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ABSTRACT

In this paper we deal with the nonlinear complementarity problem on the product space S of several unit simplices. To find a solution a variable dimension algorithm developed by van der Laan and Talman for proper labellings is adapted. This generalization deals with unproper labellings to utilize the complementarity conditions in the problem. In this way the algorithm combines lower dimensional movements on the boundary of S because of the complementarity, and in the interior because of the structure of the algorithm.
Computational results confirm the usefulness of the algorithm.

Keywords: simplicial algorithm, triangulation, nonlinear complementarity problem, labelling

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1. Introduction

Given n+1 closed subsets $C_1,\dots,C_{n+1}$ of the n-dimensional unit simplex $S^n = \{x \in R_+^{n+1} | \Sigma_{i=1}^{n+1} x_i = 1\}$ we consider the problem of finding an x^* in S^n such that for all i either $x_i^* = 0$ or $x^* \in C_i$. In the following this problem will be called the Intersection Probem (IP). This problem arises quite naturally in different fields such as nonlinear programming and economics. For instance, given a continuous function z from S^n into R^{n+1} satifying $x^T z(x) = 0$ for all $x \in S^n$, the Nonlinear Complementarity Problem (NLCP) on S^n consists in finding a point x^* in S^n such that $z(x^*) \leq 0$. Such a solution x^* is complementary to $z(x^*)$, that is $x_i^* z_i(x^*) = 0$ for $i = 1,\dots,n+1$. A function z satisfying these conditions is the exess demand function of an exchange economy with n+1 commodities, in which case S^n is the price space. Defining

$$C_i = \{x \in S^n | z_i(x^*) \geq 0\} \quad i = 1,\dots,n+1$$

we have that a solution to the NLCP corresponds to a solution to the IP. Both problems are equivalent to finding a Brouwer fixed point $f(x^*) = x^*$ of a continuous function f from S^n into S^n.

Many simplicial algorithms have been introduced to find a Brouwer fixed point (e.g., Scarf [8], Eaves [1], Kuhn and MacKinnon [3], Lüthi [6], and van der Laan and Talman [4]). These algorithms are based on a subdivision of S^n into n-dimensional simplices (n-simplices) and on a labelling assigning to each point of S^n an integer in the set $I^{n+1} = \{1,\dots,n+1\}$. A simplex of the subdivision is said to be completely labelled if its n+1 vertices jointly bear all labels in I^{n+1}. The labelling is constructed in such a way that a completely labelled simplex yields an approximate solution. To ensure both the existence of such a completely labelled simplex and the finite convergence of the algorithm to such a simplex, an additional boundary condition, called properness, is imposed on the labelling. A labelling is said to be (Scarf-)proper if each point $x = (x_1,\dots x_{n+1})$ on the boundary of S^n carries a label i for which $x_i = 0$. An example of a labelling ℓ used to solve the IP is

$$(1.1) \qquad \ell(x) = \min\{i | x \in C_i,\ i = 1,\dots, n+1\}.$$

To ensure that they carry proper and unique labels, points in the boundary can be labelled according to

(1.2) $\quad \ell(x) = \min\{i \mid x_i = 0 \text{ and } x_{i+1\,(\mathrm{mod}\ n+1)} > 0\}.$

Notice that the labels on the boundary are artificial in that they bear no relation to the sets C_i. Simplicial algorithms use the information provided by the labelling to generate a path of adjacent simplices terminating at a completely labelled simplex.

The approximation generally improves by decreasing the grid size (or mesh) of the simplicial subdivision. If an approximation is found to be insufficient the subdivision is refined to get a better approximate solution. If after a refinement the algorithm can be restarted at or close to the previously found approximate solution, the algorithm is said to be a restart method. We distinguish two types of restart methods.

The restart method due to Kuhn and MacKinnon introduces an additional dimension by embedding S^n into the set $S^n\times[0,1]$. The latter set is subdivided into (n+1)-simplices with vertices either on the "real" level $S^n\times\{0\}$ or on the "artificial" level $S^n\times\{1\}$. Verices on the real level are labelled according to (1.1) and (1.2) whereas vertices on the level $S^n\times\{1\}$are artificially labelled so as to enable the restart and to ensure the finite convergence of the algorithm. The path of simplices generated by the algorithm and leading to a completely labelled simplex on the real level consists of adjacent (n+1)-simplices in the subdivision of $S^n\times[0,1]$ whose common facets bear I^{n+1} as label set.

Van der Laan and Talman have introduced a second type of restart method. It avoids the introduction of an additional dimension and the embedding of S^n into $S^n\times[0,1]$. Instead it generates a path of adjacent simplices of varying dimension. This path starts at an arbitrary grid point representing a 0-simplex and terminates with a completely labelled n-simplex. The attractiveness of this type of restart method lies in the fact that movements with simplices in S^n of varying dimension is typically faster then movements with (n+1)-simplices in $S^n\times[0,1]$.

Lüthi [6] avoids the artificial labelling on the boundary of S^n and uses the labelling rule (1.1) also on the boundary of S^n. This labelling is not necessarily proper so that the existence of a completely labelled n-simplex is no longer guaranteed. However, Lüthi observed that when all vertices of a simplicial subdivision of S^n are labelled according to (1.1), lower dimensional simplices on the boundary of S^n yield an approximate solution if they satisfy the so-called completeness condition. A t-simplex σ $(0 \leq t \leq n)$ is called complete if for each index $i \in I^{n+1}$ either one of the vertices of σ carries label i or the t-simplex lies on the boundary $x_i = 0$. For the NLCP or IP, solutions often lie on the boundary of S^n. If this is the case, a complete simplex on the boundary of S^n generally yields a more accurate approximate solution than the completely labelled n-simplex generated by an algorithm using a proper labelling. To find a complete simplex Lüthi gave a first type restart algorithm for the case of a general labelling on S^n. His method allows for lower dimensional movements on the boundary of $S^n \times [0,1]$. As mentioned before, such a lower dimensional movement is typically faster than a movement with full-dimensional simplices.

In this paper we are concerned with the existence of solutions to problems on the product space $S = S^{n_1} \times \ldots \times S^{n_N}$ of N unit simplices S^{n_i}. For example, given a continuous function $z\colon S \to \Pi_{j=1}^{N} R^{n_j+1}$, $x \to z(x) = [z_1(x), z_2(x), \ldots, z_N(x)]$ verifying $x_j^T z_j(x) = 0$ for $j = 1,\ldots,N$ and for all $x = [x_1, x_2, \ldots, x_N] \in S$, the NLCP on S consists in finding a solution x^* in S, i.e., a point x^* such that $z(x^*) \leq 0$. This problem arises e.g., when computing noncooperative equilibria in game theory. Unless the equilibrium strategies are completely mixed an equilibrium point lies on the boundary of the strategy space. Defining the closed sets $C_{jk} = \{x \in S \mid z_{jk}(x) = \max_{(i,h)} z_{ih}(x)\}$, $k = 1,\ldots,n_j$, $j=1,\ldots,N$, where $z_{jk}(x)$ is the k-th component of $z_j(x)$, x^* is a solution to the NLCP if and only if for at least one j, $x^* \in C_{jk}$ or $x^*_{jk} = 0$ for all $k = 1,\ldots,n_j$, i.e. if x^* is a solution to the IP on S.

In van der Laan and Talman [5] a second type restart algorithm was developed to find an approximate solution of the NLCP on S by using a proper labelling which assigns to each point of S a pair of integers in $I = \{(j,k) \mid k = 1,\ldots,n_j+1,\ j = 1,\ldots,N\}$. In a simplicial subdivision of S, the algorithm searches for a simplex whose vertices carry the labels $(j,1),\ldots,(j,n_j)$ for at least one j.

In this paper we extend this algorithm on S for the case of a general labelling on S. To do so, the extension of the property of completeness is required, defined earlier for simplices in S^n. The algorithm proves constructively the existence of a complete simplex in S. When the labelling is induced by the sets C_{jh}, then the existence of a solution to the IP follows from taking a sequence of simplicial subdivisions with mesh tending to zero. A corollary of the simplicial version of this result is the generalized Scarf lemma on S^n due to Freund [2]. When applied to the special case $S = S^n$, the algorithm combines the advantages of the original van der Laan and Talman algorithm for proper labellings on S^n with the advantage of Lüthi's method. Lower dimensional movements can now occur both on the boundary and in the interior of S^n. Both factors favorably influence the efficiency of the algorithm on S^n. A first type restart algorithm on S with labels out of the set I, however, does not exist.

At the end of the paper, we discuss the introduction of vector labellings in the algorithm. As is well known, vector labels yield more accurate approximate solutions than scalar labels. We conclude with results showing that our algorithm for general labellings performs very well. These numerical results seem to substantiate the case for the elimination of artificial labels on the boundaries of S^n or of S.

The paper is organized as follows. The van der Laan and Talman algorithm on S^n for a proper labelling is reviewed in section 2. In section 3 combinatorial results for a general labelling on S are given. These results are proved constructively in section 4, by giving an extension of van der Laan and Talman's algorithm on S for proper labellings. The end of section 4 discusses the implementation of vector labelling and some examples and computational results are presented in section 5.

2. The van der Laan and Talman algorithm for proper labellings on S^n.

In this section we review the algorithm of van der Laan and Talman for finding a completely labelled simplex in a simplicial subdivision of S^n which is properly labelled. The algorithm first subdivides

S^n into n-simplices according to the Q triangulation with grid size d^{-1} where d is a positive integer. The grid points of this triangulation form the set $\{y \in S^n | y = m/d, m \text{ integer}\}$. The matrix Q is given by

$$Q = [q(1), \ldots, q(n+1)] = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & & & 0 \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \ddots & \cdot \\ 0 & 0 & & \cdot & -1 \\ -1 & 0 & \cdots & & 1 \end{bmatrix}.$$

An n-simplex of this triangulation (or simplicial subdivision) is the convex hull of n+1 grid points in S^n, $y^1, \ldots, y^{n+1}$, verifying $y^{i+1} = y^i + q(\pi_i)/d$, $i = 1, \ldots, n$, for a permutation $\pi = (\pi_1, \ldots, \pi_{n+1})$ of I^{n+1}. Notice that since $y^1 = y^{n+1} + q(\pi_{n+1})/d$ an n-simplex has n+1 different representations in terms of leading vertex, y^1, and a permutation of I^{n+1}, π.

A t-simplex, $0 \le t \le n$, in a simplicial subdivision is the convex hull of t+1 vertices of an n-simplex in the subdivision. We use the notation $\tau(y^1, \ldots, y^{t+1})$ for a t-simplex τ having vertices $y^1, \ldots, y^{t+1}$. A (t-1)-simplex σ, obtained by deleting one vertex of a t-simplex, τ, is called a facet of τ. In particular, the facet $\sigma(y^1, \ldots, y^{i-1}, y^{i+1}, \ldots, y^{t+1})$ of the simplex $\tau(y^1, \ldots, y^{t+1})$ is called the facet of τ opposite vertex y^i.

To compute a completely labelled simplex the van der Laan and Talman algorithm partitions S^n into relatively open regions $\mathring{A}(T)$ defined by

$$\mathring{A}(T) = \{y \in S^n | y = v + \sum_{j \in T} \alpha_j q(j),\ \alpha_j > 0\}$$

where v is an arbitrary grid point in which the algorithm starts and where T is a subset of I^{n+1} of cardinality at most n ($|T| \le n$). A(T) denotes the closure of $\mathring{A}(T)$. The regions $\mathring{A}(T)$ are illustrated in Figure 1. Observe that $A(\phi) = \mathring{A}(\phi) = \{v\}$ and that A(T) may be empty if the initial point v lies on the boundary of S^n.

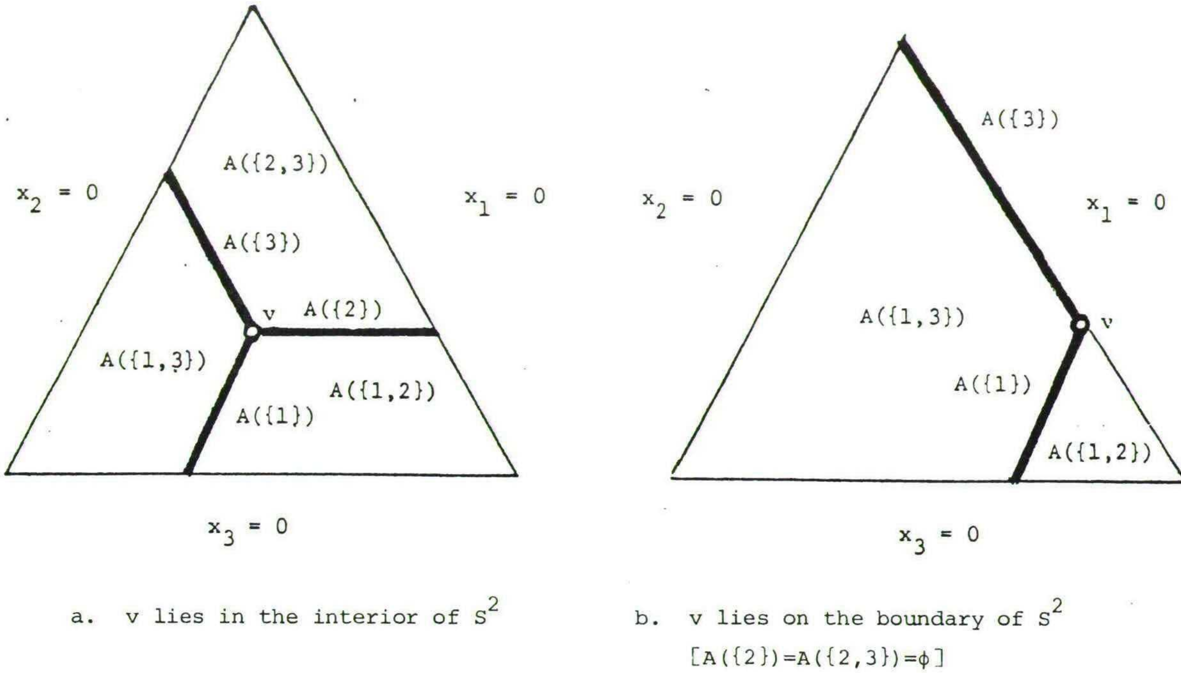

Figure 1.

A nonemty region A(T) is a t-dimensional convex compact subset of S^n with $t = |T|$. The Q triangulation subdivides each nonempty A(T) into t-simplices. Each such t-simplex is uniquely characterized by a nonnegative integer vector $a = (a_1, \ldots, a_{n+1})$ verifying $a_j = 0$ for $j \notin T$ and by a permutation π of T. Such a simplex is denoted $\sigma(a,\pi)$. The t+1 vertices of the t-simplex $\sigma(a,\pi)$ in A(T) are

$$(2.1) \qquad y^i = v + \sum_{j \in T} a_j q(j)/d \qquad \text{for } i = 1,$$

$$= y^{i-1} + q(\pi_{i-1})/d \qquad = 2, \ldots, t+1.$$

Each n-simplex σ in S^n lies in a unique region A(T). Hence even if σ has n+1 different representations in terms of a leading vertex and a permutation of I^{n+1}, its representation in terms of a permutation π of a set T ($|T|$) = n) and a nonnegative integer vector a verifying $a_j = 0$ for $j \notin T$ is unique.

An important property of our simplicial subdivision of each region A(T) is that any facet of at-simplex in A(T) is the facet of at most one other t-simplex in A(T) ($t = |T|$). Two t-simplices in A(T) sharing a common facet are called adjacent. Let us identify the t-simplex $\bar{\sigma}$ in A(T) sharing the facet τ opposite vertex y^i in the t-simplex σ ($1 \le i \le t+1$). Denote by $\bar{y}$ the vertex opposite facet τ in $\bar{\sigma}$. It should be clear from (2.1) that the only possibility for $\bar{y}$ is given by

$$
\begin{aligned}
(2.2) \qquad \bar{y} &= y^{t+1} + q(\pi_1)/d && \text{if } i = 1, \\
&= y^{i-1} + q(\pi_i)/d && 2 \le i \le t, \\
&= y^1 - q(\pi_t)/d && i = t+1.
\end{aligned}
$$

The expression of the characteristics $\bar{a}$ and $\bar{\pi}$ of the simplex $\bar{\sigma} = \bar{\sigma}(\bar{a},\bar{\pi})$ in terms of the characteristics a and π of the adjacent simplex $\sigma = \sigma(a,\pi)$ is easily decuced from (2.2) and appears in Table 1.

i	$\bar{a} = (\bar{a}_1,\ldots,\bar{a}_{n+1})$	$\bar{\pi} = (\bar{\pi}_1,\ldots,\bar{\pi}_t)$
1	$\bar{a}_h = a_h+1$ for $h = \pi_1$ $= a_h$ otherwise	$(\pi_2,\ldots,\pi_t,\pi_1)$
2,...,t	$\bar{a}_h = a_h$ for $h = 1,\ldots,n+1$	$(\pi_1,\ldots,\pi_{i-2},\pi_i,\pi_{i-1},\ldots,\pi_t)$
t+1	$a_h = a_h-1$ for $h = \pi_t$ $= a_h$ otherwise	$(\pi_t,\pi_1,\ldots,\pi_{t-1})$

Table 1.

Observe that $\bar{\sigma}$ lies in A(T) if and only if $\bar{y} \geq 0$ and $\bar{a} \geq 0$. If either $\bar{y} \not\geq 0$ or $\bar{a} \not\geq 0$ then facet τ lies on the boundary of A(T) and there is no t-simplex in A(T) adjacent with σ through facet τ.

Now consider a proper labelling of S^n verifying (1.2). Let t again denote the cardinality of T. A (t-1)-simplex is called T-complete if its t vertices jointly bear all labels in T. Any t-simplex σ in A(T) has at most two T-complete facets. Any facet of σ lies in at most one other t-simplex in A(T), which if it exists, can be found by applying the rules of Table 1. If the facet of σ lies on the boundary of A(T) then the facet belongs to a unique t-simplex in A(T) and Table 1 does not determine an adjacent simplex. The t-simplex in A(T) having T-complete facets therefore form chains of adjacent t-simplices with common T-complete facets. Every chain is either a loop or has two terminal simplices. A terminal simplex $\sigma = \sigma(a,\pi)$ (i) is $(T\cup\{k\})$-complete for some $k \notin T$ or (ii) has a T-complete facet in the boundary of A(T).

We consider these cases in turn.

(i) If $t = n$ σ is completely labelled. If $t < n$ then σ is a facet of a unique (t+1)-simplex $\bar{\sigma} = \bar{\sigma}(a,\bar{\pi})$ in $A(T\cup\{k\})$ with $\bar{\pi} = (\pi_1,\ldots,\pi_t,k)$. $\bar{\sigma}$ is a terminal simplex in a chain of (t+1)-simplices in $A(T\cup\{k\})$ having common $(T\cup\{k\})$-complete facets.

(ii) It can be shown that properness condition (1.2) implies that if τ is a T-complete facet on the boundary of A(T) then τ lies in $A(T\setminus\{k\})$ with $k = \pi_t$. If $t > 1$, then τ is a terminal (t-1)-simplex in a chain of (t-1)-simplices in $A(T\setminus\{k\})$ having common $(T\setminus\{k\})$-complete facets. If $t = 1$, then $T\setminus\{k\} = \phi$ and $\tau = \{v\}$. Notice that $T = \{\ell(v)\}$ for this case to occur.

Except for the terminal simplex {v} each terminal simplex either is completely labelled or uniquely determines a terminal simplex of a new chain. Chains can thus be linked yielding loops or paths with two terminal simplices.

Except for the terminal simplex {v} each terminal simplex of such a path of linked chains is completely labelled. There thus is a unique path which has the intial point v as terminal simplex. The van der Laan and Talman algorithm follows this path to its other terminal simplex which will be completely labelled. The algorithm is illustrated in Figure 2.

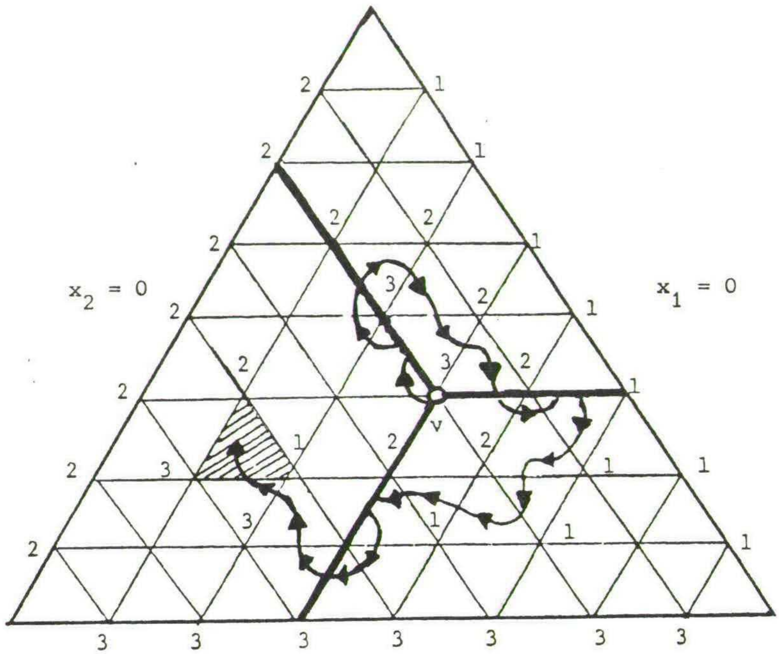

Figure 2. $x_3 = 0$

3. Complete simplices on S.

The algorithm given in the previous section searches for a completely labelled simplex in a simplicial subdivision of S^n which is properly labelled. The lemma of Scarf says that such a completely labelled simplex exists. This lemma was generalized by Freund [2], who states that in a generally labelled simplicial subdivision of S^n there is a complete simplex, that is a t-simplex σ such that for all $i \in I^{n+1}$ either σ lies in the facet $S^n_i = \{x \in S^n | x_i = 0\}$ of S^n or one of its vertices has label i.

In this section we deal with general labellings on $S = \Pi^N_{j=1} S^{n_j}$. Firstly, we give a generalization of the above mentioned Freund's lemma. This lemma will be proved constructively in the next section. Secondly we prove the existence of a solution to the IP on S by taking a sequence of simplicial subdivisions with mesh tending the zero. Finally we consider the accuracy of an approximate solution to the IP as a solution to the NLCP.

To recall some notation, a point $x \in S$ is denoted by $x = (x_1,\dots,x_N)$, where $x_j \in S^{n_j}$, $j = 1,\dots,N$. The k-th element of x_j is denoted by x_{jk}, $k = 1,\dots,n_j+1$. Further S_{jh} will be the (j,h)-th boundary of S, that is $S_{jh} = \{x \in S \mid x_{jh} = 0\}$. Finally, I is the set of pairs of integers $\{(j,h) \mid h = 1,\dots,n_j+1,\ j = 1,\dots,N\}$.

A general labelling on S is a function $\ell: S \to I$. A labelling function ℓ on S is proper if ℓ verifies

$$(3.1) \qquad \ell(x) = \text{lexicomin}\,\{(j,h) \mid x_{jh} = 0 \text{ and } x_{jh+1 (\mathrm{mod}\, n_j+1)} > 0\}$$

for all $x \in \text{bd}\, S$. Now, let G be a simplicial subdivision of S and let $\tau(y^1,\dots,y^{t+1})$ be a t-dimensional simplex of G.
Then $L(\tau) = \{\ell(y^1),\dots,\ell(y^{t+1})\}$ will be the labelset of τ and $I(\tau) = \{(j,h) \in I \mid x_{jh} = 0 \text{ for all } x \in \tau\}$ is called the index set of τ, i.e. $(j,h) \in I(\tau)$ implies that $\tau \subset S_{jh}$.

<u>Definition 3.1.</u> A t-simplex τ of a subdivision G of S is complete if there is a $j \in I^N$ such that $\{(j,1),\dots,(j,n_j+1)\} \subset L(\tau) \cup I(\tau)$.

Definition 3.1 states that τ is a complete simplex if there is a $j \in I^N$ with for all $h \in I^{n_j+1}$, at least one of the vertices of τ carries label (j,h) or $\tau \subset S_{jh}$.
If τ is complete for some $j \in I^N$, we say that τ is j-complete. Observe that when $N = 1$ a complete simplex in the sense of definition 3.1 is a complete simplex of a triangulation of S^n. Moreover, when ℓ is proper we must have that for each $h \in I^{n_j+1}$, (j,h) is a label of at least one of the vertices of τ. In case that ℓ is proper in the sense of Sperner, i.e. $\ell(x) \in \{(j,h) \mid x \notin S_{jh}\}$ when $x \in \text{bd}\, S$, van der Laan and Talman [5] have shown that there is a $j \in I^N$ for which there is a complete simplex. In the next section we adapt their algorithm to prove lemma 3.2.

<u>Lemma 3.2.</u> (Generalized Simplicial Scarf lemma on S).
Given a general labelling $\ell: S \to I$ and a triangulation G of S, there

is at least one complete simplex.

The lemma is illustrated in figure 3, in which the 2-simplices σ_1 and σ_2, 1-simplices τ_1, τ_2, τ_3 and τ_4 and the 0-simplices $\{a\}$ and $\{b\}$ are complete.

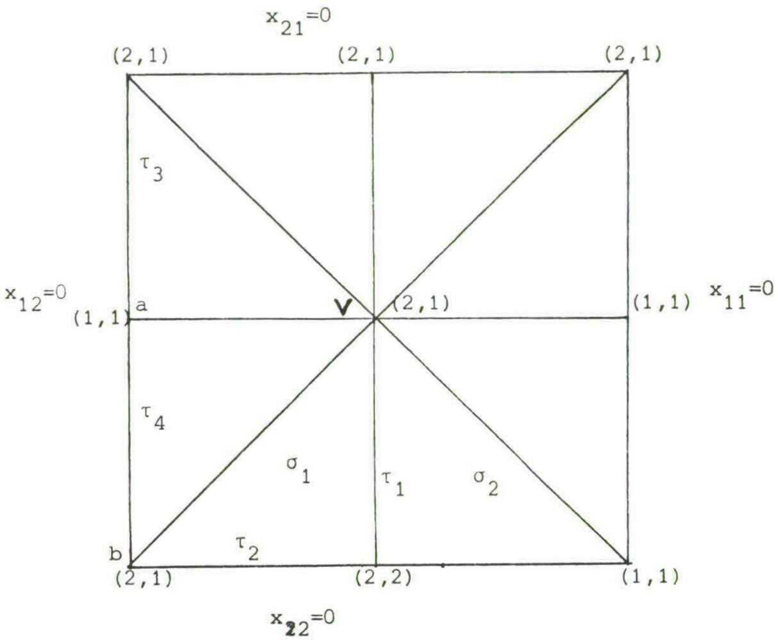

Figure 3. Generalized Simplicial Scarf lemma on S for N=2, $n_1=n_2=2$.

Some features should be observed. Firstly, σ_2 is 2-complete because τ_1 is 2-complete. The label of the vertex of σ_2 opposite τ_1 does not matter. The same holds for σ_1. Also τ_3 and τ_4 are complete because $\{a\}$ is 1-complete. Observe however, that τ_4 is not 2-complete, although $\{b\}$ is 2-complete. The simplex τ_2 is 2-complete because $L(\tau_2) = \{(2,1), (2,2)\}$. Moreover $I(\tau_2) = \{(2,2)\}$. Therefore τ_2 remains 2-complete, even when the label of the vertex of τ_2 opposite $\{b\}$ should be changed. So, the feature of a complete simplex may differ. In particular, it is possible that a j-complete simplex contains lower-dimensional h-complete faces with h not necessarily equal to j.

Lemma 3.2. enables us to prove the next theorem, which states the existence of a solution to the Intersection Problem on S.

Theorem 3.3. (Generalized Intersection Theorem).
Let $C = \{C_{jh},\ h = 1,\dots,n_j+1,\ j = 1,\dots,N\}$ be a collection of (maybe empty) closed subsets of S which cover S, i.e. $S = \cup_{j,h} C_{jh}$. Then there is an x in S and an index $j \in I^N$ such that for all $h \in I$

$$x^* \in C_{jh} \text{ or } x^*_{jh} = 0.$$

Proof. The proof follows from lemma 3.2. by taking a limit argument. Firstly, let $\{G_k,\ k = 0,1,2,\dots\}$ be a sequence of triangulations with mesh G_k going to zero, when k goes to infinity. A triangulation which can be refined arbitrarily will be given in the next section. Now, any point $x \in S$ is labelled by an element out of the set $\{(j,h) \in I \mid x \in C_{jh}\}$. By lemma 2.4, each G_k has at least one complete simplex, say σ_k. Since k goes to infinity, there is a $j \in I^N$ for which there is an infinite subsequence of j-complete simplices. Since S is compact, this subsequence has an infinite subsequence converging to a point $x^* \in S$.
Now suppose that for some (j,h), $x^*_{jh} > 0$. Then, for each j-complete simplex σ_k of the subsequence converging to x^* with k sufficiently large, $x_{jh} > 0$ for any $x \in \sigma_k$. Hence σ_k must have a vertex w with $\ell(w) = (j,h)$, implying that $w \in C_{jh}$. Since the sets C_{jh} are closed this implies that $x^* \in C_{jh}$. Hence , for all $h \in I^{n_j+1}$, $x^*_{jh} \in C_{jh}$ or $x^*_{jh} = 0$.

□

Now, let z be a continuous function from $S \to \Pi_{j=1}^{N} R^{n_j+1}$ verifying $x_j^T z_j(x) = 0$ for all $j=1,\dots,N$. Then theorem 3.3 implies that there is a point $x^* \in S$ such that $z(x^*) \le 0$. Let C_{jh} be defined by

$$C_{jh} = \{x \in S \mid z_{jh}(x) = \max_{i,k} z_{ik}(x)\}.$$

Then theorem 3.3. guarantees that there is an x^* and an index j such that for all $h = 1,\ldots,n_j+1$, $x^*_{jh} \in C_{jh}$ or $x^*_{jh} = 0$. By definition of the sets C_{jh} this implies

$$z_{jh}(x^*) \geq z_{ik}(x^*) \qquad \text{for all } (i,k) \in I$$

and hence $z_{jh}(x^*) = 0$ for all $h \in I^{n_j+1}$ with $x^*_{jh} > 0$, since $x^{*T}_j z_j(x^*) = 0$. Clearly this implies $z_{ik}(x^*) \leq 0$ for all $(i,k) \in I$. Since $x^T_i z_i(x) = 0$ for all $i \in I^N$ and all $x \in S$, we obtain $z_{ik}(x^*) = 0$ for all (i,k) with $x^*_{ik} > 0$.

The next lemma gives bounds for the accuracy of an approximate solution to the IP as a solution to the NLCP find x^* such that $z(x^*) \leq 0$.

Lemma 3.4. For some given positive ε let mesh G be so small that $|z_{jh}(x)-z_{jh}(y)| < \varepsilon$ for all $(j,h) \in I$, when x and y lie in the same simplex of G. Then, for any $\bar{x}$ in a complete simplex $\sigma(y^1,\ldots,y^{t+1})$ of G holds

$$z_{jh}(\bar{x}) < 2\,\varepsilon \text{ for all } (j,h) \in I. \tag{3.1}$$

Proof. Since $\sigma(y^1,\ldots,y^{t+1})$ is complete there is an index j such that for all $h \in I^{n_j+1}$, $x_{jh} = 0$ for all $x \in \sigma$ or σ has a vertex, say $y^{j,h}$, with $\ell(y^{j,h}) = (j,h)$. Now, suppose that there is an $\bar{x}$ in σ with $z_{ik}(\bar{x}) > 2\varepsilon$ for some $(i,k) \in I$. Then, for all $h \in I^{n_j+1}$ with $x_{jh} > 0$, $x \in \text{int}\,\sigma$, we must have $z_{ik}(y^{j,h}) \geq \varepsilon$ since $\bar{x}$ and $y^{j,h}$ both lie in σ. Since $\ell(y^{j,h}) = (j,h)$, we must have $z_{jh}(y^{j,h}) \geq z_{ik}(y^{j,h})$. Therefore $z_{jh}(y^{j,h}) > \varepsilon$ for all $h \in I^{n_j+1}$ with $x_{jh} > 0$, $x \in \text{int}\,\sigma$. Hence $z_{jh}(x) > 0$ for all these (j,h), so that $x^T_j z_j(x) > 0$, contradicting $x^T_j z_j = 0$. This proves the lemma.

□

By taking the mesh of the triangulation small enough, (3,1) holds for arbitrarily small ε. Therefore, a complete simplex yields an approximate solution to the NLCP with respect to z.

4. An algorithm for general labellings on S.

The algorithm in section 2 is extended in van der Laan and Talman [5] to find a complete simplex in a triangulation of S for proper labellings in the sense of Sperner. In this section we extend the algorithm for general labellings on S. It should be observed, however, that a complete simplex of S for a general labelling can be found by the original algorithm when it is applied on $\bar{S}$, where $\bar{S} = \Pi_{j=1}^{N} \bar{S}^{n_j}$ and $\bar{S}^{n_j} = \{x \in R^{n_j+1} \mid \Sigma_{h=1}^{n_j+1} x_h = 1,\ x_h \geq -d_j\}$ with d_j the grid size of the triangulation of S^{n_j}. Extending the labelling on S to a (Scarf) proper labelling on $\bar{S}$, the original algorithm for a Scarf proper labelling (see section 2 for the case N = 1) finds a complete simplex in $\bar{S}$. Since the labelling is proper, for such a complete simplex σ there is a j such that for all $h \in I^{n_j+1}$, σ has a vertex which carries label (j,h). However, the intersection of σ with S is a complete simplex τ in S in the sense of definition 3.1., that is there is a j with for all $h \in I^{n_j+1}$, τ has a vertex with label (j,h) or $x_{jh} = 0$ for all $x \in \tau$. Clearly, for all (j,h) being a label of a vertex of σ outside S, $x_{jh} = 0$ for all $x \in \sigma \cap S = \tau$. So, for general labellings a complete simplex can be found by applying the orginal algorithm on $\bar{S}$. In this section we adapt the original algorithm on S to find a complete simplex without using the extension from S to $\bar{S}$. Since the NLCP often has a solution on bd S, the new algorithm may avoid many replacement steps outside S.

To motivate our adaption of the original algorithm, we first consider the case N = 1, n = n_1 = 2. Let us consider the labelling $\ell(y) = j$ for all $y \in S^2$ ($1 \leq j \leq 3$). The unique complete simplex for labelling is the 0-simplex {e(j)} where e(j) denotes the j-th unit vector. It seems therefore appropriate to extend A({j}) to a 1-dimensional piecewise linear simplicial path, $\tilde{A}(\{j\})$, going from the starting point v to vertex e(j). The extended regions $\tilde{A}(\{j\})$ are illustrated in Figure 4. The regions $\tilde{A}(\{i,j\})$, $i \neq j$, coincide with the original regions A({i,j}). Now consider an arbitrary labelling. The extended algorithm proceeds in the regions $\tilde{A}(\{i,j\})$ as the original algorithm operates on the regions

$A(\{i,j\})$. A chain of simplices in $\widetilde{A}(\{i,j\})$ having common facets with labels i and j leads either to a completely labelled simplex in S^2 or to a facet with labels i an j on the boundary of $\widetilde{A}(\{i,j\})$. In the latter case the algorithm continues in a lower dimensional region, $\widetilde{A}(\{i\})$ or $\widetilde{A}(\{j\})$, or a complete 1-simplex is found lying in the facet $\{x \in S^n | x_h = 0,\ h \neq i,j\}$, etc.
Figure 4 illustrates the algorithm for this lower dimensional case.

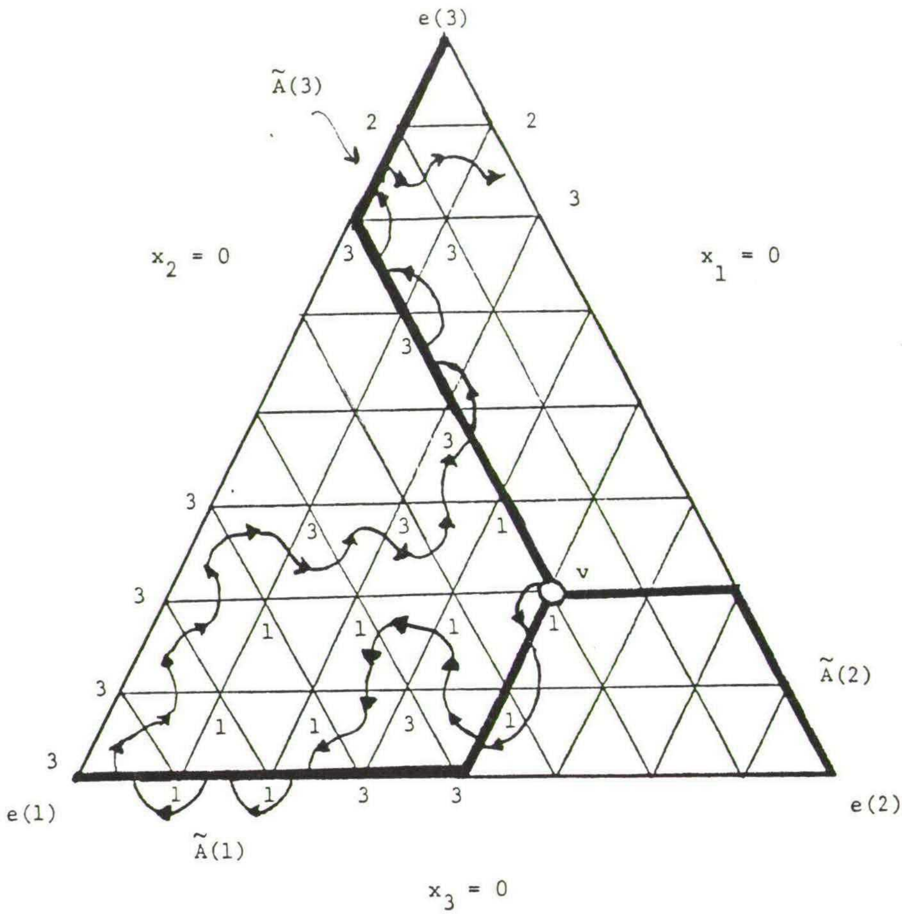

Figure 4.

For general dimension of S^n the region $\tilde{A}(T)$, $|T| \le n$, is defined as follows:

$$(4.1) \qquad \tilde{A}(T) = \bigcup_{\substack{|U \cup T| \le n \\ U \cap T = \emptyset}} B(T,U)$$

with $B(T,U) = A(T \cup U) \cap \{x \in S^n | x_j = 0 \text{ for } j \in U\}$.

Since a nonempty $A(T \cup U)$ is $|T \cup U|$-dimensional and convex and since $B(T,U)$ is obtained by intersecting $A(T \cup U)$ with the boundary $x_j = 0$ for $j \in U$, each nonempty $B(T,U)$ is $|T|$-dimensional and convex. Each nonempty $\tilde{A}(T)$ thus consists of a union of $|T|$-dimensional convex pieces. The regions $\tilde{A}(T)$ are illustrated in Figure 5 for the case $n = 3$.

The Q triangulation subdivides each nonempty region $B(T,U)$ into t-dimensional simplices ($t = |T|$). To describe these simplices precisely, we introduce the notation $p(h)$ for the fist index of the sequence $(h-1, h-2, \ldots, 1, n+1, \ldots, h)$ not in U. Recall that $B(T,U)$ is defined only for sets T and U verifying $T \cap U = \emptyset$ and $|T \cup U| \le n$. A simplex in $B(T,U)$ can be characterized by a quadruple T, U, a, π where $a \in R^{n+1}$ is a nonnegative integer vector verifying $a_j = 0$ for $j \notin T \cup U$ and where $\pi = (\pi_1, \ldots, \pi_t)$ is a permutation of T. Such a simplex will then be denoted $\sigma(T,U,a,\pi)$. Its vertices $y^1, \ldots, y^{t+1}$ are grid points in S^n verifying:

$$(4.2) \qquad \text{i)} \quad y^1 = v + \sum_{j \in T \cup U} a_j q(j)/d,$$

$$\text{ii)} \quad y^{i+1} = y^i + r(\pi_i)/d \text{ for } i = 1, \ldots, t$$

$$\text{where } r(j) = \sum_{h = p(j)+1 \,(\text{mod } n+1)}^{j} q(h)$$

$$\text{iii)} \quad U \subset \{i \in I^{n+1} | y^1_i = 0\} \subset T \cup U.$$

Conversely, any t+1 grid points $y^1, \ldots, y^{t+1}$ in S^n verifying statements (4.2) for subsets T and U of I^{n+1} such that $|T \cup U| \le n$ and $T \cap U = \emptyset$, for a nonnegative integer vector $a \in R^{n+1}$, and for a permutation π of T are the vertices of a simplex of the Q triangulation lying in $B(T,U)$. Conditions (i) and (ii) ensure that $y^1, \ldots, y^{t+1}$ are the vertices

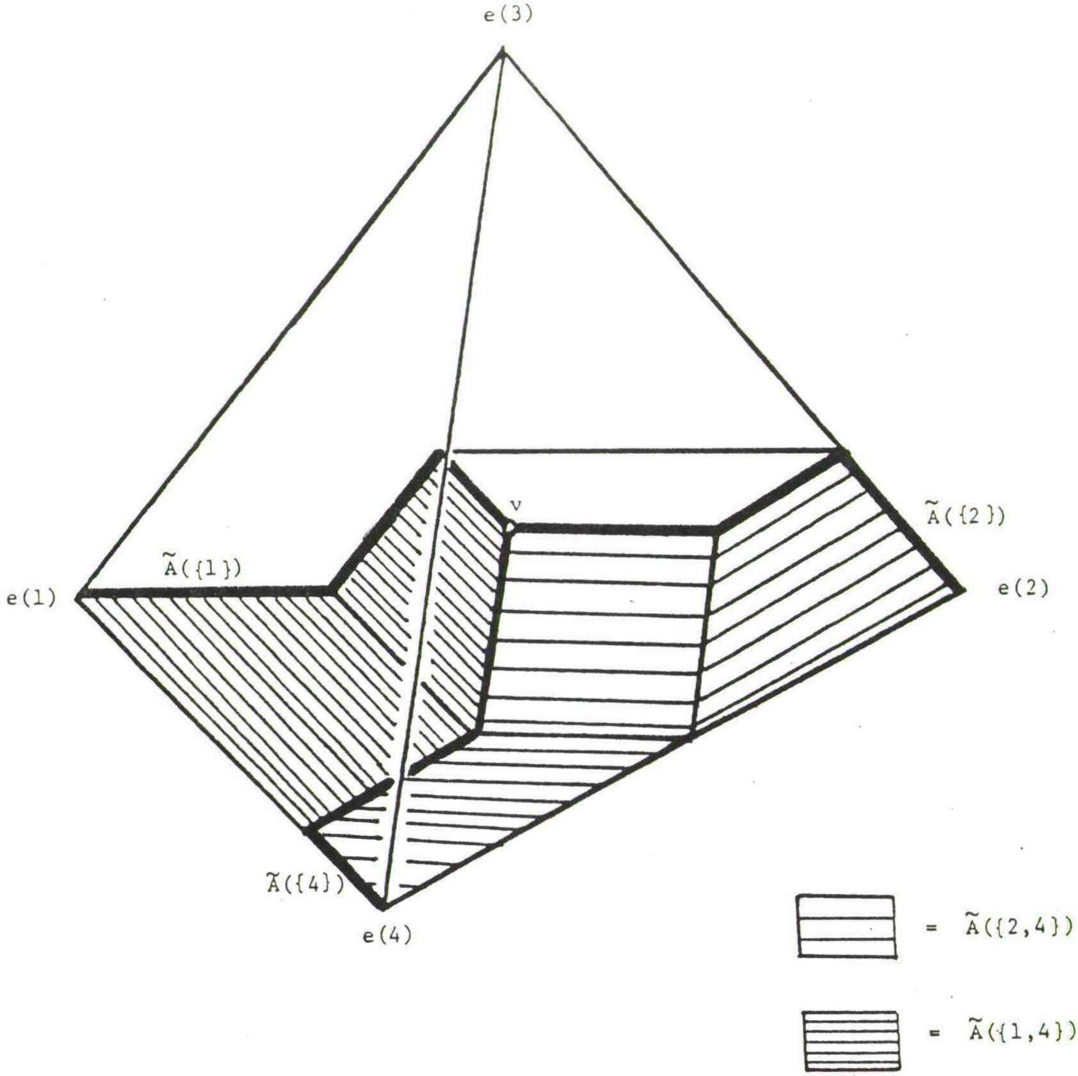

Figure 5.

of a simplex of the Q triangulation lying in $A(T \cup U)$. To show that the simplex also lies in $B(T,U)$ observe that the vector $r(j)$, defined in (ii), has exactly two non-zero coordinates, namely $r_j(j) = 1$ and $r_{p(j)}(j) = -1$. The indices of all coordinates between $p(j)$ and j are in U. Condition (iii) then identifies U as the indices of the facets of S^n, $\{x \in S^m | x_i = 0\}$ for $i \in U$, containing $\sigma(T,U,a, \pi)$.
This simplex therefore lies in $B(T,U)$.

The new algorithm operates in the regions $\tilde{A}(T)$ as the original algorithm operates in the regions $A(T)$. Therefore consider the chains of t-simplices in $B(T,U)$ having common T-complete facets ($t = |T|$). Every chain which is not a loop has two terminal simplices. Such a terminal simplex either is complete (and yields an approximate solution) or leads to a unique chain in a different region $B(T',U') \subset \tilde{A}(T')$ with $|T'| = |T| + 1$, $|T|$ or $|T|-1$. The chains of simplices can thus be linked into variable dimension simplicial loops and paths with two terminal simplices.

The 0-simplex $\{v\}$ is the only terminal simplex which may not be complete. The algorithm starts at the 0-simplex $\{v\}$ and follows the simplicial path of linked chains having the 0-simplex $\{v\}$ as a terminal simplex to its other terminal simplex. The latter simplex is complete.

Wo now turn to the general case $S = \prod_{j=1}^{N} S^{n_j}$. So, we have to give a triangulation of S. As in [5] (see also [9]) we use a triangulation which depends on the starting point v, which must be a grid point. First define regions $A(T)$. Then each region $A(T)$, $T \subset I$, is triangulated such that the union of all simplices yields a triangulation of S. So, let $d = (d_1, \ldots, d_N)$ be a vector of N positive numbers such that d_j^{-1} is an integer, $j = 1, \ldots, N$. We call d the grid size vector of the triangulation. Then the set $\overset{o}{G}$ of grid points of the triangulation is the set of vectors x in S such that x_{jh} is a nonnegative multiple of d_j, $h = 1, \ldots, n_j+1$, $j = 1, \ldots, N$. One of these points, to be denoted v, will be the starting point of the algorithm. Further, Q is redefined as the block diagonal matrix

$$Q = \begin{bmatrix} Q_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & 0 \\ 0 & & \cdot & \cdot & 0 & Q_N \end{bmatrix}$$

with Q_j the $(n_j+1) \times (n_j+1)$ matrix as given in section 2. In the following, for $h = 1, \ldots, n_j+1$, the $(\Sigma_{i=1}^{j-1}(n_i+1)+h)$-th column of Q is denoted by $q(j,h)$, i.e., $q(j,h) = e(j,h) - e(j,h-1(\mathrm{mod}(n_j+1)))$, where $e(j,h)$ is the $(\Sigma_{i=1}^{j-1}(n_i+1)+h)$-th unit column, $h = 1, \ldots, n_j+1$, $j = 1, \ldots, N$.

A subset $T \subset I$ is called feasible if for all $j \in I^N$, at least one index $(j,h) \in I$ is not in T. Clearly T is feasible if and only if the rank of the matrix consisting of the column vectors $q(j,h)$, $(j,h) \in T$ is equal to the cardinality $|T|$ of T. Now S is partitioned in cones with apex v defined by

$$\mathring{A}(T) = \{x \in S | x = v + \textstyle\sum_{(j,h) \in T} \alpha_{jh} q(j,h),\ \alpha_{jh} > 0\}$$

for all feasible $T \subset I$. As in section 2 each non empty region $A(T)$ with $A(T)$ the closure of $\mathring{A}(T)$ is subdivided into $|T|$-dimensional simplices $\sigma(a,\pi)$ with vertices $y^1,\dots,y^{t+1}$ $(t = |T|)$ in S given by

$$(4.3) \qquad y^1 = v + \textstyle\sum_{(j,h) \in T} a_{jh}\, Dq(j,h), \qquad a_{j,h} \text{ nonneg. integer}$$

$$y^{i+1} = y^i + Dq(\pi_i), \qquad i = 1,\dots,t,$$

where $(\pi_1,\dots,\pi_t)$ is a permutation of the elements of T and D is the diagonal matrix with (j,h)-th diagonal element equal to d_j, $h = 1,\dots,n_j+1$, $j = 1,\dots,N$.
Let $G(T)$ be the collection of all the $|T|$-simplices $\sigma(a,\pi)$ in $A(T)$. Then $G = \cup_T G(T)$ yields a triangulation of S, satisfying that for $T' \subset T$, $G(T')$ is induced by $G(T)$ in the sense that each simplex of $G(T')$ is the intersection of $A(T')$ and at least one simplex in $A(T)$ (see [9, chapter 6]). As for the case $N = 1$, we extend the regions $A(T)$ to the regions $\tilde{A}(T)$ by $\tilde{A}(T) = \cup B(T,U)$ with

$$B(T,U) = A(T \cup U) \cap \{x \in S | x_{jh} = 0 \text{ for all } (j,h) \in U\},$$

where the union is taken over all U such that $T \cap U = \emptyset$ and $T \cup U$ is feasible. The set $\tilde{A}(T)$ consists of a union of $|T|$-dimensional convex pieces $B(T,U)$. Each nonempty $B(T,U)$ is triangulated by G in $|T|$-simplices. To describe the simplices of this triangulation, let $p(j,h) = (j,b(j,h))$ be the first index of the sequence $((j,h-1),(j,h-2),\dots,(j,1),(j,n_j+1),\dots,(j,h))$ not in U and let $s(j,h)$ represent the sequence $((j,b(j,h)+1),\dots,(j,n_j+1),$ $(j,1),\dots,(j,h))$ if $b(j,h) > h$ and the sequence $((j,b(j,h)+1),\dots,(j,h))$

if not, $(j,h) \in I$. Then a t-simplex in $B(T,U)$ is characterized by a quadruple (T,U,a,π) with a in $\Pi_j R^{n_j+1}$ a nonnegative integer vector verifying $a_{j,h} = 0$ when $(j,h) \notin T \cup U$ and π a permutation of the t elements of T. The vertices of $\sigma(T,U,a,\pi)$ are grid points of S satisfying

i) $y^1 = v + \Sigma_{(j,h) \in I} a_{j,h} Dq(j,h)$

ii) $y^{i+1} = y^i + Dr(\pi_i)$ for $i = 1,\dots,t$

where $r(j,h) = \Sigma_{(j,k) \in s(j,h)} q(j,k)$

iii) $U \subset \{(j,h) \in I \mid y^1_{j,h} = 0\} \subset T \cup U$,

where in ii) the sum is over all values (j,k) in the sequence $s(j,h)$. For each feasible $T \cup U$ the t-simplices in $B(T,U)$ with T-complete facets form chains of adjacent t-simplices with common T-complete facets. Each chain in a $B(T,U)$ which is not a loop has two terminal simplices. Each terminal simplex is either complete or the starting simplex $\sigma(v)$ or yields a terminal simplex in a different $B(T',U')$. More precisely, the following possibilities for a terminal simplex $\sigma(T,U,a,\pi)$ in $B(T,U)$ can occur.

i) σ is $(T \cup \{(j,h)\})$-complete for some $(j,h) \notin T$. Then
   a) $(j,h) \notin U$ implies that σ is either complete or a facet of a terminal simplex $\bar{\sigma}$ in $B(T \cup \{(j,h)\},U)$;
   b) $(j,h) \in U$ implies that σ is a facet of a terminal simplex $\bar{\sigma}$ in $B(T \cup \{(j,h)\},U \setminus \{(j,h)\})$.

ii) σ has a T-complete facet τ in the boundary of $B(T,U)$. Then with i the index of the vertex of σ opposite τ
   a) $i = 1$ implies τ is either complete or a facet of a terminal t-simplex $\bar{\sigma}$ in $B(T,U \cup \{p(\pi_1)\})$;
   b) $2 \le i \le t$ implies τ is a terminal simplex in $B(T \setminus \{p(\pi_i)\}, U \cup \{p(\pi_i)\})$;
   c) $i = t+1$ and $y^1_{\pi_t} > 0$ implies τ is either a terminal simplex in

$B(T\setminus\{\pi_t\}, U)$ or for some $(j,h) \in U$, τ is a facet of a terminal simplex in $B(T,U\setminus\{(j,h)\})$ (see step 3c below);

d) $i = t+1$ and $y^1_{\pi_t} = 0$ implies that τ is a terminal simplex in $B(T\setminus\{\pi_t\},U\cup\{\pi_t\})$.

Moreover, $\sigma(v)$ is the facet of only one terminal simplex $\bar{\sigma}$ of a chain. The representation of $\bar{\sigma}$ in terms of the new T, U, a and π can be easily obtained from the quadruple (T,U,a,π) of σ.
Therefore, the simplices in $\tilde{A}(T)$ with T-complete facets for varying T form chains of adjacent simplices of varying dimension when linked together. Each chain which is not a loop has exactly two terminal simplices. Just one terminal simplex is the simplex $\sigma(v)$ whereas all other terminal simplices are complete so that there is one chain of simplices which links $\sigma(v)$ with a complete simplex. The steps of the algorithm to follow this chain are as follows, where $P(j) = \{(j,1),\ldots,(j,n_j+1)\}$, $j \in I^N$. Recall that $p(j,h) = (j,b(j,h))$, for all $(j,h) \in I$.

Step 0. [Initialization].

Set $T = \emptyset$, $U = \{(j,h) \in I \mid v_{j,h} = 0\}$, $a = 0$, $\pi = \emptyset$ and $\bar{y} = v$.

Step 1. [Computation of label of incoming vertex].

Compute $\ell(\bar{y})$. Let $\ell(\bar{y})$ be equal to (j,h). Proceed to one of the following subcases.

(a) $\ell(\bar{y}) \notin T\cup U$. If $|(T\cup U)\cap P(j)| = n_j$, then $\sigma(T,U,a,\pi)$ is j-complete; stop.
If not, set $i = t+1$ and go to step 2.

(b) $\ell(\bar{y}) \in T$. Determine the vertex y^i of $\sigma(T,U,a,\pi)$ verifying $\ell(y^i) = \ell(\bar{y})$ and $y^i \neq \bar{y}$. Go to step 3.

(c) $\ell(\bar{y}) \in U$. Identify the first element, say (j,k) in the sequence $((j,h+1),\ldots,(j,n_j+1),(j,1),\ldots,(j,h))$ not belonging to U. If $(j,k) \notin T$, then set $i = t+1$. If $(j,k) \in T$ let i be the index such that $\pi_i = (j,k)$. Proceed to step 2.

Step 2. [Increase in number of elements of T with possible decrease in number of elements of U].

Set $\bar{T} = T \cup \{\ell(\bar{y})\}$,

$$\bar{U} = U \qquad \text{if } \ell(\bar{y}) \notin U,$$
$$= U \setminus \{\ell(\bar{y})\} \qquad \ell(\bar{y}) \in U,$$
$$\bar{\pi} = (\pi_1, \ldots, \pi_t, \ell(\bar{y})) \qquad \text{if } i = t+1,$$
$$= (\pi_1, \ldots, \pi_{i-1}, \ell(\bar{y}), \pi_i, \ldots, \pi_t) \qquad 1 \le i \le t.$$

Let $\bar{y} = y^i + Dr(\bar{\pi}_i) (= \bar{y}^{i+1})$. Set $T = \bar{T}$, $U = \bar{U}$, $\pi = \bar{\pi}$, and return to step 1.

Step 3. [Replacement of y^i in $\sigma(T,U,a,\pi)$].

As long as the conditions below are not verified, a and π are updated as in Table 2. Let $\bar{y}$ be the new vertex of $\sigma(T,U,\bar{a},\bar{\pi})$. Set $a = \bar{a}$, $\pi = \bar{\pi}$, and return to step 1.

The exceptions to the above rule are:

(a) $i = 1$ and $y^{t+1}_{p(\pi_1)} = 0$ [Increase in number of elements of U]. Let π_1 be equal to (j,h). Set $\bar{U} = U \cup \{p(\pi_1)\}$. If $|(T \cup U) \cap P(j)| = n_j$ then $\sigma(T,\bar{U},a,\pi)$ is complete; stop. If not then the updates $\bar{a}$ and $\bar{\pi}$ are computed as in Table 2, while $\bar{y} = y^{t+1}$. Set $U = \bar{U}$, $a = \bar{a}$, $\pi = \bar{\pi}$, and return to step 1.

(b) $2 \le i \le t$ and $y^{i-1}_{p(\pi_i)} = 0$. Go to step 4.

(c) $i = t+1$ and $a_{j,\ell} = 0$, where $\pi_t = (j,h)$ and $\ell = b(j,h)+1 \pmod{n_j + 1}$. Go to step 4 if $y^1_{\pi_t} = 0$. If $y^1_{\pi_t} > 0$, let (j,k) be the index of the last zero element in the sequence $(a_{j,h'} \mid (j,h') \in s(p(\pi_t)))$. If $(j,k) = \pi_t$ go to step 4. If $(j,k) \neq \pi_t$ go to step 5.

Step 4. [Decrease in the number of elements of T with possible decrease in the number of elements of U].

Set $\bar{T} = T \setminus \{\pi_{i-1}\}$

$$\bar{U} = U \cup \{\pi_{i-1}\} \qquad \text{if } y^1_{\pi_{i-1}} = 0$$
$$= U \qquad \text{if } y^1_{\pi_{i-1}} > 0,$$
$$\bar{\pi} = (\pi_1, \ldots, \pi_{i-2}, \pi_i, \ldots, \pi_t)$$

Let $\bar{y}^{\bar{i}}$ be the vertex in $\sigma(\bar{T},\bar{U},a,\bar{\pi})$ such that $\ell(\bar{y}^{\bar{i}}) = \pi_{i-1}$.
Set $T = \bar{T}$, $U = \bar{U}$, $\pi = \bar{\pi}$ and $i = \bar{i}$, and return to step 3.

Step 5. [Decrease in the number of elements of U].
Set $\bar{U} = U\backslash\{j,k\}$,

$$\bar{a}_{j,h'} = a_{j,h'}-1 \qquad \text{for } (j,h') \in \bar{s}(j,h)$$
$$= a_{j,h'} \qquad \text{otherwise}$$

$\bar{\pi} = (\pi_t,\pi_1,\ldots,\pi_{t-1})$.
Let $\bar{y} = y^1-Dr(\bar{\pi}_1)$ $(= \bar{y}^1)$. Set $U = \bar{U}$, $a = \bar{a}$, $\pi = \bar{\pi}$, and return to step 1.

i	$\bar{a}$	$\bar{\pi}$
1	$\bar{a}_{j',h} = a_{j',h}+1$ $(j',h) \in s(\pi_1)$ $= a_{j',h}$ otherwise	$(\pi_2,\ldots,\pi_t,\pi_1)$
$2,\ldots,t$	$\bar{a} = a$	$(\pi_1,\ldots,\pi_{i-2},\pi_i,\pi_{i-1},\pi_{i+1},\ldots,\pi_t)$
$t+1$	$\bar{a}_{j',h} = a_{j',h}-1$ $(j',h) \in s(\pi_t)$ $= a_{j',h}$ otherwise	

Table 2. Replacement step when the vertex y^i of $\sigma(T,U,a,\pi)$ has to be replaced (T and U do not change).

The algorithm just described shows that lemma 3.2 is true for the triangulation given above. Recall that this triangulation was obtained by first triangulating each nonempty $B(T,U)$ in $|T|$-simplices. However, any triangulation of S induces a triangulation of the $B(T,U)$'s when the starting point is chosen as one of the vertices of S. Doing so, chains of T-complete simplices can be linked together for varying T, with one chain leading from one of the vertices of S to a complete simplex. This gives a constructive proof of lemma 3.2 for any triangulation of S.

To approximate solutions of the NLCP on S the algorithm described above can also be applied for vector labelling. Then a point x in S

receives the vector label $\ell(x) = z(x)+e$ instead of an integer label corresponding to the index with the largest z-value (see section1). Here e denotes the $\Sigma_j(n_j+1)$-vector of ones.
We call a k-simplex $\sigma(y^1,\ldots,y^{k+1})$, with $k = t$ or $t-1$, T-complete if the system of linear equations

$$\sum_{i=1}^{k+1} \lambda_i \ell(y^i) + \sum_{(j,h) \notin T} \mu_{j,h} e(j,h) = e$$

has a nonnegative solution λ_i^*, $i = 1,\ldots,k+1$, $\mu_{j,h}^*$ $(j,h) \notin T$. A complete simplex, i.e. $\mu_{j,h}^* > 0$ then $(j,h) \in U$ clearly yields an approximate solution $\Sigma_i \lambda_i^* y^i / \Sigma \lambda_j^*$ to the NLCP. In general a T-complete t-simplex σ has a line-segment of solutions with two endpoints. At an endpoint either one of the λ_i's is zero so that the facet opposite the corresponding vertex is also T-complete or one of the $\mu_{j,h}$'s, $(j,h) \notin T$, is zero which implies that σ is also $T \cup \{(j,h)\}$-complete for some $(j,h) \notin T$. Therefore, the T-complete t-simplices in $\tilde{A}(T)$ for varying T again induce chains of adjacent simplices with common T-complete facets of variable dimension. Exactly one chain has $\sigma(v)$ as one of its endpoints wheras all other endpoints are complete simplices. The path which leads from $\sigma(v)$ to a complete simplex can be followed by alternating replacementsteps as described for the integer labelling algorithm above and linear programming steps in the system above to determine that either a vertex has to be replaced or the label set T should be extended with a new label. When for some $(j,h) \in T$, a T-complete facet lies in $\tilde{A}(T\setminus\{(j,k)\})$, the unit vector column $e(j,k)$ is reintroduced into the system of linear equations and the algorithm continues as for integer labelling in $A(T\setminus\{(j,k)\})$.

5. Applications and numerical results.

In this section we discuss two applications of the intersection theorems on S^n and S respectively. First, consider the quadratic programming problem with quadratic (and linear) constraints (QPQC) defined by

$$(5.1) \qquad \min\{Q_{n+1}(x) \mid Q_i(x) \le 0, \quad i = 1,\ldots,n,\ x \in P\}$$

where each $Q_i(x)$, $i = 1,\ldots,n$ is a quadratic convex function on R^m and where P is a nonempty polyhedral in R^m (see Phan-Huy-Hoa [7]). Then, let $x(u)$ be the solution of the quadratic programming problem

$$\min\{\sum_{i=1}^{n+1} u_i Q_i(x) \mid x \in P\}$$

where $u \in S^n$. Suppose that x is a continuous function on S^n. Finally, define the sets C_i, $i=1,\ldots,n+1$, by

$$C_i = \{u \in S^n \mid Q_i(x(u)) = \max_j Q_j(x)) \geq 0\} \qquad i=1,\ldots,n$$

and

$$C_{n+1} = \{u \in S^n \mid Q_i(x(u)) \leq 0,\ i=1,\ldots,n\}.$$

Then each nonempty C_1 is a closed subset of S^n and according to theorem 3.3 there is a $\hat{u}$ in S^n such that for each $i \in I^{n+1}$ either $\hat{u}_i = 0$ or $\hat{u}$ belongs to C_i.

Clearly, if $\hat{u}_{n+1}$ is positive then $x(\hat{u})$ solves problem (5.1). When there is a y in P such that $Q_i(y) < 0$, $i = 1,\ldots,n$, it can be shown that $\hat{u}_{n+1}$ must be positive. To apply the algorithm of section 4 we assign to $u \in S^n$ in case of integer labelling a label according to (1.1) and in case of vector labelling the label $\ell(u) = z(u)+e$, where

$$z(u) = (Q_1(x(u)),\ldots,Q_n(x(u)),0)^T.$$

Although $u^T z(u)$ is not always equal to zero, a complete simplex is still an approximate solution to problem (5.1). A function z' which satisfies $u^T z'(u) = 0$ for all u in S^n is the following one:

$$z'(u) = (u_{n+1}Q_1(x(u)),\ldots,u_{n+1}Q_n(x(u)),\ -\sum_{i=1}^{n} u_i Q_i(x(u))).$$

However this function seems to be less natural than the function z, although z' induces an NLCP on S^n. Both the integer and vector labelling version of the algorithm described in section 4 are applied to the three QPQC problems given in [7]. Table 3 gives the cumulative number of function evaluations and linear programming steps (in case of vector labelling) to obtain a complete simplex in a triangulation of S^n with grid size 5.10^{-4}. The initial grid size is 5.10^{-1}. At each restart the grid size is refined with a factor 10. The restarting point is the grid point closest to the barycenter in case of integer labelling and to the approximate solution $\Sigma\lambda_i^* y^i / \Sigma\lambda_i^*$ in case of vector labelling found in the previous stage. It should be observed that the results are much better than in [7].

The second application concerns the computation of equilibrium strategy vectors of a noncooperative N person game. Let n_j+1 be the number of strategies of player j, j=1,...,N, and let J denote the product of index sets I^{n_j+1}, j=1,...,N. A vector $k=(k_1,\ldots,k_n)$ in J will denote the (pure) strategy vector in which player j plays his k_j-th pure strategy, j=1,...,N. Furthermore, for $k \in J$, let $a_j(k)$ be the loss to player j if strategy k is played, j=1,...,N.
We assume that for each k in J and j in I^N, $a_j(k)$ is positive. The set S^{n_j} can be considered as the (mixed) strategy space for player j, $j \in I_N$, so that

$$S = \prod_{j=1}^{N} S^{n_j}$$

is the strategy space of the noncooperative game, i.e. if $x \in S$, then x_{jh} denotes the probability that player j uses his h-th pure strategy, $h=1,\ldots,n_j+1$, $j \in I^N$.

The expected loss $p^j(x)$ to player j if strategy x in S is played is given by

$$p^j(x) = \sum_{k \in J} a^j(k) \prod_{i=1}^{N} x_{ik_i},$$

and the marginal loss to player j if he plays his h-th pure strategy and the other players stick on strategy x is given by

$$m_h^j(x) = \sum_{\substack{k \in J \\ k_j=h}} a^j(k) \prod_{\substack{i=1 \\ i\neq j}}^{N} x_{ik_i}.$$

Observe that for each x in S

$$(5.2) \qquad p^j(x) = \sum_{h=1}^{n_j+1} x_{jh}\, m_h^j(x) \qquad j=1,\ldots,N.$$

A point $\bar{x}$ in S is an equilibrium strategy vector of the game if for each player j

$$p^j(\bar{x}) \le m_h^j(\bar{x}) \qquad h=1,\ldots,n_j+1.$$

Let for $(j,h) \in I$ the sets C_{jh} be defined by

$$C_{jh} = \{x \in S \,|\, p^j(x) - m_h^j(x) = \max_{(i,g) \in I} \{p^i(x) - m_g^i(x)\}\}.$$

Then each C_{jh} is closed and the union over all C_{jh} cover S, so that according to the generalized intersection theorem there is an x^* in S and an index $j \in I^N$ such that for all $h \in I^{n_j+1}$

$$x^* \in C_{jh} \quad \text{or} \quad x^*_{jh} = 0.$$

Because of (5.2) such a point x^* must be an equilibrium strategy vector. Both the integer and vector labelling algorithm described in section 4 has been applied to three different noncooperative N-person game. In case of integer labelling a point x receives as label the index (j,h) if (j,h) is the first (lexicographic) index for which x belongs to C_{jh}. For vector labelling a point x in S receives the label $\ell(x) = z(x) + e$, where

$$z_{jh}(x) = p^j(x) - m_h^j(x) \qquad h = 1,\ldots,n_j+1; \qquad j \in I^N.$$

In Table 4 the cumulative number of function evaluations and linear programming steps are given to obtain an approximate solution with $\max_{j,k} z_{jk}(x) < 10^{-10}$. At each restart the grid size is refined with a factor two. Game 2 has also been solved in van der Laan and Talman [7]. Observe that the results given in Table 4 are much better.

OPQC problems

			IL	VL	
problem	n	d	FE	LP	FE
1	2	5.10^{-1}	4	5	4
		5.10^{-2}	7	7	6
		5.10^{-3}	10	9	8
		5.10^{-4}	15	11	11
2	2	5.10^{-1}	5	5	5
		5.10^{-2}	7	11	11
		5.10^{-3}	11	13	13
		5.10^{-4}	18	15	16
3	4	5.10^{-1}	2	2	2
		5.10^{-2}	4	5	5
		5.10^{-3}	7	7	7
		5.10^{-4}	13	9	10

Table 3. Cumulative number of Function Evaluations (FE) and Linear Programming (LP) steps, for Integer Labelling (IL) and Vector Labelling (VL). For the data see [7].

Noncooperative games.

			IL	VL	
Game	N	n_j+1	FE	LP	FE
1	3	2	377	206	205
2	3	3	269	33	34
3	4	2	165	117	127

Table 4. Cumulative number of Function Evaluations and Linear Programming steps to obtain an approximate solution with

$\max_{j,k} z_{jk} < 10^{-10}$. N is the number of players, n_j+1 the number of strategies of each player. The data are given in the appendix.

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Appendix. Data noncooperative games.

Game 1: $N = 3$, $n_j+1 = 2$, $j = 1,2,3$. The number in the (j,k)-th row and the (i_{k_1}, i_{k_2})-th column denotes the loss for player j when he plays his k-th pure strategy and player k_h, $h = 1,2$ plays his i_{k_h}-th pure strategy with $k_1, k_2 \neq j$ ordered such that $k_1 < k_2$.

	(1,1)	(1,2)	(2,1)	(2,2)
(1,1)	1	2	8	5
(1,2)	8	8	2	2
(2,1)	4	2	2	1
(2,2)	2	6	1	3
(3,1)	4	1	4	2
(3,2)	8	8	2	1

Game 2: $N = 3$, $n_j+1 = 3$, $j = 1,2,3$.

	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
(1,1)	2	3	4	2	3	3	4	1	5
(1,2)	1	1	4	3	4	1	6	8	2
(1,3)	4	7	2	4	5	5	3	6	4
(2,1)	5	6	7	4	8	9	3	5	1
(2,2)	1	1	3	3	2	1	2	2	4
(2,3)	2	3	6	5	3	6	7	5	8
(3,1)	1	3	5	1	6	2	1	2	4
(3,2)	2	6	5	3	3	7	8	5	5
(3,3)	5	2	2	4	6	5	8	1	3

Game 3: $N = 4$, $n_j+1 = 2$, $j = 1,\dots,4$. The number in the (j,k)-th row and the $(i_{k_1}, i_{k_2}, i_{k_3})$-th column denotes the loss for player j when he plays strategy k and player k_h, $h = 1,2,3$ plays i_{k_h} with $k_1, k_2, k_3 \neq j$ ordered such that $k_1 < k_2 < k_3$.

	(1,1,1)	(1,1,2)	(1,2,1)	(1,2,2)	(2,1,1)	(2,1,2)	(2,2,1)	(2,2,2)
(1,1)	3	3	4	2	3	3	4	1
(1,2)	4	1	4	3	1	1	6	8
(2,1)	4	6	2	4	5	3	3	6
(2,2)	5	2	7	4	8	6	3	5
(3,1)	1	6	3	3	3	3	1	2
(3,2)	2	2	6	5	4	6	3	5
(4,1)	6	3	5	1	3	2	3	2
(4,2)	2	6	5	3	4	7	1	5

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