On convex, cone-interior processes

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ON CONVEX, CONE-INTERIOR PROCESSES.

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T production function

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1. On convex, cone-interior processes.*)

In economic theory, the production technology is represented either by production (multi-) functions, or by production sets. The analysis of production functions with corresponding cost functions has received a considerable impulse from studies by Shephard (1953) and Uzawa (1964). The results are generalized to production multifunctions or correspondences by Shephard (1970). The relation between the input-output structure in a special linear program and its dual structure is studied by Ruys (1972). The set approach in production theory is introduced by Koopmans (1951) and elaborated by Debreu (1959).

It is possible, however, to derive (production) multifunctions from a (production) set, and vice versa. The theory of multifunctions derived from a convex cone is developed by Rockafellar (1967 and 1970). Such a multifunction is called a convex process and is a generalization of a linear transformation. This idea is extended by Rockafellar (1972) to the concept of a polyhedral convex process, which can be applied on a linear production technology. Both the inverse and the dual or cost structure follow from properties of the polyhedral convex processes.

In this paper, (which is based on the pioneering work of Rockafellar) a generalization of a polyhedral convex process is designed and called a convex cone-interior process. It is well known that the graph of linear transformation is equal to a certain subspace, and the graph of a polyhedral convex process is by definition equal to a convex cone; in the case of a convex cone-interior process, the graph is equal to a convex set which contains and is contained in a (convex) cone. The difference between the last two concepts is, firstly, the fact that in a convex, cone-interior process the behavior in a finite neighbourhood of the origin need not to coincide with the be-

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*) The author is indebted to Prof. Rockafellar and Dr. Weddepohl for their remarks and suggestions.

havior of the process in the infinite, and secondly, the fact that a convex cone-interior process may be strictly convex. This last fact will cause, however, some troubles in boundary cases.

A subclass of convex, cone-interior process is of special interest in production theory: the case in which the recession cone is equal to an orthant, and the process is point-starred. In this case the space is bipartitioned into a space of inputs and a space of outputs in a production process.
Another bipartition can be made if both private and public goods are present in the economy. The operations on and the properties of convex, cone-interior processes allow for a bipartition in commodities with the characteristics of public or private goods.
The theory of convex, cone-interior processes can also be applied in consumption theory, if the space is bipartitioned in a commodity space and a (one or even n-dimensional) utility space.

A <u>convex, cone-interior process</u> $T:R^m \to R^n$, is said to be a multifunction (or correspondence), whose graph

$$G(T) := \{(y,x) \mid x \in T(y)\} \subset R^{m+n}$$

is a closed, convex and unbounded set, which contains the origin on its boundary.
Equivalently may be required that G(T) is a closed and convex set, which contains a cone with vertex zero and which is contained in a closed halfspace with zero on its boundary. The graph, therefore, has a convex cone closure and a non-empty interior cone. One may say that the process is characterized on the infinite by the interior-cone and near the origin by the cone-closure of the graph. If both cones coincide, the process reduces to a closed, convex (cone) process defined by Rockafellar (1970), requiring that the graph is a convex cone

containing the origin. A subclass of the class of convex (cone) processes is formed by the convex polyhedral processes. These processes again contain the set of the linear transformations (or processes) as a subclass.

A proper convex, cone-interior process is said to be a convex, cone interior process whose graph does not contain lines (has zero lineality) and contains an interior-cone with a nonempty interior. The graph being a halfspace (and the process being a linear transformation) is herewith excluded.

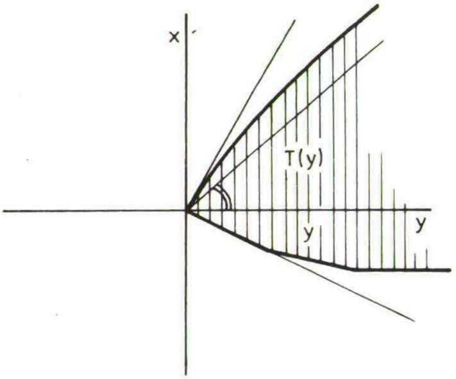


Fig. 1

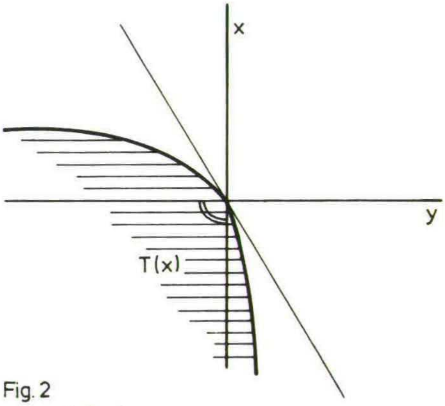


Fig. 2

Examples of convex, cone-interior processes

It is evident that on some closed, convex, unbounded set in R^n containing the origin as boundary point, at least as many cone-interior processes may be defined, as there are bipartitions of the R^n-space. Given some process, however, the graph is determined unambiguously. Other processes are generated by inverse and polarity operations to be defined. New processes are also generated by operations, such as addition, on different processes. To derive some properties of cone-interior processes, it is necessary to define the following operations on convex cone-interior processes:

<u>inverse process</u> : the inverse of a convex, cone-interior $T: R^m \to R^n$ is the process $T^{-1}: R^n \to R^m$, defined by $T^{-1}(x) := \{y \mid x \in T(y)\}$.

<u>scalar multiplication</u> : $(\lambda T)(y) := \lambda T(y) = \{\lambda x \mid x \in T(y)\}$, for a positive scalar λ;

<u>addition</u> : $(T_1 + T_2)(y) := T_1(y) + T_2(y)$;

<u>inverse addition</u> : $(T_1 \# T_2)(y) := \cup \{T_1(y_1) \cap T_2(y_2) \mid y_1+y_2=y\}$;

<u>conjunction</u> : $(T_1 \wedge T_2)(y) := T_1(y) \cap T_2(y)$;

<u>disjunction</u> : $(T_1 \vee T_2)(y) := \cup \{T_1(y_1)+T_2(y_2) \mid y_1+y_2=y\}$;

<u>multiplication</u> : $(T_2 T_1)(z) := T_2[T_1(z)] = \cup\{T_2(y) \mid y \in T_1(z)\}$;

<u>dual addition</u> : $(T_1 \text{♀} T_2)(y) :=$
$= \{\lambda_1 T_1(y) \cap \lambda_2 T_2(y) \mid \exists \lambda_1, \lambda_2 \geqq 0^+ : \lambda_1+\lambda_2=1\}$

<u>dual inverse addition</u> : $(T_1 \text{♀\#} T_2)(y) :=$
$= \{T_1(y_1) \cap T_2(y_2) \mid \exists \lambda_1 \geqq 0^+, \lambda_2 \geqq 0^+ : \lambda_1+\lambda_2=1$ and $y = \lambda_1 y_1 = \lambda_2 y_2\}$

The last two operations on processes, are based on the so called dual addition of two sets X_1 and X_2 in R^n, defined by:

$$X_1 \text{♀} X_2 := \cup \{\lambda_1 X_1 \cap \lambda_2 X_2 \mid \lambda_1 \geqq 0^+, \lambda_2 \geqq 0^+, \lambda_1+\lambda_2 = 1\}$$

$$= \{x = \lambda x_1 = \lambda x_2 \mid x_1 \in X_1, x_2 \in X_2, \lambda_1 \geqq 0^+, \lambda_2 \geqq 0^+, \lambda_1+\lambda_2 = 1\}$$

$$= \{x_1 \text{♀} x_2 \mid x_1 \in X_1, x_2 \in X_2\}.$$

(The notation $\lambda \geqq 0^+$ means that λX is taken to be 0^+X, the recession cone, rather than $\{0\}$ when $\lambda = 0$. A set 0^+X is said to be a <u>recession cone</u> of X if $X + \lambda(0^+X) = X$, for $\lambda \geqq 0$).

With the exception of inverse and dual addition, these operations are defined by Rockafellar (1972) for convex polyhedral processes. Examples are given in fig. 3 and 4.

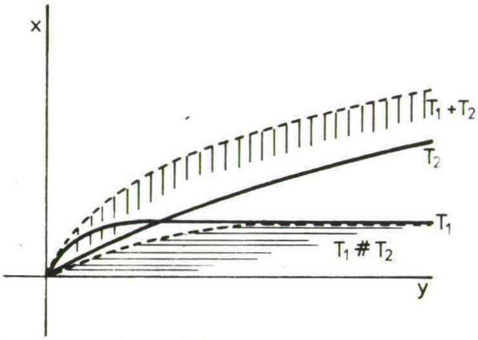


Fig. 3. Addition (+) and inverse addition (#)

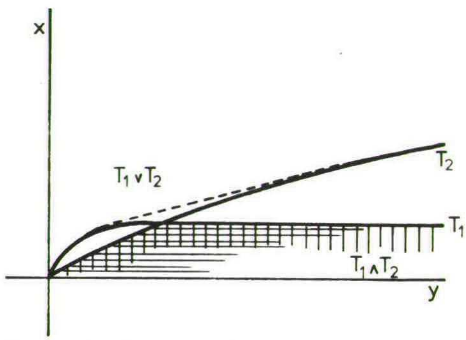


Fig. 4. Conjuntion ($\wedge$) and disjunction (v)

The economic relevance of these definitions may be deduced from the interpretation of T as a production process with input y. Suppose inputs are public goods for both processes T_1 and T_2 (i.e. inputs can be used by both processes at the same time in their full extent);
if the output are private goods, then addition of processes is required (e.g. action time in hours as input, two furnaces T_1 and T_2, amount of calories per hour as output); if the outputs are produced by complementary processes, then conjunction is required (e.g. time as input, furnace T_1 and distribution system T_2, temperature as output).

Suppose inputs are private goods for both processes (i.e. the quantity of input used by process T_1 is lost for process T_2); if the outputs are also private goods, then disjunction is relevant (e.g. the quantity of labor as input, a labor-intensive technique T_1 and a capital intensive technique T_2, both having the same commodity as output);
if the outputs are complementary, then inverse addition is necessary (e.g. labor as input in the fuel industry T_1 and car industry T_2, with transported ton-miles as output).

These examples are purposely constraint to two dimensions. One well known more dimensional composed process is the set of feasible solutions in a programming problem, which may be thought generated by conjunction of a number of constraints T_i, given a set of resources y. Relevant properties for the choice of operation on production processes are: the inputs being either public (giving no sense to substitution), or private (implying substitution) and the processes being either substitutes (implying addition of outputs), or complements (implying intersection of outputs).

Before analyzing the properties of the operations just defined, some properties of convex, cone-interior processes are derived. The following closures and openings of a process will be usefull:

If $T:R^m \to R^n$ is a convex, cone-interior process, then:

the <u>star closure</u> of T is said to be the process
$T_s(y) := \{x \in R^n \mid x = \lambda z, \text{ for some } 0 \leqq \lambda \leqq 1, z \in T(y), y \in \text{Dom } T\}$;

the <u>aureole closure</u> of T is said to be the process
$T_a(y) := \{x \in R^n \mid x = \lambda z, \text{ for some } \lambda \geqq 1, z \in T(y), y \in \text{Dom } T\}$;

the <u>cone closure</u> of T is said to be the process
$T_c(y) := \{x \in R^n \mid (y,x) \in \text{Cone } G(T), \text{ for } y \in \text{Dom } T\}$;

the <u>cone opening</u> of T is said to be the process
$T_o(y) := \{x \in T(y) \mid (y,x) \in \text{Conint } G(T), \text{ for } y \in \text{Dom } T\}$; where the cone closure of the Graph T is said to be the set:

Cone $G(T) := \{(y,x) \mid (y,x) = (\lambda w, \lambda z)$, for some $\lambda \geqq 0$, $(w,z) \in G(T)\}$;

the interior cone of the Graph T is said to be the set
Conint $G(T) := \{(y,x) \mid (\lambda y, \lambda x) \in G(T)$, for all $\lambda \geqq 0\}$.

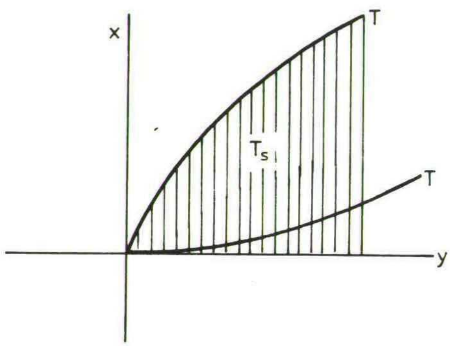


Fig.5 The star closure $T_s(y)$

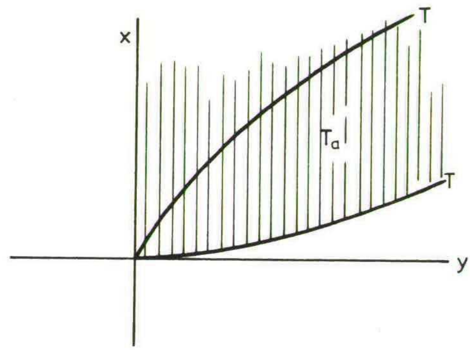


Fig.6. The aureole closure $T_a(y)$

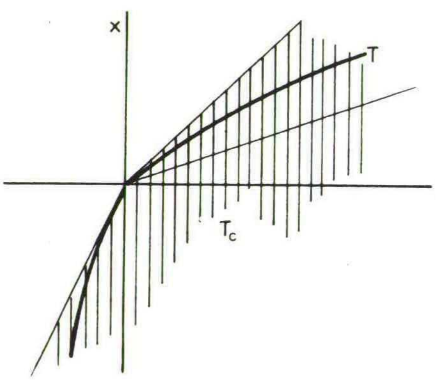


Fig.7 The cone closure $T_c(y)$

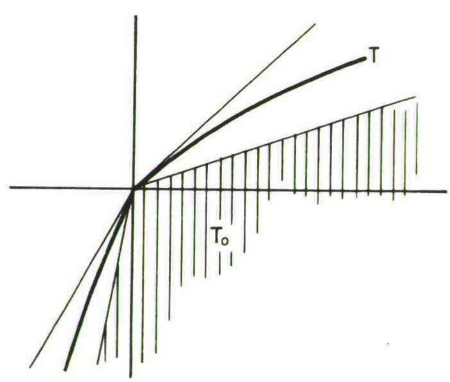


Fig.8 The cone-opening $T_o(y)$

A process T is called point-starred if $T = T_s$, and is called point-aureoled if $T = T_a$. Both processes are of considerable interest as they have a unique orientation on the whole domain of definition. A process T is said to be max-oriented if it is possible to maximize the absolute value of an objective function over any image of T on Dom T \ {0}. A process T is called min-oriented if it is possible to minimize the absolute value of an objective function over any image of T on Dom T \ {0} to a non-zero value.

It is evident that T is max-oriented, if T is point-starred, and that T is min-oriented if T is point-aureoled. If T is both max- and min-oriented, then T is point-compact (see fig. 5 or 6). If T is max-oriented on a subset of Dom T and min-oriented on another subset (see fig. 7 or 8), then T is partly max-, partly min-oriented.

Theorem 1. Properties of convex, cone-interior processes, $T: R^m \to R^n$

1. $T^{-1}: R^n \to R^m$ is a convex, cone interior process, such that $\text{Dom } T^{-1} = \text{Range } T$, and $\text{Range } T^{-1} = \text{Dom } T$.

2. T is concave on Dom T, i.e.
$\lambda T(y_1) + (1-\lambda)T(y_2) \subseteq T(\lambda y_1 + (1-\lambda)y_2)$, for $y_1, y_2 \in \text{Dom } T$ and $0 \leqq \lambda \leqq 1$.

3. T is point-convex, i.e.
$T(y)$ is a convex set for all $y \in \text{Dom } T$.

4. T is point-closed, i.e. $T(y)$ is a closed set for all $y \in \text{Dom } T$.

5. T is continuous, i.e.
T is an upper hemi-continuous ($T^{-1}(A)$ is closed in Range T^{-1}, for each closed set A in Dom T^{-1}), and T is lower hemi-continuous ($T^{-1}(A)$ is open in Range T^{-1}, for each open set A in Dom T^{-1}).

6. $G(T_o)$ = Conint $G(T)$ is the <u>recession cone</u> of $G(T)$, i.e. $G(T) + \lambda G(T_o) = G(T)$, for $\lambda \geqq 0$.

7. $T_o(0)$ is the recession cone for all $T(y)$, $y \in$ Dom T.

<u>Proof</u>:

1. $G(T^{-1}) = \{(y,x) | y \in T^{-1}(x)\} = \{(y,x) | \in T(y)\} = G(T)$.

2. Choose $x_1 \in T(y_1)$ and $x_2 \in T(y_2)$. As $G(T)$ is convex, any convex combination of (y_1,x_1) and (y_2,x_2) belongs to $G(T)$:
$(\lambda y_1+(1-\lambda)y_2,\ \lambda x_1+(1-\lambda)x_2) \in G(T)$.
Therefore $(\lambda x_1+(1-\lambda)x_2) \in T(\lambda y_1+(1-\lambda)y_2)$, or $\lambda T(y_1) + (1-\lambda)T(y_2) \subseteq T(\lambda y_1+(1-\lambda)y_2)$, for $0 \leqq \lambda \leqq 1$.

5. Upper hemi continuity follows from convexity and closedness of $G(T)$. The image $T^{-1}(A)$ is therefore convex if A is convex, and with any limit point in A can be associated a limit point in $T^{-1}(A)$, as T^{-1} is also point closed. If A contains all its limit points in Dom T^{-1}, then also $T^{-1}(A)$ relative to Range T^{-1}.
Lower hemi continuity follows from convexity and closedness of $G(T)$. Choose any $y_o \in T^{-1}(x_o)$, for $x_o \in A :=$ $\{x \in \text{Dom } T^{-1} \,|\, |x-x_o| < \delta\}$. As T^{-1} is concave there exists an open neighbourhood of y_o, which is a convex combination of the set $T^{-1}(x_o)$ with $T^{-1}(x)$, for arbitrary $x \in A$, which is contained in $T^{-1}(A)$:
$N(y_o) := \{\lambda T^{-1}(x_o) + (1-\lambda)T^{-1}(x) | x \in A \text{ and } 0 \leqq \lambda \leqq 1\} \subseteq$ $\{T^{-1}(\lambda x_o+(1-\lambda)x) | x \in A \text{ and } 0 \leqq \lambda \leqq 1\} = T^{-1}(A)$.

6. As $0 \in G(T)$, the interior cone is equal to the recession cone or asymptotic cone of $G(T)$, which is by definition the set $\{(w,z) | (y+\lambda w,\ x+\lambda z) \in G(T)$, for all $\lambda \geqq 0$, $(y,x) \in G(T)\}$.

7. The graph of the process $T_o(y)$, the cone opening of $T(y)$, is

by definition equal to the interior cone of G(T). As T_o is a convex cone process, $T_o(0)$ is either a cone, or {0}. In the first case $T_o(y)$ indicates in which T(y) is unbounded; in the second case T(y) is bounded in all directions.

In order to analyse some properties of operations on processes, it is necessary to realize that it is generally not true that the sum of two closed sets is closed and that the projection of a closed set is closed (see Rockafellar 1970).
If, however, the union of the recession cones of two closed sets is a pointed cone, then the sum of the two sets is also closed. This restriction on the graphs of the processes is quite plausible, as it implies i.a. that the sum of two processes does not result in a degenerate process, whose graph is a halfspace or the full space.

The second closedness problem is prevented if one requires that the projection of the graph on a subspace is closed. This is always the case if the recession cone does not contain an axis of the projected subspace, or if the graph is a polyheder. The problem arises only if there exists a translate of $T_o(0)$ or $T_o^{-1}(0)$ which is asymptotically supporting the graph (see fig. 9)

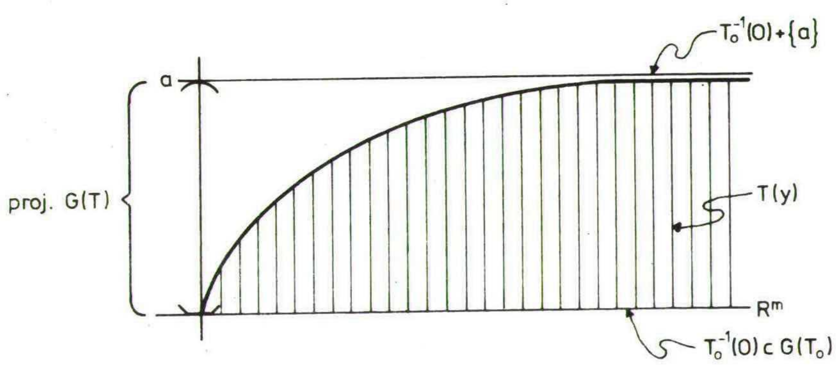


Fig. 9 A process T(y) asymptotic parallel to R^m

An asymptotic support of a closed convex set X is a hyperplane L, which is the boundary of a halfspace containing X, such that any open neighbourhood of L does intersect the interior of X. The process T(y) is called asymptotic parallel to R^m, if there does exist an asymptotic support of the Graph T parallel to some nonzero subspace of R^m.
The projection of Graph T into R^m is closed, if and only if T(y) is not asymptotic parallel to R^m. It is evident that if $T^{-1}_o(0) = \{0\}$, or if $T^{-1}_o(0) \cap \text{Int } G(T_o) \neq \emptyset$, then T(y) is not asymptotic parallel to R^m.

If G(T) is a polyhedral cone and T a convex polyhedral process, none of these closedness problem arises, as the sum of two closed polyhedral sets is always closed and the image of a closed polyhedral set under a projection is also closed.

Theorem 2 Properties of operations on convex, coneinterior processes.

The class of convex, cone-interior processes is closed under the operations of addition, inverse addition, conjunction, disjunction, scalar multiplication and multiplication, under the following conditions for the respective operations.

1. addition : [Cone $G(T_1)$ + Cone $G(T_2)$] is a pointed cone, and neither $T_1(y)$, nor $T_2(y)$ is asymptotic parallel to R^m, for $y \in \text{Dom } T_1 \cap \text{Dom } T_2$.

2. inverse addition : [Cone $G(T_1)$ + Cone $G(T_2)$] is a pointed cone, and neither $T^{-1}_1(x)$, nor $T^{-1}_2(x)$ is asymptotic parallel to R^n, for $x \in \text{Range } T_1 \cap \text{Range } T_2$.

3. conjunction : Conint $G(T_1) \cap$ Conint $G(T_2) \neq \emptyset$.

4. disjunction : [Cone $G(T_1) \cup$ Cone $G(T_2)$] is a pointed cone.

5. multiplication : neither T_1, nor T^{-1}_2 is asymptotic parallel to R^m for $y \in \text{Range } T_1 \cap \text{Dom } T_2$.

Proof:

1. $G(T_1+T_2)$ may be construceted as follows from $G(T_1)$ and $G(T_2)$. The sets K_1 and K_2 in $R^m \times R^m \times R^n$ are defined by:

   $K_1: = \{(y_1,0,x_1)|(y_1,x_1) \in G(T_1)\}$

   $K_2: = \{(0,y_2,x_2)|(y_2,x_2) \in G(T_1)\}$

   Both sets are closed, convex, unbounded and contain the origin on their boundary. As [Conint K_1 $\cup$ Conint K_2] $\subseteq$ [Cone K_1 + Cone K_1], which is a pointed cone, the sum $K_1 + K_2$ is also closed, convex, unbounded and containing the origin on its boundary. The same is true for the intersection of $K_1 + K_2$ with the subspace $\{(y_1,y_2,w)|y_1 = y_2\}$. As neither $T_1(y)$, nor $T_2(y)$ is asymptotic parallel to R^m, the same properties are valid for its image under the projection $R^m \times R^m \times R^n \to R^m \times R^n$. This image is equal to:

   $$\{(y,x)|x = x_1 + x_2, \exists(y,x) \in G(T_1), (y,x_2) \in G(T_2)\}$$

   $$= \{(y,x)|x = T_1(y) + T_2(y)\} = G(T_1+T_2).$$

2. An analogous reasoning is followed to construct $G(T_1 \# T_2) = \{(y,x)|y = y_1 + y_2, \exists(y_1,x) \in G(T_1), (y_2,x) \in G(T_2)\}$ from $G(T_1)$ and $G(T_2)$.
   In this case, the sets K_1 and K_2 in $R^m \times R^n \times R^n$ are defined by:

   $K_1: = \{(y_1, x_1, 0)|(y_1,x_1) \in G(T_1)\}$

   $K_2: = \{(y_2, 0, x_2)|(y_2,x_2) \in G(T_2)\}$.

3. $G(T_1 \wedge T_2) = G(T_1) \cap G(T_2)$.

4. $G(T_1 \vee T_2) = G(T_1) \cup G(T_2) = G(T_1) + G(T_2)$.

5. $G(T_2T_1)$ may be constructed from $G(T_1)$ and $G(T_2)$ by forming the sets K_1 and K_2 in $R^\ell \times R^m \times R^m \times R^n$:

   $K_1: = \{(y, x_1, 0, 0)|(y,x_1) \in G(T_1)\}$

   $K_2: = \{(0, 0, x_2, z)|(x_2,z) \in G(T_2)\}$

The intersection of $K_1 + K_2$ with the subspace $\{y, x_1, x_2, z) | x_1 = x_2\}$ is a closed, convex, unbounded set, containing the origin on its boundary. Given the conditions, this is also true for its image under the projection $R^\ell \times R^m \times R^m \times R^n \to R^\ell \times R^n$. This image is equal to:

$$\{(y,z) | z \in T_2(x) \Rightarrow x \in T_1(y)\}$$

$$= \text{Graph} \cup \{T_2(x) | x \in T_1(y)\} = G(T_2 T_1).$$

Theorem 3. Properties of the inverse operation on processes.

The class of convex cone-interior processes is closed and reflexive under the inverse operation. The inverse operation obeys the laws:

1. $(T_1 + T_2)^{-1}(x) = T^{-1}_1(x) \# T^{-1}_2(x);$

2. $(T_1 \# T_2)^{-1}(x) = T^{-1}_1(x) + T^{-1}_2(x);$

3. $(T_1 \wedge T_2)^{-1}(x) = T^{-1}_1(x) \wedge T^{-1}_2(x);$

4. $(T_1 \vee T_2)^{-1}(x) = T^{-1}_1(x) \vee T^{-1}_2(x).$

The inverse operation reverses the orientation of a process.

Proof:

$$G(T) = G(T^{-1}) = G[(T^{-1})^{-1}]$$

1. $G[(T_1+T_2)^{-1}] = G(T_1+T_2) = \{(y,x) | x \in T_1(y) + T_2(y)\} =$

$\cup \{(y,x) | x = x_1 + x_2,\ (y,x_1) \in G(T_1),\ (y,x_2) \in G(T_2)\} =$

$\cup \{(y,x) | x = x_1 + x_2,\ y \in T^{-1}_1(x_1) \cap T^{-1}_2(x_2)\} =$

$G(T^{-1}_1 \# T^{-1}_2)(x).$

2. An analogous argument is valid.

3. $G(T_1 \wedge T_2)^{-1} = G(T_1 \wedge T_2) = G(T_1) \cap G(T_2) = G(T_1^{-1}) \cap G(T_2^{-1}) =$

   $G(T_1^{-1} \wedge T_2^{-1})$.

4. $G(T_1 \vee T_2)^{-1} = G(T_1) \cup G(T_2) = G(T_1^{-1}) \cup G(T_2^{-1}) = G(T_1^{-1} \vee T_2^{-1})$.

It may be noticed that the rationale to call the operation $\#$ inverse addition, can be found in the first and second law. Inverse addition and dual addition of sets are closely related, as is shown in the following:

<u>Theorem 4. Related operations on sets, resp. processes</u>.

With two sets X_1 and X_2 in R^n, two processes T_1 and T_2 from R_+^1 into R^n are related and defined by:
$T_1(\lambda) := \lambda X_1$, $T_2(\lambda) := \lambda X_2$. Then

$$(T_1 + T_2)(1) = X_1 + X_2;$$

$$(T_1 \# T_2)(1) = X_1 \text{ ♀ } X_2;$$

$$(T_1 \wedge T_2)(1) = X_1 \cap X_2;$$

$$(T_1 \vee T_2)(1) = \text{Conv } (X_1 \cup X_2).$$

2. Dual processes of convex, cone interior processes.

On convex, cone-interior processes, a duality operation can be defined which is closely related with the following duality operation on sets.
The upper dual set X_+^* of a set X in R^n is said to be the set:

$$X_+^* := \{p \in R^{n*} \mid px \geqq 1, \text{ for all } x \in X\}.$$

The lower dual set X_-^* of a set X in R^n is said to be the set:

$$X_-^* := \{p \in R^{n*} \mid px \leqq 1, \text{ for all } x \in X\}.$$

Some properties of this duality or polarity operation which are relevant in this context are mentioned here. The proof and the properties in a more general context can be found in Weddepohl (1972.

Theorem 5. Properties of the duality operation on sets.

If X and Y are closed, convex, unbounded sets, containing the origin on their boundary, then:

1. $X^* = X_-^*$ and $X_+^* = \emptyset$;

2. X^* is a closed, convex, unbounded set, containing the origin on its boundary;

3. $X^{**} = X$

4. if X is a cone, then $X = X^o$, where $X^o := \{p \mid px \leqq 0, \text{ for all } x \in X\}$;

5. $X^{o} = (\text{Cone } X)^{o} \subseteq X^{*}$;

6. $X \subseteq Y \Leftrightarrow X^{*} \supseteq Y^{*}$;

7. $(X+Y)^{*} = X^{*} \,\text{♀}\, Y^{*}$;

8. $(X\text{♀}Y)^{*} = X^{*} + Y^{*}$;

9. $(X\cup Y)^{*} = X^{*} \cap Y^{*}$;

10. $(X\cap Y)^{*} = \text{Conv } (X^{*} \cup Y^{*})$.

Based on the polarity operation above, the following adjoint correspondence for convex processes can be obtained. Let $T:R^{m} \to R^{n}$ be any convex, cone-interior process.
The upper dual process $T_{+}^{*}:R^{n*} \to R^{m*}$ is said to be the multi-function:

$$T_{+}^{*}(p) := \{q \mid qy \geqq px-1, \text{ for all } x \in T(y), \text{ for all } y\}.$$

The lower dual process $T_{-}^{*}:R^{n*} \to R^{m*}$ is said to be the multi-function:

$$T_{-}^{*}(p) = \{q \mid qy \leqq px+1, \text{ for all } x \in T(y), \text{ for all } y\}.$$

Theorem 6.1 below shows that the difference between both processes is only caused by signs. It still has sense to make such a distinction, because the upper duality operation is best suited for max-oriented processes (or max-oriented restrictions of such processes) and the lower duality operation is apt for min-oriented (restrictions of) processes. This may be checked from fig. 10. It also generalizes now the adjoint of a linear operator. The following properties are easily checked:

Theorem 6. Properties of the duality operation on processes.

If T is any convex, cone-interior process, then:

1. $T_+^*(p) = \{q \mid (-q,p) \in [G(T)]^*\} = -T_-^*(-p)$;

   $T_-^*(p) = \{q \mid (q,-p) \in [G(T)]^*\} = -T_+^*(-p)$;

2. Both T_+^* and T_-^* are convex, cone-interior processes;

3. $(T_-^*)_+^* = (T_+^*)_-^* = T$;

4. $(T_+^*)^{-1} = (T^{-1})_-^*$;

   $(T_-^*)^{-1} = (T^{-1})_+^*$;

5. $T_+^*(p).y \geqq p.T(y)-1$;

   $T_-^*(p).y \leqq p.T(y)+1$.

If T is any convex, polyhedral process, then:

1. $T_+^*(p).y \geqq p.T(y)$;

   $T_-^*(p).y \leqq p.T(y)$;

If T is any linear transformation, then:

1. $T_+^*(p).y = T_-^*(p).y = p.T(y)$.

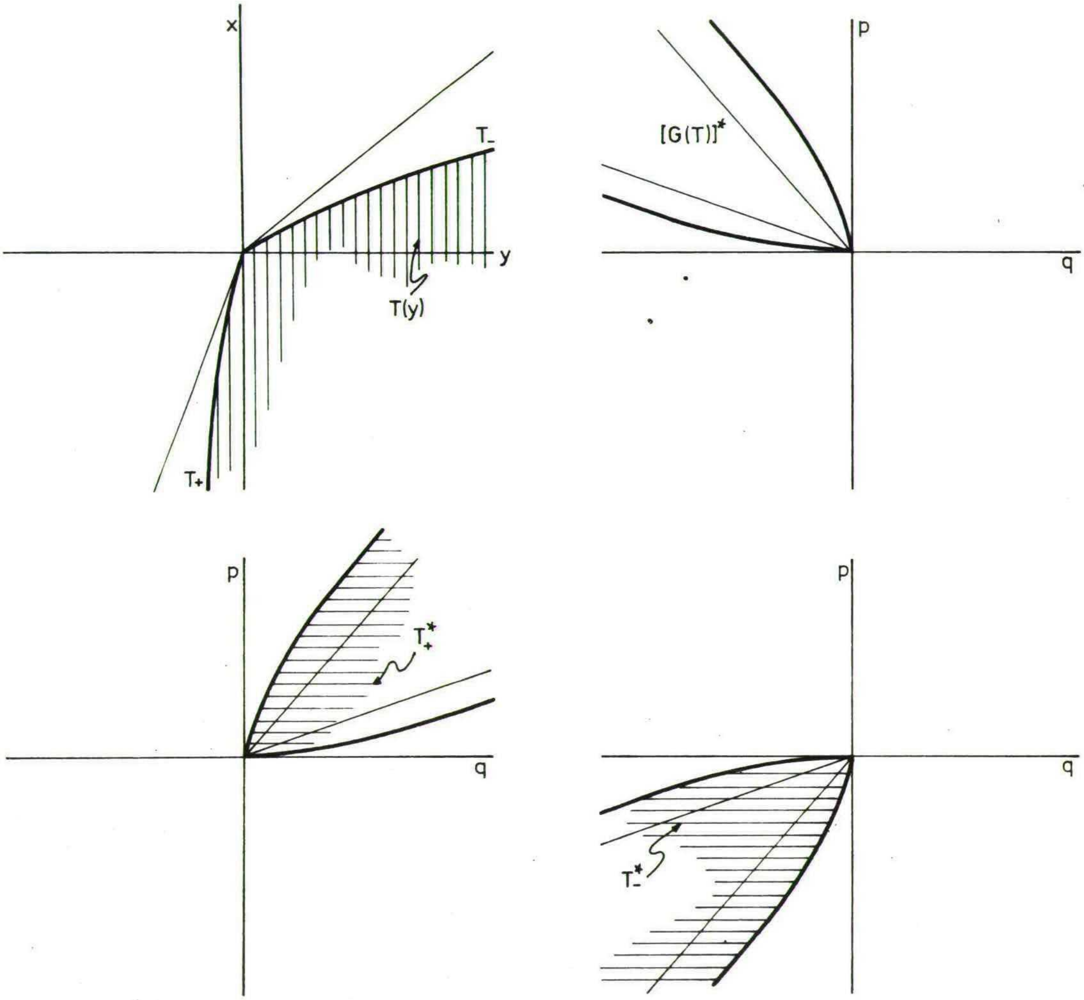


Fig. 10 Duality operations related with T

From properties (6.3) and (6.4 follows that both the inverse operation and the duality operation are orientation-reversing.

It may be noticed again that the definition of a convex, polyhedral process and its adjoint processes are given by Rockafellar (1972).

The economic interpretation of the above dual process is based, of course, on the interpretation of the process T. If T is a production process, which assigns a set of output-quantities to an input-quantity y, then $T^{*}(p)$ assigns a set of input-prices q to a given price of the output, p. The set of input-prices are such that the firm can at most make a profit equal to 1, if an input-price is choosen from $T^{*}(p)$. From the properties can be derived that if the technology is linear and the production set a polyhedral cone, then profit can at most be equal to zero. (This is a familiar result in economic theory.) If "the market" offers an input-price in the interior of $T^{*}(p)$, then profit will be less than 1 for a cone-interior process; if the market price does not belong to $T^{*}(p)$, then profit will be greater than 1.

But in the case that the input-prices are known, it is more efficient to compute the inverse dual process, $T^{*-1}(q)$, which assigns a set of output-prices feasible for the technology if profit should not exceed 1.

The above interpretation of a cone-interior process as a production process with input y and output x, makes sense if the process is max-oriented or point-starred. If the process is partly max-oriented, partly min-oriented (see fig. 10), then the interpretation should be reversed for the min-oriented part: y becomes output and x becomes input. This is the reason for the boundedness of the price-sets in the dual process, which will not arise in a point-starred process (see fig. 11).

The other interpretations given above, viz. public goods versus private goods, may be translated on a similar way in the dual or valuation space. Also the operations defined above can be translated, as is shown in the following theorem.

Theorem 7. Properties of operations on dual processes.

If T_1 and T_2 are convex, cone-interior processes, such that the operations mentioned generate a convex, cone-interior process, then:

1. $T_1 \subseteq T_2 \iff T_1^* \supseteq T_2^*$;

2. $(T_1 + T_2)^* = T_1^* \overset{\circ}{\#} T_2^*$;

3. $(T_1 \# T_2)^* = T_1^* \overset{\circ}{+} T_2^*$;

4. $(T_1 \vee T_2)^* = T_1^* \wedge T_2^*$;

5. $(T_1 \wedge T_2)^* = T_1^* \vee T_2^*$;

6. $(T_2 T_1)^* = T_1^* T_2^*$;

7. $(\lambda T(y))^* = T^*(\lambda p)$, for $\lambda > 0$

Proof: *)

1. $G(T_1) \subseteq G(T_2) \iff G(T_1)^* \supseteq G(T_2)^*$

$$\iff G(T_1^*) \supseteq G(T_2^*)$$

2. $G(T_1^* \overset{\circ}{\#} T_2^*) = G[\cup\{T_1^*(p_1) \cap T_2^*(p_2) \mid p = p_1 \overset{\circ}{+} p_2\}$

$$= \{(p,q) \mid \exists\ \lambda_1, \lambda_2 \geqq 0^+ : \lambda_1 + \lambda_2 = 1,\ p = \lambda_1 p_1 = \lambda_2 p_2,\ q = q_1 = q_2,$$

*) The orientation signs are omitted both in the statements and in the proofs, as they don't matter and may be substituted according to the desired orientation. Only in this proof is therefore supposed to be valid the equality $G(T^*) = [G(T)]^*$, to simplify the argumentation.

$p_1x_1+q_1y_1 \leqq 1,\ p_2x_2+q_2y_2 \leqq 1,\ \forall (x_1,y_1) \in G(T_1(y_1)),$

$\forall (x_2,y_2) \in G(T_2(y_2))\}$

$= \{(p,q)\,|\,px_1+\lambda_1 q\ y_1 \leqq \lambda_1,\ px_2+\lambda_2 q\ y_2 \leqq \lambda_2,$ for any

$\lambda_1,\lambda_2 \geqq 0^+,\ \lambda_1+\lambda_2 = 1,$ for all $(x_1,y_1) \in G(T_1(y_1)),$

$(x_2,y_2) \in G(T_2(y_2)\}$

$= \{(p,q)\,|\,p(x_1+x_2)+qy \leqq 1,$ for $y=y_1=y_2,\ x_1 \in T_1(y_1),$

$x_2 \in T_2(y_2)\}$

$= \{x,y)\,|\,x \in T_1(y)+T_2(y)\}^*$

$= G[\ (T_1+T_2)^*]\ ;$

3. $G(T_1^* \overset{\circ}{+} T_2^*) = G[\cup\{\lambda_1 T_1^*(p) \cap \lambda_2 T_2^*(p)\,|\,\lambda_1,\lambda_2 \geqq 0^+,\ \lambda_1+\lambda_2=1\} =$

$= \{(p,q)\,|\,\lambda_1 px_1+qy_1 \leqq \lambda_1,\ \lambda_2 px_2+qy_2 \leqq \lambda_2,\ \forall (x_1,y_1) \in G(T_1)$

$(x_2,y_2) \in G(T_2)\}$

$= \{(p,q)\,|\,px+q(y_1+y_2) \leqq 1,\ \forall x \in T_1(y_1) \cap T_2(y_2)\}$

$= \{(p,q)\,|\,px+qy \leqq 1,\quad x \in T_1(y)\ \#\ T_2(y)\}$

$= G[\ (T_1\ \#\ T_2)^*]\ ;$

4. $G(T_1^* \wedge\ T_2^*) = G(T_1^*) \cap G(T_2^*)$

$= [\,G(T_1^*)^* \cup G(T_1^*)^*]^* = [\,G(T_1) \cup G(T_2)]^*$

$= G[\ T_1 \vee T_2)^*]\ .$

5. Analogously.

6. The set $G(T_1T_2)$ may be constructed form $G(T_2T_1)$ by forming the following convex sets in $R^\ell \times R^m \times R^m \times R^n$:

$K_1 := \{k_1 = (x_2, y_1, 0, 0) \mid (x, y_1) \in G(T_1)\};$

$K_2 := \{k_2 = (0, 0, y_2, z) \mid (y_2, z) \in G(T_2)\};$

$\bar{K} := \{(x, y_1, y_2, z) \mid y_1 = y_2\}.$

The set $(K_1 + K_2) \cap \bar{K}$ is projected on $R^\ell \times R^n$ to give $G(T_2T_1)$.

$\bar{K}^* := \{p, q_1, q_2, r) \mid q_1 = q_2\}.$

The set $(K_1 + K_2)^* \cap \bar{K}^*$ is projected on $R^{\ell *} \times R^{n*}$ to give $G(T_2T_1)^*$.

$K_1^* = \{k_1^* := (p, q_1, q_2, r) \mid px + q_1y + q_2 0 + r0 \leqq 1,\ \forall\, k_1 \in K_1\};$

$K_2^* = \{k_2^* := (p, q_1, q_2, r) \mid p0 + q_1 0 + q_2 y_2 + rz \leqq 1, \forall k_2 \in K_2\};$

$$K_1^* \mathbin{\overset{\circ}{+}} K_2^* = \{k^* = \lambda_1 k_1^* = \lambda_2 k_2^* \mid \lambda_1, \lambda_2 \geqq 0^+, \lambda_1 + \lambda_2 = 1, k_1^* k_1 \leqq 1, \forall k_1 \in K_1, k_2^* k_2 \leqq 1, \forall k_2 \in K_2\} =$$

$$= \{k \mid k^* k_1 \leqq \lambda_1, k^* k_2 \leqq \lambda_2, \forall k_1 \in K_1, \forall k_2 \in K_2, \exists \lambda_1, \lambda_2 \geqq 0^+, \lambda_1 + \lambda_2 = 1\}$$

$$= \{(p, q_1, q_2, r) \mid px + q_1 y + q_2 y_2 + rz \leqq 1, \forall k_1 \in K_1, \forall\, k_2 \in K_2\}$$

The set K_1^* projected on $R^{\ell *} \times R^{m*}$ gives $G(T_1^*)$; the set K_2^* projected on $R^{m*} \times R^{n*}$ gives $G(T_2^*)$; the set $(K_1^* \mathbin{\overset{\circ}{+}} K_2^*) \cap \bar{K}^*$ projected on $R^{\ell *} \times R^{n*}$ gives $G(T_1^* T_2^*)$.
As $(K_1 + K_2)^* = K_1^* \mathbin{\overset{\circ}{+}} K_2^*$, both graphs $G(T_2^* T_1^*)$ and $G(T_1T_2)^*$ are equal.

The polarity operation T^* introduced above is completely defined in the dual spaces: the process T^* from R^{n*} into R^{m*}. Other polarity operations can be defined, which apply the duality operation on each image set of a process. These polarity operations are therefore called <u>point-duality operations</u>, and may be applied on any multifunction.

Let $T: R^m \to R^n$ any multifunction or process.
The <u>upper polar multifunction</u> $[T]_+^* : R^m \to R^{n*}$ is said to be the multifunction defined by:

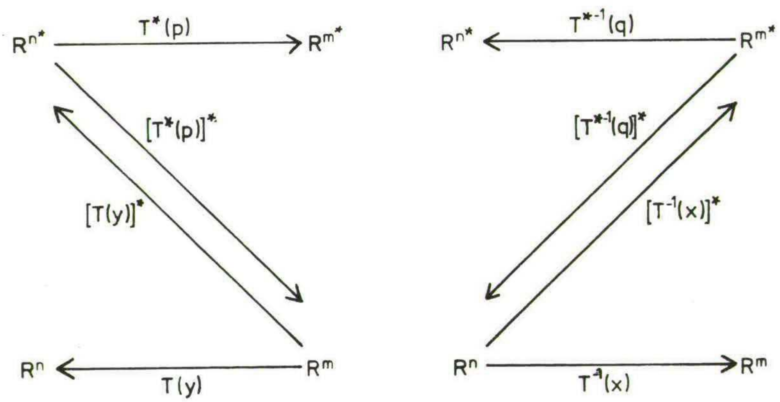
$$[T(y)]_+^* = \{p \mid px \geqq 1, \text{ for all } x \in T(y)\}.$$

The lower polar multifunction $[T]^{*}_{-} : R^{m} \rightarrow R^{n*}$ is said to be the multifunction defined by:

$$[T(y)]^{*}_{+} = \{p \mid px \leqq 1, \text{ for all } x \in T(y)\}.$$

The difference between both polar processes is more than a question of signs, as in the case for the dual processes T^{*}_{+} and T^{*}_{-}.
Also the properties deviate much from the original process: the polar process of a convex, cone-interior process need not to have convex graph, for example (see fig. 11 and 12). A relation between both duality concepts is derived in (7.5) and (8.5).

The various processes defined here are related, as is shown in the following diagram:



The relation between both diagonal processes can be assessed if one takes the inverse of one, e.g. of $[T(y)]^{*}$. Then both $[T(p)]^{*-1}$ and $[T^{*}(p)]$ are defined from a subspace in R^{n*} into R^{m} by (if the lower polar is choosen):

$$[T(p)]^{*-1} = \{y \mid p.T(y) \leqq 1\};$$

$$[T^{*}(p)]^{*} = \{y \mid T^{*}(p).y \leqq 1\}.$$

Both processes are shown in fig. 12; their graphs have there the boundary in common.

Finally, some special convex, cone-interior processes T will be defined, which are important in economics, and some properties derived.
A process $T:Y \to X$ is said to be quasi-homothetic, if for each nonzero y_1, $y_2 \in Y$, there exist a positive scalar μ such that

$$T(y_1) = \mu\, T(y_2).$$

A process $T:Y \to X$ is said to be homothetic, if for each $\lambda > 0$ there exists a $\mu > 0$, such that for any nonzero $y \in Y$

$$T(\lambda y) = \mu\, T(y).$$

(This implies that Dom T is an orthant).
A process $T:Y \to X$ is called positively homogeneous of degree k, if for all $\lambda > 0$, $T(\lambda y) = \lambda^{k} T(y)$, for any nonzero $y \in Y$.
A process $T:Y \to X$ is called starred, if T is point-starred and Dom T is an orthant.
A process $T:Y \to X$ is called aureoled, if T is point-aureoled, and Range T is an orthant.

If a positively homogeneous process T of degree $k \leqq 1$, resp. $k \geqq 1$, has a closed and convex graph, it is a starred, resp. aureoled, convex cone-interior process. Some properties of starred and aureoled processes are derived below.

Theorem 8. Properties of a starred, convex, cone-interior process.

1. $T(y) \subseteq T(\lambda y) \subseteq \lambda T(y)$, for $\lambda \geqq 1$.

2. $T_o(0) \subseteq T(y)$.

3. $(T_1 \# T_2) \subseteq (T_1 \wedge T_2) \subseteq (T_1 \vee T_2) \subseteq (T_1 + T_2)$.

4. $T^{-1}(x)$ is aureoled.

5. $T^*(p)$ is aureoled.

6. $T^*_+(p) = [T^{-1}(\{p\}^*_+)]^*_+$.

Proof.

1. As T is point starred ($T=T_s$), $0 \in T(y)$ for all $y \in Y$; as Dom T is a cone, the set $\{(y,0) | y \in Y\}$ belongs to the recession cone $G(T_o)$ of $G(T)$.
Choose any $(y,x) \in G(T)$; then $(y,x) + \lambda(y,0) \in G(T)$, for $\lambda \geqq 0$, or $(\lambda y,x) \in G(T)$, for $\lambda \geqq 1$.
Secondly, choose an $x \in T(\lambda y)$, for some $\lambda \geqq 1$ and $y \in Y$. This set is defined, as Y is a cone. $(\lambda y,x) \in G(T)$, $(0,0) \in G(T)$, and by convexity of $G(T)$ any convex combination such as $(y,x/\lambda) \in G(T)$. It follows that $x \in \lambda T(y)$.

4. Choose any $(y,x) \in G(T)$; for any $\lambda \geqq 1$, $(\lambda y,x) \in G(T)$ by (1). Therefore $\lambda y \in T^{-1}(x)$, for $\lambda \geqq 1$, implying that $T^{-1}(x)$ is point-aureoled, and because Dom T = Range T^{-1}, also aureoled.

5. As Y is an orthant, $y \in Y \Rightarrow -y \notin Y$. Therefore, if $q \in T^*_+(p)$ and $\lambda \geqq 1$, $(\lambda q)y \geqq qy \geqq px-1$, implying that $(\lambda q) \in T^*_+(p)$. Also $q \in T_-(p)$ and $\lambda \geqq 1$ imply $(\lambda q)y \leqq qy \leqq px+1$, or $(\lambda q) \in T^*_-(p)$. Thus $T^*(p)$ is point-aureoled, and as $\{q | qy \leqq p0 + 1$, for $y \in Y\}$ is an orthant, $T^*(p)$ is aureoled.

6. $[T^{-1}(\{p\}_{+}^{*})]_{+}^{*} = \{y | T(y) \cap \{p\}_{+}^{*} \neq \emptyset\}_{+}^{*} =$

$= \{y | \exists x \in T(y) : px \geqq 1\}_{+}^{*} =: \bar{Y}_{+}^{*} =$

$= \{q | qy \geqq 1, \ \forall y \in \bar{Y}\};$

This set is equal to $\{q | qy \geqq px-1, \forall\ x \in T(y), \forall\ y\} = T_{+}^{*}(p)$, for each p, as both conditions are equivalent.
Assume the last condition is not met and that there exists a $\bar{y}$, such that for some $\bar{x} \in T(y)$, $q\bar{y} < p\bar{x}-1$, for a given p. If also $\bar{y} \in \bar{Y}$ and $p\bar{x} \geqq 1$, then a contradiction follows. If $p\bar{x} < 1$, then $\bar{y} \in \bar{Y}$ implies that $\exists \bar{\bar{x}} \in T(\bar{y})$ such that $p\bar{\bar{x}} \geqq 1$. As $T(\bar{y})$ is point-starred, $q\bar{y} < p\bar{x}-1 < p\bar{\bar{x}}-1$, which also contradicts $q\bar{y} \geqq 1$.
Assume that the first condition is not met, and $\exists\bar{y} \in \bar{Y}: q\bar{y} < 1$. For a given p, and for any q. As $\bar{y} \in Y$, $\exists\bar{x}: p\bar{x} \geqq 1$.
If $q \in T_{+}^{*}(p)$, then $0 \leqq p\bar{x}-1 \leqq q\bar{y} < 1$, implying that q has a finite value and $T_{+}^{*}(p)$ is not aureoled, which contradicts a property of $T_{+}^{*}(p)$.

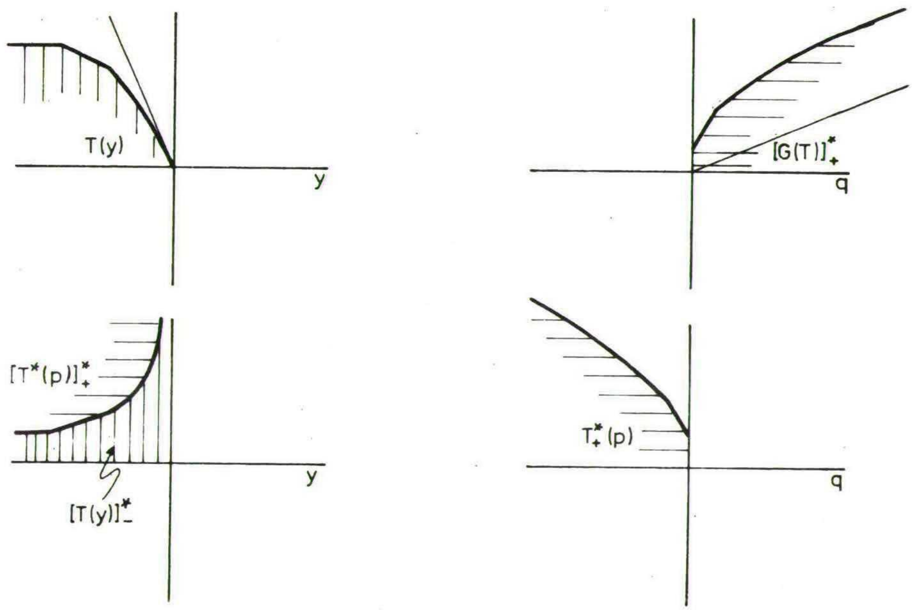


Fig. 11 A starred process T(y) and its dual process $T_{+}^{*}(p)$

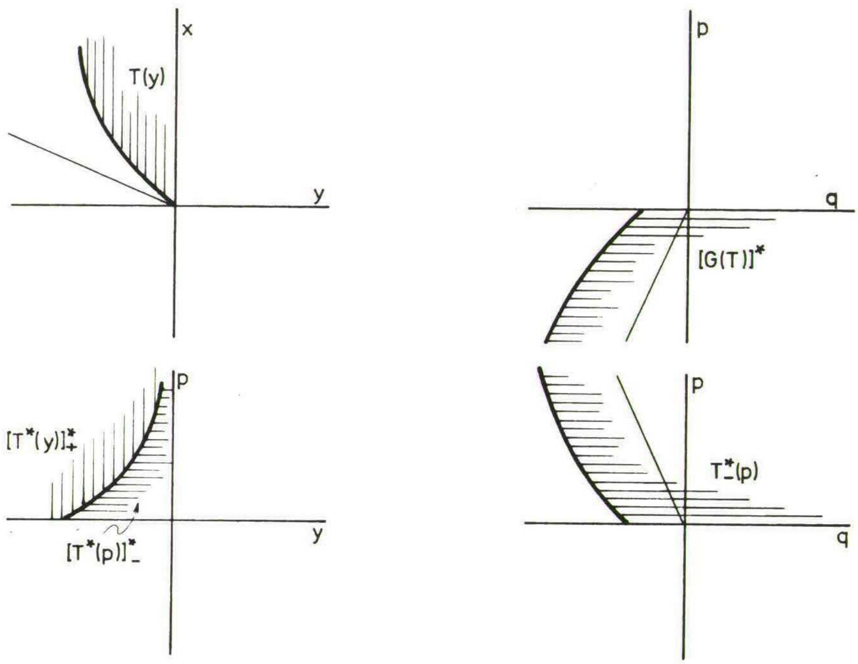


Fig.12 An aureoled process T(y) and its dual process $T^*_-(p)$

Property (8.6) was established by Ruys (1972) for the much simpler case of a linear technology defined by a regular matrix in the context of a linear programming problem. The property gives an indication how the concepts of a dual set, a polar process and a dual process are related.

Analogous arguments can be given to show the following properties of aureoled sets.

<u>Theorem 9. Properties of an aureoled, convex cone-interior process $T:Y \to X$.</u>

1. $T(y) \supseteq T(\lambda y) \supseteq \lambda T(y)$, for $\lambda \geqq 1$ and $\lambda y \in Y$.

2. $T_o(0) \supseteq T(y)$.

3. $(T_1 \# T_2) \supseteq (T_1 \vee T_2) \supseteq (T_1 \wedge T_2) \supseteq (T_1 + T_2)$.

4. $T^{-1}(x)$ is starred.

5. $T^*(p)$ is starred.

6. $T^*_-(p) = [T^{-1}(\{p\}^*_-)]^*_-$.

Finally, it may be stressed that <u>time</u> can be introduced in this theory through the multiplication operation. Analogous to the linear theory, eigenvalues may be defined, to analyze dynamic properties. This has to be done, yet.

M.v.d.B

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