > 0,\ xy - 2^2 = 1\} > 0$ if $p_1p_2 < p_3^2$.

(ii) $\inf\{h(x,y,z) \mid x,y > 0,\ xy - z^2 = 1\} = 0$ if $p_1p_2 \geqq p_3^2$; this infimum is assumed iff $p_1p_2 > p_3^2$. □ (3.14)

The resemblance of the transforms (3.10) and the Eberlein shear transform is also manifest in the analogue of theorem 3.3. Let be $w = (x-y)/2$, $t = (x+y)/2$ as in (3.5). Then

$$\|V^{-1}AV\| = \|(b_1+b_2)t + (b_1-b_2)w + b_3z\|^2 + K =: g(w,z;A) + K \quad (3.15)$$

With these new variables w,z and $t = t(w,z) = (1+z^2+w^2)^{\frac{1}{2}}$ we get the analogue of theorem 3.3 by simple calculations:

THEOREM 3.7. Grad $g(0,0;A) = \sum_{\ell=1}^{k} (c_{2\ell-1,2\ell-1}-c_{2\ell,2\ell}, 2c_{2\ell-1,2\ell})^T$.
Moreover the parallel identical shear transformation $V^{-1}AV$ gives an optimal norm-reduction iff $\sum_{\ell=1}^{k} (\tilde{c}_{2\ell-1,2\ell-1}-\tilde{c}_{2\ell,2\ell}, 2\tilde{c}_{2\ell-1,2\ell}) = (0,0)$, where $(c_{ij}) = C(V^{-1}AV)$. □

3.2. Since grad $g(0,0;A) = \sum_{\ell=1}^{k} (c_{2\ell-1,2\ell-1}-c_{2\ell,2\ell}, 2c_{2\ell-1,2\ell})$, a prologue transformation [19] with a well chosen direct sum $Q = \mathrm{diag}(Q_1,Q_2,\dots,Q_k)$ of orthogonal shears $Q_\ell, \ell=1,\dots,k$, may enlarge the gradient. Let be $(c'_{ij}) = C(Q^TAQ)$ then

$$v'_\ell := \begin{vmatrix} c'_{2\ell-1,2\ell-1}-c'_{2\ell,2\ell} \\ 2c'_{2\ell-1,2\ell} \end{vmatrix} = Q_\ell^{-2} \begin{vmatrix} c_{2\ell-1,2\ell-1}-c_{2\ell,2\ell} \\ 2c_{2\ell-1,2} \end{vmatrix} =: Q_\ell^{-2} v_\ell \quad (3.16)$$

Each v'_ℓ has a same direction by an appropriate choice of the Q_ℓ. Then

$$\|g(0,0;Q^TAQ)\| = \|\sum_{\ell=1}^{k} v'_\ell\| = \sum_{\ell=1}^{k} \|v'_\ell\| = \sum_{\ell=1}^{k} \|v_\ell\| . \quad (3.17)$$

The vectors $v_\ell \in \mathbb{R}^2$ will be rectified with simultaneous Jacobi annihilations applied to C such that

$$c'_{2\ell-1,2\ell} = 0,\ c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell} = \|v_\ell\|,\ \ell = 1,\dots,k. \quad (3.18)$$

The preconditioning $A \to A' = Q^TAQ$ simplifies the performance of the first steepest descent iteration for the minimization of g.
A lower bound of $\|A\|_E^2 - \|V^{-1}\ Q^TAQV\|$ is given in
THEOREM 3.8. Let be $Q = \mathrm{diag}(Q_1,\dots,Q_k)$ a direct sum of orthogonal shears such that $(c'_{ij}) = C(Q^TAQ)$ satisfies (3.18). Then there exists a diagonal matrix $V = \mathrm{diag}(x,x^{-1},x,\dots,x^{-1})$ such that

$$\|A\|_E^2 - \|V^{-1}\ Q^TAQV\|_E^2 \geq \frac{1}{8}\|A\|_E^{-2} \sum_{\ell=1}^{k} \|v_\ell\|^2 . \quad (3.19)$$

PROOF. Let be $A' = Q^TAQ$ and $B' = \begin{vmatrix} b'_1 & b'_2 & b'_3 \end{vmatrix}$ similar to (3.12). Then $\|V^{-1}A'V\|_E^2 := \|xb'_1 + x^{-1}b'_2\|^2$ + constant. The minimizing x gives

$$\|A\|_E^2 - \|V^{-1}Q^TAQV\|_E^2 = (\|b_1'\|-\|b_2'\|)^2.$$

Since, assuming b_1', $b_2' \neq 0$,

$$\|b_1'\| - \|b_2'\| = (\|b_1'\| + \|b_2'\|)^{-1}(\|b_1'\|^2-b_2'\|^2) = \frac{1}{2}(\|b_1'\|+\|b_2'\|^{-1}g_w(0,0;A')$$

we obtain from $\|b_1'\| + \|b_2'\| \leqq \|A\|_E \sqrt{2}$ and

$$g_w(0,0;A') = \sum_{\ell=1}^{k} (c'_{2\ell-1,2\ell-1}-c'_{2\ell,2\ell}) = \sum_{\ell=1}^{k} \|v_\ell\|$$

that

$$\|A\|_E^2 - \|V^{-1}A'V\|_E^2 \geqq \frac{1}{8} \|A\|_E^{-2} (\sum_{\ell=1}^{k} \|v_\ell\|)^2 \geqq \frac{1}{8} \|A\|_E^{-2} \sum_{\ell=1}^{k} \|v_\ell\|^2 .$$

In case $b_2' = 0$ and $b_1' \neq 0$ the same bound holds: take $x^2 < 1 - \frac{1}{8} \|A\|_E^{-2}\|b_1'\|^2$.□
For the main theorem 3.10 finally we need a modification of a lemma in [3].
THEOREM 3.9. There exists a set of k disjunct index pairs (ℓ_j,m_j) with $\ell_j \neq m_j$, $j = 1,\ldots,k$ such that

$$\sum_{j=1}^{k} ((c_{\ell_j,\ell_j}-c_{m_j,m_j})^2 + 4c_{\ell_j,m_j}^2) \geqq \frac{4}{n-1} \|C(A)\|_E^2 . \qquad (3.20)$$

PROOF. Since $\sum_{\ell=1}^{n} c_{\ell\ell} = 0$, $\sum_{\ell\neq m} (c_{\ell\ell}-c_{mm})^2 = (2n-1) \sum_{\ell=1}^{n} c_{\ell\ell} - \sum_{\ell\neq m} c_{\ell\ell}c_{mm}$ and $0 = (\sum_{\ell=1}^{n} c_{\ell\ell})^2 = \sum_{\ell=1}^{n} c_{\ell\ell}^2 + \sum_{\ell\neq m} c_{\ell\ell}c_{mm}$, we have for $n \geqq 2$: $\sum_{\ell\neq m} (c_{\ell\ell}-c_{mm})^2 = 2n \sum_{\ell=1}^{n} c_{\ell\ell}^2 \geqq 4 \sum_{\ell=1}^{n} c_{\ell\ell}^2$. Consequently

$$\sum_{\ell\neq m} ((c_{\ell\ell}-\ell_{mm})^2 + 4c_{\ell m}^2) \geqq 4 \|c(A)\|^2 .$$

Hence the mean of $\sum_{j=1}^{k} ((c_{\ell_j,\ell_j}-c_{m_j,m_j})^2 + 4c_{\ell_j,m_j}^2)$ over the sets ω of k distinct index pairs (ℓ_j,m_j) satisfies (3.20) for there are $n!/(k!2^k)$ such sets ω and each pair (ℓ,m) occurs in $(n-2)!/((k-1)!2^{k-1})$ such sets ω.□

A consequence of the theorems 3.8 and 3.9 is

THEOREM 3.10. Let a sequence $\{A_j\}$ starting with $A_0 = A$ be constructed recursively by

$$A_j = (P_jQ_jV_j)^{-1} A_{j-1} P_jQ_jV_j, \quad j = 0,1,\dots$$

where in each step k disjunct indexpairs $(\ell_1^j,m_1^j),\dots,(\ell_k^j,m_k^j)$ are selected according to rule (3.20). P_j is a permutation matrix with $P_j(\ell_1^j,m_1^j,\dots,\ell_k^j,m_k^j)^T = (1,2,\dots,n)^T$, Q_j is a preconditioning orthogonal block-diagonal matrix as described in subsection 3.2 and $V_j = \mathrm{diag}(x_j,x_j^{-1},\dots,x_jx_j^{-1})$ reduces the Euclidean norm of $(P_jQ_j)^{-1}A_{j-1}\, P_jQ_j$ as described in theorem 3.8. Then $\{A_j\}$ converges to normality.

PROOF. Since $\Delta_j = \|A_{j-1}\|_E^2 - \|A_j\|_E^2 \to 0(j\to\infty)$ and, as follows from theorem 3.8 and theorem 3.9,

$$\Delta_j \geq \frac{1}{8}\|A\|_E^{-2} \sum_{\ell=1}^{k} \|v_\ell^j\|^2 \geq \frac{1}{2(n-1)} \|A\|_E^{-2} \|C(A_{j-1})\|_E^2 ,$$

we conclude $C(A_j) \to 0(j\to\infty)$.□

REMARK 3.11. Evidently the same choice for P_jQ_j together with the optimal normreducing $V_j = \mathrm{diag}(S_1^j,S_2^j,\dots,S_k^j)$ where $S_\ell^j = S^j$, $\ell = 1,\dots,k$ as described in theorem 3.5 provides a sequence $\{A_j\}$ that so much the more converges to normality.

Finally we mention that each S_ℓ^j is row congruent with the shear $S_\ell^jU_\ell^j$ that diagonalizes the symmetric part of the current matrix. Veselic [22] proved that a sequence of normreductions interrupted by Jacobi iterations for the diagonalization of $A_j + A_j^T$ effects that $\lim_{j\to\infty} A_j = D + K$, with $D = \mathrm{diag}(\mathrm{Re}(\lambda_j))$, $K = -K^T$ and $DK = KD$. Then $D_{ii} \neq D_{jj}$ implies $K_{ij} = 0$.□

REMARK 3.12. For concreteness we indicate the parallelization of a cyclic version of the normreducing process with the caterpillar permutation P. Then the timeconsuming search in (3.20) is avoided.

(i) Annihilate the elements $c_{2\ell-1,2\ell}$, $\ell = 1,\dots,k$ of C(A) with $A' := Q^T(AQ)$. The updating of the column pairs $2\ell-1,2\ell$ can be performed simultaneously: $A \to A\,\mathrm{diag}(Q_1,\dots,Q_k)$; once this is done the updating of the row pairs $2\ell-1$, 2ℓ can be carried out concurrently: $AQ \to \mathrm{diag}(Q_1^T,\dots,Q_k^T)\,(AQ)$ with k processors $G_1,\dots,G_k$,

(ii) Compute with processor G_ℓ:

$$e_\ell := \sum_{m=1}^{k} (a'_{2\ell-1,2m})^2, \quad f_\ell := \sum_{m=1}^{k} (a'_{2\ell,2m-1})^2$$

Let be $E := \sum_{\ell=1}^{k} \ell_\ell$, $F = \sum_{\ell=1}^{k} f_\ell$ and

$$x = \begin{cases} 1 & E = F = 0 \\ (1-\frac{1}{2}\|A\|_E^{-2})^{\frac{1}{2}}, & E \neq 0,\ F = 0 \\ F/(1-\frac{1}{2}\|A\|_E^{-2}F)^{\frac{1}{2}}, & E = 0,\ F \neq 0 \\ E/F, & E,F \neq 0 . \end{cases}$$

The column and row updates $A' \to \tilde{A} := V^{-1}A'V$ with $V = \mathrm{diag}(x, x^{-1}, \ldots, x, x^{-1})$ can be performed concurrently as in (i).

(iii) Execute the caterpillar permutation $A := P^TAP$.

After n-1 of these steps the original order has been restored.

The analysis of this parallel process already leads to many problems of design and it makes clear the importance and difficulties of the dataflow and communication in the implementation of parallel algorithms.

4. REFERENCES

1. Brent, R.P. and F.L. Luk, The Solution of Singular-value and Symmetric Eigenvalue Problems on Multiprocessor Arrays, SIAM J. Sci. Stat. Comput., 6, 1985, pp. 69-84.

2. Dollinger, E., Ein linear konvergentes zyklisches jacobi ähnliches Verfahren für beliebige reele Matrizen, Num. Math., 38, 1981, pp. 245-253.

3. Eberlein, P.J., A Jacobi-like Method for the Automatic Computation of Eigenvalues and Eigenvectors of an Arbitrary Matrix, J. Soc. Indust. Appl. Math., 10, 1962, pp. 74-88.

4. Forsyth, G.E. and P. Henrici, The cyclic Jacobi Method for Computing the Principal Values of a Complex Matrix, Trans. Am. M.S., 94, 1960, pp. 1-23.

5. Elsner, L. and M.H.C. Paardekooper, On Measures of Non-normality of Matrices, forthcoming in Lin. Alg. Appl.

6. Hari, V., On the Quadratic Convergence of the Paardekooper Method I, Glasnik Matem., 17, 1982, pp. 183-195.

7. Hari, V., On the Global Convergence of the Eberlein Method for Real Matrices, Num. Math., 39, 1982, pp. 361-369.

8. Hestenes, M.R., Inversion of Matrices by Biorthogonalization and Related Results, J. Soc. Indust. Appl. Math., 6, 1958, pp. 51-90.

9. Jacobi, C.G.J., Über ein leichtes Verfahren die in der Theorie der Seculär störungen vorkommenden Gleichungen numerisch aufzulösen, Crelle's J., 30, 51-94.

10. Kempen, H.P.M. van, On the Convergence of the Classical Jacobi Method for Real Symmetric Matrices with Nondistinct Eigenvalues, Num. Math., 9, 1966, pp. 11-18.

11. Kempen, H.P.M. van, On the Quadratic Convergence of the Serial Cyclic Jacobi Method, Num. Math., 9, 19-22.

12. Luk, F.T., Computing the Singular-value decomposition on the ILLIAC IV, ACMTOMS, 6, 1980, pp. 524-539.

13. Modi, J.J. and J.D. Pryce, Efficient Implementation of Jacobi's Diagonalization Method on the DAP, Num. Math., 46, 1985, pp. 443-454.

14. Paardekooper, M.H.C., An Eigenvalue Algorithm Based on Norm-reducing Transformations, Techn. Un. Eindhoven, Thesis, 1969.

15. Paardekooper, M.H.C., An Eigenvalue Algorithm for Skew Symmetric Matrices, Num. Math., 17, 1971, pp. 189-202.

16. Paardekooper, M.H.C., Sameh's Parallel Eigenvalue Algorithm revisited, Research Memorandum, 1986, Tilburg University.

17. Ruhe, A., On the Quadratic Convergence of a Generalization of the Jacobi Method for Arbitrary Matrices, BIT, 8, 1968, 210-231.

18. Rutishauser, H., The Jacobi-method for Real Symmetric Matrices, Num. Math., 9, 1966, 1-10.

19. Sameh, A.H., On Jacobi and Jacobi-like Algorithms for a Parallel Computer, Math. Comput., 25, 1971, pp. 579-590.

20. Schönhage, A., On the Quadratic Convergence of the Jacobi Process, Num. Math., 6, 1964, pp. 410-412.

21. Stewart, G.W., A Jacobi-like Algorithm for Computing the Schur Decomposition of a Nonhermitian Matrix, SIAM J. Sci. Stat. Comput., 6, 1985, pp. 853-864.

22. Veselić, K., A Convergent Jacobi Method for Solving the Eigenproblem of Arbitrary Real Matrices, Num. Math., 25, 1976, pp. 179-184.

23. Wilkinson, J.H., Note on the Quadratic Convergence of the Cyclic Jacobi Process, Num. Math., 4, 1962, pp. 296-300.

24. Goldstine H.H. and L.P. Horowitz, A procedure for the Diagonalization of Normal Matrices, J. Assoc. Comp. Mach., 6, 1959, pp. 176-195.

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