A generalisation and some properties of Markowitz' portfolio selection method

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Summary

A proof of the validity of Markowitz' critical line method is given for a more general situation than discussed by Markowitz. Next it is shown that in the Markowitz' case the critical line in the (μ,σ^2) plane is strictly convex and an everywhere differentiable function if the covariance matrix is positive definite, so refuting a statement by Fama and Miller.

1. Introduction.

Markowitz developed the critical line method for the following portfolio selection problem (cf.[3],[4]). Suppose an investor wants to invest an amount b in the securities 1,...,n. He invests an amount x_j (≥0) in security j, so

$$(1.1) \qquad \sum_{j=1}^{n} x_j = b \ .$$

The yearly revenue of a portfolio $X'=(x_1,\dots,x_n)$ is a stochastic variable $\underline{r}(X)$ with expected value $E\underline{r}(X)=\mu(X)$ and variance $\sigma^2(\underline{r}(X)) = \sigma^2(X)$. Besides the constraint (1.1) other constraints may exist, restricting the feasible options to a set $\mathcal{X} \subset \mathbb{R}^n$.

In order to get a first selection Markowitz introduces the notion of an efficient portfolio.

A feasible portfolio is efficient if:

a) no feasible portfolio exists with larger or equal expectation and smaller variance of the revenue,

and

b) no feasible portfolio exists with smaller or equal variance and larger expectation of the revenue.

This means that a portfolio $X=\bar{X}$ is efficient if and only if it is a solution of both

$$(1.2) \qquad \min_X \{\sigma^2(X) \,|\, \mu(X) \geq \mu(\bar{X}) \wedge X \in \mathcal{X}\}$$

and

$$(1.3) \qquad \max_X \{\mu(X) \,|\, \sigma^2(X) \leq \sigma^2(\bar{X}) \wedge X \in \mathcal{X}\} \ .$$

Markowitz derived an algorithm to compute all efficient portfolio's and the corresponding efficient (μ,σ^2) points, assuming $\mu(X)$ linear, $\sigma^2(X)$ quadratic and all constraints linear. In section 2 we show that the theorem on which this algorithm is based can be reformulated for a much more general situation.

Furthermore Markowitz derived some properties of the set of efficient points, but his remarks on differentiability properties of this line are not very explicit. In section 3 we show that if the variance $\sigma^2(X)$ is a strictly convex function the line of efficient points in the (μ, σ^2) plane is strictly convex and differentiable everywhere.

2. A general theorem for the computation of efficient portfolio's.

Theorem.

Let

i. the set of feasible portfolio's be defined by $\mathcal{X} = \{X \mid \forall_{i \in \mathcal{J}} h_i(X) \geq 0\}$, with $\mathcal{J}$ an indexset, $h_i(x)$ concave and continuous differentiable[1)], $\mathcal{X}$ compact with non vacuous interior,

ii. the expected value $\mu(X)$ of the revenue be concave, continuous differentiable on $\mathcal{X}$,

iii. the variance of X be continuous differentiable on $\mathcal{X}$,

then $X=\bar{X}$ is efficient if and only if,
either

a) there exists a $\bar{\lambda}>0$, such that

$$(2.1) \quad \min_X \{\sigma^2(X) - \bar{\lambda}\, \mu(X) \mid X \in \mathcal{X}\} = \sigma^2(\bar{X}) - \bar{\lambda}\mu(\bar{X}),$$

or
b)

$$(2.2) \quad \max_X [\mu(X) \mid \sigma^2(X) = \min_Y \{\sigma^2(Y) \mid Y \in \mathcal{X}\}] = \mu(\bar{X}),$$

or
c)

$$(2.3) \quad \min_X [\sigma^2(X) \mid \mu(X) = \max_Y \{\mu(Y) \mid Y \in \mathcal{X}\}] = \sigma^2(\bar{X}).$$

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1) By continuous differentiable we mean that all partial derivatives exist and are continuous. Strictly speaking, these conditions and the concavity conditions can be somewhat weakened.

Proof.

We first show that condition a) is sufficient. Suppose $\bar{X}$ is not efficient; this implies

$$(2.4)\quad \underset{X^*\neq\bar{X}}{\overset{X^*\in\mathcal{X}}{\exists}} \{\mu(X^*)\geq\mu(\bar{X}) \wedge \sigma^2(X^*)<\sigma^2(\bar{X})\} \vee \{\sigma^2(X^*)\leq\sigma^2(\bar{X}) \wedge \mu(X^*)>\mu(\bar{X})\},$$

or

$$(2.5)\quad \underset{X^*\neq\bar{X}}{\overset{X^*\in\mathcal{X}}{\exists}} \sigma^2(X^*)-\bar{\lambda}\mu(X^*) < \sigma^2(\bar{X})-\bar{\lambda}\mu(\bar{X}),$$

contradicting a). So $\bar{X}$ must be efficient.

If $X=\bar{X}$ suffices (2.2), then

$$(2.6)\quad \sigma^2(\bar{X}) = \min_X \{\sigma^2(X)\,|\,X\in\mathcal{X}\} = \sigma^2_{min}$$

and

$$(2.7)\quad \mu(\bar{X}) = \max_X \{\mu(X)\,|\,\sigma^2(X)=\sigma^2_{min} \wedge X\in\mathcal{X}\}.$$

Thus $X=\bar{X}$ is efficient with minimum variance on $\mathcal{X}$. In the same way $X=\bar{X}$ sufficing (2.3) implies

$$(2.8)\quad \mu(\bar{X})= \max_X \{\mu(X)\,|\,X\in\mathcal{X}\} = \mu_{max},$$

$$(2.9)\quad \sigma^2(\bar{X})= \min_X \{\sigma^2(X)\,|\,\mu(X)=\mu_{max} \wedge X\in\mathcal{X}\}.$$

In other words $X = \bar{X}$ is efficient with maximum expected value on $\mathcal{X}$.

Secondly we prove that the conditions are necessary. If $X=\bar{X}$ is efficient, it solves both (1.2) and (1.3), so it is a solution of

$$(2.10)\qquad \max_X \{-\sigma^2(X) \mid \mu(X)-\mu(\bar{X})\geq 0 \wedge \forall_{i\in \mathcal{J}}\, h_i(X) \geq 0\},$$

and of

$$(2.11)\qquad \max_X\{\mu(X) \mid \sigma^2(\bar{X})-\sigma^2(X)\geq 0 \wedge \forall_{i\in \mathcal{J}}\, h_i(X)\geq 0\}.$$

We now differentiate between two situations:
1) Slater's condition is satisfied, and
2) Slater's condition is not satisfied.

1) If Slater's condition is satisfied the Kuhn-Tucker conditions are not only sufficient, but also necessary. So in the case of problem (2.10): there exist numbers $\bar{\lambda}_1$ and $\bar{t}_{i1}$ ($i \in \mathcal{J}$) such that

$$(2.12)\qquad -\nabla\sigma^2(\bar{X}) + \bar{\lambda}_1\nabla\mu(\bar{X}) + \sum_{i\in \mathcal{J}} \bar{t}_{i1}\nabla h_i(\bar{X})=0$$

$$(2.13)\qquad \mu(\bar{X}) - \mu(\bar{X})\geq 0$$

$$(2.14)\qquad h_i(\bar{X})\geq 0 \qquad (i\in \mathcal{J})$$

$$(2.15)\qquad \bar{\lambda}_1\geq 0,\ \bar{t}_{i1}\geq 0 \quad (i\in \mathcal{J})$$

$$(2.16)\qquad \bar{\lambda}_1(\mu(\bar{X})-\mu(\bar{X})) + \sum_{i\in \mathcal{J}} \bar{t}_{i1}h_i(\bar{X}) = 0\ .$$

In the same way, for problem (2.11), there exist numbers $\bar{\lambda}_2$ and $\bar{t}_{i2}$ ($i\in \mathcal{J}$) such that

$$(2.17) \qquad \nabla\mu(\bar{X}) - \bar{\lambda}_2\nabla\sigma^2(\bar{X}) + \sum_{i\in\mathcal{J}} \bar{t}_{i2}\nabla h_i(\bar{X}) = 0$$

$$(2.18) \qquad \sigma^2(\bar{X}) - \sigma^2(\bar{X}) \geq 0$$

$$(2.19) \qquad h_i(\bar{X}) \geq 0 \qquad (i\in\mathcal{J})$$

$$(2.20) \qquad \bar{\lambda}_2 \geq 0 \;,\; \bar{t}_{i2} \geq 0 \qquad (i\in\mathcal{J})$$

$$(2.21) \qquad \bar{\lambda}_2(\sigma^2(\bar{X}) - \sigma^2(\bar{X})) + \sum_{i\in\mathcal{J}} \bar{t}_{i2}h_i(\bar{X}) = 0 \;.$$

Combining (2.12) and (2.17) leads to

$$(2.22) \quad -(1+\bar{\lambda}_2)\nabla\sigma^2(\bar{X}) + (1+\bar{\lambda}_1)\nabla\mu(\bar{X}) + \sum_{i\in\mathcal{J}} (\bar{t}_{i1}+\bar{t}_{i2})\nabla h_i(\bar{X}) = 0.$$

We define

$$(2.23) \qquad \bar{\lambda} = \frac{1+\bar{\lambda}_1}{1+\bar{\lambda}_2} \;,\quad \bar{t}_i = \frac{1}{1+\bar{\lambda}_2}(\bar{t}_{i1}+\bar{t}_{i2}) \qquad (i\in\mathcal{J}) \;,$$

then (2.22) can be rewritten as

$$(2.24) \qquad -\nabla\sigma^2(\bar{X}) + \bar{\lambda}\nabla\mu(\bar{X}) + \sum_{i\in\mathcal{J}} \bar{t}_i\nabla h_i(\bar{X}) = 0.$$

We conclude the existence of numbers $\bar{\lambda}$ and $\bar{t}_i$ ($i\in\mathcal{J}$) satisfying (2.24), (2.19) and

$$(2.25) \qquad \bar{\lambda} > 0 \;,\; \bar{t}_i \geq 0 \qquad (i\in\mathcal{J})$$

$$(2.26) \qquad \sum_{i \in \mathcal{J}} \bar{t}_i h_i(\bar{X}) = 0 \ ,$$

but this means that there exists a $\bar{\lambda}>0$, such that $X=\bar{X}$ solves the problem

$$(2.27) \qquad \max_X \ \{-\sigma^2(X) + \bar{\lambda}\ \mu(X) \mid \forall_{i\in\mathcal{J}}\ h_i(X) \geq 0\},$$

which is identical to (2.1).

2) If Slater's condition is not satisfied, this means that either $\mu(X)-\mu(\bar{X})\geq 0$ or $\sigma^2(\bar{X})-\sigma^2(X)\geq 0$ doesn`t have an interior point because $\mathcal{X}$ has a non vacuous interior. In the first case $\mu(\bar{X})$ equals the maximum μ_{max} of $\mu(X)$ on $\mathcal{X}$ and the efficient portfolio $\bar{X}$ solves (2.3); in the second case $\sigma^2(\bar{X})$ equals the minimum σ^2_{min} of $\sigma^2(X)$ on $\mathcal{X}$ and the efficient portfolio $\bar{X}$ solves (2.2). If (2.6) has a unique solution, finding the corresponding efficient portfolio is equivalent to solving (2.1) for $\bar{\lambda} = 0$. Analogous if (2.8) has an unique solution, finding the corresponding efficient portfolio is equivalent to solving (2.1) for a sufficiently large value of $\bar{\lambda}$.

3. The set of efficient (μ,σ^2) points in the Markowitz' case.

We now specialize to the original portfolio selection problem of Markowitz. Suppose the yearly revenue of one dollar invested in security j equals $\underline{r}_j$ with $E\underline{r}_j=\mu_j$; the covariance matrix of the $\underline{r}_j$ is $\mathcal{C}$. If $M' = (\mu_1,\ldots,\mu_n)$, then

$$(3.1) \qquad \mu(X) = M'X,$$

$$(3.2) \qquad \sigma^2(X) = X'\mathcal{C}X .$$

The constraints are

$$(3.3) \qquad \mathcal{A}X \leq B$$

$$(3.4) \qquad X \geq 0 .$$

If the feasible set $\mathcal{X}$ has a non vacuous interior the efficient portfolio's can be found by applying the theorem of section 2 in which the left hand side of (2.1) now reduces to

$$(3.5) \qquad \min_X \{ X'\mathcal{C}X - \bar{\lambda} M'X \mid \mathcal{A}X \leq B \wedge X \geq 0 \}.$$

The points $(\bar{\mu},\bar{\sigma}^2)$ corresponding to efficient portfolio's constitute the efficient points in the (μ,σ^2) plane, sometimes called the critical line of the problem. Suppose we start with $\lambda=0$ and next raise λ, we get different efficient portfolio's, provided that we exclude the degenerate case in which there exists only one efficient portfolio. For specific values of λ, there is a change in the basis; suppose these values are $\bar{\lambda}_1,\ldots,\bar{\lambda}_k$ and corresponding efficient solutions are $\bar{X}_1,\ldots,\bar{X}_k$. We form the (sub)sequence $\bar{X}_{j_1},\ldots,\bar{X}_{j_h}$ from $\bar{X}_1,\ldots,\bar{X}_k$ for which the $(\bar{\mu},\bar{\sigma}^2)$ combinations are different. This (sub)sequence is called the set of corner portfolio's. We have

$$(3.6) \qquad M'\bar{X}_{j_i} < M'\bar{X}_{j_{i+1}}$$

and

$$(3.7) \qquad \bar{X}'_{j_i}\mathcal{C}\bar{X}_{j_i} < \bar{X}'_{j_{i+1}}\mathcal{C}\bar{X}_{j_{i+1}} .$$

The critical line in the (μ,σ^2) plane has the following properties.

a. Between the (μ,σ^2) points of two adjacent corner portfolio's, it is part of a strictly convex parabola.
b. On the segments mentioned in a, the relation

$$(3.8)\qquad \left(\frac{d\sigma^2}{d\mu}\right)_{(\bar{\mu},\bar{\sigma}^2)} = \bar{\lambda}$$

holds.
c. For $\mathcal{C}$ positive definite, the critical line is on the open interval $(\mu_{min},\ldots,\mu_{max})$ a differentiable, strictly convex function for which (3.8) holds.

Properties a and c differ from the properties of the critical line usually mentioned in the literature. Property b is well known. We shall now proof the properties a and c.

Proof of property a.

We consider a part of the critical line between two adjacent corner portfolio's, so the efficient portfolio's that are convex combinations of these corner portfolio's. For simplicity we note these corner portfolio's as X_i and X_{i+1} instead of X_{j_i} and $X_{j_{i+1}}$.

The efficient portfolio's corresponding with this part of the critical line can be written as:

$$(3.9)\qquad \bar{X} = \alpha(X_i - X_{i+1}) + X_{i+1} \qquad \alpha \in [0,1] .$$

With (3.1) and (3.2) it follows:

$$(3.10)\qquad \mu(\bar{X}) = \alpha M'(X_i - X_{i+1}) + M'X_{i+1}$$

and

$$(3.11)\qquad \sigma^2(\bar{X}) = \alpha^2(X_i - X_{i+1})'\mathcal{C}(X_i - X_{i+1}) + 2\alpha(X_i - X_{i+1})'\mathcal{C}X_{i+1} + X_{i+1}'\mathcal{C}X_{i+1} .$$

From (3.10) it is easy to derive

$$(3.12)\qquad \alpha = \frac{\mu(\bar{X}) - M'X_{i+1}}{M'(X_i - X_{i+1})} ,$$

so

$$(3.13)\quad \sigma^2(\bar{X}) = \frac{(X_i-X_{i+1})'\mathcal{C}(X_i-X_{i+1})}{\{M'(X_i-X_{i+1})\}^2}\,\mu(\bar{X})^2 +$$

$$-2\left\{\frac{M'X_{i+1}(X_i-X_{i+1})'\mathcal{C}(X_i-X_{i+1})}{\{M'(X_i-X_{i+1})\}^2} - \frac{(X_i-X_{i+1})'\mathcal{C}X_{i+1}}{M'(X_i-X_{i+1})}\right\}\mu(\bar{X})+$$

$$+\left\{\frac{M'X_{i+1}}{M'(X_i-X_{i+1})}X_i - \frac{M'X_i}{M'(X_i-X_{i+1})}X_{i+1}\right\}'\mathcal{C}\left\{\frac{M'X_{i+1}}{M'(X_i-X_{i+1})}X_i - \frac{M'X_i}{M'(X_i-X_{i+1})}X_{i+1}\right\}.$$

The coefficient of $\mu(\bar{X})^2$ is positive, because (3.6) gives

$$(3.14)\qquad \{M'(X_i-X_{i+1})\}^2 > 0,$$

and (3.7) leads to

$$(3.15)\quad (X_i-X_{i+1})'\mathcal{C}(X_i-X_{i+1}) = \sigma^2(X_i-X_{i+1}) = \sigma^2(\underline{r}(X_i)-\underline{r}(X_{i+1})) \geq$$
$$\geq (\sigma(\underline{r}(X_i)-\sigma(\underline{r}(X_{i+1})))^2 > 0.$$

So it follows directly that $\sigma^2(\bar{X})$ is a strictly convex function of $\mu(\bar{X})$. on the interval between two adjacent corner portfolio's.

<u>Proof of property c</u>.

Because of properties a. and b., property c only has to be proved for points on the critical line corresponding to corner portfolio's.
For efficient portfolio's $X = \bar{X}$ with $\mu_{min} < \mu(\bar{X}) < \mu_{max}$ there exist numbers $\bar{\lambda}$ and $\bar{t}_i$ ($i \in \mathcal{J}$) satisfying (2.19),(2.24),(2.25) and (2.26). Specializing to the problem of this section, combining the Lagrange multipliers of the conditions (3.3) in $U' = (u_1,\dots,u_m)$, those of (3.4) in $V' = (v_1,\dots,v_n)$ and adding slackvariables $y_1,\dots,y_m$ to (3.3), (2.24) and (2.19) reduce to

$$(3.16)\qquad -2\mathcal{C}\bar{X} - \mathcal{A}'\bar{U} + \bar{V} = -\bar{\lambda}M$$

and

$$(3.17) \qquad \mathcal{A}\bar{X} + \bar{Y} = B$$

$$(3.18) \qquad \bar{X} \geq 0 \ .$$

We now give an expression which holds for every efficient portfolio. Denote the basic variables of X by X_b and the corresponding parts of M, $\mathcal{C}$ and $\mathcal{A}$ by M_{b_1}, $\mathcal{C}_{b_1}$ and $\mathcal{A}_{b_1}$, then as will be shown later on, $\bar{X}_b$ can be written as

$$(3.19) \qquad \bar{X}_b = A + \bar{\lambda} D$$

with

$$(3.20) \qquad D = \tfrac{1}{2} [\mathcal{C}_{b_1}^{-1} - \mathcal{C}_{b_1}^{-1} \mathcal{A}_{b_1}' (\mathcal{A}_{b_1} \mathcal{C}_{b_1}^{-1} \mathcal{A}_{b_1}')^{-1} \mathcal{A}_{b_1} \mathcal{C}_{b_1}^{-1}] M_{b_1} \ .$$

Substituting (3.19) into (3.1) and (3.2), we get

$$(3.21) \qquad \mu(\bar{X}) = M_{b_1}' A + \bar{\lambda} M_{b_1}' D$$

$$(3.22) \qquad \sigma^2(\bar{X}) = A' \mathcal{C}_{b_1} A + 2 \bar{\lambda} A' \mathcal{C}_{b_1} D + \bar{\lambda}^2 D' \mathcal{C}_{b_1} D \ .$$

Furthermore we will show

$$(3.23) \qquad M_{b_1}' D \neq 0 \ .$$

Using (3.21) and (3.23) it is easy to verify property c.
Because of (3.21) and (3.23) there is a one to one correspondence between $\mu(\bar{X})$ and $\bar{\lambda}$. For efficient portfolio's, being convex combinations of two adjacent corner portfolio's, the basis is the same, so differences in the values of $\mu(\bar{X})$ and $\sigma^2(\bar{X})$ are only due to $\bar{\lambda}$. Property c holds for these portfolio's. Let the values $\bar{\mu}_h$ and $\bar{\lambda}_h$ correspond to a corner portfolio. If we take the limits $\bar{\mu} \uparrow \bar{\mu}_h$ and $\bar{\mu} \downarrow \bar{\mu}_h$, according to properties a and b , the corresponding limits of $\bar{\lambda}$ also exist and have as a limitvalue both $\bar{\lambda}_h$, so the lefthand derivative of the critical line for $\bar{\mu} \uparrow \bar{\mu}_h$ equals the righthand derivative for $\bar{\mu} \downarrow \bar{\mu}_h$. Thus the function is differentiable in $\bar{\mu} = \bar{\mu}_h$ with

derivative $\bar{\lambda}_h$, which means that (3.8) also holds for points on the critical line, corresponding with corner portfolio's.

Because $\left(\frac{d\sigma^2}{d\mu}\right)_{(\bar{\mu},\bar{\sigma}^2)}$ monotonically increases for increasing $\bar{\mu}$, the critical line is strictly convex.

Remark.

According to property c the statement of E.F. Fama and M. H. Miller ([1],p.243) that the critical line needs not to be differentiable everywhere doesn't hold if the covariance matrix is positive definite.

Appendix A.

Proof of the formulae (3.19) and (3.20).

We rewrite the equations (3.16) and (3.17), omitting the bars, to get variables X,Y,U and V, as follows

(A.1)

X'	Y'	U'	V'	
$-2\,\mathcal{C}$	$\mathcal{O}$	$-\mathcal{A}'$	$\mathcal{I}$	$-\bar{\lambda}M$
$\mathcal{A}$	$\mathcal{I}$	$\mathcal{O}$	$\mathcal{O}$	B

.

Let

(A.2) $\bar{Z}_b' = (\bar{X}_b' \,, \bar{Y}_b' \,, \bar{U}_b' \,, \bar{V}_b')$

be the feasible basic solution belonging to the efficient portfolio, then (A.1) can be partitioned into

(A.3)

X_b'	X_{nb}'	Y_b'	Y_{nb}'	U_b'	U_{nb}'	V_b'	V_{nb}'	
$-2\,\mathcal{C}_{b_1}$	$-2\,\mathcal{C}_{nb_1}$	$\mathcal{O}$	$\mathcal{O}$	$-\mathcal{A}_{b_1}'$	$-\mathcal{A}_{b_2}'$	$\mathcal{O}$	$\mathcal{I}$	$-\bar{\lambda}M_{b_1}$
$-2\,\mathcal{C}_{b_2}$	$-2\,\mathcal{C}_{nb_2}$	$\mathcal{O}$	$\mathcal{O}$	$-\mathcal{A}_{nb_1}'$	$-\mathcal{A}_{nb_2}'$	$\mathcal{I}$	$\mathcal{O}$	$-\bar{\lambda}M_{b_2}$
$\mathcal{A}_{b_1}$	$\mathcal{A}_{nb_1}$	$\mathcal{O}$	$\mathcal{I}$	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$	B_{b_1}
$\mathcal{A}_{b_2}$	$\mathcal{A}_{nb_2}$	$\mathcal{I}$	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$	B_{b_2}

.

The matrix $-2\,\mathcal{C}$ is partitioned into the square matrices $-2\,\mathcal{C}_{b_1}$ and $-2\,\mathcal{C}_{nb_2}$ corresponding to basic and non-basic variables x_j and into $-2\,\mathcal{C}_{b_2}$ and $-2\,\mathcal{C}_{nb_1}$ with $\mathcal{C}_{b_2} = \mathcal{C}_{nb_1}'$. $\mathcal{A}_{b_1}$ and $\mathcal{A}_{nb_1}$ represent the active constraints, $\mathcal{A}_{b_2}$ and $\mathcal{A}_{nb_2}$ the non-active constraints. Therefore we get identity matrices in the fourth place of the Y_b' column and the third place of the Y_{nb}' column. The other partitions are evident.

The matrix of basic vectors is

$$\text{(A.4)}\quad \mathcal{B} = \begin{bmatrix} -2\mathcal{C}_{b_1} & \mathcal{O} & -\mathcal{A}'_{b_1} & \mathcal{O} \\ -2\mathcal{C}_{b_2} & \mathcal{O} & -\mathcal{A}'_{nb_1} & \mathcal{I} \\ \mathcal{A}_{b_1} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{A}_{b_2} & \mathcal{I} & \mathcal{O} & \mathcal{O} \end{bmatrix} .$$

To facilitate computations we reshuffle rows and columns into

$$\text{(A.5)}\quad \mathcal{B}_v = \begin{bmatrix} -2\mathcal{C}_{b_1} & -\mathcal{A}'_{b_1} & \mathcal{O} & \mathcal{O} \\ \mathcal{A}_{b_1} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ -2\mathcal{C}_{b_2} & -\mathcal{A}'_{nb_1} & \mathcal{I} & \mathcal{O} \\ \mathcal{A}_{b_2} & \mathcal{O} & \mathcal{O} & \mathcal{I} \end{bmatrix} .$$

The values of the basic variables are

$$\text{(A.6)}\quad \bar{Z}_{bv} = \mathcal{B}_v^{-1} \begin{bmatrix} 0 \\ B_{b_1} \\ 0 \\ B_{b_2} \end{bmatrix} - \bar{\lambda}\, \mathcal{B}_v^{-1} \begin{bmatrix} M_{b_1} \\ 0 \\ M_{b_2} \\ 0 \end{bmatrix}$$

with $\bar{Z}'_{bv} = (\bar{X}'_b, \bar{U}'_b, \bar{V}'_b, \bar{Y}'_b)$. In order to get an explicit expression for $\bar{X}_b$ we compute $\mathcal{B}_v^{-1}$:

$$(A.7)\quad \mathcal{B}_v^{-1} = \left[\begin{array}{c|c} \begin{bmatrix} -2\mathcal{C}_{b_1} & -\mathcal{A}'_{b_1} \\ \mathcal{A}_{b_1} & \mathcal{O} \end{bmatrix}^{-1} & \mathcal{O} \\ \hline -\begin{bmatrix} -2\mathcal{C}_{b_2} & -\mathcal{A}_{nb_1} \\ \mathcal{A}_{b_2} & \mathcal{O} \end{bmatrix} \begin{bmatrix} -2\mathcal{C}_{b_1} & -\mathcal{A}'_{b_1} \\ \mathcal{A}_{b_1} & \mathcal{O} \end{bmatrix}^{-1} & \begin{bmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{I} \end{bmatrix} \end{array}\right].$$

Because $\mathcal{B}_v$ has an inverse, $\begin{bmatrix} -2\mathcal{C}_{b_1} & -\mathcal{A}'_{b_1} \\ \mathcal{A}_{b_1} & \mathcal{O} \end{bmatrix}^{-1}$ exists and since $\mathcal{C}$ positive definite $\mathcal{C}_{b_1}^{-1}$ exists and also $(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}$ (cf. [2] pp 107-109), so

$$(A.8)\quad \begin{bmatrix} -2\mathcal{C}_{b_1} & -\mathcal{A}'_{b_1} \\ \mathcal{A}_{b_1} & \mathcal{O} \end{bmatrix}^{-1} =$$

$$\left[\begin{array}{c|c} -\tfrac{1}{2}\mathcal{C}_{b_1}^{-1} + \tfrac{1}{2}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1}(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1} & \mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1}(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1} \\ \hline -(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1} & -2(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1} \end{array}\right].$$

Substitution of (A.8) into (A.7) and the result into (A.6) gives

$$(A.9)\quad \bar{X}_b = \mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1}(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}B_{b_1} +$$

$$+\bar{\lambda}[\tfrac{1}{2}\mathcal{C}_{b_1}^{-1} - \tfrac{1}{2}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1}(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}]M_{b_1},$$

with

$$D = \tfrac{1}{2}\,[\mathcal{C}_{b_1}^{-1} - \mathcal{C}_{b_1}^{-1}\mathcal{A}_{b_1}'(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}_{b_1}')^{-1}\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}]M_{b_1}, \tag{A.10}$$

as was to be proved.

Appendix B.

Proof of formula (3.23).

We use the fact that an efficient portfolio with expected value $\bar{\mu}$ solves problem (2.10), which in this case reduces to,
maximize

$$(B.1)\qquad -X'\mathcal{C}X$$

subject to

$$(B.2)\qquad \mathcal{A}X \leq B$$

$$(B.3)\qquad M'X \geq \bar{\mu}$$

$$(B.4)\qquad X \geq 0.$$

The Kuhn-Tucker conditions with Lagrange multipliers $\bar{U}$, $\bar{\lambda}_1$ and $\bar{V}$ and slackvariables $\bar{Y}$ and $\bar{y}_{n+1}$ are

$$(B.5)\qquad -2\mathcal{C}\bar{X} \qquad - \mathcal{A}'\bar{U} + M\bar{\lambda}_1 + \bar{V} = 0$$

$$(B.6)\qquad \mathcal{A}\bar{X} + \bar{Y} \qquad = B$$

$$(B.7)\qquad M'\bar{X} \qquad -\bar{y}_{n+1} \qquad = \bar{\mu}$$

$$(B.8)\qquad \bar{X}'\bar{V} + \bar{Y}'\bar{U} + \bar{y}_{n+1}\cdot\bar{\lambda}_1 = 0\ .$$

For the equations (B.5), (B.6), (B.7), (A.2) completed with $\bar{\lambda}_1$, forms a basic solution. Reordering in the same way as (A.5), the matrix of basic vectors changes into

$$(B.9)\qquad \mathcal{B}_v^* = \begin{bmatrix} \mathcal{B}_v & K \\ L' & 0 \end{bmatrix}$$

with

$$(B.10)\qquad L' = (\ M'_{b_1}\quad O'\quad O'\quad O')$$

and

(B.11) $K' = (M'_{b_1}\ \mathcal{O}'\ M'_{b_2}\ \mathcal{O}')$.

$\mathcal{B}^*_v$ has an inverse, so $(\mathcal{B}^*_v)^{-1}$ exists, just as $\mathcal{B}_v^{-1}$ and $(L'\mathcal{B}_v^{-1}K)^{-1}$ (cf. again [2], pp. 107-109). Now

$$(B.12)\qquad (\mathcal{B}^*_v)^{-1} = \begin{bmatrix} \mathcal{B}_v^{-1} - \mathcal{B}_v^{-1}K(L'\mathcal{B}_v^{-1}K)^{-1}L'\mathcal{B}_v^{-1} & \mathcal{B}_v^{-1}K(L'\mathcal{B}_v^{-1}K)^{-1} \\ (L'\mathcal{B}_v^{-1}K)^{-1}L'\mathcal{B}_v^{-1} & -(L'\mathcal{B}_v^{-1}K)^{-1} \end{bmatrix} .$$

Substitution of (B.10), (A.7) and (B.11) in $-(L'\mathcal{B}_v^{-1}K)^{-1}$ gives

$$(B.13)\qquad \tfrac{1}{2}\,[M'_{b_1}\{\mathcal{C}_{b_1}^{-1} - \mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1}(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\}M_{b_1}]^{-1} ,$$

which is, but for a constant, the reciproke of the left hand side of (3.23), cf. (A.10).

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