# Regular two-graphs and extensions of partial geometries

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# REGULAR TWO-GRAPHS AND EXTENSIONS OF PARTIAL GEOMETRIES

W. H. Haemers

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by

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ABSTRACT.

We study so called two-graph geometries. These are geometries that carry a regular two-graph, but also constitute one-point extensions of partial geometries. First we develop some theory, then we go through lists of known regular two-graphs and partial gemetries in order to find examples. Some are found, including one that extends the partial geometry with parameters $s=4$, $t=17$, $\alpha=2$.

## 1. INTRODUCTION.

Extensions of t-designs, especially one-point extensions, have been studied a lot in the past. More recently, people became interested in extensions of finite geometries, such as generalised quadrangles (there exist various papers on this subjuct; Cameron [6] gives a survey) or, more generally, partial geometries (see Hobart & Hughes [17]). The points of a partial geometry carry a strongly regular graph, whilst regular two-graphs are, in a certain sense, extensions of strongly regular graphs by one point. For this reason it seemed worthwhile to investigate a combination of these objects, being one-point extensions of partial geometries with the structure of a regular two-graph on the points. We call such structures two-graph geometries. The present paper is a first attempt to study these geometries.

The reader is assumed to be familiar with the theory of designs and strongly regular graphs (see for instance Cameron & Van Lint [7]). We shall briefly survey the relevant results on two-graphs and partial geometries. A *two-graph* consists of a finite set $\Omega$ together with a set $\Delta$ of triples (called *coherent* triples) from $\Omega$, such that every 4-subset of $\Omega$ contains an even number of coherent triples. Let $\nabla$ denote the set of non-coherent triples. Then $(\Omega,\nabla)$ is also a two-graph, called the *complement* of $(\Omega\ \Delta)$. The two-graph $(\Omega,\Delta)$ is *empty [complete]* if $\Delta$ $[\nabla]$ is empty; $(\Omega,\Delta)$ is *regular* if every pair of points from $\Omega$ is contained in a constant number $a$ of coherent triples. For any point $\omega \in \Omega$ of a regular two-graph $(\Omega,\Delta)$, the matrix $A_\omega$, defined by

$$(A_\omega)_{\beta\gamma} = \begin{cases} -1 & \text{if } \{\beta,\gamma,\omega\} \in \Delta, \text{ or } \omega \in \{\beta,\gamma\}, \\ 0 & \text{if } \beta = \gamma, \\ 1 & \text{otherwise,} \end{cases}$$

has just two eigenvalues $\rho_1$ and $\rho_2$ $(\rho_1 > \rho_2)$. These eigenvalues have opposite sign and are odd integers if $\rho_1 \neq -\rho_2$, furthermore

$$|\Omega| = 1-\rho_1\rho_2 \ , \quad a = -(\rho_1+1)(\rho_2+1)/2 \ .$$

The *derived graph* $\Gamma_\omega$ of $(\Omega,\Delta)$ with respect to $\omega$ has vertex set $\Omega\backslash\{\omega\}$, two vertices $\beta$ and $\gamma$ being adjacent if $\{\beta,\gamma,\omega\} \in \Delta$. So by deleting row and column $\omega$ from $A_\omega$, we obtain the $(-1,1,0)$ adjacency matrix of $\Gamma_\omega$. For any $\omega$, the derived graph of a regular two-graph (not complete or empty) is strongly regular with parameters $(V,K,\Lambda,M)$, where

$$V = |\Omega|-1 = -\rho_1\rho_2 \ , \quad K = a = -(\rho_1+1)(\rho_2+1)/2 \ ,$$

$$(1) \qquad \Lambda = 1-(\rho_1+3)(\rho_2+3)/4 \ , \ M = K/2 = -(\rho_1+1)(\rho_2+1)/4 \ .$$

Conversely, to a strongly regular graph with parameters $(V,K,\Lambda,K/2)$ there corresponds a regular two-graph. A clique (or coherent set) of $(\Omega,\Delta)$ is a subset c of $\Omega$, such that every triple from c is coherent. Clearly, for $\omega \in c$, $c\backslash\{\omega\}$ is a clique (in the normal sense) in $\Gamma_\omega$. A clique c of $(\Omega,\Delta)$ satifies:

$$(2) \qquad |c| \leq 1-\rho_2 \ .$$

Two-graphs have been introduced by G. Higman, and were studied mainly by Seidel and Taylor [24] [25] [28].

A design is denoted by the pair $(\Phi,B)$, were $\Phi$ is the set of points and B is the set of blocks. An *anti-flag* of a design $(\Phi,B)$ is a pair $(\varphi,b)$ with $\varphi \in \Phi$, $b \in B$ and $\varphi \notin b$.

A *partial geometry* $pg(s,t,\alpha)$ is a $1-(V,s+1,t+1)$ design $(\Phi,B)$, where any two distinct lines (= blocks) meet in at most one point, such that for every anti-flag $(\varphi,b)$ there are precisely $\alpha$ points on b collinear with $\varphi$.
It follows that $V = |\Phi| = (s+1)(st+\alpha)/\alpha$, $|B| = (t+1)(st+\alpha)/\alpha$. If we interchange the roles of points and lines we obtain the *dual* partial geometry $pg(t,s,\alpha)$. The *point graph* of a $pg(s,t,\alpha)$ has vertex set $\Phi$; two vertices are adjacent if they are collinear. The point graph of a $pg(s,t,\alpha)$ is strongly regular with parameters $(V,K,\Lambda,M) = (V,\ s(t+1),\ s-1+t(\alpha-1),\ \alpha(t+1))$. A *one-point extension* of $pg(s,t,\alpha)$ is a design for which the derived design with respect to any point is a $pg(s,t,\alpha)$.

Partial geometries were introduced by Bose [1] and have been studied a lot. Some general references are: Brouwer & Van Lint [2] and De Clerck [3].

## 2. TWO-GRAPH GEOMETRIES.

1. DEFINITION. A *two-graph geometry* is a 2-$(v,k,\lambda)$ design $(\Omega,C)$ satisfying the following properties:

i. two distinct blocks of C have at most two points in common (therefore blocks are called circles),

ii. any set of four points contains an even number of cocircular triples,

iii. $v = 1 + (k-1)(2\lambda-1)$.

2. PROPOSITION. Let $\Delta$ be the set of cocircular triples of a two-graph geometry $(\Omega,C)$. Then $(\Omega,\Delta)$ is a regular two-graph with eigenvalues

$$\rho_1 = 2\lambda-1,\ \rho_2 = 1-k.$$

Proof. By i and ii $(\Omega,\Delta)$ is a two graph. Since $(\Omega,C)$ is a 2-$(v,k,\lambda)$ design, $(\Omega,\Delta)$ is regular with $a = -(\rho_1+1)(\rho_2+1)/2 = \lambda(k-2)$. Using $v = |\Omega| = 1+(k-1)(2\lambda-1) = 1-\rho_1\rho_2$ and $\rho_1 > 0 > \rho_2$ the values of $\rho_1$ and $\rho_2$ follow.

Note that we did not use property iii to prove that $(\Omega,\Delta)$ is a regular two-graph, it is only used to compute $\rho_1$ and $\rho_2$. In fact, once $(\Omega,\Delta)$ is defined, property iii can be replaced by:

iii'. the circles of C are maximal cliques of $(\Omega,\Delta)$.

Herein maximal means that the bound $-\rho_2+1$ given in (2) is met. As usual, the number of circles is denoted by b and the number of circles through a fixed point by r. Then $\lambda(v-1) = r(k-1)$ and $bk = vr$ yield

$$r = \lambda(2\lambda-1) = \rho_1(\rho_1+1)/2\ ,$$

$$b = \lambda(2\lambda-1)^2 - 2\lambda(2\lambda-1)(\lambda-1)/k = \rho_1^2(\rho_1+1)/2 + \rho_1(\rho_1^2-1)/2(\rho_2-1)\ .$$

We call a regular two-graph *geometric* if it corresponds to a two-graph geometry. Clearly, for a two-graph to be geometric the following divisibility condition must be satified:

(3) $$2(-\rho_2+1) \mid \rho_1(\rho_1^2-1) .$$

The following result is straight forward.

3. PROPOSITION. A regular two-graph with eigenvalues $\rho_1$ and $\rho_2$ is geometric if and only if there exists a set C of cliques of size $(1-\rho_2)$, such that every coherent triple is covered by a unique clique of C.

A regular two-graph with $\rho_2 = -1$ is empty (no triple is coherent). Also for the next case, $\rho_2 = -3$, two-graph geometries are nothing special, because of the following result.

4. PROPOSITION. Let C be the set of all 4-cliques of a regular two-graph $(\Omega,\Delta)$ with $\rho_2 = -3$. Then $(\Omega,C)$ is the unique two-graph geometry corresponding to $(\Omega,\Delta)$.

Proof. In a regular two-graph with eigenvalues $\rho_1$ and $\rho_2$ each coherent triple is contained in exactly $\Lambda = 1-(\rho_1+3)(\rho_2+3)/4$ cliques of size 4, by use of (1). This number $\Lambda$ equals 1 if $\rho_2 =-3$, hence Proposition 3. gives the result.

Seidel's [22] determination of all regular two-graphs with $\rho_2 = -3$ leads to

5. COROLLARY. Two-graph geometries with $\rho_2 = -3$ (i.e. $k = 4$) exist if and only if $\rho_1 = 1, 3, 5$ or $9$ (i.e. $\lambda = 1, 2, 3$ or $5$) and are unique

Note that two-graph geometries with $\rho_1 = 1$ are degenerate: there is just one circle of size $k = v$, and the two-graphs are complete.

Let $(\omega,c)$ be an anti-flag of a design $(\Omega,C)$. The *anti-flag graph* $\Gamma_{\omega,c}$ has vertex set c, two vertices $\beta$ and $\gamma$ $(\beta \neq \gamma)$ are adjacent whenever $\omega$, $\beta$ and $\gamma$ are covered by a block of C.

6. PROPOSITION. A 2-$(v,k,\lambda)$ design $(\Omega,C)$ with block intersection sizes at most 2, is a two-graph geometry if and only if each anti-flag graph is the disjoint union of two complete graphs of size $k/2$.

Proof. Suppose $(\Omega,C)$ is a two-graph geometry. Let $(\omega,c)$ be an anti-flag of $(\Omega,C)$ and let $\beta$, $\gamma$ and $\delta$ be three distict points of c. Since $\{\beta,\gamma,\delta,\omega\}$ contains an even number of cocircular triples, the subgraph of $\Gamma_{\omega,c}$ induced by $\beta$, $\gamma$ and $\delta$ is either a triangle or has just one edge. Thus $\Gamma_{\omega,c}$ is the complete graph or the disjoint union of two complete graphs. Conversely, it is easily seen that any 4-set contains 0, 2 or 4 cocircular triples if each anti-flag graph is the disjoint union of two complete graphs.
Next fix $c \in C$. For $\omega \in \Omega\backslash c$, let $m_\omega$ denote the size of a component of $\Gamma_{\omega,c}$. Counting in two ways the total number of triples $(\omega,\beta,\gamma)$ with $\omega \in \Omega\backslash c$, and $\beta,\gamma$ adjacent vertices in $\Gamma_{\omega,c}$ gives

$$\sum_{\omega \notin c} (m_\omega(m_\omega-1) + (k-m_\omega)(k-m_\omega-1)) = k(k-1)(\lambda-1)(k-2)$$

The left hand side is at least $\ell = (v-k)k(\frac{1}{2}k-1)$ with equality if and only if $m_\omega = k/2$ for all $\omega \notin c$. This proves the result, because $\ell$ equals the right hand side, precisely when $v = 1+(k-1)(2\lambda-1)$.

By definition, a 2-design is a one-point extension of a partial geometry $pg(s,t,\alpha)$ if and only if any two distinct blocks meet in at most 2 points and each anti-flag graph is regular of degree $\alpha$. Therefore we have:

7. THEOREM. A two-graph geometry $(\Omega,C)$ with eigenvalues $\rho_1$ and $\rho_2$ is a one point extension of a partial geometry with parameters

$$s = -\rho_2-1, \quad t = (\rho_1-1)/2, \quad \alpha = (-\rho_2-1)/2.$$

So, only partial geometries with $s = 2\alpha$ occur. Clearly, the point graph of the partial geometry with respect to $\omega \in \Omega$ (say) is the derived graph $\Gamma_\omega$ of $(\Omega,\Delta)$. Such strongly regular graphs satisfy $K = 2\Lambda$ ($(V,K,\Lambda,M)$ is the set of parameters), which is equivalent to $s = 2\alpha$.

Clearly, the anti-flag graph of a one-point extension of a pg(s,t,1) (i.e. a generalised quadrangle) consists of disjoint edges. So by Proposition 6 we have:

8. PROPOSITION. A one-point extension of a pg(2,t,1) is a two graph geometry

This result needs not to be true for $\alpha > 1$. For instance, there exists one-point extensions of pg(4,1,2) for which some anti-flag graphs are hexagons. So they are not two-graph geometries. But no other exceptions are known (to us). The two-graph geometries corresponding to the above proposition have $\rho_2 = -3$. So by Corollary 5 we have the following result due to Beukenhout [3] (see also Thas [29]).

9. COROLLARY. One-point extensions of pg(2,t,1) exist and are unique.

## 3. REGULAR TWO-GRAPHS AND PARTIAL GEOMETRIES.

Next we investigate known or feasible regular two-graphs and partial geometries with $s = 2\alpha$. We more or less follow the surveys of Seidel [24] and De Clerck [10]. Since the point graph of a partial geometry with $s = 2\alpha$ is strongly regular graph with $K = 2M$, the regular two-graph exists if the partial geometry exists. The converse, however, is not true.

Case 1: $\rho_1 = -\rho_2-2$, or equivalently, $t = \alpha-1$.
The regular two-graph corresponds to a regular symmetric Hadamnard matrix with constant diagonal. The corresponding partial geometries are duals of block designs with $\lambda = 1$. In such a partial geometry any two lines meet. This implies that two circles of the two-graph geometry can only have no or two points in common. The two-graph geometry is therefore a quasi-symmetric block design. The divisibility condition (3) leads to $\rho_1 = 1$, 3 or 9. The case $\rho_1 = 1$, $\rho_2 = -3$ is treated in Corollary 5. For the other two cases the parameters $(\rho_1, \rho_2, v, b, k, r, \lambda, s, t, \alpha)$ are (3,-5,16,16,6,6,2,4,1,2) and (9,-11,100,375,12,45,5,10,4,5). The first one is a 2-(16,6,2) design. There exist precisely three such designs, but only one satisfies condition

2.1.ii, viz. the unique 2-(16,6,2) design with characteristic 3, see Cameron [5]. Nothing is known about the second case. Mavron & Shrikhande [21] also found the mentioned possibilities in their classification of quasi-symmetric block designs with block intersection sizes 0 and 2 and an additional requirement, a little weaker than condition 2.1.ii.

Case 2: $\rho_1 = -\rho_2$, or equivalently, $t = \alpha$.
The regular two-graphs are the ones associated to conference matrices with integral eigenvalues. The partial geometries pg(s,s/2,s/2) are dual nets; they coresponding to (s-2)/2 mutually orthogonal latin squares of order s+1. The parameters are:

$$v = \rho_1^2+1 \;,\quad b = \rho_1(\rho_1^2+1)/2 \;,\quad k = \rho_1+1 \;,\quad r = \rho_1(\rho_1+1)/2 \;,$$

$$\lambda = (\rho_1+1)/2 \;,\quad s = \rho_1-1 \;,\quad t = \alpha = (\rho_1-1)/2 \;.$$

It was observed by Fisher [14] that such two-graph geometries can be constructed from the inversive plane over the field with $\rho_1$ elements. So they exist whenever $\rho_1$ is an odd prime power. The required set of circles is just one orbit of the group generated by the inversions acting on the blocks (circles) of the inversive plane. Wilbrink [32] proved that the corresponding two-graphs are the Paley two-graphs. The derived partial geometry is the corresponding half of the affine plane derived from the original inversive plane. The lines of this partial geometry are the only maximal cliques in the Paley graph (the Paley graph is the derived graph of the Paley two-graph with respect to any point), see Blokhuis [0]. This implies that the Paley two-graph is geometric in a unique way; the circles are just all maximal cliques.

The above cases together with Corollary 5 cover all two-graph designs whose corresponding partial geometry is improper (i.e. $\alpha = 1$, t or t+1). For the remaining cases $\alpha < t$ holds, which implies $\rho_1 > -\rho_2$.

Case 3: $\rho_1 = -\rho_2+2$, or equivalently, $t = \alpha+1$.

The regular two-graphs are the complements of the ones considered in Case 1. Partial geometries with $t = \alpha+1$ are classified by De Clerck [9]. The parameters for this case are:

$$v = (\rho_1-1)^2, \quad b = \rho_1(\rho_1^2-1)/2, \quad k = \rho_1-1, \quad r = \rho_1(\rho_1+1)/2,$$

$$\lambda = (\rho_1+1)/2, \quad s = \rho_1-3, \quad t = (\rho_1-1)/2, \quad \alpha = (\rho_1-3)/2.$$

A $pg(2\alpha,\alpha+1,\alpha)$ can be constructed from a projective plane of order $2\alpha+2$ posessing a hyperoval. Such planes do exist if the order is a power of 2. If $\alpha = 1$ the the two-graph geometry exists (Corollary 5). If $\alpha = 2$ or 4 the partial geometry does not exist (by De Clerck [9] and Lam et al [19] respectively). The regular two-graph is known for many more values of $\rho_1 = 2\alpha+3$ than the corresponding partial geometry, including for $\alpha = 2$ and $\alpha = 4$. The symplectic two-graphs, for instance, belong to this case. By De Clerck, Gevaert & Thas [12] they are not geometric if $\rho_1 = 9$ or 17. In fact, for no other value of $\alpha$ existence of such a two-graph geometry is settled. The smallest candidate has parameters $\rho_1 = 9$, $\rho_2 = -7$, $v = 64$, $b = 360$, $k = 8$, $r = 45$, $\lambda = 5$, $s = 6$, $t = 4$, $\alpha = 3$. By Mathon [20], there are just two such partial geometries. Storme [27] and Tonchev [31] proved by computer that one of the two (the one corresponding to the hyperoval in the plane of order 8) cannot be extended to a two-graph geometry.

Case 4: $\rho_1 = 2^m-1$, $\rho_2 = -2^{m-1}-1$, or equivalently $\alpha = 2^{m-2}$, $t = 2^{m-1}-1$. By De Clerck, Dye & Thas [11] partial geometries and (hence) regular two-graphs with these parameters are known if m is even. The two-graphs exist for all $m > 1$, they are the complements of the orthogonal two-graphs $\Omega^*(2m,2)$, see Seidel [23]. By condition (3) only $\rho_1 = 3$, 7 and 15 are possible. The first possibility exists (Corollary 5), the second one doesn't (see Case 3) and the remaining one has parameters: $\rho_2 = -9$, $v = 136$, $k = 10$, $b = 1632$, $r = 120$, $\lambda = 8$, $s = 8$, $t = 7$, $\alpha = 4$. Tonchev [31] showed by computer that the known $pg(8,7,4)$ (see Cohen [8] and Haemers & Van Lint [16] for other ways to construct this partial geometry) does not extend to a two-graph geometry. Since $pg(8,7,4)$ is conjectured to be unique, the two-graph geometry probably does not exist.

Case 5. $\rho_1 = 2^m+1$, $\rho_2 = -2^{m-1}+1$, or equivalently, $t = 2^{m-1}$, $\alpha = 2^{m-2}-1$.

For the other parameters we find

$$v = 2^{2m-1}-2^{m-1}\ , \quad b = (2^{2m}-1)(2^m+1)\ , \quad k = 2^{m-1}\ ,$$

$$r = (2^m+1)(2^{m-1}+1)\ , \quad \lambda = 2^{m-1}+1\ , \quad s = 2^{m-1}-2\ .$$

The corresponding regular two-graphs are the orthogonal two-graphs $\Omega^-(2m,2)$, see Seidel [23]. Corollary 5 takes care of $m = 3$. For $m > 3$ existence of the partial geometry is still open. De Clerck & Tonchev [13] showed that for $m = 4$ the corresponding pg(6,8,3) can only have automorphisms of order 2 and 3, leaving not much hope for finding the two-graph geometry.

Case 6: $\rho_1 = \rho_2^2$, or equivalently, $t = 2\alpha(\alpha+1)$.
Regular two-graphs with these eigenvalues were constructed by Taylor [28] whenever $\sqrt{\rho_1}$ is an odd prime power. The remaining parameters are:

$$v = \rho_1\sqrt{\rho_1}+1\ , \quad b = \rho_1(\rho_1+1)(\rho_1-\sqrt{\rho_1}+1)/2\ , \quad k = \sqrt{\rho_1}+1\ ,$$

$$r = \rho_1(\rho_1+1)/2\ , \quad \lambda = (\rho_1+1)/2\ , \quad s = 2\alpha = \sqrt{\rho_1}-1\ , \quad t = (\rho_1-1)/2.$$

Again the first one exists ($\rho_1 = 9$, $\rho_2 = -3$) by Corollary 5. For $\rho_2 = -5$ and $\rho_2 = -7$ Spence [26] proved that the derived strongly regular graph is not geometric. For $-\rho_2 > 7$ nothing is known. Also the Ree groups provide regular two-graphs with these eigenvalues whenever $-\rho_2$ is an odd power of 3. We have no idea whether these two-graphs can be geometric.

Case 7. $\rho_2 = -5$, or equivalently, $\alpha = 2$.
Then $k = 6$, $s = 4$, $\alpha = 2$. For $\rho_1$, 15, 19, 35 and 55 are the only possible values that have not been considered before. If $\rho_1 = 15$, 19 the regular two-graph nor the partial geometry is known to exist. For $\rho_1 = 35$ a two-graph geometry is realised in the next section. For $\rho_1 = 55$ there is a unique regular two-graph and a unique derived strongly regular graph. Nevertheless, existence of the partial geometry and the two-graph geometry is yet unsolved.

## 4. AN EXCEPTIONAL TWO-GRAPH GEOMETRY.

In this section we construct a two graph geometry $(\Omega,C)$ with parameters: $\rho_1 = 35$, $\rho_2 = -5$, $v = 176$, $b = 18480$, $k = 6$, $r = 630$, $\lambda = 18$, $s = 4$, $t = 17$, $\alpha = 2$. The regular two graph $(\Omega,\Delta)$ is the one having the Higman-Sims group HS acting on $\Omega$ as a 2-transitive automorphism group, see Taylor [28]. The partial geometry (with respect to any point) is the one constructed by the author [15]. The group of the two-graph geometry will be the Mathieu group $M_{22}$, which is a subgroup of HS and the corresponding action on $\Omega$ is rank 3. For the construction we need some properties of this action.

10. LEMMA The action of $M_{22}$ on $(\Omega,\Delta)$ satisfies:

i. There exists an orbit C of size 18480 on the 6-cliques of $(\Omega,\Delta)$.

ii. Every triple from $\Delta$ is contained in a 6-clique of C.

Proof. Fix a point $\omega \in \Omega$. The subgroup of $M_{22}$ stabilizing $\omega$ is $A_7$. It is an automorphism group of $\Gamma_\omega$ (the full automorphism group of $\Gamma_\omega$ is $P\Sigma U(5,2)$). We can define $\Gamma_\omega$ on the edges of the Hoffman-Singleton graph (for short HoSi), where two edges are adjacent whenever they are disjoint and posses an interconnecting edge (see Hubaut [18]). The group of automophisms of HoSi, that fixes (setwise) a distinguished 15-coclique is $A_7$. Its action on the edges is the action on $\Gamma_\omega$, just mentioned. This description is worked out in some detail in [15], in order to construct pg(4,17,2). Using this description, the following facts are straigtforward:

- The 5-cliques of $\Gamma_\omega$ are one-factors in Petersen subgraphs of HoSi.
- The group $A_7$ has two orbits on the Petersen subgraphs of HoSi. The sizes are 105 (the 'special Petersen graphs' in [15]) and 420, respectively.
- The subgroup of $A_7$ that stabilizes any Petersen subgraph P of HoSi acts transitively on the one-factors of P.

So $A_7$ has two orbits on the 5-cliques of $\Gamma_\omega$; one of size 630 (the lines of pg(4,17,2)), and one of size 2520. Any such 5-cliques is, together with $\omega$, a 6-cliques of $(\Omega,\Delta)$. Thus (remember that $M_{22}$ acts transitively on $\Omega$) there are $176\times3150/6 = 92400$ 6-cliques, and $M_{22}$ acts either transitively on $\Omega$, or has two orbits; one of size 18480 and one of size 73920, respectively. The first option, however, cannot occur, since 92400 doesn't divide the order of $M_{22}$ (= 443520). This proves i. Next let $\{\omega,\beta,\gamma\} \in \Delta$. Then $\{\beta,\gamma\}$ is an edge of $\Gamma_\omega$, which is contained in a (unique) 5-clique of the smaller orbit (i.e. a line of pg(4,17,2)). Hence $\{\omega,\beta,\gamma\}$ is contained in a 6-clique of C, proving ii.

11. THEOREM. With C as in Lemma 10, $(\Omega,C)$ is a two-graph geometry.

Proof. By i of the lemma, the cliques of C cover at most 18480.20 = 369600 coherent triples. This, however, is precisely the total number of coherent triples. Therefore by ii every triple is covered exactly ones by a clique of C, so $(\Omega,C)$ is a two-graph geometry by Proposition 3.

It is clear that the parameters of this two-graph geometry are the ones mentioned above. There must be several ways to prove Lemma 10. For instance, another proof could go along the lines of the construction by Calderbank & Wales [4] of pg(4,17,2); instead of HoSi, they start from the Steiner system S(5,8,24). An approach that does not use one of these two construction methods of pg(4,17,2), would give a new way to describe this partial geometry.

It has been checked (using a computer) that there is a unique way to extend the pg(4,17,2) to a two-graph geometry such that all automorphisms ($A_7$) of the partial geometry are preserved.

## 5. CONCLUDING REMARKS.

Apart from the existence questions mentioned in Section 3, there are several other problems that look interesting. We mention a few.
We don't know examples of non-isomorphic two-graph geometries with the same parameters. For $\rho_2$ = -3 they are unique (Corollary 5). Is is premature to conjecture that two-graph geometries are necessarely unique, but it seems save to do so for the sporadic one of the previous section, because of the remark at the end (it is even conceivable that the regular two-graph and the partial geometry are unique).

The relation between two-graphs and Seidel switching leads to a class of (-1,1,0) incidence matrices of a two graph geometry in the following way. Consider the incidence matrix N of $(\Omega,C)$. Let $\Gamma$ be a (strong) graph in the switching class of $(\Omega,\Delta)$. Each circle of C corresponds to a disjoint union of two complete graphs in $\Gamma$. Sign the non-zero entries of each column of N with + and - according to the partition of the corresponding circle, just described. The matrix obtained in this manner has some interesting properties; for instance, its rank equals the multiplicity of $\rho_2$. It is not clear how this can be explored.

If for a two-graph geometry, $2(-\rho_2+1)$ devides $\rho_1^2-1$ (compare with (3)), then it is feasible that a subset of the circles forms a partial geometry $pg(-\rho_2,(\rho_1-1)/2,(-\rho_2+1)/2)$ $(= pg(s+1,t,\alpha+1))$. For Fisher's two-graph geometries (Section 3, Case 2) it would mean that a subset of the circles is the dual of a $2\text{-}((\rho_1^2+1)/2,(\rho_1+1)/2,1)$ design. If $\rho_1 = 3$, this is possible, however, Thas [30] proved that it is impossible for $\rho_1 > 3$. An affirmative answer for the sporadic example of Section 4 would give a new partial geometry pg(5,17,3).

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