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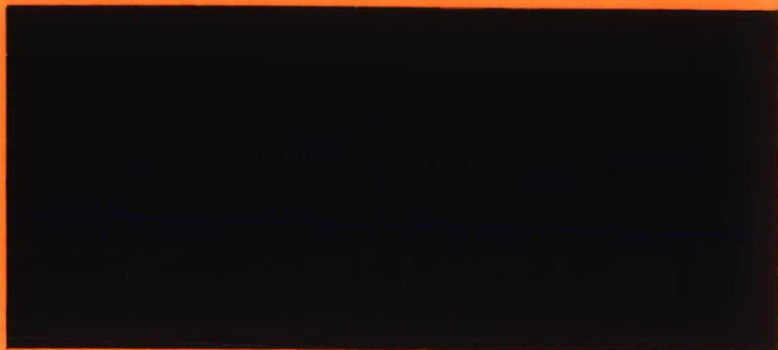
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CONSISTENT SETS OF ESTIMATES FOR REGRESSIONS
WITH CORRELATED OR UNCORRELATED MEASUREMENT
ERRORS IN ARBITRARY SUBSETS OF ALL VARIABLES*

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Abstract

We consider the single equation errors-in-variables model and assume that a researcher is willing to specify an upper bound on the variance covariance matrix of measurement errors in the endogenous and exogenous variables. The measurement errors may show any pattern of correlations. It is shown that as a result the set of ML estimates is bounded by an ellipsoid. When, in addition, the variance covariance matrix of the errors is constrained to be diagonal, the set of ML estimates is shown to be bounded by the convex hull of 2^{ℓ} points (ℓ being the number of error-ridden exogenous variables), lying on the surface of the ellipsoid. The results are applied to an empirical example and extensions to a simultaneous equations system are briefly discussed.

1. <u>Introduction</u>

Over the last decade the problem of measurement errors in the independent variables of a regression equation has attracted renewed interest among econometricians. In the fifties and sixties, the problem was considered to be more or less hopeless due to its inherent under-identification (e.g., Theil [20]). Apart from instrumental variables, the most frequently cited textbook solution was Wald's method of grouping (Wald [22]). Recent insight into the properties of the method of grouping can be interpreted as making this method worthless (Pakes [18]). Since about 1970, new approaches to the problem have been explored, basically along three lines, viz. embedding the error-ridden equation into a set of mutiple equations (e.g., Zellner [23], Goldberger [8]), into a set of simultaneous equations (e.g., Hsiao [10], Geraci [7]), and using the dynamics of the equation, if present (e.g., Maravall and Aigner [16]). In view of the underidentification of the basic model, it is clear that all these methods invoke additional information of some kind. If this information takes the form of exact or stochastic knowledge of certain parameters in the model, the construction of consistent estimators is fairly straightforward (e.g. Fuller [6], Kapteyn and Wansbeek [11]). For an overview of the state of the art, see Aigner et al. [1].

An approach somewhat orthogonal to the ones described above has been to take the model as it is and to use prior ideas about the size of the measurement errors to diagnose how serious the problem is. Examples are Blomqvist [3], Hodges and Moore [9] and Davies and Hutton [5]. Leamer [14] starts from the opposite direction by asking how serious the measurement error problem has to be in order to render the data useless for inference, that is to say, when measurement error is large enough to make it impossible to put bounds on regression parameters. In an empiri-cal example, he shows that even very small measurement errors in some explanatory variables would open up the possibility of perfectly colli-near explanatory variables and hence make the data useless for statisti-cal inference (at least without additional prior information).

The most systematic analysis of the information loss caused by measurement error is due to Klepper and Leamer [12]. They start out by

invoking a minimal amount of prior information and then ask the question under what conditions it is still possible to make some inferences regarding the vector of unknown regression parameters β. In the special case where the measurement errors are assumed uncorrelated and the k+1 estimates of β, obtained by regressing each of the k+1 variables involved (i.e. the one dependent variable and the k independent variables) on the remaining k variables, are all in the same orthant, one can bound the ML estimates of β. In that case, the convex hull of the k+1 regressions contains all possible ML estimates and any point in the hull is a possible ML-estimate. If the k+1 regressions are not all in the same orthant then the set of ML estimates is unbounded.

In that case Klepper and Leamer [12] introduce extra prior information which allows them to bound the set of maximum likelihood estimates. The prior information comes in two forms. Firstly, a researcher is supposed to be able to specify a maximum value of R^2 if all exogenous variables were measured accurately. It is shown that if this maximum is low enough, one can again bound the set of ML estimates by a convex hull. Secondly, if the R^2 bound does not help in bounding the estimates, a researcher is assumed to be able to give upper and lower bounds for the measurement error variances. If the upper bound is tight enough, so that the true explanatory variables cannot be perfectly collinear, the set of maximum likelihood estimates is shown to be bounded by an ellipsoid. In the derivation of the ellipsoid, based on a result in Leamer [13], it is assumed that all exogenous variables are measured with error. Obviously, this is restrictive.

Bekker, Kapteyn, Wansbeek [2] have generalized Klepper and Leamer's result to the case where the variance covariance matrix of the measurement errors may be singular, but they still assumed, as did Klepper and Leamer, that the endogenous variable is measured without error or that the measurement error in the endogenous variables is uncorrelated with the errors in the exogenous variables. In this paper we relax this assumption, which turns out to be a non-trivial exercise. Not only are there many cases where a non-zero correlation between errors in the endogenous variable and in the explanatory variables is likely (for instance when all variables in an equation are deflated by the same imperfect price index), but the importance of the generalization also lies in the possibility to extend the analysis to more complicated models

than just the linear regression model. Section 2 presents this result.

Although Klepper and Leamer [12] assume throughout their paper that all measurement errors are uncorrelated, they do not exploit that information in the derivation of the ellipsoid. For any point in the ellipsoid we can find an Ω (the variance covariance matrix of the errors in the explanatory variables) that yields this point as an ML estimate, but such an Ω need not be diagonal. In Section 3 we investigate the consequences of the extra requirement that Ω is diagonal. In that case the ML estimates are bounded by a polyhedron, which need not be convex. Of course, the polyhedron lies within the ellipsoid. The convex hull of the polyhedron is determined by 2^ℓ vertex points that all lie on the ellipsoid, where ℓ is the number of nonzero measurement error variances. These points can be computed easily and then used to find, for all elements of β, intervals that bound the ML estimates. Generally, these intervals are tighter than the ones obtained from the ellipsoid.

In Section 4, an empirical example illustrates how the various types of prior restrictions affect the bounds on the ML estimates. Section 5 concludes by briefly discussing extensions to simultaneous equations models. All proofs are collected in two appendices.

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2. The Model and the Ellipsoid

Throughout we deal with the following model:

(2.1) $\quad \eta = \Xi \beta_0 + \varepsilon$

(2.2) $\quad y = \eta + u$

(2.3) $\quad X = \Xi + V \;;$

(2.1) is the classical linear model, which relates the n-vector of dependent variables η to the $n \times k$-matrix of explanatory variables Ξ and the n-vector of disturbances ε. We assume that the distribution of ε is independent of Ξ and satisfies $E\varepsilon = 0$, $E\varepsilon\varepsilon' = \sigma_0^2 I$. The k-vector of parameters β_0 and σ_0^2 are unknown and have to be estimated.

Both η and Ξ are unobservable. Instead, y and X are observed and u and V therefore are the errors of measurement in y and X. We assume that u and V are uncorrelated with Ξ, η and ε and that $Eu = 0$, $EV = 0$. Moreover, letting u_i be the i-th element of u and v_i' the i-th row of V, we assume that

$$E \begin{pmatrix} u_i \\ v_i \end{pmatrix} (u_i, v_i') = \Phi \equiv \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Omega \end{bmatrix}$$

for all i and that (u_i, v_i') is stochastically independent of (u_j, v_j') for $i \neq j$.

Let Φ be known and define β and σ^2 by

(2.4) $\quad \beta \equiv (A-\Omega)^{-1}(Ab-\Phi_{21})$

(2.5) $\quad \sigma^2 \equiv \frac{1}{n} y'y - \Phi_{11} - \beta'(A-\Omega)\beta \;,$

where $A \equiv \frac{1}{n} X'X$, $b \equiv (X'X)^{-1}X'y$. Under a variety of assumptions, (β, σ^2) will be a consistent estimate of (β_0, σ_0^2). Of course, if $\Phi = 0$, (β, σ^2) reduces to the OLS-estimate (b, s^2), where $s^2 \equiv \frac{1}{n} y'y - b'Ab$.

Although Φ will usually be unknown, it seems reasonable to assume that a researcher will be able to specify bounds for Φ, i.e.,

$$(2.6) \qquad 0 \leq \Phi \leq \Phi^* \equiv \begin{bmatrix} \Phi_{11}^* & \Phi_{12}^* \\ \\ \Phi_{21}^* & \Omega^* \end{bmatrix},$$

where Φ^* is specified by the researcher.[1] This bound on Φ will be used to derive bounds on the estimates β defined by (2.4). We assume that Φ^* is symmetric and that

$$(2.7) \qquad 0 \leq \Phi^* < B \equiv \frac{1}{n} \begin{bmatrix} y'y & y'X \\ X'y & X'X \end{bmatrix},$$

thereby guaranteeing the existence of the estimate β and also the positiveness of the estimate σ^2 for any choice of Φ satisfying (2.6)[2]. The latter can be shown easily by writing the positive definite matrix $(B-\Phi)^{-1}$ as

$$(2.8) \qquad (B-\Phi)^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & (A-\Omega)^{-1} \end{bmatrix} + \sigma^{-2} \binom{-1}{\beta}(-1, \beta'),$$

so that

$$(2.9) \qquad \sigma^2 = \{e_1'(B-\Phi)^{-1}e_1\}^{-1} > 0,$$

where e_1 is the first unit vector. Furthermore, if we denote the estimate (β, σ^2) by (b^*, s^{*2}) if $\Phi = \Phi^*$, it is readily established that, as a consequence of the boundedness of Φ, also σ^2 is bounded:

$$(2.10) \qquad s^2 \geq \sigma^2 \geq s^{*2} > 0 \ .$$

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1) The notation $C \leq D$ means that $D-C$ is a positive semidefinite matrix; $C < D$ means $D-C$ is positive definite.
2) Note that Φ^* has to be strictly less than B. Among other things, this excludes the possibility that the true explanatory variables in Ξ are perfectly collinear. If Ξ could have less than full column rank, no bounds for β exist.

We may now ask the question whether we can also delimit the set of esti-
mates β given that Φ satisfies (2.6). The answer to that question is
contained in proposition 1:

Define

$$(2.11) \quad F^* \equiv (A-\Omega^*)^{-1} - A^{-1}.$$

Then we have

Proposition 1: The set of solutions β satisfying (2.4), with Φ satis-
fying (2.6), is given by:

$$(2.12) \quad (\beta-\tfrac{1}{2}(b+b^*))'F^{*-}(\beta-\tfrac{1}{2}(b+b^*)) \leq \tfrac{1}{4}(s^2-s^{*2})$$

$$(2.13) \quad F^*F^{*-}(\beta-\tfrac{1}{2}(b+b^*)) = \beta - \tfrac{1}{2}(b+b^*) .$$

where F^{*-} is an arbitrary g-inverse of F^*.
This bound is minimal, i.e., for any β satisfying (2.12) and (2.13) the-
re exists a Φ such that (2.4) and (2.6) hold true.

Proof: See Appendix A.

Equation (2.12) describes a cylinder and (2.13) presents a pro-
jection of the cylinder onto the space spanned by F^*. Thus (2.12) and
(2.13) describe an ellipsoid in the space spanned by F^*. It is rather
easy to show (see Appendix A) that

$$(2.14) \quad s^2-s^{*2} = (b^*-b)' F^{*-} (b^*-b) + \phi^*_{11}-\phi^*_{12}\Omega^{*-}\phi^*_{21}.$$

The non-negative definiteness of ϕ^* implies that

$$(2.15) \quad \phi^*_{11} > \phi^*_{12} \Omega^{*-} \phi^*_{21}.$$

If (2.15) holds as an equality, i.e.

(2.16) $\phi_{11}^* = \phi_{12}^* \; \Omega^{*-} \; \phi_{21}^*$,

then (2.12) and (2.14) imply that both b and b* lie on the surface of the ellipsoid and the centre of the ellipsoid is located at the midpoint of the segment joining b and b*. See Figure 1.
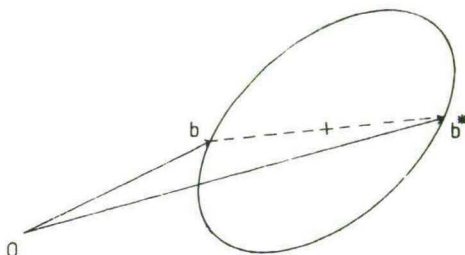

Figure 1:   The ellipsoid when (2.16) holds

If (2.16) holds, the measurement error u_i in y is linearly dependent upon the measurement errors v_i in the exogenous variables, in the sense that the mean square of their difference is zero. To see this, define $\lambda = \Omega^{*-} \phi_{21}^*$, so that (2.16) is equivalent to

(2.17) $(-1, \lambda') \; \phi^* = 0$.

This implies, in conjunction with (2.6):

(2.18) $0 < (-1, \lambda') \; \phi \; \binom{-1}{\lambda} < (-1, \lambda') \; \phi^* \; \binom{-1}{\lambda} = 0$,

so that $(-1, \lambda') \; \phi = 0$, which is equivalent to $E(u_i - \lambda' v_i)^2 = 0$. That is, the measurement error in y is a fixed linear combination of the measurement errors in X with probability one. One particular case in which this holds is where $\phi_{21} = 0$ and $\phi_{11} = 0$, i.e. no measurement errors in y.

If we let Φ_{11}^* increase, keeping all other elements of Φ^* constant, so that (2.15) becomes a strict inequality, $s^2 - s^{*2}$ increases according to (2.14). As b, b^* and F^* do not depend on Φ_{11}^*, this means that the ellipsoid expands. In that case b and b^* are no longer on the surface of the ellipsoid, but the midpoint of the line joining b and b^* is still the center of the ellipsoid. See Figure 2. The intuitive explanation
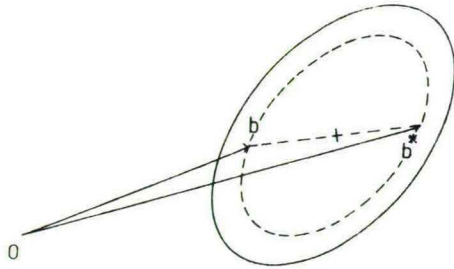

Figure 2: The ellipsoid when (2.15) is a strict inequality

for this is that if Φ_{11}^* increases, we do not only allow more measurement error in y (which is indistinguishable from errors in the equation anyway) but also more covariance between the errors in y and X. Thus, the bound on Φ becomes less tight and the ellipsoid expands.

If the number of regressors exceeds two, it will be hard in practice to represent the ellipsoid given by (2.12) and (2.13) in a transparant way. For that reason it is useful to derive bounds for linear functions of β. Let ψ be a known vector, then bounds for $\psi'\beta$ are implied by the following proposition.

Proposition 2: The maximum and minimum of $\psi'\beta$, with ψ fixed and β satisfying (2.12) and (2.13), are given by

(2.19) $\qquad \psi'\hat{\beta} = \tfrac{1}{2}\psi'(b+b^*) \pm \tfrac{1}{2}\sqrt{(s^2-s^{*2}).\psi'F^*\psi}.$

Proof: See Appendix A.

3. Uncorrelated measurement errors

In this section we assume that, in addition to the bounds on ϕ as given in (2.6), a researcher is also willing to assume that ϕ^* and ϕ are <u>diagonal</u>. That is, measurement errors in different variables are uncorrelated.

The first thing to notice is that in this case the measurement error in the regressand is completely indistinguishable from the error in the equation. Therefore it is of no consequence for the set of estimates β. Since ϕ is diagonal, $\phi_{21} = 0$ and the estimator β is simply given by

(3.1) $\beta = (A-\Omega)^{-1}Ab,$

where Ω is diagonal and bounded by

(3.2) $0 \leq \Omega \leq \Omega^* < A.$

Clearly, the set of estimates is unchanged if we choose $\phi_{11}^* = \phi_{11} = 0$. Consequently the ellipsoid (2.12)-(2.13), only depends on Ω^*. We will refer to (2.12)-(2.13), with $\phi_{21}^*, = 0$ and $\phi_{11}^* = 0$, as "the ellipsoid spawned by Ω^*". This ellipsoid is still a bound for the set of estimators β, but it is no longer a minimal bound if Ω and Ω^* are restricted to be diagonal.

In order to derive a more satisfactory bound we define the following points

(3.3) $\beta_\delta \equiv (A-\Omega_\delta^*)^{-1}Ab,$

where $\Omega_\delta^* = \Omega^*\Delta = \Delta\Omega^* = \Delta\Omega^*\Delta$, with $\Delta = \text{diag}(\delta)$ and δ a vector with ones and zeros as elements. If Ω^* has ℓ non-zero diagonal elements then there are 2^ℓ different matrices Ω_δ^*, which all satisfy (3.2). Clearly the 2^ℓ solutions β_δ are bounded by the ellipsoid spawned by Ω^*. We shall refer to the β_δ as "generated by Ω^*."

Proposition 3: All β_δ lie on the surface of the ellipsoid spawned by Ω^*

Proof: See Appendix B.

Having established that all β_δ lie on the surface of the ellipsoid spawned by Ω^*, we next show that β lies in the convex hull of the 2^ℓ points β_δ that are generated by Ω^*.

Proposition 4: If Ω and Ω^* are diagonal and satisfy (3.2), then the set of estimates β satisfying (3.1) is contained in the convex hull of the 2^ℓ points β_δ generated by Ω^*

Proof: See Appendix B.

Thus, the diagonality of Ω further reduces the region where β may lie when measurement error is present. In practical applications, the most obvious use of this result is to compute all 2^ℓ points β_δ and to derive the interval in which each coefficient lies. These intervals will in general be smaller than the ones obtained from Proposition 2 by choosing for ψ the k unit vectors successively. Proposition 4 is similar to a result given by Chamberlain and Leamer [4] (employing a result by Leamer and Chamberlain [15]) that bounds the posterior mean by 2^k regressions if the prior covariance matrix is diagonal. In terms of the present framework, their proof assumes that Ω is non-singular (so $\ell=k$, among other things).

An example shows that the convex polyhedron need not be a minimal bound for β. Consider the case of two regressors, where all variables (including the regressand) are subject to measurement error and the 3×3 matrix Φ^* is diagonal. If Φ is not restricted to be diagonal, the set of estimates β is bounded by the ellipsoid spawned by Φ^* given in Proposition 1. As has been observed in Section 2, the ellipsoid spawned by Ω^* is the same ellipsoid with a smaller radius. If Φ is restricted to be diagonal all 4 vertex points ($\ell = 2$) lie on the surface of this latter ellipsoid.

Let a and c be the vertex points (besides b and b^*):

(3.4) $a \equiv (A - \Omega^*_{\delta_1})^{-1} A b$

$$(3.5) \qquad c \equiv (A - \Omega^*_{\delta_2})^{-1} A b,$$

where $\delta_1 = (1,0)'$, $\delta_2 = (0,1)'$. Assume without loss of generality that $b > 0$. Let us follow the path from b to a. Note that

$$(3.6) \qquad \beta - b = A_1^{-1} \omega_1 \beta_1 + A_2^{-1} \omega_2 \beta_2,$$

where ω_1 and ω_2 are the diagonal elements of Ω, A_1^{-1} and A_2^{-1} are the first and second column of A^{-1}, respectively, and β_1 and β_2 are the two elements of β. Going from b to a, we set $\omega_2 = 0$ and let ω_1 go from 0 to ω_1^*. So $\beta - b = A_1^{-1} \omega_1 \beta_1$. As $b_1 > 0$, $\omega_1 \beta_1 > 0$ and has as its maximum $\omega_1^* a_1$. As $(A^{-1})_{11} > 0$ the line has a positive angle with e_1. Analogously, the line from b to c has a positive angle with e_2. A possible case is given in Figure 3, with $a > 0$, and $c > 0$. Going from a to b^* we have $\beta - a = = (A - \Omega^*_{\delta_1})_2^{-1} \omega_2 \beta_2$. As $a_2 > 0$, $\omega_2 \beta_2 > 0$ with maximum $\omega_2^* b_2^*$. As $(A - \Omega^*_{\delta_1})_{22}^{-1} > 0$, the line has a positive angle with e_2. Analogously, the line from c to b^* has a positive angle with e_1. So if Φ is restricted to be diagonal we end up with the shaded area in Figure 3 (the outer ellipsoid gives the bound for the estimates if Φ is not restricted to be diagonal).
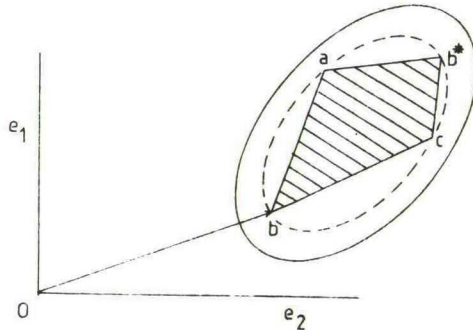

Figure 3. The convex hull when Ω is diagonal and the vertices are in the same orthant

Now assume $c_1 < 0$. The line from a to b^* has again a positive angle with e_2, but the line from c to b^* has a negative angle with e_1. This is so since $\beta - c = (A - \Omega^*_{\delta_2})^{-1}_1 \omega_1 \beta_1$, and $c_1 < 0$, so $\omega_1 \beta_1 < 0$. See Figure 4.
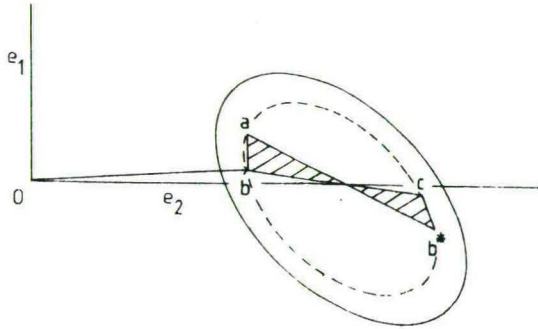

Figure 4. The convex hull when Ω is diagonal and the vertices are not in the same orthant

Now all β's are within the shaded area, which is clearly not convex. The wasp-waist is on e_2: in (3.6), choose ω_1 and ω_2 such that $\beta_1 = 0$, then we can next vary ω_1 at will without affecting β, as $\beta_1 = 0$.

4. An Empirical Example

In Van de Stadt, Kapteyn, Van de Geer [21] (SKG from now on) a model of preference formation is constructed and estimated. The central relationship of the model is the following one:

$$(4.1) \quad \mu_i = \beta_0 + \beta_1 \mu_i(-1) + \beta_2 fs_i(-1) + \beta_3 fs_i + \beta_4 y_i + \beta_5 y_i^* + \beta_6 fs_i^* + \varepsilon_i$$

The index i refers to the i-th household in the sample; μ_i is a measure of the household's present wants ($\exp(\mu_i)$ is the income the household head would consider just about "sufficient to make ends meet"); $\mu_i(-1)$ is the same measure observed one year ago for the same household; fs_i is the log of the present number of household members ("log-family size") whereas $fs_i(-1)$ is log-family size one year ago; y_i is the present after tax household log-income. The starred variables are sample means of log-incomes and log-family sizes in the "social group" to which household i belongs. A social group is a set of households with identical characteristics (the age of the household head is in the same age bracket, the household heads have a similar education and they live in a town of similar size); ε_i is a random disturbance term. See SKG for further details.

Thus relation (4.1) explains the level of a household's present financial wants by its family size, both present and lagged one period, its present log-income (habit formation), by present log-income and log-family size in the household's social group (preference interdependence), and by the level of financial wants one year ago (habit formation).

Since ε_i is allowed to show negative serial correlation, $\mu_i(-1)$ may correlate negatively with ε_i. This is equivalent to allowing a measurement error in $\mu_i(-1)$. [1] The variables y_i^* and fs_i^* are proxies for reference group effects and may therefore be expected to suffer from measurement error; fs_i and $fs_i(-1)$ are crude proxies of the effects of

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1) As a matter of fact, ε_i has the form $u_i - \beta_1 u_i(-1) + v_i$, where u_i and $u_i(-1)$ are uncorrelated with each other or with v_i; v_i may be serially correlated. If we write $\mu_i(t) = \eta_i(t) + u_i(t)$, $t = 0,-1$, then we can rewrite (4.1) as $\eta_i = \beta_0 + \beta_1 \eta_i(-1) + \beta_2 fs_i(-1) + \beta_3 fs_i + \beta_4 y_i + \beta_5 y^* + \beta_6 fs^* + v_i$, where $\eta_i(-1)$ is assumed uncorrelated with v_i. If we replace $\eta_i(-1)$ by $\mu_i(-1)$, as in (4.1), we obtain a model in which the covariance of $\mu_i(-1)$ with ε_i equals $-\beta_1 \sigma_u^2$.

family composition on financial wants, which can therefore also be expected to suffer from measurement error. Finally, y_i may be subject to measurement error as well.

Table 1. Sample means and covariances of the observed variables.

Variable	Mean	Covariance with						
		μ_i	$\mu_i(-1)$	$fs_i(-1)$	fs_i	y_i	y_i^*	fs_i^*
μ_i	10.11	.1260						
$\mu_i(-1)$	10.07	.1123	.1348					
$fs_i(-1)$	1.01	.0876	.0922	.2706				
fs_i	1.00	.0887	.0889	.2559	.2751			
y_i	10.31	.1238	.1212	.0881	.0924	.1783		
y_i^*	10.30	.0606	.0593	.0523	.0533	.0782	.0828	
fs_i^*	1.00	.0434	.0443	.0873	.0880	.0515	.0535	.0972

Table 2. Specification of Φ^*.

Variable	μ_i	$\mu_i(-1)$	$fs_i(-1)$	fs_i	y_i	y_i^*	fs_i^*	% error
μ_i	.0154							35
$\mu_i(-1)$		.0165						35
$fs_i(-1)$			.0061	.0061				15
fs_i			.0061	.0061				15
y_i					.0040			15
y_i^*						.0130	.0100	40
fs_i^*						.0100	.0150	40

The sample means, standard deviations and correlations of all variables involved are given in Table 1. Our specification of Φ^* is given in Table 2. The column headed "% error" indicates the standard deviation of the measurement errors (the square root of the diagonal of Φ^*) as a percentage of the sample standard deviation of the corresponding observed variables. The specification of Φ^* represents the prior ideas of the authors of SKG. The upper bounds on the measurement errors in the proxies y_i^* and fs_i^* are chosen relatively high and so are the bounds on the subjective measures μ_i and $\mu_i(-1)$. Since the proxies y_i^* and fs_i^* are constructed in a similar way, as sample means per social group, a substantial correlation in measurement error seems likely. The bounds on the "objective" variables fs_i, $fs_i(-1)$ and y_i are considerably tighter. The reason for the perfect correlation between the measurement errors in fs_i and $fs_i(-1)$ is that most of it represents the crudity of the specification of family composition effects on subjective wants by means of log-family size. This crudity will be more or less the same in both periods. Secondly, there is some ambiguity in the definition of a household. Not only persons living with a family, but also others supported by the family for at least 50% are counted as members. The latter criterion is rather loose, but it seems likely that if a respondent applies the criterion incorrectly in one year, then he will make the same mistake the next year. For the rest, the elements in Φ^* are set equal to zero. Given the analysis in the preceding sections, it should be clear that without further restrictions, the corresponding elements of Φ can still be non-zero.

We present extreme values for the elements of β (using Proposition 2 with ψ equal to the successive unit vectors, or by using Proposition 4) for four cases.

(i) Φ^* is as given in Table 2.

(ii) $\Phi_{11}^* = 0$. For the rest Φ^* is as given in Table 2. The intervals for β should be tighter than in the previous case.

(iii)   The off-diagonal elements in Table 2 are set equal to zero. However, Φ can, of course, still be non-diagonal.

(iv)   As Case (iii), with $\Phi_{11}^* = 0$. The intervals for β should be tighter than in the previous case.

(v)   As Case (iii), but diagonality is imposed on Φ. Again, this should
       narrow the intervals relative to the previous case.

In Table 3 the values of b and b^* are presented, along with the extreme
values of β for the five cases considered.

For all specifications of Φ^*, $B-\Phi^*$ is positive definite. As a re-
sult, s^{*2} is always positive, as it should be. The various columns in
Table 3 are pretty much according to expectation. The intervals for β_1
are a great deal wider in Case (i) than in Case (ii). In Case (ii) we
see that β_5 and β_6 can switch signs depending on the choice of Φ. In
Case (i) the interval for β_2 becomes so wide that this parameter may
reverse signs as well. Similarly, Case (iii) gives rise to wider inter-
vals than Case (iv). Comparing (iii) and (iv) to (i) and (ii) makes it
clear that, in this example, the diagonal Φ^* generates wider intervals.
Now, β_3 may reverse signs as well. Finally, imposing diagonality on Φ
narrows the interval dramatically. No parameter estimate reverses signs.

The example illustrates two points. First, it is important to use
prior information economically. If one "knows" that Φ is diagonal, this
knowledge should be used. Otherwise the computed intervals may be much
wider than the intervals that correspond to one's prior knowledge. Se-
condly, allowing for measurement error in the endogenous variable (and
correlation between this error and the errors in the exogenous varia-
bles) has a non-trivial influence on the intervals for the β_i.

Table 3. Extreme values of β [1)]

	$\Phi = 0$	non-diagonal Φ^*				diagonal Φ^*		
	b	b^*	(i)	(ii)	b^*	(iii)	(iv)	(v)
β_1	.509 (.026)	.772	.912 .369	.790 .491	.781	.926 .364	.801 .489	.796 .490
β_2	-.013 (.032)	-.079	.029 -.121	-.005 -.087	-.129	.264 -.406	.115 -.257	-.005 -.136
β_3	.066 (.031)	.095	.123 .038	.104 .057	.149	.428 -.213	.285 -.070	.153 .061
β_4	.298 (.031)	.149	.418 .029	.331 .117	.135	.427 .006	.334 .100	.321 .122
β_5	.071 (.029)	.047	.270 -.152	.175 -.057	.073	.441 -.297	.278 -.133	.133 .037
β_6	-.031 (.025)	-.025	.124 -.180	.056 -.112	-0.48	.251 -.330	.122 -.201	-.020 -.075
s^{*2}			.0021	.0175		.0019	.0173	
s^2	.0242							

1) Standard errors in parentheses. Each cell in the columns (i)–(v) contains the extreme values for the elements of β.

5. Conclusion

As illustrated in Section 4, it is very simple to apply Proposi-
tions 2 and 4 to empirical problems, and the analysis could easily be
incorporated in regression packages. Since the propositions cover a wide
range of cases, the researcher has considerable freedom to express his
prior ideas about Ω as precisely or as vaguely as he wants. The result
of the analysis will then summarize succinctly the sensitivity of esti-
mation outcomes for assumptions about the quality of the data used.

It appears that the framework developed in this paper will allow
for extensions to more complicated models. Consider for example the j-th
structural equation in a linear simultaneous equations system:

(5.1) $\qquad y_j = Y_j \alpha_0 + \Xi_j \gamma_0 + \varepsilon_j,$

where Y_j and Ξ_j are matrices of endogenous and exogenous variables res-
pectively, included as explanatory variables in this equation; y_j is the
vector of endogenous variables to be explained by this equation and ε_j
is a vector of errors. Let Ξ be the matrix of all exogenous variables in
the system. Then 2-SLS amounts to GLS applied to

(5.2) $\qquad \Xi_j' y_j = \Xi_j' Y_j \alpha_0 + \Xi_j' \Xi_j \gamma_0 + \Xi_j' \varepsilon_j.$

If Ξ is measured with error, this model becomes similar to (2.1)-(2.3).
Since Ξ occurs on both sides of the equation, the measurement errors
in the left and right hand side variables will in general be correlated.
For the special case where $\gamma_0 = 0$, it is easy to show that Proposition 1
can be applied directly to derive an ellipsoid for a consistent estimate
of α_0, defined analogous to β (cf. (2.4)). (Bekker, Kapteyn and
Wansbeek [2] have derived the same ellipsoid without reference to Propo-
sition 1, assuming that all exogenous variables are measured with er-
ror.) Proposition 1 is not applicable when $\gamma_0 \neq 0$. For that more ge-
neral case further research is needed.

Appendix A: Proofs of propositions 1 and 2 and of (2.14)

We first establish two lemmas and a corollary.

Lemma 1 Let $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ be a symmetric matrix and let C_{22}^- be a generalized inverse of C_{22}, then $C \geqslant 0$ if and only if

(A.1) (i) $C_{22} \geqslant 0$

       (ii) $C_{22} C_{22}^- C_{21} = C_{21}$

       (iii) $C_{12} C_{22}^- C_{21} \leqslant C_{11}$

Proof: If $C \geqslant$ then,

       (i)   trivial.

       (ii)   Let $A' = (0, I - C_{22} C_{22}^-)$, then $A' C A = 0$, so
                $C A = 0$, or $(I - C_{22} C_{22}^-) C_{21} = 0$,

       (iii)   Let $B' = (I, - C_{12} C_{22}^-)$,
                then $B' C B = C_{11} - C_{12} C_{22}^- C_{21} \geqslant 0$.
                if (i), (ii) and (iii) hold true, then

$$C = \begin{bmatrix} I & C_{12} C_{22}^{'-} \\ 0 & I \end{bmatrix} \begin{bmatrix} C_{11} - C_{12} C_{22}^- C_{21} & 0 \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ C_{22}^- C_{21} & I \end{bmatrix} \geqslant 0.$$

Q.E.D.

A similar result has been mentioned by Ouellette [17]. We also note that, according to lemma 2.2.4 in Rao and Mitra [19], (ii) and (iii) are invariant under the choice of g-inverse.

Corollary. Let C be a symmetric matrix, with C^- a generalized inverse of C, then the following three statements are equivalent:

(A.2) $\begin{bmatrix} 1 & x' \\ x & C \end{bmatrix} \geqslant 0$

(A.3) $xx' \leqslant C$

(A.4)   (i) $C \geq 0$

         (ii) $CC^- x = x$

         (iii) $x' C^- x \leq 1$

Proof: Apply Lemma 1 twice.

Lemma 2. Let C and C^* be positive definite symmetric matrices, c^2 and c^{*2} positive scalars and let γ and γ^* be vectors, then

$$(A.5) \qquad \begin{bmatrix} \gamma'C\gamma + c^2 & \gamma'C \\ C\gamma & C \end{bmatrix} \leq \begin{bmatrix} \gamma^{*'}C^*\gamma^* + c^{*2} & \gamma^{*'}C^* \\ C^*\gamma^* & C^* \end{bmatrix}$$

if and only if

(A.6)   (i) $C^{-1} - C^{*-1} \geq 0$

         (ii) $(C^{-1} - C^{*-1})(C^{-1} - C^{*-1})^- (\gamma^* - \gamma) = \gamma^* - \gamma$

         (iii) $(\gamma^* - \gamma)' (C^{-1} - C^{*-1})^- (\gamma^* - \gamma) \leq c^{*2} - c^2.$

Proof: Premultiply (A.5) by $A = \begin{bmatrix} 1 & -\gamma' \\ 0 & I \end{bmatrix}$ and postmultiply by A'. This implies that (A.5) is equivalent to

$$(A.7) \qquad \begin{bmatrix} c^2 & 0 \\ 0 & C \end{bmatrix} \leq \begin{bmatrix} (\gamma^* - \gamma)' C^*(\gamma^* - \gamma) + c^{*2} & (\gamma^* - \gamma)' C^* \\ C^*(\gamma^* - \gamma) & C^* \end{bmatrix}.$$

Since the matrices on both sides of the inequality sign are positive definite, (A.7) is equivalent to

$$(A.8) \qquad \begin{bmatrix} c^{*-2} & -c^{*-2}(\gamma^* - \gamma)' \\ -c^{*-2}(\gamma^* - \gamma) & c^{*-1} + c^{*-2}(\gamma^* - \gamma)(\gamma^* - \gamma)' \end{bmatrix} \leq \begin{bmatrix} c^{-2} & 0 \\ 0 & c^{-1} \end{bmatrix},$$

or

$$(A.9) \qquad c^{*-2} \begin{pmatrix} -1 \\ \gamma^* - \gamma \end{pmatrix} \begin{pmatrix} -1 \\ \gamma^* - \gamma \end{pmatrix}' \leq \begin{bmatrix} c^{-2} & 0 \\ 0 & c^{-1} - c^{*-1} \end{bmatrix}.$$

Finally, using the collory, we find (A.9) to be equivalent to

(A.10) (i) $C^{-1} - C^{*-1} \geqslant 0$

(ii) $(C^{-1} - C^{*-1})(C^{-1} - C^{*-1})^{-}(\gamma^* - \gamma) = (\gamma^* - \gamma)$

(iii) $c^{*-2}(\gamma^* - \gamma)'(C^{-1} - C^{*-1})^{-}(\gamma^* - \gamma) + c^{*-2}c^2 < 1.$

<div align="right">Q.E.D.</div>

Proof of Proposition 1: There holds:

(A.11) $(B-\Phi) = \begin{bmatrix} \beta'(A-\Omega)\beta+\sigma^2 & \beta'(A-\Omega) \\ (A-\Omega)\beta & (A-\Omega) \end{bmatrix} < \begin{bmatrix} b'Ab+s^2 & b'A \\ Ab & A \end{bmatrix} = B.$

Since $A > 0$, $A-\Omega > 0$, $\sigma^2 > 0$, $s^2 > 0$, we can apply Lemma 2 to show that (A.11) is equivalent to

(A.12) (i) $(A-\Omega)^{-1} - A^{-1} \geqslant 0$

(ii) $\{(A-\Omega)^{-1} - A^{-1}\}\{(A-\Omega)^{-1}-A^{-1}\}^{-}(\beta-b) = \beta - b$

(iii) $(\beta-b)'\{(A-\Omega)^{-1} - A^{-1}\}^{-}(\beta-b) < s^2 - \sigma^2.$

The corollary implies that (A.12) is equivalent to

(A.13) $\begin{bmatrix} s^2 - \sigma^2 & \beta' - b' \\ \beta - b & (A-\Omega)^{-1} - A^{-1} \end{bmatrix} \geqslant 0.$

Similarly we find that $(B-\Phi^*) < (B-\Phi)$ is equivalent to

(A.14) $\begin{bmatrix} \sigma^2 - s^{*2} & \beta' - b^{*'} \\ \beta-b^* & (A-\Omega^*)^{-1} - (A-\Omega)^{-1} \end{bmatrix} \geqslant 0.$

Adding (A.13) and (A.14) yields:

(A.15) $\begin{bmatrix} s^2 - s^{*2} & 2\beta' - b' - b^* \\ 2\beta - b - b^* & F^* \end{bmatrix} \geqslant 0.$

Application of Lemma 1 yields (2.12) and (2.13).

The second part of the proof is constructive. For each $\tilde{\beta} \neq b$, satisfying (A.15) we construct a $\tilde{\sigma}^2$ and $\tilde{\Omega}$ that satisfy (A.13) and (A.14). Define

(A.16) $\tilde{\sigma}^2 \equiv \frac{1}{2}\{s^2 + s^{*2} - (\tilde{\beta}-b)'F^{*-}(\tilde{\beta}-b) + (\tilde{\beta}-b^*)'F^{*-}(\tilde{\beta}-b^*)\}.$

Since b satisfies (2.12) and (2.13) we have

$$(A.17) \quad \tfrac{1}{4}\begin{pmatrix} -1 \\ F^{*-}(b-b^*) \end{pmatrix}' \begin{bmatrix} s^2-s^{*2} & 2\tilde{\beta}-b'-b^{*'} \\ 2\tilde{\beta}-b-b^* & F^* \end{bmatrix}\begin{pmatrix} -1 \\ F^{*-}(b-b^*) \end{pmatrix} =$$

$$\tfrac{1}{4}\{(b-b^*)'\ F^{*-}(b-b^*) - (s^2-s^{*2})\} + s^2 - \tilde{\sigma}^2 \leqslant s^2 - \tilde{\sigma}^2.$$

Furthermore,

$$(A.18) \quad \tfrac{1}{2}\begin{bmatrix} s^2 - s^{*2} & 2\tilde{\beta}' -b' - b^{*'} \\ 2\tilde{\beta} - b - b^* & F^* \end{bmatrix}\begin{pmatrix} -1 \\ F^{*-}(b-b^*) \end{pmatrix} = -\begin{pmatrix} s^2 - \tilde{\sigma}^2 \\ \tilde{\beta} - b \end{pmatrix},$$

which implies

$$(A.19) \quad \begin{pmatrix} s^2-\tilde{\sigma}^2 \\ \tilde{\beta}-b \end{pmatrix}' \begin{bmatrix} s^2-s^{*2} & 2\tilde{\beta}'-b'-b^{*'} \\ 2\tilde{\beta}-b-b^* & F^* \end{bmatrix}^- \begin{pmatrix} s^2-\tilde{\sigma}^2 \\ \tilde{\beta}-b \end{pmatrix} \leqslant s^2 - \tilde{\sigma}^2.$$

Using the corollary, we find for $\tilde{\beta} \neq b$ that

$$(A.20) \quad \begin{bmatrix} s^2-s^{*2} & 2\tilde{\beta}'-b'-b^{*'} \\ 2\tilde{\beta}-b-b^* & (A-\Omega^*)^{-1}-A^{-1} \end{bmatrix} \geqslant (s^2-\tilde{\sigma}^2)^{-1}\begin{pmatrix} s^2-\tilde{\sigma}^2 \\ \tilde{\beta}-b \end{pmatrix}\begin{pmatrix} s^2-\tilde{\sigma}^2 \\ \tilde{\beta}-b \end{pmatrix}' =$$

$$\begin{bmatrix} s^2-\tilde{\sigma} & \tilde{\beta}'-b \\ \tilde{\beta}-b & (s^2-\tilde{\sigma}^2)^{-1}(\tilde{\beta}-b)(\tilde{\beta}-b)' \end{bmatrix} \geqslant 0.$$

If we now choose $\tilde{\Omega}$ such that

$$(A.21) \quad (A-\tilde{\Omega})^{-1} = A^{-1} + (s^2-\tilde{\sigma}^2)^{-1}\ (\tilde{\beta}-b)(\tilde{\beta}-b)',$$

then, clearly, both (A.13) and (A.14) are satisfied. <u>Q.E.D.</u>

<u>Proof of (2.14)</u>: Let x be a scalar and let

$$\tilde{\Phi} \equiv \begin{bmatrix} \phi_{11}^* - x^2 & \phi_{12}^* \\ \phi_{21}^* & \Omega^* \end{bmatrix}.$$

It follows from Lemma 1 that $\tilde{\Phi} > 0$ if and only if

(A.22) $x^2 \leqslant \phi_{11}^* - \phi_{12}^* \, \Omega^{*-} \, \phi_{21}^*.$

On the other hand $\tilde{\Phi} > 0$ if and only if $B - \tilde{\Phi} \leqslant B$. Partitioning of the matrices $B - \tilde{\Phi}$ and B just as in (A.11) and application of Lemma 2 shows that $B - \tilde{\Phi} \leqslant B$ if and only if

(A.23) $x^2 \leqslant s^2 - s^{*2} - (b-b^*)' \, F^{*-} (b-b^*).$

Clearly (A.22) is equivalent to (A.23) and thus

(2.14) $s^2 - s^{*2} = (b-b^*)' \, F^{*-} (b-b^*) + \phi_{11}^* - \phi_{12}^* \, \Omega^{*-} \, \phi_{1*}^*.$ <u>Q.E.D.</u>

<u>Proof of Proposition 2</u>: Given that F^* is symmetric and positive semi-definite, the corollary implies that (2.12) and (2.13) are equivalent to

(A.24) $(\beta-\tfrac{1}{2}(b+b^*)) \, (\beta-\tfrac{1}{2}(b+b^*))' \leqslant \tfrac{1}{4} \, (s^2-s^{*2}) \, F^*.$

This implies

(A.25) $(\psi'\beta - \tfrac{1}{2}\psi'(b+b^*))^2 \leqslant \tfrac{1}{4} \, (s^2-s^{*2}) \, \psi' \, F^* \, \psi,$

for any given vector ψ. This makes it clear that (2.19) gives the extreme values of $\psi' \beta$.   <u>Q.E.D.</u>

<u>Appendix B</u>: <u>Proofs of Proposition 3 and 4</u>

<u>Proof of proposition 3</u>:
Clearly $\Omega_\delta^* = \Omega_\delta^* \, \Omega^{*-} \, \Omega_\delta^*$ and $\Omega_\delta^* = \Omega^* \, \Omega^{*-} \, \Omega_\delta^*.$
If we define

$$F_\delta^* \equiv (A-\Omega_\delta^*)^{-1} - A^{-1},$$

then

(B.1) $\qquad F_\delta^* (A-\Omega^*) \; \Omega^{*-} \; AF_\delta^* =$

$$= (A-\Omega_\delta^*)^{-1} \; \Omega_\delta^* \; A^{-1} \; (A-\Omega^*) \; \Omega^{*-} \; AA^{-1} \; \Omega_\delta^* \; (A-\Omega_\delta^*)^{-1} =$$

$$= F_\delta^* \, .$$

So $(A-\Omega^*) \; \Omega^{*-} \; A$ is a g-inverse of F_δ^* for every δ; in particular it is a g-inverse of F^*. As

(B.2) $\qquad F_\delta^* = F^* (A-\Omega^*) \; \Omega^{*-} \; \Omega_\delta^* \; (A-\Omega_\delta^*)^{-1}.$

it follows that

(B.3) $\qquad F_\delta^* \; F^{*-} \; F_\delta^* = F_\delta^*$

for any g-inverse F^{*-}. As $2\beta_\delta - b - b^* = (2F_\delta^* - F^*)Ab$, and using (2.14) with $\phi_{11}^* = 0$ and $\phi_{21}^* = 0$, it follows that (2.12) becomes an equality if we substitute β_δ for β. $\qquad\qquad$ Q.E.D.

Lemma 3. <u>Let A be a positive-definite matrix, k a vector and μ a scalar, $0 < \mu < 1$. Then</u>

(B.4) $\qquad (A+\mu kk')^{-1} = \lambda A^{-1} + (1-\lambda)(A+kk')^{-1},$

<u>where</u>

(B.5) $\qquad 1 > \lambda = \dfrac{1 - \mu}{1 + \mu \; k'A^{-1}k} > 0.$

<u>Proof</u>: Straightforward

$\qquad$ Without loss of generality, we assume that the first ℓ diagonal elements of Ω^*, ω_1^*, $\omega_2^*, \ldots, \omega_\ell^*$, are non-zero ($\ell < k$) and the remaining $k-\ell$ elements are zero. Let us index the 2^ℓ vectors δ by a subscript j, with $j=1,\ldots,2^\ell$. A typical element of δ_j is δ_{ij} $1,\ldots k$. We order the δ_j in such a way that, for $j \leq 2^m$ and $0 \leq m \leq \ell-1$, $\delta_{j+2^m} = \delta_j - e_{m+1}$, with e_{m+1} the $(m+1)$-th unit vector. In Fig. 5 we give an example for $k=4$ and $\ell=3$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\delta_1 \qquad \delta_2 \qquad \delta_3 \qquad \delta_4 \qquad \delta_5 \qquad \delta_6 \qquad \delta_7 \qquad \delta_8$$

Figure 5. The ordering of $\underline{\delta}_j$ for k=4, ℓ=3.

Define $K_j \equiv A - \sum_{i=1}^{\ell} \delta_{ij} \omega_i^* e_i e'_i$ (this would be denoted as $A - \Omega_\delta^*$ in section 3, with $\delta = \delta_j$). Then we have that $K_{j+2^m} = K_j + \omega_{m+1}^* e_{m+1} e'_{m+1}$.

Lemma 4. Let μ_i, i=1,...,2^m, be scalars satisfying $\mu_i > 0$, $\sum_i \mu_i = 1$, then there exist scalars λ_j, j=1,...,2^m, satisfying $\lambda_j > 0$, $\sum_j \lambda_j = 1$, such that

(B.6) $\left\{ \sum_{i=1}^{2^m} \mu_i K_i \right\}^{-1} = \sum_{j=1}^{2^m} \lambda_j (K_j)^{-1}$, for all $0 < m < \ell$

Proof: The proof is by induction. Assume (B.6) holds for $m < \ell - 1$ then we show that it also holds for m+1.

(B.7) $\sum_{i=1}^{2^{m+1}} \mu_i K_i = \sum_{i=1}^{2^m} \mu_i K_i + \sum_{i=1}^{2^m} \mu_{i+2^m} K_{i+2^m} = \sum_{i=1}^{2^m} (\mu_i + \mu_{i+2^m}) K_i +$

$, \left\{ \sum_{i=1}^{2^m} \mu_{i+2^m} \right\} w_{m+1}^* e_{m+1} e'_{m+1}.$

Lemma 3 implies

(B.8) $\left\{ \sum_{i=1}^{2^{m+1}} \mu_i K_i \right\}^{-1} = \lambda \left\{ \sum_{i=1}^{2^m} (\mu_i + \mu_{i+2^m}) K_i \right\}^{-1} +$

$$(1-\lambda)\left\{ \sum_{i=1}^{2^m} (\mu_i + \mu_{i+2^m})(K_i + w_{m+1}^* e_{m+1} e'_{m+1}) \right\}^{-1} =$$

$$= \lambda\left\{ \sum_{i=1}^{2^m} (\mu_i + \mu_{i+2^m}) K_i \right\}^{-1} + (1-\lambda)\left\{ \sum_{i=1}^{2^m} (\mu_i + \mu_{i+2^m}) K_{i+2^m} \right\}^{-1}.$$

with $0 < \lambda < 1$. Assuming that the proposition holds for m, (B.8) implies that it holds also for m+1. Furthermore, (B.6) holds if m=0.

$$\underline{\text{Q.E.D.}}$$

$\underline{\text{Proof of Proposition 4}}$: Consider $K \equiv A - \Omega$ Given that $0 < \Omega < \Omega^*$ and that Ω and Ω^* are diagonal we can write K as

$$(B.9) \qquad K = \sum_{j=1}^{2^\ell} \mu_j K_j,$$

where $\mu_j > 0$, $\sum_j \mu_j = 1$

According to (3.1) and Lemma 4 we have,

$$(B.10) \qquad \beta = (A-\Omega)^{-1} Ab = K^{-1} Ab = \left\{ \sum_{j=1}^{2^\ell} \mu_j K_j \right\}^{-1} Ab = \sum_{j=1}^{2^\ell} \lambda_j K_j^{-1} Ab = \sum_{j=1}^{2^\ell} \lambda_j \beta_{\delta_j},$$

with $\lambda_j > 0$, $\sum_j \lambda_j = 1$.

$$\underline{\text{Q.E.D.}}$$

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