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APPLIED TO CYCLIC POLLING SYSTEMS

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Abstract

The computation scheme of the power-series algorithm for the evaluation of the joint queue length distributions for a broad class of multi-queue systems is extended in order to become applicable to polling systems with non-negligible switching times. Several properties of the mean waiting times at the various queues are discussed, in particular concerning their light-traffic and heavy-traffic behaviour.

Keywords: Bernoulli schedules, switching times, waiting times, heavy-traffic limits.

1. Introduction

The power-series algorithm is a new tool which is being developed for the numerical analysis of a large class of multi-queue systems which do not possess some kind of product form solution, but which have the structure of a multi-dimensional birth-death process, possibly with one or more supplementary variables (Blanc [1,2]). An important application of this algorithm is the evaluation of performance measures for polling systems. These systems consist of several stations (queues) which are attended by a single server. They are often used to model computer-communication systems, for instance, local area networks with one common communication channel (see Takagi [7,8] for surveys on polling systems). The power-series algorithm has been used to study cyclic polling systems with Bernoulli or limited service disciplines in Blanc [2,3] in the case of negligible switching times between the stations. The aim of the present paper is to extend the algorithm such that systems with non-negligible switching times can also be handled. For this purpose, the simplest polling models with respect to the algorithm will be discussed in this paper, namely models with cyclic polling, Poisson arrival streams, exponential service and switching time distributions and Bernoulli disciplines at the queues. However, the method developed in this paper is readily extended to models with a visit order determined by a general polling table, and with Coxian service and switching time distributions. Cyclic polling models with Bernoulli schedules were previously studied by Servi [6] and Tedianto [9] among others.

The power-series algorithm is based on power-series expansions of the state probabilities and moments of the joint queue length distributions as functions of the load of the system, in light traffic. By applying extrapolation methods such as the epsilon algorithm (Wynn [10], Blanc [2]) the heavy traffic behaviour of performance measures can also be studied. For polling systems with zero switching times the procedure leads to a recursive scheme for the computation of the coefficients of the power-series expansions. For systems with non-negligible switching times the coefficients can be calculated recursively for all states except the empty states. The cause of this phenomenon is the fact that there is no longer a unique empty state as in models with zero switching times, because the server continues to move along the queues when no jobs are present in the system. In order to determine the coefficients for the empty states a set of s linear equations has to be solved (here s stands for the number of stations) at each step of the algorithm. It will be shown that explicit solutions to these sets of equations can be given.

The organisation of the paper is as follows. The multi-queue model for cyclic polling systems will be described in section 2. Section 3 contains a discussion on the generalization of the power-series algorithm to polling systems with non-negligible switching times. In section 4 some properties of the waiting times are discussed: the pseudo-conservation law and some light- and heavy-traffic limits. Some of these properties have been derived analytically, others are conjectures based on numerical evidence. Section 5 contains some numerical examples for polling systems with 6 stations, and a discussion on the performance of the algorithm. Section 6 is concerned with some concluding remarks.

2. Model description and notations

The system consists of s queues and a single server. Jobs arrive at queue j according to a Poisson process with rate λ_j, j=1,...,s. Each queue may contain an unbounded number of jobs. The server inspects the queues in a cyclic order (queue 1 up to queue s and then again queue 1, etc.). The number of jobs which are served during a visit of the server to a certain queue depends on the service discipline at that queue. In this paper we will take as service disciplines Bernoulli schedules, which include 1-limited and exhaustive service. A Bernoulli schedule is a vector of s probabilities $(q_1,\dots,q_s)$, which are used as follows. When the server arrives at a queue, at least one job is served, unless the queue is empty (in which case the server directly proceeds to the next queue). After the completion of a service at queue j the server starts serving another job at this queue with probability q_j if queue j has not yet been emptied; otherwise the server proceeds to the next queue. At each queue jobs are served in order of arrival. Service times of jobs arriving at queue j are assumed to be exponentially distributed with rate μ_j, j=1,...,s. The times which are needed for switching from queue j-1 to queue j are assumed to be exponentially distributed with rate ν_j, j=1,...,s (read here and below queue s for queue 0).

The sum of the arrival processes at the various queues is a Poisson process with rate $\Lambda = \sum_{j=1}^{s} \lambda_j$. The first two moments β_1 and β_2 of the distribution of the service time of an arbitrary job are given by:

$$\beta_1 = \frac{1}{\Lambda} \sum_{j=1}^{s} \lambda_j/\mu_j, \qquad \beta_2 = \frac{2}{\Lambda} \sum_{j=1}^{s} \lambda_j/\mu_j^2. \tag{2.1}$$

The load ρ_j offered at queue j, j=1,...,s, and the total *offered load* ρ to the system are defined by

$$\rho_j := \lambda_j/\mu_j, \qquad \rho := \sum_{j=1}^{s} \rho_j. \tag{2.2}$$

The first two moments σ_1 and σ_2 of the total switching time during one cycle of the server along the queues are given by:

$$\sigma_1 = \sum_{j=1}^{s} \frac{1}{\nu_j}, \qquad \sigma_2 = 2 \sum_{j=1}^{s} \sum_{h=1}^{j} \frac{1}{\nu_j \nu_h}. \tag{2.3}$$

Kühn [5] has derived general conditions for stability of cyclic polling systems. For the present models with Bernoulli schedules these conditions read (see also Tedianto [9]): for j=1,...,s,

$$\lambda_j \, [(1-q_j)\, \sigma_1 + 1/\mu_j] < 1 - \rho + \rho_j. \tag{2.4}$$

These conditions can be summarized in the following condition:

$$x := \rho + \sigma_1 \max_{j=1,..,s} \{\lambda_j(1-q_j)\} < 1. \tag{2.5}$$

We will call x the *occupancy* of the system. Because this quantity x will be used as a variable in power-series expansions, the arrival rates will be written as

$$a_j \, x = \lambda_j, \qquad j = 1,...,s. \tag{2.6}$$

It will be assumed that the systems are in steady state and hence (2.5) will hold. Let N_j denote the number of jobs in queue j (waiting or being served), j=1,...,s. The supplementary variables H, indicating the queue to which the server is switching or to which the server is attending, and Z, where $Z=0$ indicates that the server is switching and $Z=1$ indicates that the server is serving, are introduced in order to transform the queue length process into a Markov process. Let $\bar{n} = (n_1,\ldots,n_s)$ be a vector with non-negative integer entries. The state probabilities of the cyclic polling system are defined as follows: for $\bar{n} \in \mathbb{N}^s$, $h=1,\ldots,s$, $\zeta=0,1$,

$$p(\bar{n},h,\zeta) := \Pr\{N_j = n_j,\ j=1,\ldots,s;\ H=h,\ Z=\zeta)\}. \tag{2.7}$$

In the next section we will develop a computation scheme for the coefficients of the power-series expansions of these state probabilities as functions of the occupancy of the system.

3. The power-series algorithm

The power-series algorithm has been described in Blanc [2] for cyclic polling systems with Bernoulli schedules with zero switching times. In this section it will be shown how the algorithm can be modified such that it can also be applied to systems with non-negligible switching times. The most important difference concerns the computation of the coefficients of the power-series expansions of the probabilities for the states in which no jobs are present in the system. There is no longer a unique empty state in systems with non-negligible switching times, because the server continues to move along the queues when no jobs are present in the system. In the following relations, let I{E} stand for the indicator function of the event E, and let $\bar{e}_j$ be a vector with zero entries except an entry of one at the j^{th} position ($j=1,..,s$). The balance equations for the state probabilities (2.7) are readily verified to be, for $h=1,\ldots,s$, $\bar{n} \in \mathbb{N}^s$,

$$\begin{aligned}[x \sum_{j=1}^{s} a_j + \nu_h]\ p(\bar{n},h,0) = {} & x \sum_{j=1}^{s} a_j\ p(\bar{n}-\bar{e}_j,h,0)\ I\{n_j>0\} \\ & + \mu_{h-1}\ p(\bar{n}+\bar{e}_{h-1},h-1,1)\ [1 - q_{h-1} + q_{h-1}\ I\{n_{h-1}=0\}] \\ & + \nu_{h-1}\ p(\bar{n},h-1,0)\ I\{n_{h-1}=0\};\end{aligned} \tag{3.1}$$

and for $h=1,\ldots,s$, $\bar{n} \in \mathbb{N}^s$, $n_h > 0$,

$$[x \sum_{j=1}^{s} a_j + \mu_h]\ p(\bar{n},h,1) = x \sum_{j=1}^{s} a_j\ p(\bar{n}-\bar{e}_j,h,1)\ I\{n_j>0\}$$

$$+ \nu_h\ p(\bar{n},h,0) + q_h\ \mu_h\ p(\bar{n}+\bar{e}_h,h,1); \tag{3.2}$$

and further it holds of course that

$$\sum_{n_1=0}^{\infty} \dots \sum_{n_s=0}^{\infty} \sum_{h=1}^{s} \sum_{\zeta=0}^{1} p(\bar{n},h,\zeta) = 1. \tag{3.3}$$

It should be noted that for all h, h=1,...,s,

$$p(\bar{n},h,1) = 0, \quad \text{if} \quad n_h = 0. \tag{3.4}$$

We introduce the following power-series expansions, for h=1,...,s, ζ=0,1,

$$p(\bar{n},h,\zeta) = x^{n_1+\dots+n_s} \sum_{k=0}^{\infty} x^k\ b(k;\bar{n},h,\zeta), \tag{3.5}$$

Substitute the power-series (3.5) into the balance equations (3.1) and (3.2). Equating the coefficients of corresponding powers of x in the resulting equations leads to the following set of equations for the coefficients of the power-series (3.5): for k=0,1,2,..., for h=1,...,s, $\bar{n} \in \mathbb{N}^s$,

$$\nu_h\ b(k;\bar{n},h,0) = \sum_{j=1}^{s} a_j\ [b(k;\bar{n}-\bar{e}_j,h,0)\ I\{n_j>0\} - b(k-1;\bar{n},h,0)\ I\{k>0\}]$$

$$+ \mu_{h-1}\ b(k-1;\bar{n}+\bar{e}_{h-1},h-1,1)\ I\{k>0\}\ [1 - q_{h-1} + q_{h-1}\ I\{n_{h-1}=0\}]$$

$$+ \nu_{h-1}\ b(k;\bar{n},h-1,0)\ I\{n_{h-1}=0\}; \tag{3.6}$$

and for k=0,1,2,..., for h=1,...,s, $\bar{n} \in \mathbb{N}^s$, $n_h > 0$,

$$\mu_h\ b(k;\bar{n},h,1) = \sum_{j=1}^{s} a_j\ [b(k;\bar{n}-\bar{e}_j,h,1)\ I\{n_j>0\} - b(k-1;\bar{n},h,1)\ I\{k>0\}]$$

$$+ \nu_h\ b(k;\bar{n},h,0) + q_h\ \mu_h\ b(k-1;\bar{n}+\bar{e}_h,h,1)\ I\{k>0\}. \tag{3.7}$$

This set of equations (3.6), (3.7) forms almost a recursive scheme. In order to make this observation clear we introduce the following partial ordering of the vectors $(k;\bar{n},h,\zeta)$. We say that $(i;\bar{m},j,\xi) \prec (k;\bar{n},h,\zeta)$ if one of the following conditions holds:

a. $i+m_1+\ldots+m_s < k+n_1+\ldots+n_s$;
b. $i+m_1+\ldots+m_s = k+n_1+\ldots+n_s$ and $i < k$;
c. $i+m_1+\ldots+m_s = k+n_1+\ldots+n_s$, $i = k$ and $\xi < \zeta$.

It is readily verified that the set of equations (3.6) and (3.7) expresses coefficients $b(k;\bar{n},h,\zeta)$ in terms of coefficients with a lower order, with the exception of the term $b(k;\bar{n},h-1,0)$ in (3.6), which only plays a role in the case that $n_{h-1}=0$. If $\bar{n} \neq \bar{0}$, an empty state, the set of coefficients $b(k;\bar{n},h,0)$, $h=1,\ldots,s$, for k and $\bar{n}$ fixed, can still be recursively computed by starting at a value $h=j$ with $n_{j-1}>0$ and by proceeding the computations of $b(k;\bar{n},h,0)$ then sequentially for $h=j+1,\ldots,s,1,\ldots,j-1$. Hence, the only states which require further attention are states with $\bar{n} = \bar{0}$ and $\zeta = 0$. For these empty states the equations (3.6) read: for $k=0,1,\ldots,$

$$\nu_h \, b(k;\bar{0},h,0) = \nu_{h-1} \, b(k;\bar{0},h-1,0) + y(k;h), \qquad h = 1,\ldots,s; \tag{3.8}$$

here, the quantities $y(k;h)$, $h=1,\ldots,s$, defined by $y(0;h):=0$ and

$$y(k;h) := \mu_{h-1} \, b(k-1;\bar{e}_{h-1},h-1,1) - \sum_{j=1}^{s} a_j \, b(k-1;\bar{0},h,0), \quad k = 1,2,\ldots,$$

consist of terms with coefficients of lower order and, hence, can be considered to be known. By summing the equations (3.8) over h, $h=1,\ldots,s$, for k fixed, it is readily seen, that these sets of equations are dependent. Therefore, we still need to use the law of total probability, cf. Blanc [1,2]. Substituting the power-series (3.5) into (3.3) and equating the coefficients of corresponding powers of x in the resulting equation leads to the following equations:

$$\sum_{h=1}^{s} b(0;\bar{0},h,0) = 1, \qquad \text{for } k = 0,$$

$$\sum_{h=1}^{s} b(k;\bar{0},h,0) = -Z(k), \qquad \text{for } k = 1,2,\ldots, \tag{3.9}$$

here,

$$Z(k) := \sum_{0<n_1+\ldots+n_s\leq k} \ldots \sum \sum_{h=1}^{s} \sum_{\zeta=0}^{1} b(k-n_1-\ldots-n_s;\bar{n},h,\zeta), \qquad k = 1,2,\ldots$$

Consider, for k fixed, the set of equations consisting of (3.9) and the equations (3.8) for $h=2,\ldots,s$. It is readily verified that the determinants $D(k)$ of these sets of equations are given by:

$$D(k) = \sigma_1 \prod_{h=1}^{s} \nu_h, \qquad \text{for } k = 0,1,2,\ldots. \tag{3.10}$$

For $k=0$, this set of equations is readily solved: for $h=1,\ldots,s$,

$$b(0;\bar{0},h,0) = \frac{1}{\sigma_1 \nu_h}. \tag{3.11}$$

It is more tedious, but straightforward, to show that for $k=1,2,\ldots,$

$$b(k;\bar{0},1,0) = -\frac{1}{\sigma_1\nu_1}\left[Z(k) + \sum_{h=2}^{s} y(k;h) \sum_{j=h}^{s} \frac{1}{\nu_j}\right]. \tag{3.12}$$

Once the coefficient $b(k;\bar{0},1,0)$ has been determined according to (3.12), the coefficients $b(k;\bar{0},h,0)$ can be sequentially obtained for $h=2,\ldots,s$ by using (3.8). Hence, relations (3.11), (3.6), (3.7), (3.12) and (3.8) form a complete scheme for computing the coefficients $b(k;\bar{n},h,\zeta)$ for $k=0,1,\ldots,$ $\bar{n} \in \mathbb{N}^s$, $h=1,\ldots,s$, $\zeta=0,1$. The coefficients of the power-series expansions of moments of the joint queue length distribution can be obtained from those of the state probabilities in the usual way, cf. Blanc [1]. The convergence of the power-series can be improved by means of the epsilon algorithm, cf. Wynn [10], Blanc [2], especially when the occupancy of a system is high. The epsilon algorithm also transforms a divergent series into a convergent series if the analytic continuation of the function defined by the series at $x = 0$ possesses only poles as singularities inside the unit circle, i.e. for $|x| \leq 1$. The latter seems to hold in all cases considered. It may happen that the power series are so strongly divergent that numerical instabilities occur. In that case a conformal mapping as discussed in Blanc [1,2] should be used (see also the remarks at the end of section 5).

4. Waiting times

This section is concerned with a discussion of the stationary distributions of the waiting times W_j of jobs arriving at queue j, j=1,...,s. The number of jobs at queue j left behind by a job departing from that queue is equal to the number of jobs that arrived at queue j during the sojourn time of the departing job. Because arrivals occur according to a Poisson process, this implies, cf. Takagi [7], for j=1,...,s,

$$E\{N_j\} = \lambda_j \,[E\{W_j\} + 1/\mu_j] = a_j x \,[E\{W_j\} + 1/\mu_j],$$

$$E\{N_j^2\} - E\{N_j\} = a_j^2 x^2 \,[E\{W_j^2\} + 2E\{W_j\}/\mu_j + 2/\mu_j^2]. \qquad (4.1).$$

The first two moments of the waiting time distributions can be obtained from the moments of the marginal queue length distributions through these relations. Let W be the waiting time of an arbitrary job, not depending on the queue at which it arrives. The mean and the standard deviation of W can be computed from:

$$E\{W\} = \sum_{j=1}^{s} \frac{\lambda_j}{\Lambda} E\{W_j\}, \qquad \sigma^2\{W\} = \sum_{j=1}^{s} \frac{\lambda_j}{\Lambda} E\{W_j^2\} - E^2\{W\}. \qquad (4.2)$$

The expected values of the waiting times for jobs in the various queues of a cyclic service system with Bernoulli schedules satisfy the following pseudo-conservation law, cf. Boxma & Groenendijk [4], Tedianto [9],

$$\sum_{j=1}^{s} \left[1 - a_j(1-q_j)\,\frac{\sigma_1 x}{1-\rho}\right] \eta_j \, E\{W_j\} =$$

$$= \frac{\rho}{1-\rho}\,\frac{\beta_2}{2\beta_1} + \frac{\sigma_2}{2\sigma_1} + \frac{\sigma_1 \rho}{1-\rho}\left[\sum_{j=1}^{s} \eta_j^2(1-q_j) + \frac{1}{2}\sum_{j=1}^{s} \eta_j(1-\eta_j)\right]. \qquad (4.3)$$

here $\eta_j := \rho_j/\rho$ is the relative offered load at queue j, j=1,...,s, cf. (2.2). This relation is useful for checking the correctness and the accuracy of the computations.

From the relations of the power-series algorithm, in particular (3.11) for $\bar{n} = \bar{0}$ and (3.6) and (3.7) for $\bar{n} = \bar{e}_j$, $j=1,\dots,s$, it follows that as $x \downarrow 0$,

$$E\{N_j\} = \eta_j x + a_j x \frac{\sigma_2}{2\sigma_1} + O(x^2), \qquad j=1,\dots,s. \tag{4.4}$$

The following light-traffic limits are obtained from (4.4) with the aid of Little's formula (4.1).

For cyclic polling systems with Bernoulli schedules it holds:

$$\lim_{x\downarrow 0} E\{W_j\} = \frac{\sigma_2}{2\sigma_1}, \qquad \textit{for } j = 1,\dots,s. \tag{4.5}$$

Note that these limits are same for each queue. These light-traffic limits seem to form lower bounds for the mean waiting times for positive values of the occupancy x.

This section is concluded with a discussion of the following heavy-traffic limits of the mean waiting times:

$$\omega_j := \lim_{x\uparrow 1} (1-x)\, E\{W_j\}, \qquad j = 1,\dots,s. \tag{4.6}$$

Note that these limits are defined in such a way that the total arrival rate to the system increases to a value at which one or more queues become instable, while the proportions between the arrival rates remain fixed, cf. (2.5), (2.6).

When the number of queues is not too large and the parameters of a system are not too asymmetrical, it is possible to obtain accurate data for performance measures even for high occupancy of such a system (x close to 1). From those results heavy-traffic limits as defined in (4.6) can be estimated. An important general property is the following:

The limit ω_j, $j=1,\dots,s$, is positive if

$$a_j(1-q_j) = \max_{i=1,\dots,s} \{a_i(1-q_i)\}, \tag{4.7}$$

and it is zero otherwise.

A set of Bernoulli parameters will be called a *balanced discipline* for a certain system if equality holds in (4.7) for each j, $j=1,\dots,s$. Special cases of systems with a balanced discipline are systems with exhaustive service at each queue ($q_j=1$, $j=1,\dots,s$). For the latter systems we have found explicit expressions for the heavy traffic limits of the mean waiting times:

If $q_j = 1$, $j=1,\dots,s$, *then the limits defined in* (4.6) *are given by:*

$$\omega_j = \frac{1-\eta_j}{\sum_{i=1}^{s} \eta_i(1-\eta_i)} \frac{\beta_2}{2\beta_1} + \frac{1}{2}\sigma_1(1-\eta_j), \qquad j = 1,\dots,s. \tag{4.8}$$

The mean waiting times in systems with exhaustive service at each queue can also be computed by solving a set of s^3 linear equations, cf. Takagi [7], Ch. 4. The heavy traffic limits (4.8) can be derived from this set of equations in a similar way as it was done in Blanc [3] for systems with zero switching times.

Also for the special case of 2 queues we have found explicit expressions for the heavy traffic limits of the mean waiting times:

If $s = 2$ *and if* $a_1(1-q_1) > a_2(1-q_2)$, *then* $\omega_2 = 0$ *and*

$$\omega_1 = \frac{(1+a_2/a_1)\,[\beta_2+2(1-q_1)\sigma_1\eta_1\beta_1] + (1-q_1)\sigma_2}{2[(1+a_2/a_1)\eta_1\beta_1+(1-q_1)\sigma_1]}, \tag{4.9}$$

$$\lim_{x\uparrow 1} E\{W_2\} = \frac{(1-q_1)\omega_1 + (1+a_2/a_1)\beta_1[q_1\eta_1+(1-q_2)\eta_2]}{(1-q_1) - (1-q_2)a_2/a_1}. \tag{4.10}$$

These rather messy expressions have been obtained on the basis of the following four observations:

1. the property described with (4.7) implies that $\omega_2 = 0$;
2. the pseudo-conservation law (4.3) leads to a linear relation between ω_1 and the finite limit of $E\{W_2\}$;
3. numerical evidence that ω_1 is independent of q_2 if $a_1(1-q_1) > a_2(1-q_2)$;
4. numerical evidence that if $q_2=1$ then:

$$\lim_{x\uparrow 1} E\{W_2\} = \omega_1 + \frac{q_1}{1-q_1}\eta_1\beta_1(1+a_2/a_1). \tag{4.11}$$

Observations 2 and 4 lead to expression (4.9) for ω_1 for $q_2=1$. But this expression holds for every value of q_2 by observation 3. By using (4.9) and again observation 2 the limit (4.10) is found. In this derivation we have used the fact that when $x=1$, then, cf. (2.5), (2.6), (2.2), (2.1),

$$1/a_1 = (1-q_1)\sigma_1 + (1+a_2/a_1)\beta_1. \tag{4.12}$$

The conjectures (4.9) and (4.10) include similar conjectures formulated in Blanc [3] for models with zero switching times (take $\sigma_1=0$ and $\sigma_2=0$). However, the motivation was less simple in the case without switching times than in the case with switching times, because the conservation law (observation 2 above) directly implies the value of ω_1 when switching times are zero, so that observation 3 becomes trivial, while observation 4 does not contain enough information for determining the finite limit of $E\{W_2\}$ for values of $q_2 < 1$. Therefore, many more numerical data were needed to find (4.10) for models with zero switching times.

Similar expressions as (4.9) and (4.10) hold when $a_1(1-q_1) < a_2(1-q_2)$ (all indices should be interchanged). In the case that $a_1(1-q_1) = a_2(1-q_2)$ the limits ω_1 and ω_2 are both positive. No expressions for ω_1 and ω_2 have been found in this case, only a linear relationship stemming from the pseudo-conservation law.

Finally, we note that the property described with (4.7) as well as the limits given in (4.8), (4.9) and (4.10) hold for general service and switching time distributions (with proper values for the moments β_1 and β_2 of the distribution of the service time of an arbitrary job and for the moments σ_1 and σ_2 of the distribution of the total switching time during one cycle of the server).

5. Examples

This section discusses some examples of cyclic polling systems for which performance measures have been computed with the aid of the power-series algorithm. Firstly, the parameters of the models will be described; then the numerical results will be discussed. Finally, some remarks on the computations will be made.

Example 1. The service rates, the arrival rates and the offered traffic are listed in table 1 for a system with 6 queues. The total offered load in this system is $\rho = 0.70$. Table 2 contains a list of Bernoulli schedules for this model. All stations are assigned 1-limited service in schedule Ba. This discipline is one of the disciplines with the highest occupancy, cf. (2.5), over all Bernoulli schedules for any given set of model parameters. Other schedules with equal Bernoulli parameters for all stations are Bb, Bc and Bo. Schedules Bd and Be are favorable to the stations with the highest service rates. For models with zero switching times it turned

Table 1. Arrival and service rates and offered loads for the model of example 1.

Queue	1	2	3	4	5	6	System	
λ_j	0.40	0.10	0.40	0.20	0.20	0.10	1.40	Λ
μ_j	2.00	1.00	4.00	4.00	1.00	2.00	2.00	β_1
ρ_j	0.20	0.10	0.10	0.05	0.20	0.05	0.70	ρ

Table 2. Bernoulli schedules (BS) for the model of example 1.

BS	q_1	q_2	q_3	q_4	q_5	q_6	$\mathrm{Max}\{\lambda_j(1-q_j)\}$
Ba	0.00	0.00	0.00	0.00	0.00	0.00	0.40
Bb	0.50	0.50	0.50	0.50	0.50	0.50	0.20
Bc	0.80	0.80	0.80	0.80	0.80	0.80	0.08
Bd	0.50	0.00	1.00	1.00	0.00	0.50	0.20
Be	1.00	0.00	1.00	1.00	0.50	1.00	0.10
Bf	0.00	0.50	0.75	0.75	0.00	0.50	0.20
Bg	0.80	0.60	0.90	0.90	0.60	0.80	0.08
Bh	0.90	0.80	0.95	0.95	0.80	0.90	0.04
Bi	0.75	0.50	0.50	0.00	0.75	0.00	0.20
Bj	0.90	0.80	0.80	0.60	0.90	0.60	0.08
Bk	0.95	0.90	0.90	0.80	0.95	0.80	0.04
Bl	0.75	0.00	0.75	0.50	0.50	0.00	0.10
Bm	0.90	0.60	0.90	0.80	0.80	0.60	0.04
Bn	0.95	0.80	0.95	0.90	0.90	0.80	0.02
Bo	1.00	1.00	1.00	1.00	1.00	1.00	0.00

Table 3. Lay-out (Lo) of polling systems for example 1: switching rates.

Lo	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	σ_1	σ_2/σ_1
L1	100.0	100.0	100.0	100.0	100.0	100.0	0.06	0.07
L2	10.0	10.0	10.0	10.0	10.0	10.0	0.60	0.70
L3	1.0	1.0	1.0	1.0	1.0	1.0	6.00	7.00
L3a	0.5	2.0	2.0	0.5	2.0	2.0	6.00	7.50
L3b	0.5	0.5	2.0	2.0	2.0	2.0	6.00	7.50
L3c	0.375	1.5	1.5	1.5	1.5	1.5	6.00	7.56
L4	0.1	0.1	0.1	0.1	0.1	0.1	60.00	70.00

Table 4. Model of example 1 with an offered load of ρ = 0.70 and with equal switching rates between the stations.

BS	Lo	x	E{W1}	E{W2}	E{W3}	E{W4}	E{W5}	E{W6}	E{W}	σ{W}
Ba	L1	.724	2.33	1.16	2.17	1.35	1.58	1.09	1.87	3.11
Bb	L1	.712	2.02	1.31	2.02	1.54	1.54	1.30	1.78	3.00
Bc	L1	.705	1.78	1.48	1.91	1.73	1.51	1.53	1.73	2.91
Bd	L1	.712	1.84	1.56	0.99	1.07	2.28	1.22	1.48	2.46
Be	L1	.706	1.16	2.34	1.33	1.45	2.11	1.34	1.48	2.43
Bf	L1	.712	1.78	1.51	1.28	1.19	2.21	1.18	1.55	2.51
Bg	L1	.705	1.67	1.59	1.52	1.52	1.84	1.45	1.61	2.59
Bh	L1	.702	1.62	1.63	1.63	1.70	1.68	1.60	1.64	2.67
Bi	L1	.712	1.57	1.49	2.33	2.47	1.33	1.79	1.89	3.38
Bj	L1	.705	1.56	1.58	2.06	2.21	1.40	1.80	1.79	3.13
Bk	L1	.702	1.56	1.63	1.94	2.10	1.43	1.79	1.75	3.01
Bl	L1	.706	1.52	1.91	1.61	1.72	1.72	1.76	1.65	2.67
Bm	L1	.702	1.53	1.79	1.70	1.84	1.59	1.78	1.67	2.75
Bn	L1	.701	1.54	1.74	1.74	1.89	1.54	1.79	1.68	2.79
Bo	L1	.700	1.56	1.68	1.78	1.95	1.49	1.79	1.69	2.81
min	L1		0.87	0.81	0.86	0.86	0.83	0.84		
max	L1		3.19	2.89	3.75	3.54	2.92	2.91		
Ba	L2	.940	19.03	2.23	17.93	3.05	3.71	2.13	11.80	18.07
Bb	L2	.820	5.15	2.29	5.15	2.88	2.91	2.29	4.08	5.66
Bc	L2	.748	3.15	2.40	3.38	2.81	2.50	2.47	2.97	3.97
Bd	L2	.820	4.69	2.99	1.63	1.75	5.85	2.16	3.26	4.74
Be	L2	.760	1.82	4.40	2.07	2.23	3.93	2.12	2.46	3.32
Bf	L2	.820	4.60	2.91	2.47	2.08	5.73	2.10	3.49	4.76
Bg	L2	.748	2.99	2.68	2.52	2.41	3.32	2.37	2.76	3.50
Bh	L2	.724	2.63	2.61	2.57	2.60	2.76	2.51	2.62	3.33
Bi	L2	.820	2.92	2.56	5.89	6.20	2.27	3.40	4.14	6.26
Bj	L2	.748	2.53	2.53	3.60	3.85	2.23	2.96	3.01	4.24
Bk	L2	.724	2.42	2.53	3.10	3.32	2.23	2.83	2.75	3.76
Bl	L2	.760	2.91	3.67	3.08	3.27	3.30	3.41	3.16	4.01
Bm	L2	.724	2.51	2.96	2.78	2.97	2.63	2.95	2.73	3.54
Bn	L2	.712	2.41	2.75	2.70	2.91	2.43	2.82	2.62	3.41
Bo	L2	.700	2.32	2.54	2.64	2.85	2.25	2.69	2.52	3.27
min	L2		1.44	1.39	1.44	1.45	1.38	1.45		
max	L2		22.58	5.27	25.51	8.21	7.15	5.29		
Bh	L3	.94	52.21	18.74	18.17	14.14	56.04	14.13	32.50	40.7
Bk	L3	.94	16.42	13.78	61.62	67.47	12.14	19.92	36.07	49.8
Bm	L3	.94	50.32	61.67	56.59	60.72	53.49	62.42	55.73	56.1
Bn	L3	.82	16.64	19.58	18.80	20.07	17.12	20.28	18.29	16.7
Bo	L3	.70	9.95	11.12	11.22	11.88	9.88	11.72	10.79	8.4
Bo	L4	.70	86.26	96.94	96.99	102.14	86.13	101.99	93.46	61.5

Table 5. Model of example 1 with offered load ρ = 0.70 and with σ_1 = 6.00.

BS Lo	x	$E\{W_1\}$	$E\{W_2\}$	$E\{W_3\}$	$E\{W_4\}$	$E\{W_5\}$	$E\{W_6\}$	$E\{W\}$	$\sigma\{W\}$
Bh L3a	.94	53.75	19.02	18.47	14.29	57.69	14.32	33.33	42.2
Bh L3b	.94	53.82	18.98	18.42	14.32	57.65	14.35	33.32	42.2
Bh L3c	.94	53.87	19.03	18.46	14.34	57.84	14.37	33.42	42.4
Bk L3a	.94	16.74	14.02	64.01	69.82	12.36	20.31	37.25	52.0
Bk L3b	.94	16.77	13.93	63.88	69.91	12.38	20.43	37.23	51.9
Bk L3c	.94	16.75	14.00	64.09	70.17	12.44	20.41	37.37	52.0
Bm L3a	.94	51.50	62.98	57.88	61.99	54.73	63.77	56.98	57.7
Bm L3b	.94	51.55	62.92	57.84	61.98	54.73	63.75	56.97	57.7
Bm L3c	.94	51.64	63.07	57.99	62.13	54.91	63.92	57.11	57.9
Bn L3a	.82	17.00	20.03	19.29	20.48	17.52	20.78	18.71	17.3
Bn L3b	.82	17.04	19.93	19.18	20.51	17.56	20.82	18.70	17.3
Bn L3c	.82	17.00	20.02	19.27	20.58	17.64	20.90	18.75	17.4
Bo L3a	.70	10.13	11.38	11.56	12.08	10.13	12.08	11.05	8.71
Bo L3b	.70	10.21	11.26	11.40	12.12	10.17	12.13	11.03	8.68
Bo L3c	.70	10.11	11.33	11.47	12.19	10.25	12.23	11.06	8.74

out that such schedules minimize E{W} in many cases, cf. Blanc [3]. Schedules Bf, Bg and Bh are such that the maximal mean visit time is the same for each station. Schedules Bi, Bj and Bk are such that $\rho_j(1-q_j)$ is constant over j, j=1,..,6. Schedules Bl, Bm, Bn and Bo are balanced disciplines. Various lay-outs for this system, reflected by the switching rates between the queues, are described in table 3. The stations are located at equal distances in L1, L2, L3 and L4. The stations 1,2,3 and the stations 4,5,6 form two groups in L3a, with longer distances between stations in different groups than between stations within the same group. Station 1 is located at a longer distance from the other stations than the distances which exist between the other stations in L3b. The stations can be considered to be arranged in one line in L3c, where the server has to travel a longer distance from station 6 back to station 1 than in between the other stations. The total mean switching time during one cycle is the same for L3, L3a, L3b and L3c. Table 4 contains the mean waiting times at the indivual queues and the mean and the standard deviation of the waiting time of an arbitrary job, for various Bernoulli schedules and symmetrical lay-outs of the system. Table 5 shows the same quantities for systems in which the switching rates between the stations are not all equal.

Example 2. Table 6 shows the influence of the presence of a relatively heavily loaded queue on the mean waiting times at queues which are four times more lightly loaded, for various Bernoulli schedules. The parameters of the system are: $s = 6$, $\mu_1 = 1$, $\mu_j = 2$, $j=2,\ldots,6$; $\lambda_1 = 0.32$, $\lambda_j = 0.16$, $j=2,\ldots,6$; so that $\rho_1 = 0.32$, $\rho_j = 0.08$, $j=2,\ldots,6$, and $\rho = 0.72$. In all examples we have chosen $q_j = q_2$, $j=2,\ldots,6$. The stations are equally spaced in the cases considered in table 6 (all switching rates are equal). In table 7 the same stations as in table 6 have been considered, but with different switching rates between the stations. Here, all switching rates are equal to 10/3, except the switching rate to the station indicated by J in the table, where $\nu_J=2/3$. Consequently, $\sigma_1 = 3.0$, $\sigma_2 = 11.7$ for the systems considered in table 7, whereas $\sigma_1 = 3.0$, $\sigma_2 = 10.5$ for the systems considered in the third block of table 6.

The Bernoulli schedules which only appear in some of the blocks in the tables 4 and 6 have been deleted from the other blocks, because the total mean switching time σ_1 in the latter blocks is such that these schedules give rise to unstable systems. The data in tables 4 and 6 show that different Bernoulli schedules may lead to a lower value of E{W} depending on the total mean switching time σ_1. For instance, E{W} is minimal over the Bernoulli schedules considered in table 4 for Be, when $\sigma_1=0.06$ (L1) and when $\sigma_1=0.60$ (L2), but for Bo, when $\sigma_1=6.00$ (L3), and E{W} is minimal over the Bernoulli schedules considered in table 6 for the schedule with $q_1=0.0$, $q_j=1.0$, $j=2,\ldots,6$, when $\sigma_1=0.03$, for the schedules with $q_1=0.5$ or $q_1=0.8$, and $q_j=1.0$, $j=2,\ldots,6$, when $\sigma_1=0.30$, and for the schedule with $q_j=1.0$, $j=1,\ldots,6$, when $\sigma_1=3.00$. It should be noted that the discipline with exhaustive service at each queue will outperform any other Bernoulli schedule with increasing σ_1 and fixed arrival and service rates, because it is the only discipline for which the occupancy is independent of σ_1, cf. (2.5). It seems to be intuitively clear that the mean waiting time $E\{W_j\}$, $j=1,\ldots,s$, is minimal over all Bernoulli schedules for the schedule with $q_j=1.0$ and $q_i=0.0$, $i\neq j$, $i=1,\ldots,s$, and maximal for the schedule with $q_j=0.0$ and $q_i=1.0$, $i\neq j$, $i=1,\ldots,s$, for any system. The values of $E\{W_j\}$, $j=1,\ldots,s$, obtained with these schedules, have been added to table 4 for the model of example 1 with lay-outs L1 and L2. It should be born in mind that the values in each of these rows (indicated by min or max) have been obtained with different schedules and cannot be realized simultaneously

Table 6. Model of example 2 with offered load ρ = 0.72. Switching times between the various queues are equally distributed. In the first block σ_1 = 0.03, in the second block σ_1 = 0.30, in the third block σ_1 = 3.00, and in the last case σ_1 = 30.0.

q_1	q_2	x	E{W_1}	E{W_2}	E{W_3}	E{W_4}	E{W_5}	E{W_6}	E{W}	σ{W}
0.0	0.0	.730	2.65	1.41	1.43	1.45	1.46	1.48	1.79	2.85
0.0	0.5	.730	2.88	1.20	1.22	1.24	1.26	1.28	1.71	2.86
0.0	1.0	.730	3.08	1.02	1.04	1.06	1.09	1.13	1.65	2.91
0.5	1.0	.725	2.59	1.33	1.36	1.39	1.44	1.50	1.74	2.82
1.0	0.0	.725	1.02	2.60	2.63	2.66	2.69	2.72	2.19	3.97
0.5	0.0	.725	1.88	1.97	1.99	2.01	2.03	2.05	1.97	3.13
0.5	0.5	.725	2.24	1.65	1.67	1.70	1.73	1.76	1.85	2.91
0.8	0.8	.722	1.85	1.88	1.92	1.96	2.01	2.06	1.93	3.11
0.8	0.9	.722	1.95	1.79	1.83	1.88	1.93	1.99	1.90	3.03
0.8	1.0	.722	2.05	1.70	1.74	1.79	1.85	1.93	1.87	2.95
0.9	0.6	.722	1.44	2.22	2.25	2.29	2.34	2.39	2.05	3.54
0.9	0.8	.721	1.59	2.07	2.11	2.16	2.21	2.27	2.00	3.35
0.9	1.0	.721	1.80	1.88	1.92	1.98	2.05	2.14	1.94	3.10
1.0	0.8	.721	1.33	2.26	2.31	2.37	2.43	2.50	2.07	3.66
1.0	1.0	.720	1.51	2.08	2.14	2.20	2.28	2.39	2.02	3.37
0.0	0.0	.816	5.19	2.15	2.17	2.18	2.19	2.21	3.04	4.50
0.0	0.5	.816	5.54	1.68	1.70	1.72	1.74	1.76	2.81	4.52
0.0	1.0	.816	5.81	1.34	1.36	1.39	1.42	1.46	2.65	4.59
0.5	1.0	.768	3.82	1.70	1.73	1.77	1.82	1.88	2.36	3.48
1.0	0.0	.768	1.30	3.83	3.86	3.88	3.91	3.94	3.15	5.05
0.5	0.0	.768	2.83	3.01	3.02	3.04	3.06	3.08	2.98	4.12
0.5	0.5	.768	3.36	2.28	2.30	2.32	2.35	2.38	2.62	3.63
0.8	0.8	.739	2.45	2.43	2.46	2.50	2.55	2.60	2.49	3.51
0.8	0.9	.739	2.56	2.27	2.31	2.36	2.41	2.47	2.42	3.38
0.8	1.0	.739	2.69	2.12	2.16	2.21	2.28	2.36	2.36	3.27
0.9	0.6	.739	1.85	2.93	2.97	3.00	3.05	3.09	2.68	4.08
0.9	0.8	.730	2.04	2.65	2.69	2.73	2.78	2.84	2.54	3.74
0.9	1.0	.730	2.28	2.32	2.37	2.43	2.50	2.59	2.39	3.37
1.0	0.8	.730	1.65	2.86	2.91	2.96	3.02	3.09	2.59	4.05
1.0	1.0	.720	1.86	2.55	2.61	2.67	2.75	2.85	2.45	3.61
0.8	0.8	.912	21.25	10.59	10.64	10.69	10.75	10.82	13.67	15.3
0.8	0.9	.912	21.75	8.06	8.13	8.20	8.29	8.39	12.07	14.8
0.8	1.0	.912	22.10	6.43	6.51	6.60	6.72	6.88	11.05	14.6
0.9	0.6	.912	7.85	25.80	25.82	25.84	25.86	25.90	20.71	25.1
0.9	0.8	.816	8.61	11.42	11.46	11.49	11.54	11.58	10.67	10.9
0.9	1.0	.816	9.33	6.84	6.90	6.98	7.09	7.23	7.67	7.2
1.0	0.8	.816	4.98	11.93	11.95	11.98	12.01	12.04	9.96	11.2
1.0	1.0	.720	5.33	7.25	7.30	7.36	7.43	7.52	6.79	6.2
1.0	1.0	.720	40.08	54.22	54.22	54.21	54.20	54.18	50.17	34.4

Table 7. Model of example 2 with offered load ρ = 0.72 and with σ_1 = 3.00.

q_1	q_2	J	x	E{W_1}	E{W_2}	E{W_3}	E{W_4}	E{W_5}	E{W_6}	E{W}	σ{W}
0.8	0.8	1	.912	21.83	10.80	10.87	10.95	11.03	11.11	14.07	15.9
0.8	0.8	2	.912	22.15	10.77	10.83	10.90	10.99	11.07	14.06	16.0
0.8	0.8	4	.912	22.14	10.85	10.92	10.88	10.95	11.04	14.07	15.9
0.8	0.9	1	.912	22.58	8.21	8.29	8.38	8.49	8.61	12.44	15.4
0.8	0.9	2	.912	22.70	8.17	8.25	8.34	8.45	8.57	12.44	15.4
0.8	0.9	4	.912	22.64	8.26	8.35	8.32	8.42	8.53	12.44	15.4
0.8	1.0	1	.912	22.97	6.53	6.62	6.74	6.89	7.08	11.40	15.3
0.8	1.0	2	.912	23.10	6.50	6.59	6.71	6.85	7.03	11.41	15.3
0.8	1.0	4	.912	23.03	6.59	6.70	6.68	6.81	6.99	11.41	15.3
0.9	0.6	1	.912	8.01	26.65	26.69	26.74	26.78	26.80	21.38	26.0
0.9	0.6	2	.912	8.05	26.58	26.61	26.66	26.71	26.76	21.35	26.0
0.9	0.6	4	.912	8.03	26.67	26.70	26.62	26.69	26.77	21.36	26.0
0.9	0.8	1	.816	8.84	11.73	11.79	11.86	11.92	12.00	11.00	11.3
0.9	0.8	2	.816	8.93	11.67	11.73	11.78	11.85	11.91	10.97	11.2
0.9	0.8	4	.816	8.89	11.80	11.86	11.75	11.81	11.88	10.98	11.2
0.9	1.0	1	.816	9.62	6.98	7.07	7.18	7.32	7.50	7.90	7.47
0.9	1.0	2	.816	9.75	6.93	7.01	7.11	7.24	7.41	7.89	7.46
0.9	1.0	4	.816	9.69	7.05	7.16	7.09	7.21	7.37	7.89	7.46
1.0	0.8	1	.816	5.01	12.30	12.36	12.41	12.48	12.54	10.30	11.6
1.0	0.8	2	.816	5.09	12.22	12.27	12.32	12.37	12.42	10.25	11.6
1.0	0.8	4	.816	5.05	12.39	12.45	12.28	12.33	12.38	10.28	11.6
1.0	1.0	1	.720	5.45	7.44	7.52	7.61	7.73	7.88	7.01	6.50
1.0	1.0	2	.720	5.58	7.37	7.44	7.52	7.61	7.73	6.97	6.43
1.0	1.0	4	.720	5.51	7.54	7.64	7.49	7.58	7.69	6.99	6.47

(see also the appendix). For the lay-outs L3 and L4 all these schedules give rise to unstable systems, which implies that the maximal value of each mean waiting time is infinite in these cases; the minimal values can be obtained by means of vacation models, but this is outside the scope of the present paper.

The data in the tables 5 and 7 show that the individual switching rates have only a minor influence on the mean waiting times. These and other data indicate that the influence of the individual switching rates is most important at a moderate occupancy of the system (x between 0.5 and 0.8), but the relative differences in the mean waiting times due to variations in the switching rates, with the total mean switching time kept fixed, have never been observed to be more than a few percentages for systems with 6 stations. The strongest influence of the switching times on the mean waiting times is through the first two moments of the total switching time during a cycle of the server along the stations, cf. (2.5), (4.3), (4.5), (4.8), (4.9), (4.10).

Most data for the above examples have been determined on the basis of 18 terms of the power-series expansions of the first two moments of the distributions of the number of jobs at the various stations, without the use of a conformal mapping (see the remark at the end of section 3), and with at most 7 steps of the epsilon algorithm. This number of terms has been imposed by the limitations on storage capacity on a specific computer. The relative errors for the data presented in the tables 4-7 is estimated to be well below 1% in most cases. Generally, the relative errors increase with increasing occupancy. The accuracy of the data has been checked on the basis of differences between the left- and the right-hand side of the pseudo-conservation law (4.3) and on the basis of differences in the results obtained with 16, 17 and 18 terms of the power-series expansions, respectively.
The data for several two-queue models which have been used to formulate the conjectures (4.9) and (4.10) have been obtained on the basis of 60 terms of the power-series expansions, with the use of a conformal mapping which is usually necessary with this amount of terms, and values of x up to 0.995.

6. Conclusions

The scope of the power-series algorithm has been extended to polling models with switching times. For the sake of simplicity, the method has been discussed on the basis of models with a cyclic polling order and exponential distributions for all random variables, but it can readily be generalized to models with arbitrary periodic polling orders and Coxian distributions. In fact, the algorithm is applicable to any polling system which can be modelled by a multi-dimensional quasi-birth-death process. This includes systems with random polling and also with dynamical strategies such as polling with priority for the longest queue.
The advantages of the power-series algorithm over techniques based on truncation of the state space and solution of large sets of balance equations are that the required quantities are iteratively computed, that

algorithms for improving the convergence of sequences can be applied, and that, once the coefficients of the power-series have been obtained, it requires little effort to compute performance measures for various values of the load of the system. The main limitation for application of the algorithm is the storage requirement which grows exponentially with the number of stations. Concepts for economic use of the available storage capacity have been discussed in Blanc [2]. The algorithm provides accurate data for moderate sized systems, which compare favorably with simulation results, and which may be helpful in finding and validating approximations for larger systems. A topic for further research is the possibility to modify the algorithm in such a way that it will generate good approximations for larger systems.

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Appendix

This appendix contains a table with values of the mean waiting times $E\{W_j\}$, $j=1,\ldots,6$, for the model of example 1 with lay-outs L1 and L2 and for the schedules with $q_j=1.0$ and $q_i=0.0$, $i\neq j$, $i=1,\ldots,6$, indicated by Ij, $j=1,\ldots,6$, and for the schedules with $q_j=0.0$ and $q_i=1.0$, $i\neq j$, $i=1,\ldots,6$, indicated by Uj, $j=1,\ldots,6$. The values of $E\{W_j\}$ for the schedules Ij and Uj, $j=1,\ldots,6$, form the rows in table 4 indicated by min and max respectively. E{W} is minimal over the Bernoulli schedules considered in table A for U5, and is even smaller than the minimal value of E{W} in table 4 (attained to with the schedule Be), when $\sigma_1=0.06$ (L1), but E{W} is minimal over the schedules considered in table A for U2, when $\sigma_1=0.60$ (L2); in the latter case it is somewhat larger than the minimal value of E{W} in table 4 (attained to with the schedule Be).

Table A. Model of example 1 with an offered load of ρ = 0.70 and with equal switching rates between the stations.

BS	Lo	x	$E\{W_1\}$	$E\{W_2\}$	$E\{W_3\}$	$E\{W_4\}$	$E\{W_5\}$	$E\{W_6\}$	E{W}	$\sigma\{W\}$
I1	L1	.724	0.87	1.53	2.85	1.82	2.05	1.40	1.85	3.45
I2	L1	.724	2.39	0.81	2.22	1.41	1.64	1.14	1.89	3.21
I3	L1	.724	2.59	1.27	0.86	1.52	1.78	1.20	1.63	2.89
I4	L1	.724	2.37	1.18	2.19	0.86	1.62	1.12	1.82	3.14
I5	L1	.724	2.62	1.41	2.46	1.63	0.83	1.32	2.00	3.53
I6	L1	.724	2.35	1.18	2.19	1.37	1.60	0.84	1.86	3.14
U1	L1	.724	3.19	1.11	1.19	1.31	1.02	1.24	1.75	3.41
U2	L1	.706	1.37	2.89	1.58	1.73	1.31	1.58	1.60	2.73
U3	L1	.724	1.30	1.43	3.75	1.58	1.22	1.47	2.05	4.27
U4	L1	.712	1.45	1.57	1.68	3.54	1.37	1.65	1.83	3.58
U5	L1	.712	1.09	1.17	1.24	1.35	2.92	1.26	1.45	2.52
U6	L1	.706	1.48	1.59	1.69	1.85	1.42	2.91	1.69	2.98
I1	L2	.940	1.44	2.87	24.04	4.32	5.13	2.64	8.97	17.7
I2	L2	.940	19.41	1.39	17.95	3.26	3.76	2.19	11.81	18.2
I3	L2	.940	21.12	2.43	1.44	3.58	4.18	2.32	7.99	15.8
I4	L2	.940	19.23	2.27	17.85	1.45	3.70	2.17	11.43	18.6
I5	L2	.940	19.75	2.58	18.20	3.62	1.38	2.46	11.92	18.6
I6	L2	.940	19.19	2.26	17.87	3.25	3.77	1.45	11.77	18.5
U1	L2	.940	22.58	1.77	1.85	2.00	1.63	1.97	7.76	16.4
U2	L2	.760	2.08	5.27	2.38	2.57	2.02	2.42	2.48	3.5
U3	L2	.940	1.97	2.19	25.51	2.35	1.87	2.25	8.78	19.3
U4	L2	.820	2.17	2.39	2.50	8.21	2.09	2.50	3.16	5.7
U5	L2	.820	1.72	1.88	1.95	2.10	7.15	2.00	2.65	4.3
U6	L2	.760	2.21	2.42	2.52	2.72	2.15	5.29	2.60	3.7

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