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Alternative production models

Some empirical relevance for postwar Belgian Economy

Research Memorandum

TILBURG INSTITUTE OF ECONOMICS
DEPARTMENT OF ECONOMETRICS

ALTERNATIVE PRODUCTION MODELS

WITH

SOME EMPIRICAL RELEVANCE FOR POSTWAR BELGIAN ECONOMY

by

J. Plasmans.

Contents

ALTERNATIVE PRODUCTION MODELS

Introduction.

Setting up a capacity model, measuring the long run growth possibilities of a certain economy, it might be instructive to formulate various production models for that economy and select one or some of them on the basis of available prior information about growth characteristics and/or of available sample information for the near past.
An important purpose of this and a subsequent paper is therefore to construct alternative production models and to test their validity for the postwar Belgian economy (1948 - 1967). The models will vary, besides the characteristics of the underlying production function, according to the prevailing market conditions (perfect/monopolistic competition) and the entrepreneurial objectives((expected) profit maximization, revenue from sales maximization, cost minimization), supposed to exist for the aggregate (non public) economy.

Since the underlying study is merely concentrated on time series analysis, much attention has to be paid to the inference of technical change, which may be:

- disembodied in the sense of applying equally and alike to all resources of labour and capital or embodied;

- neutral in the sense of leaving undisturbed the balance between certain economic variables or nonneutral(*).

(*) Introducing t as an index for the state of technology in the following homogeneous production function:
(1) $Q = F(L,K,t)$, we may distinguish, in general, the following four classes of neutral technological progress, based upon the invariancy between certain variables:
a) Product augmenting t.p. as e.g. Hicks-neutrality:
(2) $Q = F[A(t)L, A(t)K] = A(t)\,F(L,K)$, labour additive or capital additive, where the increase in production is proportional to the amount of labour, resp. capital used.
b) Labour augmenting such as Harrod neutrality:
(3) $Q = F[A(t)L, K]$ and labour combining:
(4) $Q = F[A(t)K + L, K]$ where the augmentation of labour, measured in efficiency units, is proportional to the amount of capital used
c) Capital augmenting such as Solow neutrality:

Obviously, the notion of "elasticity of substitution" between labour and capital, defined as the percentage change of the capital-labour ratio with respect to a one percent change of the marginal rate R of substitution of capital for labour, will play a central role in the following discussion. Mathematically, the elasticity of substitution is defined here as the following nonnegative number:

(7) $\sigma = \frac{d \log K/L}{d \log R}$ with marginal rate of substitution:

$$R = - \frac{dK}{dL} = \frac{\partial Q/\partial L}{\partial Q/\partial K} = \frac{\partial K}{\partial L}$$, which is assumed to

decrease as substitution proceeds, or R decreases as K/L decreases. It has to be stressed that the factor inputs are supposed to be changed in such a way (represented by R) that output is kept constant. So, σ can be studied as a property of the isoquants on a production surface. In this respect, only variations along these isoquants will be studied but at the end this approach will be enlarged by taking variations into account both along the constant production lines and along the production expansion line (a ray from the origin).

The plan of the paper is as follows:

- In the first section, alternative production models are briefly analyzed when the underlying production functions show unitary, constant and varying elasticity of substitution during the sample period. A monotonic transformation of the above homogeneous production functions is studied in a final paragraph.

- In the second section, the impact of technical progress will be studied, e.g. within the framework of a C.E.S. vintage model of production, while

- In the final section some preliminary statistical estimations will be discussed in order to check the production models presented on their relevance for postwar Belgian economy. A more thorough comparison between various statistical estimations will be reserved for a subsequent paper.

--- (5) $Q = F[L, A(t)K]$ and capital combining

d) <u>Input decreasing</u>: labour and capital decreasing where the reduction in labour, resp. capital input is proportional to output.

The above classes are discussed in detail by M.J. Beckmann and R. Sato, [2] and [3]; this discussion also includes important combinations such as <u>factor augmenting technical progress</u>:

(6) $Q = [A(t) L, B(t)K]$.

I. Some neo-classical production models with disembodied technical progress.

According to the four paragraphs of this section, the underlying production relationships of the models under discussion are:

a) Cobb- Douglas (C.D) production function:

(1.1) $Q_t = A\, L_t^{\alpha}\, K_t^{\beta}\, e^{\lambda t}$ where

A is a parameter denoting the scale on which an economy is operating (scale transformation of inputs L_t and K_t into output Q_t)

α and β are the labour and capital coefficients whose sum, representing the degree of homogeneity of the function, can show:

- either increasing returns to scale ($\alpha + \beta > 1$) so that output is increased by a larger proportion than a given proportional increase in all inputs (economies of scale).
- or decreasing returns to scale ($\alpha + \beta < 1$) where output is increased by a smaller proportion (diseconomies of scale)
- or constant returns to scale ($\alpha + \beta = 1$) where output is increased by the same proportion as inputs.

λ is a parameter resulting from disembodied technical progress. In fact, α and β are the partial elasticities of production with respect to their respective factor inputs, which becomes clear from the (positive) marginal products of labour and capital:

(1.2)
$$\frac{\partial Q_t}{\partial L_t} = \alpha\, A L_t^{\alpha-1}\, K_t^{\beta}\, e^{\lambda t} = \alpha\, \frac{Q_t}{L_t}$$

$$\frac{\partial Q_t}{\partial K_t} = \beta\, A L_t^{\alpha}\, K_t^{\beta-1}\, e^{\lambda t} = \beta\, \frac{Q_t}{K_t}$$

or

(1.3)
$$\alpha = \frac{\partial Q_t/Q_t}{\partial L_t/L_t} = \frac{\partial \log Q_t}{\partial \log L_t} \quad \text{and} \quad \beta = \frac{\partial Q_t/Q_t}{\partial K_t/K_t} = \frac{\partial \log Q_t}{\partial \log K_t}$$

so that $\alpha + \beta$ measures the total percentage change in output for a given change in labour and capital.

b) Constant Elasticity of Substitution (C.E.S.) production function (*):

(1.4) $$Q_t = A\,[\,(1-\delta)\,L_t^{-\rho} + \delta K_t^{-\rho}]^{-\frac{\nu}{\rho}}\, e^{\lambda t}$$ where

$A_t = Ae^{\lambda t}$ denoting efficiency of a certain technology t for the combination of labour and capital (variations in A_t are indications of Hicks neutral technological progress)

δ distribution parameter indicating the degree to which the technology is capital intensive and defined over the interval $0 < \delta < 1$ (the larger this capital intensity parameter is, the larger the K/L ratio for all factor prices).

ρ substitution parameter which can be written as $\rho = \frac{1}{\sigma} - 1$, with σ the elasticity of substitution (7) measuring the ease with which the technology permits labour to be substituted for capital. (**)

(*) Function (1.4) with (1.1) as a special case ($\rho = 0$) will be derived in appendix A.

(**) As was the case for the C.D. function, it is easily seen that the (positive) marginal product of each factor is proportional to its average product, because, by differentiating:

(1.5) $$Q_t^{-\frac{\rho}{\nu}} = A^{-\frac{\rho}{\nu}}\,[(1-\delta)\,L_t^{-\rho} + \delta K_t^{-\rho}]\, e^{-\frac{\lambda\rho t}{\nu}}$$ with respect to L_t:

(1.6) $$-\frac{\rho}{\nu}\, Q_t^{-\frac{\rho}{\nu}-1}\,\frac{\partial Q_t}{\partial L_t} = A^{-\frac{\rho}{\nu}}\,[-\rho(1-\delta)L_t^{-\rho-1}]\; e^{-\frac{\lambda\rho t}{\nu}}$$ or

$$\frac{\partial Q_t}{\partial L_t} = (1-\delta)\;\nu A^{-\frac{\rho}{\nu}}\; e^{-\frac{\lambda\rho t}{\nu}}\; Q_t^{\frac{\rho+\nu}{\nu}}\; L_t^{-(\rho+1)}$$

(1.7) $$\frac{\partial Q_t}{\partial K_t} = \delta\nu\, A^{-\frac{\rho}{\nu}}\, e^{-\frac{\lambda\rho t}{\nu}}\, Q_t^{\frac{\rho+\nu}{\nu}}\, K_t^{-(\rho+1)}$$, which shows that, under

constant returns to scale (ν=1), the proportionality factor is a function of t, δ,A,λ and ρ.

c) Modified C.E.S.-functions allowing for changing returns to scale and variable elasticity of substitution (V.E.S.)

(1.8) $Q_t^D = Ae^{\lambda t} \left[(1-\delta)\, L_t^{-\rho} + \delta K_t^{-\rho} \right]^{-\nu/\rho}$ with Q_t^D a deflated or inflated value of Q_t when the returns to scale function is assumed to depend on output.

Various production functions with variable elasticity of substitution may be defined. Almost all are based upon the empirical behaviour of σ. So, if σ is assumed to be linearly dependent on capital intensity:

(1.9) $$\sigma_t = 1 + b \frac{K_t}{L_t}$$

the V.E.S. production function under constant returns to scale becomes:

(1.10) $$Q_t = Ae^{\lambda t} K_t^{\frac{1}{1+c}} \left(L_t + \frac{b}{1+c} K_t\right)^{\frac{c}{1+c}}$$ (*)

When an underlying theory is introduced (say profit maximization or cost minimization), it may be assumed that, for homogeneous production functions of degree one, the output labour ratio is dependent on the relative price ratio and on the capital-labour ratio; then the p.f. becomes:

(1.11) $$Q_t = A\, e^{\lambda t} \left[(1-\delta) \left(\frac{K_t}{L_t}\right)^{-m\rho} L_t^{-\rho} + \delta K_t^{-\rho} \right]^{-\frac{1}{\rho}}$$

with $m = o$ if $\frac{Q_t}{L_t}$ does not depend on $\frac{K_t}{L_t}$. (*)

d) Homothetic production functions:

(1.12) $Q_t = G\,[F(L_t, K_t, t)]$, where G is a monotonic transformation of a homogeneous C.E.S. or V.E.S. production function F of finite degree. Through such a transformation, the interpretation of the elasticity of substitution is enlarged to a class of convex isoquants,

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(*) Derivation: see appendix B.

varying only in scale not in shape (so same σ from isoquant to isoquant). These constant production lines will constitute a homogeneous field on the production surface.(*)
To derive alternative production models, use will be made of some general assumptions regarding the market conditions. Indeed, in order to start as general as possible, it will be assumed that there is "imperfect" competition (monopolistic competition, oligopoly or pure monopoly) on both factor markets and on the product market. Therefore, prices of production and factor inputs are assumed to depend on the quantity of products supplied and on the quantities of production factors demanded. These dependencies are formalized in a demand function for the product and in supply functions for the factors of production, which, for simplicity, are supposed to be loglinear, only involving own-prices, i.e. the demand for production depends only on the price of production itself and not on the prices paid for any of the factors and similarly, the supply of each factor depends only on the price paid for that factor and not on the price of the product or the price paid for any other factor.

So, the aggregate demand function for production and the aggregate supply functions for labour and capital may be expressed conveniently as homogeneous functions where the degrees of homogeneity are equal to the (constant) elasticities of demand and supply (**):

(1.14) $Q_t = k_o \, p_t^{\eta_o}$

$L_t = k_1 \, w_t^{\eta_1}$ and $K_t = k_2 \, r_t^{\eta_2}$ with

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(*) This homogeneous field of production isoquants emerges because, for given factor prices (perfect competition), the capital labour ratio at period t remains constant irrespective of the production level on which the economy is situated or

(1.13) $\dfrac{\partial \log (K_t/L_t)}{\partial \log Q_t} = 0$ See derivation of (1.12) in appendix C.

(**) For simplicity, the price elasticities of demand and supply are supposed to be constant over time so that they are invariant for changes in product demand and factor supplies (or elasticities are in fact mean elasticities). This assumption can, however, easily be released by providing the η's in (1.14) with a subscript t.

η_o, η_1, η_2 the constant price elasticities of demand and supply ($\eta_o \leq 0$; $\eta_1, \eta_2 \geq o$)

p_t, w_t, r_t resp. the price of production, the wage rate of labour and the cost of using one unit of capital services (prices expressed in indexes).(*)

From (1.14), the total revenue from scales of the production and the total expenditure on factors L and K may be expressed as:

$$p_t\, Q_t = \left(\frac{1}{k_o}\right)^{\frac{1}{\eta_o}} Q_t^{\frac{1}{\eta_o}+1} = s_o\, Q_t^{\ell_o} \qquad (\ell_o \leq 1)$$

(1.16) $$w_t\, L_t = \left(\frac{1}{k_1}\right)^{\frac{1}{\eta_1}} L_t^{\frac{1}{\eta_1}+1} = s_1\, L_t^{\ell_1} \qquad (\ell_1 \geq 1)$$

$$r_t\, K_t = \left(\frac{1}{k_2}\right)^{\frac{1}{\eta_2}} K_t^{\frac{1}{\eta_2}+1} = s_2\, K_t^{\ell_2} \qquad (\ell_2 \geq 1)\ (**)$$

where the relating ℓ_i ($i = o,1,2,$) is equal to one if there is perfect competition in the corresponding market.

(*) Since the degree m of monopoly (monopolistic power) for a certain economy may be measured by the inverse (absolute) price elasticity (see Lerner), or $m_i = \frac{1}{|\eta_i|}$ ($i = 0,1,2$), all kinds of market structures ranging from perfect competition ($m_i = o$ as $|\eta_i| = \infty$) to pure monopolism ($m_i = \infty$ as $\eta_i = o$) might be represented as a point on the non-negative half-line. But since price formation under short run monopolistic equilibrium (marginal cost equal to marginal return) is generally reflected by:

(1.15) $$m_i = \frac{1}{|\eta_i|} = \frac{\text{price - marginal cost}}{\text{price}}$$, the upper bound of m_i is generally given by $m_i = 1$, i.e. $|\eta_i| = 1$, otherwise there exists a negative marginal cost. Then, "perfect" monopolism corresponds to a value $m_i = 1$ ($o \leq m_i \leq 1$ or $|\eta_i| \geq 1$).

(**) Ideally, it would be preferrable to consider the rental price (return) of a unit of capital, say r'_t , defined as

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With help of the above three quantity - price relationships, we shall discuss now alternative production models, varying (besides the underlying production function) according to the objective the producing economy has in mind i.e.

1. Maximization of deterministic profit
2. Maximization of median profit in comparison to maximization of the mathematical expectation of profit
3. Maximization of total revenue from sales (oligopolistic industries)
4. Minimization of total factor costs (regulated economies)

A. Cobb- Douglas Models.

§ A 1. Classical Profit Maximization

For each period of time, the economy is placed before the following situation (★):

(1.17) $\max \pi = pQ - wL - rK$ subject to

$$Q = AL^{\alpha} K^{\beta} e^{\lambda t}$$

which leads to the following necessary profit maximizing conditions (equilibrium conditions in the short run), utilizing equations (1.16):

(1.18)
$$\frac{\partial \pi^{\star}}{\partial Q} = \ell_o p - \mu = o$$
$$\frac{\partial \pi^{\star}}{\partial L} = -\ell_1 w + \mu\alpha \frac{Q}{L} = o$$
$$\frac{\partial \pi^{\star}}{\partial K} = -\ell_2 r + \mu\beta \frac{Q}{K} = o$$
$$\frac{\partial \pi^{\star}}{\partial \mu} = AL^{\alpha}K^{\beta}e^{\lambda t} - Q = o$$

($\pi^{\star}$ = Lagrange function and μ = Lagrange parameter)

(1.17) $r'_t = P^K_t (i + d)$, where i and d are the interest and depreciation rates, in stead of $P^K_t = r_t$ the cost of using one capital unit (price of equipment). This would not complicate matters however since the total expenditure for capital, $r_t K_t$ in (1.16), is simply substituted by $r'_t K_t$ then.

(★)For ease of notation, we dropped the index t whereever possible.

From system (1.18), it follows immediately that:

(1.19) $\alpha = \frac{\ell_1 wL}{\ell_o pQ}$ and $\beta = \frac{\ell_2 rK}{\ell_o pQ}$, so that the partial elasticities of production with respect to labour and capital are precisely equal to the shares of resp. nominal wages and nominal capital in total nominal production if there exists perfect competition on all markets or if all ℓ_i are equal (*).

Sufficient conditions for a maximum of model (1.17), implying a decrease of the marginal products as each factor changes, can be derived from the usual second order conditions for maximum and the (strict) convexity of the production function:

$$\frac{\partial^2\pi^*}{\partial L^2} = p\ell_o\alpha\ (\ell_o\alpha - \ell_1)\frac{Q}{L^2} < o\ ,\quad \frac{\partial^2\pi^*}{\partial K^2} = p\ell_o\beta(\ell_o\beta - \ell_2)\ \frac{Q}{K^2} < o \quad \text{and}$$

(1.20)

$$\ell_o\alpha\ell_o\beta\ (\ell_o\alpha - \ell_1)(\ell_o\beta - \ell_2)\ \frac{Q^2}{L^2K^2} > (\ell_o\alpha\ell_o\beta)^2\ \frac{Q^2}{L^2K^2} \quad (**)$$

from which it follows that <u>only decreasing returns</u> to scale ($\alpha+\beta<1$) are compatible with profit maximization under conditions of perfect competition in all markets and a strict convex production function (see convexity of isoquants).

When imperfect competition, a decision regarding the economies of scale is not so easy to make because, from the sufficiency conditions (1.20), it follows that, for $\ell_1,\ell_2 \geq 1, \alpha,\beta > o$ and for $o \leq \ell_o \leq 1$ (ℓ_o should be nonnegative, i.e. the demand for production should be elastic because otherwise output will be restricted until either the production ceases or the demand for its product ceases - see also footnote on p. 7 regarding emerging negative marginal cost):

(*) From the marginal productivity conditions (1.2) it is also clear then that the marginal products are equal to the relative prices $\frac{w}{p}$ and $\frac{r}{p}$.

(**) Note that the first and third conditions of (1.20) are equivalent to the (strict) concavity of the Lagrange function π^* (positive definiteness of the matrix of 2nd order derivatives of $-\pi^*$).

(1.21) $\alpha < \frac{\ell_1}{\ell_o}$, $\beta < \frac{\ell_2}{\ell_o}$ and $\alpha \frac{\ell_o}{\ell_1} + \beta \frac{\ell_o}{\ell_2} < 1$, or at the minimum level (perfect competition on factor markets : $\ell_1 = \ell_2 = 1$):

(1.22) $\alpha + \beta < \frac{1}{\ell_o}$, so that for the case of perfect competition on the factor markets and monopolistic competition on the product market (which is an often occurring situation, especially since most industries are established in the neighbourhood of large consumption centres like regions with a dense population, etc.) there will be always possibility for non-decreasing returns to scale.

To specify the imperfectness and the unreliability of profit maximization, i.e. the incapability of the entrepreneurs to adjust inputs to satisfy the necessary conditions for profit maximization, random disturbance terms v_{1t} and v_{2t} are introduced into the equations for derived demand for capital and labour (economic disturbances corresponding to managerial capacity, relevance of perfect profit maximization in a certain period, etc...), while random disturbance terms u_t are super-imposed to the production function in order to reflect factors as exogenous effects (foreign strikes, war...), will, effort, special technical knowledge etc. which are characteristic for a certain time period.
So, the randomnization of the production model, involved in (1.18), may be carried out, rewriting the necessary conditions as follows:

$$Q_t = AL_t^{\alpha} K_t^{\beta} e^{\lambda t}$$

(1.23) $$L_t^{\ell_1} = Q_t^{\ell_o} \frac{\alpha \ell_o s_o}{\ell_1 s_1}$$

$$K_t^{\ell_2} = Q_t^{\ell_o} \frac{\beta \ell_o s_o}{\ell_2 s_2}$$; or in the above mentioned stochastic specification:

$$\log Q_t = \log A + \alpha \log L_t + \beta \log K_t + \lambda t + \varepsilon_t$$

$$(1.24) \quad \log L_t = \frac{\ell_o}{\ell_1} \log Q_t + \frac{1}{\ell_1} \log \frac{\alpha \ell_o s_o}{\ell_1 s_1} + \frac{1}{\ell_o} v_{1t}$$

$$\log K_t = \frac{\ell_o}{\ell_2} \log Q_t + \frac{1}{\ell_2} \log \frac{\beta \ell_o s_o}{\ell_2 s_2} + \frac{1}{\ell_2} v_{2t}$$

from which it is clear that the random disturbance terms are introduced in exponential form into model (1.23). They are supposed to be serially independently distributed with finite moments (*).

Passing, it may be interesting to note that a set of equations basically similar to (1.24) is arrived at if the disturbance terms v_{1t} and v_{2t}, together with other ones, v_{ot}, do not possess the interpretation given previously, but represent the imperfections in the specification of the demand and supply equations (1.16) (e.g. other influencing prices than own-prices, relevance of exogenous components, etc....):

$$Q_t = A L_t^{\alpha} K_t^{\beta} e^{\lambda t} u_t$$

$$p_t Q_t = s_o Q_t^{\ell_o} v'_{ot}$$

$$(1.25) \quad w_t L_t = s_1 L_t^{\ell_1} v'_{1t}$$

$$r_t K_t = s_2 K_t^{\ell_2} v'_{2t}$$

or transforming to model (1.23), we get the same specification as the log-transformed model (1.24), except for the error terms, which become:

(*) Note that if $u_t = e^{\varepsilon t}$ with ε_t a NID - disturbance term with zero mean and constant variance σ^2, the mathematical expectation of u_t is $E(u_t) = e^{E(\varepsilon_t) + \frac{\sigma^2}{2}} = e^{\frac{1}{2}\sigma^2}$ and its median: $M(u_t) = e^{M(\varepsilon_t)} = 1$. We say that u_t is lognormally distributed. The lognormal p.d.f. shows positive skewness since $M(u_t) = E(u_t)\, e^{-\frac{1}{2}\sigma^2} \leq E(u_t)$ so that equality only holds if the producer has exact a priori knowledge about his technical(production) relation (then $\sigma^2 = o$ and equivalence with anticipated profit maximization: see A2). In general, if $E(\varepsilon_t) = o$, $E(u_t) = \phi(1)$ where $\phi(.)$ is the moment generating function of ε_t.

(1.26) $v_{1t} = -\log\left(\frac{v'_{1t}}{v'_{ot}}\right)$ and $v_{2t} = -\log\left(\frac{v'_{2t}}{v'_{ot}}\right)$ (*)

In the light of the above interpretation of the error terms u_t, v_{1t} and v_{2t}, it is pretty clear that $\log L_t$ and $\log K_t$ are, in fact, not completely independent of $\log u_t = \varepsilon_t$, since each input is a function of all disturbances of the system. Therefore, classical least squares of the production function parameters are biased and, in general, even inconsistent. Requiring the assumption that v_{1t} and v_{2t} are normally distributed, independently of ε_t, may still yield consistent maximum likelihood estimates, while, if there exists perfect competition on all markets, the factor shares method even devises unbiased and efficient estimators of the production function parameters. Discussion of further estimation problems is postponed until the final section and a subsequent paper, but it may be clear from the above reasoning that there are both statistical and economic reasons for accepting an alternative objective where probabilistic elements, being beyond the control of the producer, and mainly regarding the stochastic nature of production, are introduced.(**)

§ A 2. Maximization of anticipated versus expected profit.

Assuming that the prices depend on output demanded and factor inputs supplied as given by equations (1.16), we may consider, as an apparently valuable hypothesis, that the producer (the economy) wants:

(*) For maximum likelihood estimation of model (1.24), we assume that all disturbance terms u_t, v'_{it} are lognormally distributed with mean $e^{\frac{1}{2}\sigma_i^2}$ (i = 0, 1, 2) and finite variances. Note also that, if v'_{ot}, v'_{1t} and v'_{2t} are supposed to follow (mutually independent) lognormal distributions, then so will their ratios $\frac{v'_{1t}}{v'_{ot}}$ and $\frac{v'_{2t}}{v'_{ot}}$ do (Theorem 2.1 of [1]).

(**) These stochastic components of production include factors as weather, unpredictable variations in machine or labour performance, etc..... Although a disturbance vector u has been intoduced into the C.D.-function, we derived all properties of the previous model as if production were non-stochastic with $u_t = 1$ ($\forall t$).

- either to maximize the anticipated profit for period t (★) or
- to maximize the mathematical expectation of this period's profit.

The difference with the previous problem lies in the two following assumptions:

(i) Entrepreneurs are maximizing anticipated, resp. expected profits instead of actual profits

(ii) A stochastic production function is directly introduced into the maximizing behaviour.

According to either objective, entrepreneurs are faced with the following model:

$$\max \overline{\Pi}_t = (\overline{p_t Q_t}) - (\overline{w_t L_t}) - (\overline{r_t K_t})$$

$$(1.27a) \qquad = (\overline{s_o Q_t^{\ell_o}}) - (\overline{s_1 L_t^{\ell_1}}) - (\overline{s_2 K_t^{\ell_2}}) \qquad \text{or}$$

$$\max \ E(\Pi_t) = E(p_t Q_t) - E(w_t L_t) - E(r_t K_t)$$

$$(1.27b) \qquad = E(s_o Q_t^{\ell_o}) - E(s_1 L_t^{\ell_1}) - E(s_2 K_t^{\ell_2})$$

subject to

$$(1.28) \quad Q_t = A \ L_t^{\alpha} \ K_t^{\beta} \ e^{\lambda t} \ u_t$$

where a - above the variable (s) indicates "anticipated value" (★★)

(★) Which might be the case if the producer had exact knowledge about his technical and commercial functions. Obviously, it may and generally will diverge from real profits (e.g. imperfections in anticipited profit maximization).

(★★) Notice that we do not introduce a priori the rather unrealistic assumption that p_t (or w_t, r_t) is statistically independent of the production function disturbance u_t nor that there should exist perfect competition on the factor markets as in fact did A. Zellner, J.Kmenta and J. Drèze [21], p. 787, for a cross-section model. See also D.Hodges [9], who relaxed, for the case of C.E.S.-functions, the strong requirement for perfect competition on the product market but left the other assumptions untouched.

The difference between the two submodels (1.27) can be reduced, under certain general assumptions, to the difference between conditional median functions and conditional mean functions. Indeed, if the production function disturbance terms $\log u_t = \varepsilon_t$ are assumed to be independently normally distributed with zero mean and finite variance σ^2, we know, from properties of the resulting lognormal distribution of u_t, that:

- the anticipated value of $s_o Q^{\ell_o}$ can nicely be defined by its conditional median value or:

$$(1.29a)\quad \overline{(s_o Q_t^{\ell_o})} = M(s_o Q_t^{\ell_o}) = s_o M(Q_t^{\ell_o}|L_t,K_t) = s_o A^{\ell_o} L_t^{\alpha\ell_o} K_t^{\beta\ell_o} e^{\ell_o\lambda t} M(e^{\ell_o\varepsilon_t})$$

$$= s_o Q_t^{\ell_o} e^{-\ell_o\varepsilon_t} \quad (\star)$$

- and the mathematical expectation of $s_o Q_t^{\ell_o}$ can be defined as the conditional mean value:

$$(1.29b)\quad E(s_o Q_t^{\ell_o}) = s_o E(Q_t^{\ell_o}|L_t,K_t) = s_o A^{\ell_o} L_t^{\ell_o\alpha} K_t^{\ell_o\beta} e^{\ell_o\lambda t} e^{\frac{1}{2}\ell_o^2\sigma^2}$$

$$= s_o Q_t^{\ell_o} e^{\frac{1}{2}\ell_o^2\sigma^2 - \ell_o\varepsilon_t}$$

Remark that the two models are only identical if $\sigma^2 = o$, i.e. if the producers exactly know their production factor combinations ex ante (also identity between anticipation and expectation in the probability limit). Obviously, this will seldom be the case. Therefore, we shall derive, for both models, the profit maximizing conditions and the reduced form equations in order to check whether L_t and K_t are

(★) Where it is assumed, initially, that u_t is statistically independent of L_t, K_t because u_t does not occur in the supply equations for factor inputs. This assumed independency will be verified later on. Notice also that the anticipated values of the nominal factor values can be put equal to their imperfectly competitive equivalents $s_1 L_t^{\ell_1}$ and $s_2 K_t^{\ell_2}$, where evt. error terms regarding the imperfections in the specification of these supply functions are cancelled out.

really statistically independent of u_t (so that consistent and even unbiased estimators for the production function parameters might be obtained).

From the profit maximizing conditions:

$$(1.30)\quad \frac{\partial \overline{\Pi}_t}{\partial L_t} = o \quad \text{and} \quad \frac{\partial \overline{\Pi}_t}{\partial K_t} = o \quad \text{resp.} \quad \frac{\partial E(\Pi_t)}{\partial L_t} = o \quad \text{and} \quad \frac{\partial E(\Pi_t)}{\partial K_t} = o,$$

we find for the "anticipations" model, after introducing error terms v_{it} $(i = 1,2)$ standing for differences between anticipated and realized prices, imperfactions in the (anticipated, resp. expected) profit maximization (managerial inertia, ignorance....):

$$\log Q_t = \log A + \alpha \log L_t + \beta \log K_t + \lambda t + \varepsilon_t$$

$$(1.31a)\quad \log L_t = \frac{1}{\ell_1} \log \alpha\frac{s_o \ell_o}{s_1 \ell_1} + \frac{\ell_o}{\ell_1} \log Q_t - \frac{\ell_o}{\ell_1} \varepsilon_t + v_{1t}$$

$$\log K_t = \frac{1}{\ell_2} \log \beta\frac{s_o \ell_o}{s_2 \ell_2} + \frac{\ell_o}{\ell_2} \log Q_t - \frac{\ell_o}{\ell_2} \varepsilon_t + v_{2t}$$

which is, apart from the error term contents, equivalent to (1.24).

The reduced form equations for the endogenous variables of the production model are:

$$(1.32)\quad \begin{bmatrix} \log Q_t \\ \log L_t \\ \log K_t \end{bmatrix} = \begin{bmatrix} 1 & -\alpha & -\beta \\ -\frac{\ell_o}{\ell_1} & 1 & o \\ -\frac{\ell_o}{\ell_2} & o & 1 \end{bmatrix}^{-1} \begin{bmatrix} \log A & \lambda \\ k_1 & o \\ k_2 & o \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} +$$

$$+ \begin{bmatrix} 1 & -\alpha & -\beta \\ -\frac{\ell_o}{\ell_1} & 1 & 0 \\ -\frac{\ell_o}{\ell_2} & o & 1 \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_t \\ -\frac{\ell_o}{\ell_1} \varepsilon_t + v_{1t} \\ -\frac{\ell_o}{\ell_2} \varepsilon_t + v_{2t} \end{bmatrix}$$

with $k_1 = \frac{1}{\ell_1} \log \alpha \frac{s_o \ell_o}{s_1 \ell_1}$ and $k_2 = \frac{1}{\ell_2} \log \beta \frac{s_o \ell_o}{s_2 \ell_2}$ or

$$\log Q_t = \frac{1}{(1-\frac{\ell_o}{\ell_1}\alpha-\frac{\ell_o}{\ell_2}\beta)}\left[\log A + \alpha k_1 + \beta k_2 + \lambda t + (1 - \frac{\ell_o}{\ell_1}\alpha - \frac{\ell_o}{\ell_2}\beta)\varepsilon_t + \alpha v_{1t} + \beta v_{2t}\right]$$

(1.33)

$$\log L_t = \frac{1}{(1-\frac{\ell_o}{\ell_1}\alpha-\frac{\ell_o}{\ell_2}\beta)}\left[\frac{\ell_o}{\ell_1}\log A + (1-\beta\frac{\ell_o}{\ell_2})k_1 + \beta\frac{\ell_o}{\ell_1}k_2 + \frac{\ell_o}{\ell_1}\lambda t + (1-\beta\frac{\ell_o}{\ell_2})v_{1t} + \beta\frac{\ell_o}{\ell_1}v_{2t}\right]$$

$$\log K_t = \frac{1}{(1-\frac{\ell_o}{\ell_1}\alpha-\frac{\ell_o}{\ell_2}\beta)}\left[\frac{\ell_o}{\ell_2}\log A + \alpha\frac{\ell_o}{\ell_2}k_1 + (1-\alpha\frac{\ell_o}{\ell_1})k_2 + \frac{\ell_o}{\ell_2}\lambda t + \alpha\frac{\ell_o}{\ell_2}v_{1t} + (1-\alpha\frac{\ell_o}{\ell_1})v_{2t}\right],$$

while the structural equations for the "expectations model", introducing the same kind of error terms v'_{it} $(i = 1,2)$ (*):

(1.31b)

$$\log Q_t = \log A + \alpha \log L_t + \beta \log K_t + \lambda t + \varepsilon_t$$

$$\log L_t = \frac{1}{\ell_1}\log\alpha\frac{\ell_o s_o}{\ell_1 s_1} + \frac{\ell_o}{\ell_1}\log Q_t + \tfrac{1}{2}\frac{\ell_o^2}{\ell_1}\sigma^2 - \frac{\ell_o}{\ell_1}\varepsilon_t + v'_{1t}$$

$$\log K_t = \frac{1}{\ell_2}\log\beta\frac{\ell_o s_o}{\ell_2 s_2} + \frac{\ell_o}{\ell_2}\log Q_t + \tfrac{1}{2}\frac{\ell_o^2}{\ell_2}\sigma^2 - \frac{\ell_o}{\ell_2}\varepsilon_t + v'_{2t},$$

(*) Remark that we treat the expected nominal factor values in the same way as the anticipated nominal factor values. Differences are symbolized bij de different error terms v'_{it} $(i = 1,2)$

which, apart from the constant and error terms, is equivalent to (1.31a).

So, putting $k_1' = \frac{1}{\ell_1} \log \alpha \frac{\ell_o s_o}{\ell_1 s_1} + \frac{1}{2} \frac{\ell_o^2}{\ell_1} \sigma^2$ and

$$k_2' = \frac{1}{\ell_2} \log \beta \frac{\ell_o s_o}{\ell_2 s_2} + \frac{1}{2} \frac{\ell_o^2 \sigma^2}{\ell_2} ,$$

we find reduced form equations for Q_t, L_t and K_t similar to (1.33), replacing k_1, k_2 and v_{1t}, v_{2t} by k_1', k_2' and v_{1t}', v_{2t}'.

Since the production function disturbance does not enter into the reduced form equations (derived demand equations) for inputs, our initial hypothesis about independency between u_t and L_t, K_t is confirmed and simple least squares estimators of the production function parameters are consistent if $E(\varepsilon_t\, v_{1t}) = E(\varepsilon_t\, v_{2t}) = E(\varepsilon_t\, v'_{1t}) = E(\varepsilon_t\, v'_{2t}) = o$. Simple least squares estimators of A, α, β and λ are also unbiased if v_{1t}, v_{2t}, v_{1t}', v_{2t}' are statistically independent of ε_t. Further estimation problems regarding this model will be discussed in the final section and in a subsequent paper.

§ A 3 Maximization of total revenue from sales.

The objective function is now the commercial function being predominant in industries with a rather considerable degree of monopoly. The problem is to maximize

(1.34) $Y_t = p_t\, Q_t$ subject to

(1.35) a minimal profit level $\Pi_t^o = s_o Q_t^{\ell_o} - s_1 L_t^{\ell_1} - s_2 K_t^{\ell_2}$ (see 1.16)

a C.D.- production function $Q_t = A L_t^{\alpha} K_t^{\beta} e^{\lambda t}$

which, for a certain period t, results in the following system of necessary conditions:

(1.36)
$$\frac{\partial Y^*}{\partial Q} = \ell_o p + \mu_1 \ell_o p - \mu_2 = o$$

$$\frac{\partial Y^*}{\partial L} = -\mu_1 \ell_1 w + \mu_2\ \alpha\ \frac{Q}{L} = o$$

$$\frac{\partial Y^*}{\partial K} = -\mu_1 \ell_2 r + \mu_2\ \beta\ \frac{Q}{K} = o$$

$$\frac{\partial Y^*}{\partial \mu_1} = \frac{\partial Y^*}{\partial \mu_2} = o \quad \text{(restrictions 1.35)}$$

where Y^* is the Lagrange function and μ_1 and μ_2 the Lagrange parameters. From (1.36) it follows that the partial elasticities of production with respect to labour and capital are respectively:

(1.37)
$$\alpha = \frac{\ell_1}{\ell_o} \quad \frac{\mu_1}{1+\mu_1} \quad \frac{wL}{pQ}$$

$$\beta = \frac{\ell_2}{\ell_o} \quad \frac{\mu_1}{1+\mu_1} \quad \frac{rK}{pQ} \quad ,$$

which are, apart from $\frac{\mu_1}{1+\mu_1}$, similar to (1.19) (deterministic profit maximization).

Also the 2nd order stability conditions are similar to the deterministic profit case.
Since $o \leq \ell_o \leq 1$, $\ell_1, \ell_2 \geq 1$ and $\frac{\mu_1}{1+\mu_1} < 1$, the elasticities of production are always smaller than the factor shares if there exists perfect competition or if all l_i's are equal, while they may be greater than or equal to the factor shares depending on the values of ℓ_i $(i = 0, 1, 2)$ and μ_1.

Model (1.36) may stochastically be specified in the same way as (1.23) and (1.24), so from (1.36), (1.37) and substituting with (1.16):

$$\log Q_t = \log A + \alpha \log L_t + \beta \log K_t + \varepsilon_t$$

$$(1.38)\quad \log L_t = \frac{\ell_o}{\ell_1}\log Q_t + \frac{1}{\ell_1}\log \alpha\,\frac{\ell_o s_o}{\ell_1 s_1}\cdot\frac{\mu_1+1}{\mu_1} + \frac{1}{\ell_1}v_{1t}$$

$$\log K_t = \frac{\ell_o}{\ell_2}\log Q_t + \frac{1}{\ell_2}\log \beta\,\frac{\ell_o s_o}{\ell_2 s_2}\cdot\frac{\mu_1+1}{\mu_1} + \frac{1}{\ell_2}v_{2t}$$

Obviously, instead of deterministic sales revenue maximization, also the expected value of total revenue from salws could be maximized. Since, however, the derivation of the production model is completely similar to (1.38) and (1.31b), it will not be discussed here.

§ A 4 Minimization of total factor costs.

This managerial goal, especially relevant for regulated industries where the producer knows, ex ante, the production demanded so that production can explicitly be fixed at an exogenously determined level, leads to the following model:

$$(1.39)\quad \min C_t = w_t L_t + r_t K_t$$

subject to a C.D.-function determining the combination of production factors in order to yield the partly exogenously fixed production(*).

The industry has to find the optimal requirement for factors of production, which may be realized by deriving the equations for their derived factor demand.

(*) The above problem of cost minimization is equivalent to maximization of the profit function:

$$(1.40)\quad \Pi_t = p_t\, Q_t - w_t\, L_t - r_t K_t \text{ subject to the conditions that}$$

$$(1.41)\quad Q_t = AL_t^{\alpha}\, K_t^{\beta}\, e^{\lambda t}$$

$Q_t = Q_t^{*}$ (fixed volume of production), which leads, under perfect competition, to the property that the marginal return is equal to the Lagrange parameter (equal to the marginal cost)(see also (1.45)). Compare this property with the previous case of sales revenue maximization which can be brought back to the maximization of the following Lagrange function:

$$(1.42)\quad Y_t^{*} = p_t\,(AL_t^{\alpha}\, K_t^{\beta}\, e^{\lambda t}) - C_t^{o} + \mu(w_t L_t + r_t K_t - C_t^{o})$$

These reduced form equations may be derived from minimization of

(1.43) $C_t = w_t L_t + r_t K_t = s_1 L_t^{\ell_1} + s_2 K_t^{\ell_2}$ (1.16)

subject to

$Q_t = Q_t^{*} u_t = A L_t^{\alpha} K_t^{\beta} e^{\lambda t} u_t$, where

Q_t^{*} is the systematic ("exogenous", non-stochastic) part of production (i.e. "planned" or "fixed" production)

u_t is a random error term with finite mean and variance.

The necessary conditions for the minimum of the Lagrange function:

(1.44) $C_t^{*} = s_1 L_t^{\ell_1} + s_2 K_t^{\ell_2} - \mu(A L_t^{\alpha} K_t^{\beta} e^{\lambda t} u_t - Q_t)$ are, besides the stochastic production function:

$$\frac{\partial C_t^{*}}{\partial L_t} = \ell_1 w_t - \mu\alpha \frac{Q_t}{L_t} = 0$$

(1.45)

$$\frac{\partial C_t^{*}}{\partial K_t} = \ell_2 r_t - \mu\beta \frac{Q_t}{K_t} = 0 \quad ,$$

which are similar to the deterministic profit maximizing conditions (1.18). In fact cost minimization is equivalent to deterministic profit maximization regarding to the equilibrium conditions (1.18) and (1.45) because it becomes clear from the footnote on p. 19 and from (1.18 - 1.44) that producer's marginal cost $\frac{\partial C_t^{*}}{\partial Q_t} = \mu$ is equal to the output price times a constant ℓ_o , measuring the degree of monopolistic competition on the product market, or, from the cost minimizing conditions (1.45) total factor costs become:

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where is has been assumed that the economy can only spend a certain amount of cost, say C_t^{o}, per period. Perfect competition now leads to the marginal equilibrium condition $\mu = 1$ which becomes clear from the necessary maximum conditions of (1.42).

$$(1.46)\quad C_t = p_t Q_t \left(\frac{\ell_o}{\ell_1}\alpha + \frac{\ell_o}{\ell_2}\beta \right),$$

and the marginal productivity conditions (1.2) for the non-stochastic portion of output, provided with an error term v_t reflecting incomplete cost minimization, may be rewritten as:

$$(1.47)\quad \frac{\partial Q^*_t/\partial L_t}{\partial Q^*_t/\partial K_t} = \frac{\partial Q_t/\partial L_t}{\partial Q_t/\partial K_t} = \frac{\partial K_t}{\partial L_t} = \frac{\ell_1 w_t}{\ell_2 r_t} = \frac{\alpha}{\beta}\frac{K_t}{L_t}\, v_t,$$ where v_t is also

supposed to have finite 1st and 2nd moments. (*)

Substitution of the C.D. production function into (1.47) and solution for the endogenous variables L and K leads to the derived demand equations of the "production" model, where production itself, the wage rate and the capital interest rate (return on capital) are treated as exogenous variables:

$$(1.48)\quad \begin{aligned} L_t &= \frac{\ell_2\alpha\; r_t K_t}{\ell_1\beta\; w_t}\, v_t = \frac{\ell_2\alpha}{\ell_1\beta}\left(\frac{r_t}{w_t}\right) A^{-\frac{1}{\beta}} e^{-\frac{\lambda}{\beta}t} L_t^{-\frac{\alpha}{\beta}} Q_t^{*\frac{1}{\beta}} v_t \\ &= \left(\frac{\ell_2\alpha}{\ell_1\beta}\right)^{\frac{\beta}{\alpha+\beta}} A^{-\frac{1}{\alpha+\beta}} e^{-\frac{\lambda}{(\alpha+\beta)}t} \left(\frac{r_t}{w_t}\right)^{\frac{\beta}{\alpha+\beta}} Q_t^{*\frac{1}{\alpha+\beta}} v_t^{\frac{\beta}{\alpha+\beta}} \\ K_t &= \frac{\ell_1\beta\; w_t}{\ell_2\alpha\; r_t} L_t v_t^{-1} = \left(\frac{\ell_1\beta}{\ell_2\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} A^{-\frac{1}{\alpha+\beta}} e^{-\frac{\lambda}{(\alpha+\beta)}t} \left(\frac{w_t}{r_t}\right)^{\frac{\alpha}{\alpha+\beta}} Q_t^{*\frac{1}{\alpha+\beta}} v_t^{-\frac{\alpha}{\alpha+\beta}} \quad (**) \end{aligned}$$

(*) Remark also the striking similarity between the above cost minimization model and the expected profit maximization model under § A 2. Indeed, one could argue that entrepreneurs attempt to minimize the costs of producing an expected level of output since, in fact, output is a stochastic variable. The problem is easily solved in terms of the above cost minimization problem by replacing Q^*_t by $E(Q_t)$ so that condition (1.47) is equivalent for both models.

(**) Rewriting the reduced form equations (1.48) as:

$$(1.48b)\quad L_t = (\ell_2\alpha)\left[A(\ell_2\alpha)^{\alpha}(\ell_1\beta)^{\beta}e^{\lambda t}\right]^{-\frac{1}{\alpha+\beta}} \frac{w_t^{\frac{\alpha}{\alpha+\beta}}\, r_t^{\frac{\beta}{\alpha+\beta}}}{w_t} Q_t^{*\frac{1}{\alpha+\beta}} v_t^{\frac{\beta}{\alpha+\beta}}$$

$$K_t = (\ell_1\beta)\left[A(\ell_2\alpha)^{\alpha}(\ell_1\beta)^{\beta}e^{\lambda t}\right]^{-\frac{1}{\alpha+\beta}} \frac{w_t^{\frac{\alpha}{\alpha+\beta}}\, r_t^{\frac{\beta}{\alpha+\beta}}}{r_t} Q_t^{*\frac{1}{\alpha+\beta}} \underline{v_t^{\frac{\beta}{\alpha+\beta}}}\, v_t^{-1},$$

Notice that, if we replace Q_t^* by $Q_t u_t^{-1}$ in (1.48), the factor quantities are no longer independent of the production function disturbances so that, in general, neither an unbiased nor a consistent estimate of the production function itself can be obtained. The production function parameters have to be estimated from (1.48) and (1.49) then.

Up to now, we have discussed a theoretical framework for C.D. production function models. In the stochastic specification (s) of each alternative model, we have introduced error terms in a multiplicative form. This has been done to provide convenient estimability properties. It may be noticed, however, that other ways of introducing error terms, say in additive form, might be considered. But this complicates the estimation of the various production models without any real gain in analyzing capacity unless we have strong prior belief that the error terms should be introduced in that particular form. In the paper announced, "multiplicative" and "additive" production models will be compared upon their estimating performance.

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and substituting into (1.39), we obtain the derived total cost function:

$$(1.49)\quad C_t = (\ell_2\alpha + \frac{\ell_1\beta}{v_t}) \left[A(\ell_2\alpha)^{\alpha} (\ell_1\beta)^{\beta} e^{\lambda t}\right]^{-\frac{1}{\alpha+\beta}} w_t^{\frac{\alpha}{\alpha+\beta}} r_t^{\frac{\beta}{\alpha+\beta}} Q_t^{*\frac{1}{\alpha+\beta}} v_t^{\frac{\beta}{\alpha+\beta}}$$

and by duality between cost and production functions(see also following section: II A), there is a unique relationship (one to one correspondence) between the empirical cost function (1.49) and the underlying production function in (1.43). Consequently, the production function parameters might be estimated from (1.49).

B. Constant Elasticity of Substitution Models

In the previous paragraph, it was assumed, by the unitary hypothesis for σ, that eventually occurring changes in the relative supplies of factors did not affect the relative factor shares, so that a relative change in (weighted) factor prices was assumed to be automatically compensated by a same movement, in opposite direction, of the production factors. Consequently,the factor shares of labour and capital in the total value added are supposed to remain constant. It is generally know, however, that labour's share in national income has considerably increased in most developed countries(*). So, there is strong evidence for a more general framework.

In first instance, we shall restrict ourselves to C.E.S.-functions, while in the next paragraphs homogeneous productions functions with variable elasticity of substitution and changing returns to scale and homothetic production functions with C.E.S. or V.E.S. functions as underlying p.f.'s will briefly be discussed.

Since the models to be derived under alternative economic goals are very similar to the corresponding C.D. - models, we shall only briefly summarize two cases:

(i) deterministic profit maximization under imperfect competition with derivation of the relating cost function and the derived reduced form equations for production and factors (see similarity between profit maximization and cost minimization under § A);

(ii) maximization of the expected (∼ anticipated) value of profit under imperfect competition.

(*) The formerly accepted belief (during prewar and early postwar period) that labour's share was indeed constant, was partly caused by the intervenience of various compensating effects as technical progress, imperfect competition, etc... and/or by the fact that factor shares are often pretty insensitive to moderate variations in σ (due to factor augmenting technical change, to changes of the capital-labour ratio, etc....), so that their constant character seemed to be consistent with a rather large number of σ's ≠ 1.

§ B 1. Deterministic profit maximization under monopolistic competition.

The problem is to maximize for each period t:

$$(1.50)\qquad \Pi = pQ - wL - rK = s_o Q^{\ell_o} - s_1 L^{\ell_1} - s_2 K^{\ell_2} \quad \text{(see (1.16))}$$

subject to C.E.S.-function (1.4), which yields the first order conditions:

$$\frac{\partial \Pi^*}{\partial Q} = \ell_o p - \mu = o$$

$$\frac{\partial \Pi^*}{\partial L} = -\ell_1 w + \mu (1-\delta)\nu A^{-\frac{\rho}{\nu}} e^{-\frac{\lambda\rho t}{\nu}} Q^{1+\frac{\rho}{\nu}} L^{-\rho-1} = o$$

$$(1.51)\qquad \frac{\partial \Pi^*}{\partial K} = -\ell_2 r + \mu \delta \nu A^{-\frac{\rho}{\nu}} e^{-\frac{\lambda\rho t}{\nu}} Q^{1+\frac{\rho}{\nu}} K^{-\rho-1} = o$$

$$\frac{\partial \Pi^*}{\partial \mu} = A \left[\delta K^{-\rho} + (1-\delta) L^{-\rho}\right]^{-\frac{\nu}{\rho}} e^{\lambda t} - Q = o$$

Substituting $\mu = \ell_o p$ (= producer's marginal cost) into the middle two equations, we obtain the relative factor shares:

$$(1.52)\qquad \begin{aligned} \frac{wL}{pQ} &= \frac{\ell_o}{\ell_1} (1-\delta)\nu A^{-\frac{\rho}{\nu}} e^{-\frac{\lambda\rho t}{\nu}} Q^{\frac{\rho}{\nu}} L^{-\rho} \\ \frac{rK}{pQ} &= \frac{\ell_o}{\ell_2} \delta\nu A^{-\frac{\rho}{\nu}} e^{-\frac{\lambda\rho t}{\nu}} Q^{\frac{\rho}{\nu}} K^{-\rho} \end{aligned} \qquad (*)$$

(*) From (1.7) and (1.52), it becomes clear that the marginal productivity conditions may be written as:

$$(1.53)\qquad \frac{\partial Q_t}{\partial L_t} = \frac{\ell_1}{\ell_o}\frac{w_t}{p_t} \quad \text{and} \quad \frac{\partial Q_t}{\partial K_t} = \frac{\ell_2 r_t}{\ell_o p_t}$$ or the marginal rate of substitution equilibrium condition (for period t) amounts to (see (7)):

$$(1.54)\qquad R_t = -\frac{dK_t}{dL_t} = \frac{\partial Q_t/\partial L_t}{\partial Q_t/\partial K_t} = \frac{\ell_1 w_t}{\ell_2 r_t} = \ell\frac{w_t}{r_t} \quad \text{with } \ell = \frac{\ell_1}{\ell_2},$$

so that the parameter ℓ is a proportionality constant between the factors' relative contributions to production and their relative rates of remuneration. If $\ell_1 < \ell_2$ ($\ell < 1$) labour is being exploited

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or

$$(1.55)\quad \frac{rK}{wL} = \frac{\ell_1}{\ell_2}\ \frac{\delta}{1-\delta}\ \frac{L^\rho}{K^\rho}$$

The equations (1.52) imply that the share of labour (resp. capital) in the total product is proportional to the elasticity of production with respect to labour (capital), where the constant of proportionality is composed of the elasticities of product demand and labour (capital) supply (see (1.7)). Moreover, it is noticed that both labour's and capital's elasticities of production are variable in contrast to the constant elasticities in the C.D.-case (1.19) (★). Remark also that the shares of income accruing to labour, resp. capital are equal to their production elasticities if the competitive degree is the same; i.e. $\ell_o = \ell_1$, $\ell_o = \ell_2$.

The second order conditions for maximum profit, requiring $d^2\Pi^\star < 0$, should satisfy (see (1.20)):

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relative to capital since the marginal product of labour relative to the marginal product of capital is less than their relative price ratio, conversely if $\ell > 1$; if $\ell_1 = \ell_2 (\ell = 1)$ either both factors are subject to inequity in the same degree or neither is at all.

(★) But under the assumption of product exhaustion , i.e. if

$$(1.56)\quad w_t L_t + r_t K_t = p_t Q_t \quad \text{(total revenue = total cost)},$$

and a homogeneous production function of degree ν, implying (Euler):

$$(1.57)\quad \nu Q_t = L_t \frac{\partial Q_t}{\partial L_t} + K_t \frac{\partial Q_t}{\partial K_t}$$ with marginal productivity conditions (1.53) we find that the partial elasticities of production w.r.t. labour and capital are no longer variable unless the elasticities of demand (for output) and/or of supply (for labour and capital) are variable:

$$(1.58)\quad \frac{\partial Q_t}{\partial L_t} \cdot \frac{L_t}{Q_t} = \frac{\ell_1}{\ell_o}\ \frac{(\ell_2 - \nu\ell_o)}{(\ell_2 - \ell_1)} \quad \text{and} \quad \frac{\partial Q_t}{\partial K_t}\ \frac{K_t}{L_t} = \frac{\ell_2}{\ell_o}\ \frac{(\ell_1 - \nu\ell_o)}{(\ell_1 - \ell_2)}$$

$$(1.59)\qquad \frac{\partial^2\Pi^\star}{\partial L^2} < 0 \qquad \frac{\partial^2\Pi^\star}{\partial K^2} < 0 \qquad \text{and} \qquad \frac{\partial^2\Pi^\star}{\partial L^2}\,\frac{\partial^2\Pi^\star}{\partial K^2} > \left(\frac{\partial^2\Pi^\star}{\partial L\partial K}\right)^2,$$

which reduces to the stability conditions for the production elasticities (see (1.21)):

$$(1.60)\qquad \frac{\partial Q}{\partial L}\,\frac{L}{Q}\ (=\alpha \text{ for C.D.}) < \frac{\ell_1+\rho}{\ell_o+\frac{\rho}{\nu}} \qquad \frac{\partial Q}{\partial K}\,\frac{K}{Q}\ (=\beta \text{ for C.D.}) < \frac{\ell_2+\rho}{\ell_o+\frac{\rho}{\nu}} \quad \text{and}$$

$$\frac{\partial Q}{\partial L}\,\frac{L}{Q}\left[\frac{\ell_o+\frac{\rho}{\nu}}{\ell_1+\rho}\right] + \frac{\partial Q}{\partial K}\,\frac{K}{Q}\left[\frac{\ell_o+\frac{\rho}{\nu}}{\ell_2+\rho}\right] < 1,$$

where the last equation states that the weighted sum of the production elasticities should be less than one in order to satisfy a stable equilibrium in an imperfectly competitive C.E.S.-world. The weights themselves include the elasticities of product demand and factor supplies, as was also the case for the imperfectly competitive C.D.-model, but also the elasticity of substitution between labour and capital (for $\rho = \frac{1}{\sigma} - 1$) and the returns to scale parameter ν. From the stability conditions (1.60), it is clear, once more, that increasing returns to scale are compatible with a profit maximization model only if there exists imperfect competition on product or factor markets.

For a stochastic specification of the profit maximizing production model, random disturbances may be introduced into the production function (technical disturbances) and into the profit maximizing conditions (1.52) representing imperfect profit maximization, or alternatively, into the product demand and factor supply equations (see (1.25)) representing imperfections in their specification (economic disturbances).

As was already indicated previously (see § A 1), the profit maximizing conditions can then be written in various, mutually equivalent, specifications as e.g.

B 1.1 : from (1.52) :

(1.61)
$$\frac{w_t L_t}{p_t Q_t} = \frac{\ell_o}{\ell_1} (1-\delta)\nu A^{-\frac{\rho}{\nu}} e^{-\frac{\lambda\rho t}{\nu}} u_t^{-\frac{\rho}{\nu}} Q_t^{\frac{\rho}{\nu}} L_t^{-\rho}$$
$$\frac{r_t K_t}{p_t Q_t} = \frac{\ell_o}{\ell_2} \delta\nu A^{-\frac{\rho}{\nu}} e^{-\frac{\lambda\rho t}{\nu}} u_t^{-\frac{\rho}{\nu}} Q_t^{\frac{\rho}{\nu}} K_t^{-\rho}$$
, where

(1.62) $Q_t = Q_t^{\star} u_t = A_e^{\lambda t} \left[\delta K_t^{-\rho} + (1-\delta) L_t^{-\rho}\right]^{-\frac{\nu}{\rho}} u_t$, with, as before,

$Q_t^{\star} = Q_t u_t^{-1}$ the non-stochastic part of production (~ anticipated production)

B 1.2 : from (1.51) and (1.62)

(1.63)
$$\frac{Q_t}{L_t} = \left(\frac{\ell_1}{\ell_o}\right)^{\frac{1}{\rho+1}} A^{\frac{\rho}{\nu(1+\rho)}} (1-\delta)^{-\frac{1}{1+\rho}} \nu^{-\frac{1}{1+\rho}} e^{\frac{\lambda\rho t}{\nu(1+\rho)}} \left(\frac{w_t}{p_t}\right)^{\frac{1}{1+\rho}} Q_t^{\frac{-\rho(1-\nu)}{\nu(1+\rho)}} u_t^{\frac{\rho}{\nu(1+\rho)}}$$
$$\frac{Q_t}{K_t} = \left(\frac{\ell_2}{\ell_o}\right)^{\frac{1}{\rho+1}} A^{\frac{\rho}{\nu(1+\rho)}} \delta^{-\frac{1}{1+\rho}} \nu^{-\frac{1}{1+\rho}} e^{\frac{\lambda\rho t}{\nu(1+\rho)}} \left(\frac{r_t}{p_t}\right)^{\frac{1}{1+\rho}} Q_t^{\frac{-\rho(1-\nu)}{\nu(1+\rho)}} u_t^{\frac{\rho}{\nu(1+\rho)}}$$

B 1.3 : from a combination of the two equations of (1.63) (division)

(1.64) $$\frac{K_t}{L_t} = \left(\frac{\ell_1}{\ell_2}\right)^{\frac{1}{\rho+1}} \left(\frac{\delta}{1-\delta}\right)^{\frac{1}{\rho+1}} \left(\frac{w_t}{r_t}\right)^{\frac{1}{\rho+1}}$$

or

B 1.4 : from (1.55) and the marginal productivity relationships (1.7):

(1.65) $$\frac{r_t}{w_t} = \frac{\ell_1}{\ell_2}\left(\frac{\delta}{1-\delta}\right)\left(\frac{K_t}{L_t}\right)^{-(\rho+1)} = \left(\frac{\ell_1}{\ell_2}\right)\frac{\partial Q_t^{\star}/\partial K_t}{\partial Q_t^{\star}/\partial L_t} = \left(\frac{\ell_1}{\ell_2}\right)\frac{\partial Q_t/\partial K_t}{\partial Q_t/\partial L_t} = \left(\frac{\ell_1}{\ell_2}\right)\frac{\partial L_t}{\partial K_t} = \frac{\ell_1}{\ell_2 R_t}$$

The above profit maximizing equations can be utilized to derive an estimate for σ :

- from (1.63) by assuming that the prices (w,r,p) are independent of the disturbance vector $u^{\frac{\rho}{\nu(1+\rho)}}$ and of the imperfections in profit maximization denoted by the multiplicative disturbance v
- from (1.64) by assuming that the prices w,r are only independent of a multiplicative error v for imperfections in profit maximization
- from (1.65) on similarly introducing a multiplicative error term, say v_t', also representing incomplete profit maximization or, for this particular case, incomplete cost minimization. (*)

The derived supply function for production and the derived demand equations for factors of production may be obtained from the cost function and the equality, at the equilibrium state, between marginal cost and price (times ℓ_o).
So, from (1.63) and the definition of total factor costs:

$$(1.66)\quad \frac{C_t}{p_tQ_t} = \frac{w_tL_t}{p_tQ_t} + \frac{r_tK_t}{p_tQ_t} =$$

$$= \left[a_t \left(\frac{w_t}{p_t}\right)^{\frac{\rho}{1+\rho}} + b_t \left(\frac{r_t}{p_t}\right)^{\frac{\rho}{1+\rho}}\right] Q_t^{\frac{\rho(1-\nu)}{\nu(1+\rho)}} u_t^{\frac{-\rho}{\nu(1+\rho)}} \quad \text{with}$$

$$a_t = \left(\frac{\ell_1}{\ell_o}\right)^{-\frac{1}{\rho+1}} A^{\frac{-\rho}{\nu(1+\rho)}} (1-\delta)^{\frac{1}{1+\rho}} \nu^{\frac{1}{1+\rho}} e^{\frac{-\lambda\rho t}{\nu(1+\rho)}} \quad \text{and}$$

$$b_t = \left(\frac{\ell_2}{\ell_o}\right)^{-\frac{1}{\rho+1}} A^{\frac{-\rho}{\nu(1+\rho)}} \delta^{-\frac{1}{1+\rho}} \nu^{-\frac{1}{1+\rho}} e^{\frac{-\lambda\rho t}{\nu(1+\rho)}}$$

so that from the Lagrange function:

(*) According to the causality principle however, we have to stick to (1.64) where the economy is assumed to react to current (and past) changes in relative factor prices by altering the factor proportions in order to achieve minimal cost or maximal profit.

$$(1.67)\qquad C_t^{\star} = w_t L_t + r_t K_t - \mu \left[A\, e^{\lambda t} \left(\delta K_t^{-\rho} + (1-\delta) L_t^{-\rho}\right)^{-\frac{\nu}{\rho}} u_t - Q_t \right] ,$$

the producer's marginal cost can be derived as (taking account of the demand function in (1.14)):

$$(1.68)\qquad \frac{\partial C_t^{\star}}{\partial Q_t} = \mu = \frac{(\nu+\rho)}{\nu(1+\rho)} p_t \left[a_t \left(\frac{w_t}{p_t}\right)^{\frac{\rho}{1+\rho}} + b_t \left(\frac{r_t}{p_t}\right)^{\frac{\rho}{1+\rho}} \right] Q_t^{\frac{\rho(1-\nu)}{\nu(1+\rho)}} u_t^{\frac{-\rho}{\nu(1+\rho)}}$$

$$+ \frac{1}{\eta_o} k_o^{-\frac{1}{\eta_o}} \left[a_t \left(\frac{w_t}{p_t}\right)^{\frac{\rho}{1+\rho}} + b_t \left(\frac{r_t}{p_t}\right)^{\frac{\rho}{1+\rho}} \right] Q_t^{\frac{\nu(1+\rho)+\rho\eta_o(1-\nu)}{\eta_o \nu (1+\rho)}} u_t^{-\frac{\rho}{\nu(1+\rho)}}$$

$$- \frac{\rho}{\nu(1+\rho)} p_t \left[a_t \left(\frac{w_t}{p_t}\right)^{\frac{\rho}{1+\rho}} + b_t \left(\frac{r_t}{p_t}\right)^{\frac{\rho}{1+\rho}} \right] Q_t^{\frac{\nu+\rho}{\nu(1+\rho)}} u_t^{\frac{-\rho(1+\nu)-\nu}{\nu(1+\rho)}} \frac{\partial u_t}{\partial Q_t}$$

Supposing that there exists perfect competition on the output market ($|\eta_o| = \infty$), the marginal cost (1.68) can be set equal to the output price p_t, so that we obtain the derived supply function in implicit form from (1.68) (replacing $Q_t = Q_t^{\star} u_t$ and cancelling the last term of the r.h.s. in (1.68)):

$$(1.69)\qquad \left[a_t \left(\frac{w_t}{p_t}\right)^{\frac{\rho}{1+\rho}} + b_t \left(\frac{r_t}{p_t}\right)^{\frac{\rho}{1+\rho}} \right] Q_t^{\star\frac{\rho(1-\nu)}{\nu(1+\rho)}} u_t^{-\frac{\rho}{1+\rho}} = \frac{\nu(1+\rho)}{(\nu+\rho)} ,$$

so that the reduced form equations for Q_t, L_t and K_t are, under the above assumed perfect competition on the product market, directly derived form raising (1.69) to power $-\frac{\nu(1+\rho)}{\rho(1-\nu)}$ (see also (1.63)):

$$Q_t^{\star} = \left[a_t' \left(\frac{w_t}{p_t}\right)^{\frac{\rho}{1+\rho}} + b_t' \left(\frac{r_t}{p_t}\right)^{\frac{\rho}{1+\rho}} \right]^{\frac{-\nu(1+\rho)}{\rho(1-\nu)}} u_t^{\frac{\nu}{1-\nu}}$$

$$(1.70)\qquad L_t = \frac{\nu(1+\rho)}{(\nu+\rho)} a_t' \left(\frac{w_t}{p_t}\right)^{-\frac{1}{1+\rho}} \left[a_t' \left(\frac{w_t}{p_t}\right)^{\frac{\rho}{1+\rho}} + b_t' \left(\frac{r_t}{p_t}\right)^{\frac{\rho}{1+\rho}} \right]^{-\frac{(\rho+\nu)}{\rho(1-\nu)}} u_t^{\frac{1}{1-\nu}}$$

$$K_t = \frac{\nu(1+\rho)}{\nu+\rho} b_t' \left(\frac{r_t}{P_t}\right)^{-\frac{1}{1+\rho}} \left[a_t' \left(\frac{w_t}{p_t}\right)^{\frac{\rho}{1+\rho}} + b_t' \left(\frac{r_t}{p_t}\right)^{\frac{\rho}{1+\rho}} \right]^{\frac{-(\rho+\nu)}{\rho(1-\nu)}} u_t^{\frac{1}{1-\nu}} , \text{ with}$$

$$a'_t = \frac{\nu + \rho}{\nu(1+\rho)} \; a_t \quad \text{and} \quad b'_t = \frac{\nu + \rho}{\nu(1+\rho)} \; b_t$$

Since the production function disturbances enter into the two derived demand equations (because cost analysis is based upon (1.63) and not upon (1.64)), only Full - Information - estimation methods may conveniently be applied on (1.70) in order to obtain consistent and asymptotically efficient estimators.

It may be remarked from system (1.70) and from (1.69) that, for constant returns to scale, supply is perfectly elastic at a price:

$$(1.71). \qquad p_t = (a_t w_t^{\frac{\rho}{1+\rho}} + b_t r_t^{\frac{\rho}{1+\rho}}) \; u_t^{-\frac{\rho}{1+\rho}}.$$

So, there are two major disadvantages to the previous approach:

(i) in general, neither consistent nor efficient estimators for the production function parameters can be obtained from direct estimation of the production function itself;

(ii) there is degeneracy for $\nu = 1$.

§ B 2. Maximization of the expected value of profits.(*)

The underlying model can be defined as:

$$(1.72)\quad \max.\ E(\Pi_t) = E(p_tQ_t - w_tL_t - r_tK_t)$$
$$= E(p_tQ_t) - w_t^{*} L_t - r_t^{*} K_t$$
$$= E(s_oQ_t^{\ell_o}) - s_1L_t^{\ell_1} - s_2K_t^{\ell_2} \quad (**)$$

subject to the stochastic production function (1.62), where $\log u_t = \varepsilon_t$ is assumed to be normally distributed with zero mean and variance σ^2.

Similarly to the C.D.- case, the economy has to determine the optimal combination of production factors so as to maximize, per period t, the mathematical expectation of profit or

$$(1.73)\quad \max\ E(\Pi_t) = s_tQ_t^{\ell_o} e^{\frac{\ell_o^2\sigma^2}{2} - \ell_o\varepsilon_t} - s_1L_t^{\ell_1} - s_2K_t^{\ell_2} \text{ subject to:}$$

$Q_t = A\left[\delta K_t^{-\rho} + (1-\delta)\ L_t^{-\rho}\right]^{-\nu/\rho} e^{\lambda t + \varepsilon_t}$, which yields the following maximizing conditions with respect to factor inputs:

$$\frac{\partial E(\Pi_t)}{\partial L_t} = s_o\, e^{\frac{1}{2}\ell_o^2\sigma^2 - \ell_o\varepsilon_t} \frac{\partial Q_t}{\partial L_t} - s_1\ell_1L_t^{\ell_1-1}$$

$$(1.74)\quad = s_o\ell_o\ \nu(1-\delta)A^{-\frac{\rho}{\nu}} e^{-\frac{\lambda\rho t}{\nu}} L_t^{-(\rho+1)} Q_t^{\frac{\rho}{\nu}+\ell_o} e^{\frac{1}{2}\ell_0^2\sigma^2-(\frac{\rho}{\nu}+\ell_o)\varepsilon_t} - s_1\ell_1L_t^{\ell_1-1} = o$$

$$\frac{\partial E(\Pi_t)}{\partial K_t} = s_o\ell_o\nu\delta A^{-\frac{\rho}{\nu}} e^{\frac{-\lambda\rho t}{\nu}} K_t^{-(\rho+1)} Q_t^{\frac{\rho}{\nu}+\ell_o} e^{\frac{1}{2}\ell_a^2\sigma^2-(\frac{\rho}{\nu}+\ell_o)\varepsilon_t} - s_2\ell_2K_t^{\ell_2-1} = o$$

(*) Since the maximization of anticipated profit follows along similar lines (only constant terms of resulting reduced form equations have to be redefined: see C.D.-case under § A 2), we shall only derive the most general model here, i.e. maximization of the mathematical expectation of profits.

(**) As is also implied for the C.D.-model § A 2, s_1, s_2, ℓ_1 and ℓ_2 may be different from the original constants in (1.16).

and after introducing error terms v_{it} (i = 1,2) representing occurring differences between expected and realized prices (mis-specification of demand and supply relationships), imperfections in the profit-maximization, etc..., we find:

$$\log Q_t = \log A + \lambda t - \frac{\nu}{\rho} \log \left[\delta K_t^{-\rho} + (1-\delta) L_t^{-\rho}\right] + \varepsilon_t$$

$$(1.75) \qquad \log L_t = \frac{1}{\ell_1+\rho} \log \frac{s_o \ell_o}{s_1 \ell_1} \nu (1-\delta) A^{-\frac{\rho}{\nu}} - \frac{\lambda\rho}{\nu(\ell_1+\rho)} t + \frac{(\frac{\rho}{\nu}+\ell_o)}{(\ell_1+\rho)} \log Q_t + \frac{1}{2} \frac{\ell_o^2\sigma^2}{(\ell_1+\rho)} - \frac{(\frac{\rho}{\nu}+\ell_o)}{(\ell_1+\rho)} \varepsilon_t + v_{1t}$$

$$\log K_t = \frac{1}{\ell_2+\rho} \log \frac{s_o \ell_o}{s_2 \ell_2} \nu\delta A^{-\frac{\rho}{\nu}} - \frac{\lambda\rho}{\nu(\ell_2+\rho)} t + \frac{(\frac{\rho}{\nu}+\ell_o)}{(\ell_2+\rho)} \log Q_t + \frac{1}{2} \frac{\ell_o^2\sigma^2}{(\ell_2+\rho)} - \frac{(\frac{\rho}{\nu}+\ell_o)}{(\ell_2+\rho)} \varepsilon_t + v_{2t}$$

Putting

$$k_1 = \frac{1}{\ell_1+\rho} \log \frac{s_o \ell_o}{s_1 \ell_1} \nu (1-\delta) A^{-\frac{\rho}{\nu}} + \frac{1}{2} \frac{\ell_o^2\sigma^2}{(\ell_1+\rho)} \quad \text{and}$$

$$(1.76) \qquad k_2 = \frac{1}{\ell_2+\rho} \log \frac{s_o \ell_o}{s_2 \ell_2} \nu\delta A^{-\frac{\rho}{\nu}} + \frac{1}{2} \frac{\ell_o^2\sigma^2}{(\ell_2+\rho)} \quad ,$$

we may derive, in a way similar to (1.32-1.33), the reduced form equations for the endogenous variables. Since, also this time, and in contradiction to profit maximization or cost minimization (see (1.70)), the production function disturbance ε_t does not enter into the derived demand equations for inputs, the inputs L_t and K_t are statistically independent of e^{ε_t} (provided also that $E(\varepsilon_t v_{it})$=o), so that at least consistent estimates of the C.E.S.- function parameters can be obtained by direct non-linear least squares or maximum likelihood methods (Marquardt, Scoring: see next paper) or by O.L.S. to some linear approximation to (1.62), say about ρ = o.

C. Generalized C.E.S.-models: changing returns to scale and variable elasticity of substitution.

C.1 Changing returns to scale (see appendix B 1)

If we write the left-hand side of (B 11) as f(Q) and if, in general, the marginal productivity conditions hold:

$$(1.77) \quad \frac{\partial Q}{\partial L} = \frac{\ell_1 w_t}{\ell_o p_t} = \frac{dQ}{df} \; \frac{\partial f}{\partial L} \quad \text{and the same for capital,}$$

σ can, under conditions of profit maximization or cost minimization, be estimated from:

$$(1.78) \quad \log \frac{Q_t}{L_t} = \sigma \log \left(\frac{\ell_1}{\ell_o}\right) \frac{A}{(1-\delta)a_o}^{\frac{1-\sigma}{\sigma a_o}} + (1-\sigma)\frac{\lambda}{a_o}t + \sigma \log \left(\frac{w_t}{p_t}\right) +$$

$$+ \frac{1}{a_o}(1-\sigma)(a_o-1) \log Q_t + \sigma \log \frac{df(Q)}{dQ} +$$

$$+ \frac{(1-\sigma)(a_o-1)}{a_o} \log [\beta_1 \log |Q_t-\alpha_1| + \beta_2 \log |Q_t-\alpha_2|] + \xi_t$$

which implies that, if equation (1.63) is used to estimate σ and there are in fact changing returns to scale as a decreasing quadratic function of output, one commits a specification error in omitting the two last terms of (1.78) which does not lead to a very bad estimator if $\log Q_t$, $\log \frac{df(Q)}{dQ}$ and the last composite variable are very strongly interrelated.

Obviously, the difficulty of changing returns to scale can be overcome by estimating σ directly from (1.64). Only, if all production function parameters have to be estimated, the quadratic function (B 7) has to substituted for ν in all C.E.S.-models discussed.

C.2 Variable Elasticity of Substitution Models.

A specification error regarding the fundamental property of C.E.S.-functions, i.e. the assumed invariancy of the elasticity of substitution w.r.t. evt. occurring changes in factor proportions, could occur as long as we stick ourselves to a constant σ. As put forward in appendix B2, this is a hardly tenable hypothesis. Also, for problems of business-cycle analysis, a preassumed constant value for σ is irrelevant because in recessions, output falls more than capital stock so that the output-capital ratio decreases and capital's share falls while in boom periods labour's share is falling and through the inverse relationship between labour's share and average productivity of labour, labour productivity invariably decreases in recessions and increases sharply when the upturn begins. A rise in σ permits more capital (the plentiful factor) to be substituted for labour (the scarce factor) at each capital-labour combination, which implies a reduction of labour's relative income share (capital using/labour saving).

If σ supposed to be a linear function of the capital-labour ratio with $a_o = 1$, the V.E.S.- production function is (see B 32):

(1.79) $Q_t = A_t K_t^{\frac{1}{1+c}} (L_t + \frac{a_1}{1+c} K_t)^{\frac{c}{1+c}} u_t$ with marginal products:

(1.80) $$\frac{\partial Q_t}{\partial L_t} = \frac{\frac{c}{1+c} Q_t}{(L_t + \frac{a_1}{1+c} K_t)} = \frac{\ell_1 w_t}{\ell_o p_t}$$ and

$$\frac{\partial Q_t}{\partial K_t} = \frac{(\frac{1}{1+c}) Q_t}{K_t} + \frac{\frac{c}{1+c} (\frac{a_1}{1+c}) Q_t}{(L_t + \frac{a_1}{1+c} K_t)}$$ or

(1.81) $$\frac{\partial Q_t}{\partial K_t} = (\frac{1}{1+c}) \frac{Q_t}{K_t} + (\frac{a_1}{1+c}) \frac{\ell_1 w_t}{\ell_o p_t} = \frac{\ell_2 r_t}{\ell_o p_t}$$ from which the prior

coefficient $\frac{a_1}{1+c}$, necessary for estimation of the V.E.S. function

(1.79), could easily be estimated if $\ell_1 = \ell_2 = \ell_o$ (same degree of competition on all markets).
Starting from the profit maximizing conditions (1.63) for $\nu = 1$, we might introduce the V.E.S.-function (B 43) into some production function model, with the constraint that equation (B38) is already fixed. The production function parameters involved in:

$$(1.82) \quad \log Q_t = \log A + \lambda t - \frac{1}{\rho} \log \left[(1-\delta) \left(\frac{K_t}{L_t}\right)^{-m\rho} L_t^{-\rho} + \delta K_t^{-\rho}\right] + \varepsilon_t$$

may then be estimated from such a model, where prior estimates of m, ρ and δ may be obtained from the profit maximizing equations (see also appendix B 2.2), derived in a similar way as under § B.

D. Homothetic Production Function Models

As these types of models are also analyzed in the appendix (C), we shall quickly overlook some particular relationships w.r.t. the underlying economic theory. By the property of inter-isoquant invariancy, there is only one isoquant map, establishing the nature of the relating elasticity of substitution, required. Returns to scale are related to the curvature of the production surface.

For the whole function, there is only one important a priori assumption: i.e. the p.f. should be homothetic, which implies that the "initial" neo-classical production function should be a homothetic function (always satisfied since it is assumed homogeneous) and that along a ray from the origin crossing the isoquant map all the isoquant slopes are equal, so that only iso-elastic transformation are possible.

Specifying the returns to scale function as (C 5) with corresponding homothetic production function (C 8) and assuming that factors are remunerated according to their weighted marginal products (profit maximization : (1.53)):

$$(1.83) \quad \frac{\partial Q}{\partial L} = \frac{dG}{dF} \frac{\partial F}{\partial L} = \frac{\ell_1 w}{\ell_o p} \text{ and } \frac{\partial Q}{\partial K} = \frac{dG}{dF} \cdot \frac{\partial F}{\partial K} = \frac{\ell_2 r}{\ell_o p}$$

factor shares can be expressed as:

$$\frac{wL}{pQ} = \frac{\ell_o}{\ell_1}\frac{F}{Q}\frac{dG}{dF}\frac{L}{F}\frac{\partial F}{\partial L} = \frac{\ell_o}{\ell_1}\frac{h(Q)}{\nu}\;\frac{L}{F}\frac{\partial F}{\partial L} = \frac{\ell_o}{\ell_1}(1-\frac{Q}{c})\frac{L}{F}\frac{\partial F}{\partial L} \quad \text{(from (4))}$$

(1.84) $$\frac{rK}{pQ} = \frac{\ell_o}{\ell_2}(1-\frac{Q}{c})\frac{K}{F}\;\frac{\partial F}{\partial K}$$, which leads for

(i) a C.D.-function for F to:

(1.85) $$\frac{wL}{pQ} = \frac{\ell_o}{\ell_1}(1-\frac{Q}{c})\alpha \quad \text{and} \quad \frac{rK}{pQ} = \frac{\ell_o}{\ell_2}(1-\frac{Q}{c})\beta ,$$

so that the factor shares for a homothetic p.f. with underlying C.D.p.f. are variable, in contrast to (1.19).

(ii)a C.E.S.- function for F to:

(1.86)
$$\frac{wL}{pQ} = \frac{\ell_o}{\ell_1}(1-\frac{Q}{c})\,A^{-\frac{\rho}{\nu}}\,\nu(1-\delta)\,L^{-\rho}F^{\frac{\rho}{\nu}} = \frac{\ell_o}{\ell_1}(1-\frac{Q}{c})\,\frac{\nu(1-\delta)}{(1-\delta)+\delta(\frac{K}{L})^{-\rho}} \quad \text{and}$$

$$\frac{rK}{pQ} = \frac{\ell_o}{\ell_2}(1-\frac{Q}{c})\quad\frac{\nu\delta}{(1-\delta)(\frac{L}{K})^{-\rho}+\delta} ,$$

from which it is clear that factor shares can rise, reach a maximum and fall as Q and K/L resp. L/K, increase (depending upon the value of ρ).

Equations (1.85) and (1.86) can easily be utilized to derive explicit functions for log Q, log L and log K (in analogy with C.D.- and C.E.S.-models).

II. Embodied versus Desembodied Technical Progress.

In general, the economy may be faced with a homogeneous production function of the form $F(E_L L, E_K K, t)$ or some monotonic transformation there of (homothetic production function). $E_L L$ and $E_K K$ represent "effective" labour and "effective" capital, with L and K the measured inputs and E_L and E_K the relating "augmentation parameters". All four quantities L, K, E_L and E_K are a function of time.

The discussion in this section will be threefold:

a) an optimal technology (besides optimal levels of factors of production) will be derived under most general conditions of maximization of a discounted profits stream over an indefinite but certain future and of a homothetic production function;

b) the functional form in which any kind of technological progress is allowed to be introduced will be determined, and finally

c) a vintage model of production, based on C.E.S. vintage functions, will be studied in a "putty-putty" sense.

A. A brief account of the (optimal) impact of technical progress (*)

Remembering that factor augmenting technical progress (t.p.) is in fact a combination of Hicks and of either Harrod or Solow neutral t.p. (an improvement highly specific to labour, resp. capital may be reflected in E_L or E_K or both), all technical progress may be viewed as factor augmenting, evt. split up regarding its effects specific to labour or capital and/or to a general increase in efficiency (non-specific factor augmenting t.p.), so that we may define a homothetic production function as:

(2.1) $Q = G(F(E_L L, E_K K))$, with a continuously differentiable underlying production function F, assumed, for the following analysis, to be homogeneous of first degree.

(*) This paragraph is based upon an illuminating article of Kamien and Schwartz, see [11].

An expression for the optimal direction of and the optimal rate of expenditure for factor augmenting technical progress will be derived under the assumptions that:

(i) the production function is described by (2.1) and $G^{-1}(Q)$ exists;

(ii) there exists perfect competition on the factor markets;

(iii) the cost function, describing the opportunities of technical progress, is based on a Kennedy innovation possibility frontier;

(iv) the economy selects rates of factor augmentation to maximize the discounted stream of profits.

1. Factor augmenting technical progress transforms the isoquants towards the origin because:

$$(2.2) \qquad \frac{\partial Q}{\partial E_L} = \frac{dG}{dF}\frac{\partial F}{\partial E_L} > o \quad \text{and} \quad \frac{\partial Q}{\partial E_K} = \frac{dG}{dF}\frac{\partial F}{\partial E_K} > o$$

, which might be illustrated by the following figure:

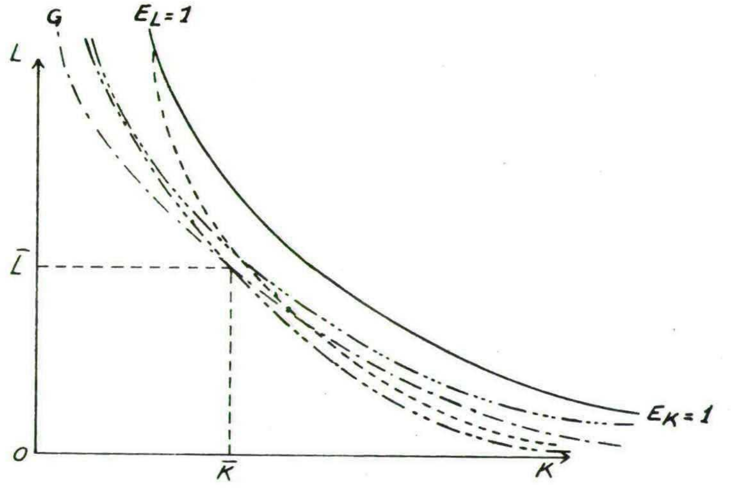

Figure 1. Isoquant transformation.

from which it is clear that there is an isoelastic transformation of the basis isoquant if $E_L = E_K$ (or if $\frac{dE_L}{dtE_L} = \frac{dE_K}{dtE_K}$ if labour and capital are not expressed in the same units), i.e. if there is Hicks neutral technical progress.

2. The change in the slope of an isoquant of (2.1), through any given point $(\bar{L}, \bar{K})$, due to factor augmenting t.p. is determined from utilization of def. (7) of the marginal rate of substitution (= absolute value of the slope of an isoquant):

$$(2.3) \qquad \frac{dR}{dt} \equiv \frac{d}{dt}\left(-\frac{dK}{dL}\right)_{\bar{L},\bar{K}} = \frac{d}{dt}\left(\frac{E_L F_1}{E_K F_2}\right) \quad \frac{d}{dt} X(E_L, E_K, L, K) = \frac{\partial X}{\partial E_L}\frac{dE_L}{dt} + \frac{\partial X}{\partial E_K}\frac{dE_K}{dt}$$

where use has been made of the property that, along an isoquant, the production is kept constant, or from (2.1):

$$(2.4) \qquad dQ = \frac{dG}{dF} F_1 E_L \; dL + \frac{dG}{dF} F_2 E_K \; dK = o \text{ with } F_1 = \frac{\partial F}{\partial (E_L L)} \text{ and } F_2 = \frac{\partial F}{\partial (E_K K)}$$

The r.h.s. partial derivatives of (2.3) are:

$$(2.5) \qquad \frac{\partial X}{\partial E_L} = \frac{E_K F_2 (F_1 + E_L L F_{11}) - E_L F_1 E_K L F_{21}}{(E_K F_2)^2}$$

$$= \frac{(F_2 F_1 - F_{12} E_K K F_2 - E_L L F_1 F_{21})}{E_K F_2^2} = \frac{F_2 F_1 - F_{12} F}{E_K F_2^2}$$

$$= -\frac{F_1}{E_K F_2}\left(\frac{FF_{12}}{F_1 F_2} - 1\right) = -\frac{F_1}{E_K F_2}\left(\frac{1-\sigma}{\sigma}\right) \qquad \text{and, similarly:}$$

$$(2.6) \qquad \frac{\partial X}{\partial E_K} = \frac{E_K F_2 E_L F_{12} - E_L F_1 F_2 - E_L F_1 E_K K F_{22}}{(E_K F_2)^2}$$

$$= \frac{E_L}{E_K^2}\left[\frac{FF_{12}}{F_2^2}\left(1 - \frac{F_1 F_2}{FF_{21}}\right)\right] = \frac{E_L}{(E_K)^2}\frac{F_1}{F_2}\left(\frac{1-\sigma}{\sigma}\right) \quad \text{, where use has}$$

been made of

(i) the definitions $F_{11} \equiv \frac{\partial^2 F}{\partial(E_L L)^2}$, $F_{22} \equiv \frac{\partial^2 F}{\partial(E_K K)^2}$, $F_{12} \equiv F_{21} \equiv \frac{\partial^2 F}{\partial(E_L L)\partial(E_K K)}$

(ii) the property that F is homogeneous of degree one, or by Euler's theorem:

(2.7) $F_1 E_L L + F_2 E_K K = F$,

where the "effective" marginal products F_1 and F_2 are homogeneous of degree zero or

(2.8) $F_{11} E_L L + F_{12} E_K K = 0$ and $F_{21} E_L L + F_{22} E_K K = 0$

(iii) the definition of the elasticity of factor substitution:

$$\sigma \equiv \frac{F_1 F_2}{F F_{12}}$$

Substituting (2.6) and (2.5) into (2.3), we observe that the change in the slope of an isoquant of (2.1) is dependent upon the elasticity of factor substitution and upon the proportional rates of factor augmentation:

(2.9) $$\frac{dR}{dt} \equiv \frac{d}{dt}\left(-\frac{dK}{dL}\right)_{\bar{L},\bar{K}} = \frac{E_L F_1}{E_K F_2} \frac{(\sigma-1)}{\sigma} \left(\frac{E'_L}{E_L} - \frac{E'_K}{E_K}\right) \text{ with } E'_L = \frac{dE_L}{dt} \text{ and } E'_K = \frac{dE_K}{dt}$$

3. From the above result, it might be true that factor augmentation may lead to a change in the <u>optimal capital - labour ratio.</u> Therefore, the optimal capital-labour ratio itself (and conditions for satisfying it) has to be determined, which might be done by deriving the optimal cost function and relating factor shares in total production cost. Under hypotheses (i-ii), the cost function is the solution of the minimization of wL + rK subject to (2.1), where Q is kept fixed at an "exogenously" determined level.

Since F is assumed to be homogeneous of degree one:

$$(2.10)\quad F\left(\frac{E_L K}{G^{-1}(Q)}, \frac{E_K K}{G^{-1}(Q)}\right) = 1 \text{ because } G^{-1}(Q) = F(E_L L, E_K K),$$

the problem:

$$(2.11)\quad \min_{L,K}\; wL + rK \equiv G^{-1}(Q)\left[\frac{wE_L L}{E_L G^{-1}(Q)} + \frac{rE_K K}{E_K G^{-1}(Q)}\right] \text{ subject to (2.1)}$$

is equivalent to the problem:

$$(2.12)\quad \min_{\hat{L},\hat{K}}\; G^{-1}(Q)\left[\frac{w\hat{L}}{E_L} + \frac{r\hat{K}}{E_K}\right] \text{ subject to } F(\hat{L},\hat{K}) = 1,$$

which has a solution of the form: $\phi(\frac{w}{E_L},\frac{r}{E_K})$, or the solution of (2.11) corresponding to the homothetic p.f.G is:

$$(2.13)\quad C(Q, \frac{w}{E_L}, \frac{r}{E_K}) = G^{-1}(Q)\,\phi(\frac{w}{E_L}, \frac{r}{E_K}),$$

which is a homothetic cost function (see duality between cost functions and production functions: footnote on p.21) with ϕ a homogeneous function of degree one.

Expressing the solution values for L and K as:

$$(2.14)\quad L^* = L^f(Q, \frac{w}{E_L}, \frac{r}{E_K}) \text{ and } K^* = K^f(Q, \frac{w}{E_L}, \frac{r}{E_K}),$$ the optimal cost

function (2.13) may be rewritten as:

$$(2.15)\quad C(Q, \frac{w}{E_L}, \frac{r}{E_K}) = wL^f(Q, \frac{w}{E_L}, \frac{r}{E_K}) + rK^f(Q, \frac{w}{E_L}, \frac{r}{E_K}),$$ with partial

derivatives w.r.t. w and r as (see also (2.13)):

$$(2.16)\quad \frac{\partial C}{\partial w} = G^{-1}(Q)\frac{\phi_1}{E_L} = L^f + w\frac{\partial L^f}{\partial w} + r\frac{\partial K^f}{\partial w} \text{ and}$$

$$\frac{\partial C}{\partial r} = G^{-1}(Q)\frac{\phi_2}{E_K} = K^f + \frac{w\partial L^f}{\partial r} + \frac{r\partial K^f}{\partial r}, \text{ with}$$

$$\phi_1 \equiv \frac{\partial\phi(\frac{w}{E_L}, \frac{r}{E_K})}{\partial(\frac{w}{E_L})} \quad \text{and} \quad \phi_2 \equiv \frac{\partial\phi(\frac{w}{E_L}, \frac{r}{E_K})}{\partial(\frac{r}{E_K})}$$

Since the output is still kept constant, the partial derivatives of (2.1), where the optimal values in (2.14) have been substituted for L and K, w.r.t. w, viz. r are zero or:

$$(2.17) \quad \frac{\partial Q}{\partial w} = \frac{dG}{dF}\left(F_1E_L \frac{\partial L^f}{\partial w} + F_2E_K \frac{\partial K^f}{\partial w}\right) = \frac{\partial Q}{\partial r} = \frac{dG}{dF}\left(F_1E_L \frac{\partial L^f}{\partial r} + F_2E_K \frac{\partial K^f}{\partial r}\right) = 0,$$

and taking account of the necessary maximum conditions for (2.11) (besides (2.1)):

$$(2.18) \quad w = \mu \frac{dG}{dF} F_1E_L \quad \text{and} \quad r = \mu \frac{dG}{dF} F_2E_K \quad (\mu = \text{Lagrange parameter}),$$

we find from substitution of (2.18) into (2.17):

$$(2.19) \quad w \frac{\partial L^f}{\partial w} + r \frac{\partial K^f}{\partial w} = w \frac{\partial L^f}{\partial r} + r \frac{\partial K^f}{\partial r} = 0,$$ or the equations in (2.16)

can in view of (2.14) and (2.19) be rewritten as:

$$(2.20) \quad \frac{\partial C}{\partial w} = G^{-1}(Q) \frac{\phi_1}{E_L} = L^{\star} \quad \text{and} \quad \frac{\partial C}{\partial r} = G^{-1}(Q) \frac{\phi_2}{E_K} = K^{\star}$$ and the

optimal capital-labour ratio amounts to:

$$(2.21) \quad \frac{K^{\star}}{L^{\star}} = \frac{\phi_2 \; E_L}{\phi_1 \; E_K}$$

The conditions for satisfying (2.21) may be derived from the optimal cost function (2.15) taking account viz. of (2.18) and of (2.20):

$$(2.22) \quad C = wL^{\star} + rK^{\star} = \mu \frac{dG}{dF}(F_1E_L \, L^{\star} + F_2E_K \, K^{\star}) = \mu \frac{dG}{dF} F^{\star} \text{ (see (2.7))}$$

$$= G^{-1}(Q)\left(\frac{w\phi_1}{E_L} + \frac{r\phi_2}{E_K}\right) = G^{-1}(Q)\, \phi \text{ (see (2.13))}$$

and the relating optimal factor shares, taking account of (2.22), (2.18) and (2.20):

$$(2.23)\quad S_L^* = \frac{wL^*}{C} = \frac{F_1E_LL^*}{F^*} = \frac{w\phi_1}{\phi E_L} \text{ and } S_K^* = \frac{rK^*}{C} = \frac{F_2E_KK^*}{F^*} = \frac{r\phi_2}{\phi E_K} \quad (*)$$

or the conditions are (from(2.23)):

$$(2.25)\quad \frac{w}{r} = \frac{E_LF_1}{E_KF_2} = X(E_L, E_K, L, K)$$

(see (2.3)), so that the time path of the optimal capital-labour ratio (2.21) can be studied from the total change in the r.h.s. of (2.25):

$$(2.26)\quad \frac{dX}{dt} = \frac{\partial X}{\partial E_L}\frac{dE_L}{dt} + \frac{\partial X}{\partial E_K}\frac{dE_K}{dt} + \frac{\partial X}{\partial L}\frac{dL}{dt} + \frac{\partial X}{\partial K}\frac{dK}{dt},$$

which should be equal to zero since the factor prices, and hence the ratio $\frac{w}{r}$ in (2.25), are kept constant (ass. (ii))(**)

Since $\frac{\partial X}{\partial E_L}$ and $\frac{\partial X}{\partial E_K}$ are already given by (2.5) and (2.6), only $\frac{\partial X}{\partial L}$ and $\frac{\partial X}{\partial K}$ have to be evaluated, which is done yet in a way similar to (2.5) and (2.6):

$$(2.27)\quad \frac{\partial X}{\partial L} = \frac{F_2E_L^2F_{11} - E_LF_1E_LF_{21}}{E_K F_2^2} = -\frac{F_2E_LF_{12}\frac{1}{L}E_KK + E_LF_1E_LF_{21}}{E_K F_2^2} \quad \text{(see (2.8))}$$

(*) It is obvious from (2.13) , (2.22) and (2.23) that factor augmentation tends to reduce the cost of producing at any given rate because:

$$(2.24)\quad \frac{\partial C}{\partial E_L} = G^{-1}(Q)\frac{\partial \phi}{\partial E_L} = -G^{-1}(Q)\frac{w\phi_1}{E_L^2} = \frac{-wL^*}{E_L} < 0 \text{ and } \frac{\partial C}{\partial E_K} = -G^{-1}(Q)\frac{r\phi_2}{E_K^2} = -\frac{rK^*}{E_K} < 0.$$

(**) The two last terms in (2.26) do not vanish, as was the case in (2.3), because this time the change of the slope of an isoquant in a given point $(\bar{L},\bar{K})$ is not measured, but rather the total change in $\frac{E_LF_1}{E_KF_2}$ when L and K are optimally adjusted to changes in E_L and E_K.

$$= - \frac{E_L F_{12}}{E_K L F_2^2} (F_2 E_K K + F_1 E_L L) = - \frac{E_L F_{12} F}{E_K L F_2^2} \quad \text{(see (2.7))}$$

$$= - \frac{E_L F_1}{E_K L F_2 \sigma} \quad \text{and analogously:}$$

(2.28) $\frac{\partial X}{\partial K} = \frac{F_2 E_L F_{12} - E_L F_1 F_{22}}{F_2^2} = \frac{E_L F_{12} F}{E_K K F_2^2} = \frac{E_L F_1}{E_K K F_2 \sigma}$, so that substitution of (2.5), (2.6), (2.27) and (2.28) into (2.26) yields:

(2.29) $\frac{dX}{dt} = \frac{E_L F_1}{E_K F_2} \frac{\sigma-1}{\sigma} \left(\frac{E_L'}{E_L} - \frac{E_K'}{E_K}\right) + \frac{E_L F_1}{E_K F_2 \sigma} \left(\frac{K'}{K} - \frac{L'}{L}\right) = 0$, where

it is assumed, as everywhere else in this note, that the economy operates in a static and certain world (no lags, or always $L = L^*$ and $K = K^*$) (*) and $K' = \frac{dK}{dt}$ and $L' = \frac{dL}{dt}$.

From (2.29), it is immediately found that the time rate of change in the optimal capital-labour ratio is related to the elasticity of substitution and the proportional time rates of labour and capital augmentation by:

(2.30) $$\frac{dK^*}{K^* dt} - \frac{dL^*}{L^* dt} = \frac{K'}{K} - \frac{L'}{L} = (1-\sigma) \left(\frac{E_L'}{E_L} - \frac{E_K'}{E_K}\right) \quad (**)$$

4. Finally, the <u>optimal direction</u> of technical change and the <u>optimal rate of expenditure for factor augmentation</u> are determined utilizing the following definition for the maximum net revenue obtainable for certain values E_L and E_K of factor augmentation:

(2.31) $$N(E_L, E_K) = \max \left[p(Q)Q - C\left(Q, \frac{w}{E_L}, \frac{r}{E_K}\right) \right] = p(Q^*)Q^* - C^*\left(\frac{w}{E_L}, \frac{r}{E_K}\right),$$

(*) In the subsequent paper announced, deplacements from equilibrium values (principally regarding the factors of production) will be briefly studied for some previously discussed production models.

(**) Note that if $\sigma = 1$ (C.D.), factor augmentation cannot influence the (optimal) capital-labour ratio and is Hicks - neutral in effect.

where $C^*(\frac{w}{E_L}, \frac{r}{E_K})$ denotes the cost of producing at the optimal level Q^* for given technology (E_L, E_K)

Introducing the rate M of expenditure for technical advance, we may assume in general that feasible combinations of proportional rates of factor augmentation are given by a (smooth concave) invention possibility frontier, implying that the greater the rate of expenditure for technical change becomes, the more rapid technical advance proceeds but with decreasing marginal returns or in general:

$$(2.32) \quad \frac{E'_L}{E_L} = g(e)\,h(M) \text{ and } \frac{E'_K}{E_K} = eh(M),$$

with $e \geq 0$, $g(e) \geq 0$, $g'(e) < 0$ and $g''(e) < 0$,
where e is equal to $\frac{E'_K}{E_K}$ for a particular given rate of spending on technical advance and h(.) is a nonnegative, monotone increasing <u>convex</u> function of M

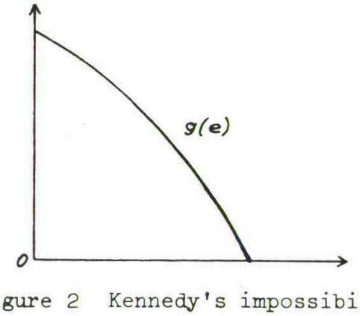

figure 2 Kennedy's impossibility frontier

The optimal direction e^* of factor augmentation and the optimal rate of expenditure M^* on it are then found from the application of Pontryagin's maximum principle, a necessary condition for fulfilment provided by the existence of continuous functions $\mu_1(t)$ and $\mu_2(t)$ such that the Hamiltonian:

$$(2.33) \quad H = e^{-it}\left[N(E_L, E_K) - M\right] + \mu_1 E_L\, g(e)\, h(M) + \mu_2 E_K\, eh(M)$$

be maximum at every point $0 \leq t < \infty$ with Hamiltonian canonical equations:

$$(2.34) \quad \mu_1' = \frac{d\mu_1}{dt} = -\frac{\partial H}{\partial E_L} = -e^{-it}\frac{\partial N(E_L,E_K)}{\partial E_L} - \mu_1 g(e)\,h(M)$$

$$\mu_2' = \frac{d\mu_1}{dt} = -\frac{\partial H}{\partial E_K} = -e^{-it}\frac{\partial N(E_L,E_K)}{\partial E_K} - \mu_2\, eh(M),$$

where i is an appropriate positive rate of discount.

From (2.31) and (2.13):

$$(2.35) \quad \frac{\partial N(E_L,E_K)}{\partial E_L} = -G^{-1}(Q^*)\frac{\partial \phi}{\partial E_L} + \left[p(Q^*) + p'(Q^*)\,Q^* - G^{-1}(Q^*)\,\phi\right]\frac{\partial Q^*}{\partial E_L},$$

and since the bracketed term on the r.h.s. of (2.35) is the first order condition for a maximum net revenue ((2.31) partially derived w.r.t. Q^*):

$$(2.36) \quad \frac{\partial N(E_L,E_K)}{\partial E_L} = -G^{-1}(Q^*)\frac{\partial \phi}{\partial E_L} = -\frac{\partial C}{\partial E_L} \quad \text{and}$$

$$\frac{\partial N(E_L,E_K)}{\partial E_K} = -G^{-1}(Q^*)\frac{\partial \phi}{\partial E_K} = -\frac{\partial C}{\partial E_K}, \quad \text{or}$$

the rate of change of maximal net revenue $N(E_L,E_K)$ due to factor augmentation is equal to the rate of cost reduction at optimal output. Combining (2.36) and (2.24), we find that the nominal factors amount to:

$$(2.37) \quad wL^* = wL = E_L\frac{\partial N(E_L,E_K)}{\partial E_L} \quad \text{and} \quad rK^* = rK = E_K\frac{\partial N(E_L,E_K)}{\partial E_K},$$

which substituted in (2.34), replacing $\frac{E_L'}{E_L}$ and $\frac{E_K'}{E_K}$ for the coefficients of μ_1 and μ_2 (see (2.32)):

$$(2.38) \quad \begin{aligned} E_L\,\mu_1' + E_L'\,\mu_1 &= -e^{-it}E_L\frac{\partial N(E_L,E_K)}{\partial E_L} = -e^{-it}wL \\ E_K\,\mu_2' + E_K'\,\mu_2 &= -e^{-it}E_K\frac{\partial N(E_L,E_K)}{\partial E_K} = -e^{-it}rK \end{aligned}$$

and taking account of the transversality conditions $\lim_{t \to \infty} \mu_i(t) = 0$ $(i = 1,2)$:

(2.39) $E_L \mu_1 = \int_t^{\infty} e^{-i\tau} \, wLd\tau$ and $E_K \mu_2 = \int_t^{\infty} e^{-i\tau} \, rKd\tau$, so that the optimal direction e^{*} of technical progress and the optimal rate M^{*} of expenditure for it are found from the first order conditions for maximum of the integral of the Hamilton function (2.33), taking account of (2.39):

(2.40)

$$g'(e^{*}) = \frac{-\int_t^{\infty} e^{-i\tau} \, rKd\tau}{\int_t^{\infty} e^{-i\tau} \, wLd\tau} \quad \text{and}$$

$$h'(M^{*}) = \frac{e^{-it}}{\left[g(e^{*}) \int_t^{\infty} e^{-i\tau} \, wLd\tau + e^{*} \int_t^{\infty} e^{-i\tau} \, rKd\tau \right]} ,$$

where the first solution equation becomes exactly the classical expression for optimal direction of t.p. (with magnitude of the slope of the innovation possibility frontier being equal to the ratio of factor shares) if capital and labour do not change (also equilibrium value for e^{*}: then Hicks neutral technical progess

$\frac{E'_L}{E_L} = \frac{E'_K}{E_K}$ or $e = g(e)$).

B. The specification of t.p. in the aggregate p.f.: "Impossibility Theorem" of Diamond and Mc. Fadden (*)

The theorem says that, under conditions of:

- constant returns to scale (so also output exhaustion with output price as numeraire)
- perfect competition on all markets
- profit maximization or cost minimization

(*) See its formulation in M. Nerlove [17], pp. 92 - 100, originally provided by K.J. Arrow.

we can define more than one neoclassical production function consistent with any given set of observations if either of the following situations occurs:

(i) the elasticity of factor substitution is equal to one and there is specific factor augmenting t.p.;

(ii) there is both capital and labour augmenting t.p. and non-specific factor augmenting t.p.;

(iii) there is only specifically capital and labour augmenting technical change being expressed in non-smooth form

This serious identification problem of production functions regarding the impact of any type of technical progress will now be made clear following lines similar to Nerlove [17].

First, two neoclassical production functions F and G are called to be "consistent with the data" if and only if:

(2.41) $Q = F(E_L^F L,\ E_K^F K,\ t) = G(E_L^G L, E_K^G K, t)$, where E_L^i and E_K^i are the augmentation parameters specifically due to labour, viz. capital $(i = F,G)$ (*)

Second, the three (crucial) above called assumptions imply for (2.41):

$$(2.42)\quad Q = \frac{\partial F}{\partial (E_L^F L)} E_L^F L + \frac{\partial F}{\partial (E_K^F K)} E_K^F K = \frac{\partial G}{\partial (E_L^G L)} E_L^G L + \frac{\partial G}{\partial (E_K^G K)} E_K^G K = S_L + S_K,$$

where $E_L^F L$, $E_K^F K$, $E_L^G L$ and $E_K^G K$ are the "effective" or "true" factor inputs and S_L and S_K the total payments to labour and capital.

Third, differentiating (2.41) w.r.t.t, taking account of (2.42):

(*) Note that these parametes E_L en E_G are different from those appearing in (2.1), because here, an improvement specific to labour viz. capital is interpreted to be reflected only in E_L, resp. E_K. The present model is also different from the previous one in that it analyses the form of t.p. in the context of a basis p.f. only, there is difference in economy goal and imperfect competition on the product market is not allowed.

$$(2.43)\quad \frac{dQ}{dt} = \frac{\partial F}{\partial (E_L^F L)} \left(L \frac{d\,E_L^F}{dt} + E_L^F \frac{dL}{dt}\right) + \frac{\partial F}{\partial (E_K^F K)} \left(K \frac{dE_K^F}{dt} + E_K^F \frac{dK}{dt}\right) + \frac{dF}{dt}$$

$$= \frac{\partial G}{\partial (E_L^G L)} \left(L \frac{dE_L^G}{dt} + E_L^G \frac{dL}{dt}\right) + \frac{\partial G}{\partial (E_K^G K)} \left(K \frac{dE_K^G}{dt} + E_K^G \frac{dK}{dt}\right) + \frac{dG}{dt}$$

Fourth, setting the rates of specific labour and capital augmentation equal to:

$$(2.44)\quad e_L^i = \frac{dE_L^i}{E_L^i dt} \quad \text{and} \quad e_K^i = \frac{dE_K^i}{E_K^i dt} \quad (\text{for } i = F, G)$$

and for non-specific factor augmenting (purely disembodied) t.p.:

$$(2.45)\quad \lambda^F = \frac{dF}{Qdt} \quad \text{and} \quad \lambda^G = \frac{dG}{Qdt}$$, which quantities (percentage changes)

may vary or remain constant during the sample period.
Then (2.43), making use of (2.42), becomes:

$$(2.46)\quad q = \frac{dQ}{dt} = s_L\,(e_L^F + \ell) + s_K\,(e_K^F + k) + \lambda^F$$

$$= s_L\,(e_L^G + \ell) + s_K\,(e_K^G + k) + \lambda^G$$

with $\ell = \frac{dL}{Ldt}$, $k = \frac{dK}{Kdt}$, $s_L = \frac{S_K}{Q}$ and $S_K = \frac{S_K}{Q} = 1 - s_L$

Or from (2.46):

$$(2.47)\quad s_L\,(e_L^F - e_L^G) + (1 - s_L)(e_K^F - e_K^G) + \lambda^F - \lambda^G = 0,$$

so that identification of either type of t.p. is fully possible if and only if $e_L^F = e_L^G$, $e_k^F = e_K^G$ and $\lambda^F = \lambda^G$.

(i) If $\sigma = 1$, the relative factor shares remain constant and (2.47) is satisfied for an infinite number of technical progress rates, unless there is no capital and labour augmenting t.p. (all e's = 0) This confirms the will known result that specific factor augmenting t.p. cannot be distinguished from purely product augmenting t.p. in the C.D. case.(*) So, the p.f. is not identified and, moreover, the bias of technical change is indeterminate.

(ii) It is clear from (2.47) the p.f. is not identified if both types of t.p. occur, even if e^i_L and e^i_K are assumed to be constant. If all rates of t.p. are assumed to be constant, we find from (2.47):

$$(2.49) \quad s_L \left[(e^F_L - e^F_K) - (e^G_L - e^G_K) \right] + (e^F_K + \lambda^F) - (e^G_K + \lambda^G) = 0, \text{ which}$$

implies:

$$(2.50) \quad e^F_L - e^F_K = e^G_L - e^G_K \quad \text{and} \quad e^F_K + \lambda^F = e^G_K + \lambda^G,$$

so that the bias of technical change is determined uniquely but neither the technical change itself (and so) nor the production function is (provided $\sigma \neq 1$).

(iii) a. If there is no specific factor augmenting t.p. (or $E^F_L = E^F_K = E^G_L = E^G_K = 1$ and so $e^F_L = e^F_K = e^G_L = e^G_K = 0$), it is clear from (2.47) that $\lambda^F = \lambda^F$ and the p.f. is completely identified.

b. But, if there is only specifically factor augmenting t.p. the p.f. is not identified unless a smoothness condition is imposed on technical change, say in terms of exponential growth (provided of course that $\sigma \neq 1$). Indeed, if

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(*) Note in this respect that the only p.f. which is both Harrod and Solow neutral, and so, at the same time Hicks neutral, is the C.D.-function with constant returns to scale:

$$(2.48) \quad Q_t = Ae^{\lambda t} L^{\alpha}_t K^{1-\alpha}_t \text{ with } \lambda = m\alpha \text{ for (constant) Harrod neutral rate}$$

m, Solow neutral rate $m' = m\frac{\alpha}{1-\alpha}$ and Hicks neutral rate $m'' = m\alpha$.

$\lambda^F = \lambda^G = 0$ and all e's are constant percentage values(*) it follows immediately from (2.47) that for non-constant factor shares ($\sigma \neq 1$) $e_L^F = e_L^G$ and $e_K^F = e_K^G$ and the p.f. is identified.

Conclusively it may be said that specifically factor augmenting and purely product augmenting t.p. may occur simultaneously in a production model with $\sigma \neq 1$, only, and probably under very strong restrictions, if at least one of the three pre-assumptions for the impossiblility theorem are not fulfilled.(**)

C. A putty - putty C.E.S. vintage model

Various types of C.E.S. vintage models may be defined:

(i) only product-augmenting technical change per vintage τ:

$$(2.51) \quad Q_{t,\tau} = Ae^{\lambda t+\mu\tau}\left[(1-\delta)L_{t,\tau}^{-\rho} + \delta K_{t,\tau}\right]^{-\frac{\nu}{\rho}}$$

(ii) pure product augmenting and specific capital and labour augmenting t.p. per vintage

---

(*) Which is so if there is smooth exponential growth in the effectiveness of measured labour and capital inputs. Note, however, that exponential technological change is a sufficient condition for identification.

(**) The Diamond- Mc. Fadden conditions given above are unduly restrictive if the growth equation (2.43) is supposed to be sectionally continuous, i.e. if it can be subdivided into a finite number of parts ("technological epochs") in each of which F and G are continuous and have finite limits as the arguments approach either endpoint of the subinterval from the interior. So, only smooth specific factor augmenting t.p. is supposed to occur within one epoch and non-smooth, structural breaks of the growth equation occur between two epochs, so that different p.f. may be estimated for different technological epochs (identification conditions fulfilled within one subperiod) (see M. Brown, [6] , pp. 114 - 118).

$$(2.52) \quad Q_{t,\tau} = Ae^{\lambda t+\mu\tau} \left[(1-\delta)\ (\gamma_1(\tau)L_{t,\tau})^{-\rho} + \delta(\gamma_2(\tau)K_{t,\tau})\right]^{-\frac{\nu}{\rho}}$$

(if at least one assumption of the impossibility theorem is not fulfilled or "technological epochs"), or less general cases, as the following one which will be discussed here.

As is often made in the recent literature since Solow, all technical improvements embodied in new physical equipment I_τ is assumed to be capital augmentive, so that the production planned to be brought about by means of capital invested in year τ and still in operation at year t and labour associated with this capital stock may be specified by:

$$(2.53) \quad Q_{t,\tau} = A_t \left[(1-\delta)\ L_{t,\tau}^{-\rho} + \delta(e^{\gamma\tau}\ K_{t,\tau})^{-\rho}\right]^{-\frac{\nu}{\rho}} \quad \text{where}$$

$A_t = Ae^{\lambda t}$ symbolizes disembodied technological progress specifying that there are some increases in efficiency which tend to increase the productivity of labour regardless of the capital associated

$\gamma > 0$ influence of technical improvements embodied in the gross investment of year τ by means of:

$$(2.54) \quad K_{t,\tau} = D_{t-\tau} I_\tau \text{ , where } D_{t-\tau} \text{ is a given survival function } (D_o=1),$$

representing the deterioration of physical equipment of vintage τ at period t.(★)

For further analysis, we shall assume that the marginal productivities, under profit maximization or cost minimization, and the market conditions are the same for all vintages, such that we find from (1.53):

(★) If capital is assumed to depreciate exponentially at rate d (so, average life time of capital is 1/d), the survival function is formalized as:

$$(2.55) \quad D_{t-\tau} = e^{-d(t-\tau)}$$, which becomes zero only if $t-\tau$ grows indefinitely. Therefore, in finite time series analysis, $D_{t-\tau}$ is usually put equal to zero if $t-\tau$ exceeds the maximal life time of capital, say θ, or $D_{t-\tau} = 0$ if $t-\tau > \theta$

(2.56) $\frac{\partial Q_{t,\tau}}{\partial L_{t,\tau}} = \frac{\ell_1 w_t}{\ell_o p_t} = \frac{\partial Q_t}{\partial L_t}$ or from (1.7)

(2.57) $L_{t,\tau} = (\frac{Q_{t,\tau}}{Q_t})^{\frac{\rho+\nu}{\nu(1+\rho)}} L_t$, which says that, under constant returns to scale, equal marginal products of labour imply equal average products of labour for all vintages (C.E.S.- case is different with C.D.- case because equal marginal productivities always imply equal average productivities for the latter).

For ease of derivation of the aggregate production function corresponding to (2.53), we shall also assume constant returns to scale.(*)

Then, if total output and total labour supply at period t are obtained from integration (summation) over all vintages:

(2.58) $Q_t = \int_{-\infty}^{t} Q_{t,\tau}\, d\tau$ and $L_t = \int_{-\infty}^{t} L_{t,\tau}\, d\tau$ (**)

and if it is assumed that labour supply is homogeneous and optimally distributed over the capital vintages (i.e. output is at a maximum requiring that the marginal productivity of labour is equal for all vintages: see (2.56)), we obtain the aggregate production function for period t, corresponding to (2.53), in a way similar to the C.D. case, utilizing (2.57) and (2.58) for $\nu = 1$:

(2.59) $Q_t = A_t \left[(1-\delta)\, L_t^{-\rho} + \delta\, J_t^{-\rho}\right]^{-1/\rho}$ with the "effective" capital stock:

(2.60) $J_t = \int_{-\infty}^{t} e^{\gamma\tau} K_{t,\tau}\, d\tau = \int_{-\infty}^{t} e^{\gamma\tau} D_{t-\tau} I_\tau\, d\tau$ (**)

(*) Notice that, although we allow for monopolistic competition on the markets, there is not any identification problem w.r.t. the aggregate p.f., even if everywhere perfect competition, because there is no labour augmentive t.p. assumed. Neither case (ii) nor case (iii) of the impossibility theorem is valid for this model.

(**) Integration or summation from $t-\theta$ to t if the maximal life time is θ periods.

Taking the marginal product of new capital as the numeraire, we finally get:

(2.61) $Q_t = Ae^{\lambda t} \left[(1-\delta)L_t^{-\rho} + \delta(e^{\gamma t}K_t)^{-\rho}\right]^{\frac{1}{\rho}}$, where K_t is the capital stock.

It is now possible to estimate disembodied and embodied technical progress simultaneously. As was already indicated before (see footnotes on p. 44 and p. 50), this is not possible for the C.D. case, which might be illustrated by the following variable returns C.D. vintage model:

(2.62) $Q_{t,\tau} = Ae^{\lambda t} L_{t,\tau} (e^{\gamma\tau} D_{t-\tau} I_\tau)^{\beta}$ with α and β the elasticities of $Q_{t,\tau}$ w.r.t. $L_{t,\tau}$ and $(e^{\gamma\tau} K_{t,\tau})$, where, taking account of the equality for all vintages of the marginal (and average) labour productivities and of the aggregate output and labour input equations (2.58), the aggregate production function is immediately derived as:

(2.63) $Q_t = Ae^{\lambda t} L_t^{\alpha} J_t^{1-\alpha}$ with $J_t = \int_{-\infty}^{t} (e^{\gamma\tau} D_{t-\tau} I_\tau)^{\frac{\beta}{1-\alpha}} d\tau$ as effective capital stock or equivalently:

(2.64) $Q_t = Ae^{(\lambda+\beta\gamma)t} L_t^{\alpha} K_t^{1-\alpha}$ with $K_t = \int_{-\infty}^{t} e^{\frac{-\beta\gamma(t-\tau)}{1-\alpha}} D_{t-\tau}^{\frac{\beta}{1-\alpha}} I_\tau^{\frac{\beta}{1-\alpha}} d\tau$

as net stock of capital, from which it is clear that, unless an independent estimate of either technical progress parameter is given, it is not possible to estimate the other one (not even under constant returns to scale). As indicated previously, this problem of underidentification of the C.D. model results to the fact that only $(\lambda+\beta\gamma)$ and α can be estimated, but not λ and $\beta\gamma$ separately.

However, both parameters of either technical progress may be estimated in a C.E.S.-model. Indeed, from (2.59) and (1.63), we find:

(2.65) $\frac{Q_t}{L_t} = (\frac{\ell_1}{\ell_o})^{\sigma} A^{(1-\sigma)}(1-\delta)^{-\sigma} e^{\lambda(1-\sigma)t} (\frac{w_t}{p_t})^{\sigma}$

from which relationship, given data on Q_t, L_t, w_t and p_t, the substitution parameter ρ(from σ) and the disembodied technical progress parameter λ can be estimated, while, from (2.61) and (1.64):

$$(2.66) \quad \frac{r_t K_t}{w_t L_t} = \frac{\ell_1}{\ell_2} \frac{\delta}{1-\delta} e^{-\gamma\rho t} \left(\frac{K_t}{L_t}\right)^{-\rho}$$ the embodied t.p. parameter γ

can be estimated if data on w_t, r_t, L_t and K_t are available.

Up to now, it has been assumed that together with the capital embodied t.p., there is only Hicks neutral disembodied progress. Owing to the often important quality changes of labour input however, it may be desirable to express the labour input in efficiency units so that disembodied t.p. is split up in a Hicks neutral part (λ) and, say a Harrod neutral part standing for factors as the increased education level (rate η), reduction in working hours (ξ), changes in the age-sex distribution of the labour force (ζ). Then, the aggregate labour input may be expressed as:

$$(2.67) \quad L'_t = L_t \, e^{(\eta-\xi+\zeta)t}$$ and the exponent in (2.65) becomes:

$(\lambda+\eta-\xi+\zeta)(1-\sigma)\, t$ instead of $\lambda(1-\sigma)t$ and in (2.66) $-\rho(\gamma-\eta+\xi-\zeta)\, t$

instead of $-\gamma\rho t$, from which it follows however that, by lack of any other data, only the results of various influences can be measured, i.e. the combined effect of all factors influencing labour qualification or the sums $(\lambda+\eta-\xi+\zeta)$ and $(\gamma-\eta+\xi-\zeta)$ are estimated, so that in fact, the p.f. is not (fully) identified at all (see also impossibility theorem)(*)

(*) It may be significant to consider both capital and labour embodied t.p., where effective labour force M_t and effective capital stock J_t bring about (potential) aggregate output:

$$(2.68) \quad Q_t = F(M_t, J_t, t) = Ae^{\lambda t} F(M_t, J_t).$$ The relevance of disembodied technological change may now be estimated from (2.68).

III Some empirical results for postwar Belgian Economy

With the help of macro data about the sample period 1948 - 1967, some of the previous production models will be statistically tested on their empirical relevance for the postwar Belgian economy.
In general, two (extreme) approaches are open:

(i) either total available labour and capital stock are combined to yield maximum production

(ii) or real production is yielded from combination of (utilized) labour and capital services

The underlying production function of the first case is in fact a production capacity function since the economy is assumed to operate at full capacity, while the second case is based upon the empirical explanation of realized production, brought about by labour employed and capital utilized. Although the latter case is not in full agreement with the neo-classical full-employment assumption, most empirically tested production models are founded upon it. The sole difficulty is the evaluation of capital services.
But since in principle idle capital cannot emerge as long as a particular smooth production structure shows substitution possibilities ($\sigma > 0$) and the marginal product of capital is positive, a two-fold approach is left open to us:

- either the underlying production function is estimated from observed employment of labour and total available capital stock
- or it is estimated from labour employed and capital utilized (implicitly assuming that idle capital arises in periods of recessions and depressions, often accompanied with a rather low elasticity of substitution).

The relationship between both approaches can easily be established from specification analysis. Indeed, suppose that the p.f. of the 2nd approach represents the "true" relationship but that it is estimated (since utilized capital stock is not easily measured) by the p.f. of the first approach. Assuming that the p.f. can be conveniently linearized, the "true" p.f. can be written in the standard linear form:

(3.1) $y = X\beta + \varepsilon$ with $E(\varepsilon) = 0$, $E(\varepsilon\varepsilon') = \sigma^2 I_T$ and

the columns of X be independent of ε,

while, say through lack of data, we can only estimate:

(3.2) $y = X^{*}\beta^{*} + \varepsilon^{*}$ with $E(\varepsilon^{*}) = 0$, $E(\varepsilon^{*}\varepsilon^{*\prime}) = \sigma^2_{*} I_T$ and

the columns of X^{*} be independent of ε(*)

The mathematical expectation of the S.L.S. estimator of β^{*} is, utilizing (3.1) and (3.2):

(3.3) $E(\hat{\beta}^{*}) = E[(X^{*\prime}X^{*})^{-1} X^{*\prime} (X\beta + \varepsilon)] = E[(X^{*\prime}X^{*})^{-1}(X^{*\prime}X)]\beta = P\beta$,

where $(X^{*\prime}X^{*})^{-1}(X^{*\prime}X)$ is the matrix of regression coefficients of the explanatory variables in the "true" (second) model on the explanatory variables in the "misspecified" (first) model.(**)

Under the above assumptions, it follows immediately from (3.3) that:

(i) $\hat{\beta}^{*}$ is a biased estimator of β with specification bias:

(3.4) bias $(\hat{\beta}^{*}) = E(\hat{\beta}^{*}) - \beta = (P - I)\beta$(***)

(*) Notice that for our production models, the only difference between X and X^{*} is that utilized capital stock occurs in the former and available capital stock in the latter observation matrix.

(**) From $E(X^{*\prime}\varepsilon) = 0$, it does not follow necessarily (unless perhaps asymptotically) that $E[(X^{*\prime}X^{*})^{-1} X^{*\prime}\varepsilon] = 0$ (because X^{*} may be stochastic).

(***) For the C.D. function in (1.24), the parameter vector of the "true" model is $\gamma = \begin{bmatrix} \log A \\ \alpha \\ \beta \\ \lambda \end{bmatrix}$, so that the "bias" of the S.L.S. estimator of the parameter vector based on capital stock is:

(3.5) "bias" $(\hat{\gamma}^{*}) = \begin{bmatrix} s_0\beta \\ s_1\beta \\ (s_2-1)\beta \\ s_3 \end{bmatrix}$, $s_i (i = 0, 1, 2, 3)$ being the regression

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(ii) $\hat{\beta}^{\star}$ is an inconsistent estimator of β , with specification inconsistency given by:

(3.7) $\quad$ inc. $(\hat{\beta}^{\star}) = \underset{T\to\infty}{\text{plim}} \; \hat{\beta}^{\star} - \beta = (M_{\star\star}^{-1} M_{\star\cdot} - I_n)\, \beta$, where it is assumed that both the (asymptatic) 2nd order moment matrix of the "misspecified" variables,

$M_{\star\star} = \underset{T\to\infty}{\text{plim}} \; \frac{(X^{\star\prime}X^{\star})}{T}$, and the (asymptotic) cross moment matrix between "misspecified" and "true" variables, $M_{\star\cdot} = \underset{T\to\infty}{\text{plim}} \; \frac{X^{\star\prime}X}{T}$, exist and n is the number of explanatory variables in X.(⋆)

It will be interesting to keep the above specification analysis in mind when interpreting the following results, representing empirical estimates of various production models under alternative market structures. Estimations have been performed, each time with the capital variable expressed as total capital stock and as utilized capital services. The data are adjusted Dulbea-data 1948 - 1952 and N.I.S.-data 1953 - 1967, and may be obtained, together with various unpublished regression results, upon request to the author. The sector considered is private non-farm industry, exclusive residential structures. End of period capital stock is constructed recursively as:

(3.8) $\quad K_t = (1 - 0.084)K_{t-1} + I_t$, where I_t is gross investment of the

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efficients in:

(3.6) $\quad \ln K_t = s_o + s_1 \ln L_t + s_2 \ln K_t^{\star} + s_3 t + \eta_t$, with K_t the (utilized) capital services and $K_t^{\star}$ the total capital stock measured at time t. Since usually, $s_1 > 0$ and $s_2 < 1$, the production elasticity of labour is likely to be larger for the 1st than for the 2nd model while the reverse is true for the production elasticity of capital.

(⋆) If the matrix P as defined in (3.3) exists and if the elements of $(X^{\star\prime}X^{\star})^{-1}(X^{\star\prime}X)$ follow (at least) the weak law of large numbers, $P = M_{\star\star}^{-1} M_{\star\cdot}$ and the specification bias of $\hat{\beta}^{\star}$ coincides with its specification inconsistency.

relating sector during the year t and it is assumed that aggregate capital stock deteriorates linearly at a mean depreciation rate of 8,4% (mean life time of postwar Belgian non-residential, non-farm capital stock is about 12 years; see balance sheets of A.S.L.K.). Following various authors investigating the evaluation of Belgian capital stock (see Kirschen, Lamfalussy, Sandee etc....) three observation series for K_t have been constructed varying according to the hypotheses about the initial capital-output ratio for 1948; i.e. $CØR_{48} = 1.15$; $CØR_{48} = 1.25$ and $CØR_{48} = 1.375$ ($CØR_{67}$ each time about 1.20).

It was assumed that the degree of underutilization of capital stock is equal to the unemployment rate, so that capital services K^u are defined as deflated capital stock figures:

(3.9) $K_t^u = K_t\ (1 - u_t)$, with u_t the unemployment rate at period t.

In the following table, the estimations for the stochastic C.D.-function are reproduced: three versions for capital stock and three versions for capital services. (*)

(*) The estimated regression coefficients are accompanied with their standard errors, the relative contribution of the explanatory variable x_j into the "explained" standard derivation of the dependent variable y or:

(3.10) $e_i = \dfrac{|\hat{\beta}_i|\sigma_{x_i}}{\sum_{j=1}^{n}|\hat{\beta}_j|\sigma_{x_j}}$, and the computed F values testing the significance of the squared multiple correlation coefficient between an explanatory variable, say x_i, and all the other x-variables:

(3.11) $F_{i(n-1)}^{(T-n)} = \dfrac{R^2_{x_i x}}{1-R^2_{x_i x}}\ \dfrac{T-n}{n-1}$ $(i = 1, 2,\ldots,n)$ to find out the multicollinear subset, the Durbin-Watson statistic, the corrected first order autocorrelation parameter ($\bar{\hat{\rho}} = \hat{\rho} + \frac{n}{T}$), the corrected R^2 and the performance index:

(3.12) $I = \sqrt{\dfrac{\hat{\varepsilon}'\hat{\varepsilon}}{y'y}}$. The computed F statistics of (3.11) are <u>only</u> reproduced if they are larger than one.

To avoid multicollinearity as much as possible, S.L.S. estimation has been carried out, wherever necessary, on relative first differences instead of on simple log transformed variables.(*) For this reason, little attention has to be paid to the sometimes low $\bar{R}^2$ values. What really matters is strong parameter estimates.

If first order autocorrelation seems to be present, the results of one autocorrelation correction (type Cochran-Orcutt), according to $\hat{\rho}$ are also given (between brackets), together with the DW-value of the transformed model.
Finally, original variable values are denoted by a sign ∿ and relative first differences without any complementary sign. This is done to avoid confusion in notation.

(*) Note that by differentiation transformation, the error terms of the original log-model are transformed according to a perfectly positive autocorrelation scheme (which might be somewhat realistic owing to the trend component involved in the log model).

[illegible]ased on total capital stock.

	1			2			3		
	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i
ct	1.69(2.34)	1.18(1.50)	-	2.05(2.52)	0.98(1.35)	-	2.34(2.73)	0.81(1.20)	-
L_t	0.72(1.01)	0.27(0.25)	69(81)	0.72(1.01)	0.27(0.24)	71(79)	0.73(1.01)	0.27(0.24)	73(77)
K^1_t	0.38(0.28)	0.32(0.37)	31(19)						
K^2_t				0.31(0.28)	0.29(0.35)	29(21)			
K^3_t							0.26(0.29)	0.26(0.33)	27(23)
	DW = 1.42		$\bar{\bar{\rho}}$ = 0.38	DW = 1.38		$\bar{\bar{\rho}}$ = 0.39	DW = 1.35		$\bar{\bar{\rho}}$ = 0.41
	$\bar{R}^2$ = 0.33		I = 0.39	$\bar{R}^2$ = 0.32		I = 0.39	$\bar{R}^2$ = 0.32		I = 0.40
	(DW = 1.86)			(DW = 1.88)			(DW = 1.90)		

Based on capital services.

	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	F_i	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	F_i	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	F_i
[illegible]	2.[illegible]3(2.33)	1.00(1.45)	-		2.59(2.44)	0.83(1.33)	-		2.71(2.62)	0.70(1.20)	-	
L_t	0.65(0.86)	0.37(0.30)	74(69)	7.12	0.67(0.86)	0.35(0.29)	78(66)	6.-	0.69(0.85)	0.34(0.29)	79(64)	[illegible].80
$K^{u_1}_t$	0.19(0.32)	0.31(0.36)	26(31)	7.12								
$K^{u_2}_t$					0.16(0.34)	0.28(0.35)	22(34)	6.-				
$K^{u_3}_t$									0.13(0.36)	0.25(0.34)	21(36)	5.80
	DW = 1.30		$\bar{\bar{\rho}}$ = 0.43		DW = 1.28		$\bar{\bar{\rho}}$ = 0.44		DW = 1.27		$\bar{\bar{\rho}}$ = 0.44	
	$\bar{R}^2$ = 0.29		I = 0.40		$\bar{R}^2$ = 0.29		I = 0.40		$\bar{R}^2$ = 0.29		I = 0.40	
	(DW = 1.89)				(DW = 1.92)				(DW = 1.94)			

Table 3.1 Cobb- Douglas functions with capital resp. as capital stock and capital services.

In order to obtain approximate values for the population regression coefficients in (3.6), we computed three S.L.S.- estimations on the relative first differences of the explanatory variables in the first model. The resulting values 0.60, 0.59 and 0.58 for s_1 and 0.95, 0.98 and 0.99 for s_2 implied that the difference between the two production models is much greater as might seem from the bias-expression (3.5). This becomes clear when comparing the estimates in the above table. Contrary to the a priori belief, the production elasticities of capital are even larger in the first model than in the second. Moreover, the higher the initial capital-output ratio, the lower the production elasticity of capital.
Provided that the elasticity of factor substitution be unity, there seems to be no reason to reject the hypothesis of constant returns to scale if the factor capital is represented by capital stock. If capital services are considered, there is even much to say for decreasing returns. But if a first order autocorrelation correction is performed, increasing returns are strongly favoured. One should however be careful with interpreting the S.L.S. estimations of the 2nd model, since the F-values computed indicate intercorrelation between labour per man year and utilized capital stock (critical F-value with 1 and 17 degrees of freedom is 4.45 for a 95% confidence interval).
None the less, a general result seems to be that "disembodied" technical progress amounts about 2 à 2,5% per year for the postwar Belgian economy (significantly differing from zero).

Incorporating an underlying optimizing theory, we may estimate derived demand equations for labour and capital inputs. They show optimal combinations of factors depending upon the assumption of the economy goal. Comparing models (1.24) from deterministic profit maximization or cost minimization, (1.31) from anticipated and expected profit maximization and (1.38) from maximization of total revenue from sales, we remark that the coefficients for log $\tilde{Q}_t$ are each time the same, i.e. viz. $\frac{\ell_o}{\ell_1}$ and $\frac{\ell_o}{\ell_2}$. The only difference between the above mentioned models resists in the contents of the constant and error terms. So the assumptions about the market structure may be directly tested from statistical estimation of the relating models. Although the models are highly interdependent, only S.L.S. estimates are presented in this paper, irrespective of possibly considerable simultaneous equation bias: (*)

a) Derived demand for labour

(3.13) $\log \tilde{L}_t = 3.67(3.52) + 0.31(0.33) \log \tilde{Q}_t$ $\quad DW = 1.35 \quad \bar{\bar{\rho}} = 0.32$

$(\hat{\sigma}_{\hat{\beta}_i})$ (0.17) (0.23) (0.01) (0.02) $\quad \bar{R}^2 = 0.97 \quad I = 0.002$

(DW = 1.46)

b)(i) Derived demand for capital stock

(3.14) $\log \tilde{K}^1_t = 0.89(0.46) + 0.94(0.97) \log \tilde{Q}_t$ $\quad DW = 0.90 \quad \bar{\bar{\rho}} = 0.57$

$(\hat{\sigma}_{\hat{\beta}_i})$ (0.29) (0.57) (0.02) (0.04) $\quad \bar{R}^2 = 0.99 \quad I = 0.002$

(DW = 1.74)

(3.15) $\log \tilde{K}^2_t = 2.20(1.35) + 0.84(0.90) \log \tilde{Q}_t$ $\quad DW = 0.60 \quad \bar{\bar{\rho}} = 0.72$

$(\hat{\sigma}_{\hat{\beta}_i})$ (0.34) (0.82) (0.03) (0.06) $\quad \bar{R}^2 = 0.98 \quad I = 0.002$

(DW = 1.76)

(3.16) $\log \tilde{K}^3_t = 3.71(2.37) + 0.72(0.83) \log \tilde{Q}_t$ $\quad DW = 0.38 \quad \bar{\bar{\rho}} = 0.84$

(0.43) (1.31) (0.03) (0.10) $\quad \bar{R}^2 = 0.96 \quad I = 0.002$

(DW = 1.61)

(ii) Derived demand for capital services

(3.17) $\log \tilde{K}^{u_1}_t = -0.03(-0.73) + 1.01(1.06) \log \tilde{Q}_t$ $\quad DW = 0.78 \quad \bar{\bar{\rho}} = 0.50$

(0.33) (0.44) (0.03) (0.03) $\quad \bar{R}^2 = 0.99 \quad I = 0.001$

(DW = 1.71)

(3.18) $\log \tilde{K}^{u_2}_t = 1.29(0.22) + 0.91(0.99) \log \tilde{Q}_t$ $\quad DW = 0.60 \quad \bar{\bar{\rho}} = 0.56$

(0.38) (0.46) (0.03) (0.04) $\quad \bar{R}^2 = 0.98 \quad I = 0.002$

(DW = 1.81)

(3.19) $\log \tilde{K}^{u_3}_t = 2.80(1.08) + 0.79(0.92) \log \tilde{Q}_t$ $\quad DW = 0.43 \quad \bar{\bar{\rho}} = 0.66$

(0.46) (0.56) (0.04) (0.04) $\quad \bar{R}^2 = 0.96 \quad I = 0.002$

(DW = 1.81)

(*) Also the production function parameters should be estimated simultaneously. Only, if the economy goal is maximization of anticipated or expected profit, consistent and even unbiased production function parameters may be obtained under the assumptions mentioned in §A2. Then, the degree of monopolistic competition may conveniently be estimated by S.L.S. of the derived demand equations (1.31).

The estimated value for $\frac{\ell_o}{\ell_1}$ in (3.13) shows that ℓ_o is substantially lower than ℓ_1 and compared with a mean estimate of $\frac{\ell_o}{\ell_2}$ of about 0.90, it follows that the price elasticity of supply of labour has been significantly lower than the price elasticity of capital supply ($\eta_1 < 1$ and $\eta_2 > 1$) for the postwar Belgian economy. This is confirmed by direct estimation of the supply and demand equations (1.14), which yields $\hat{\eta}_0 = 1.84$(*), $\hat{\eta}_1 = 0.22$ and $\hat{\eta}_2 = \pm 1.20$. This experience implies that perfect competition is far from realistic for postwar Belgian non-form, non-residentialprivate industry, particularly for the labour factor market. The results about the derived demand for capital, involved in equations (3.14) until (3.19), show that there exists an inverse relationship between the ratio $\frac{\ell_o}{\ell_2}$ and the initial capital-output ratio and, compared with the estimated C.D.-functions, we remark that $\frac{\ell_o}{\ell_2}$ should depend in some positive way upon the elasticity of output w.r.t. capital (irrespective of course, of evt. occurring specification errors regarding σ...). The above results also indicate that the impact of the price elasticity of production is somewhat similar to that of the price elasticity of capital.

But since labour's share in total nominal production of the Belgian private non-farm non-residential industry has considerably increased during the postwar period: 46% in 1948, 47% in 1955, 49% in 1960, 52% in 1965 and 53% in 1967, there is a serious reason to investigate whether the production structure of the corresponding industry has known an elasticity of substitution significantly lower than unity or not.

Assuming profit maximization or cost minimization, σ has been estimated from (1.64) in two ways:

(*) $\hat{\eta}_o$ is positive, since for postwar time-series analysis, there always seems to be a <u>positive</u> relationship between total output and the relating price index. As long as own prices instead of relative (e.g. w.r.t. the wage rate) or composite prices are considered as the price variable, real demand functions can hardly be constructed for time series analysis (intrinsically inflationary character of postwar demand relationships). Therefore, ℓ_o will generally be greater than unity.

(i) by simple least squares where, due to imperfections in the optimizing behaviour of the economy, multiplicative error terms e^{v_t} are introduced in (1.64) with $E(v_t) = 0$ and $E(v_t^2) = \sigma_v^2$. Since, however, these error terms are in general dependent upon the relative factor price ratio, consistent estimates of σ may be obtained

(ii) by instrumental variables utilizing $\log \tilde{Q}_t$ as an instrumental variable for $\log \frac{\tilde{w}_t}{\tilde{r}_t}$, which might be very conceivable if there exist fluctuations of aggregate demand (reason for p.f. estimation) so that the output of the economy may be considered as independent of the random effects occuring in the profit maximizing (or cost minimizing) conditions (1.51). Then $\log \tilde{Q}_t$ is assumed to be independently distributed of v_t, and since it is, in general, (positively) correlated with $\log \frac{\tilde{w}_t}{\tilde{r}_t}$, it might act as a valuable instrumental valiable for it. Then consistent estimates of σ are obtained using the general formula:

(3.20) $\tilde{\beta} = (Z'X)^{-1} Z'y$ with estimated variance-covariance matrix

(3.21) $\tilde{\Sigma}_{\tilde{\beta}\tilde{\beta}} = \hat{\sigma}^2 (Z'X)^{-1} (Z'Z) (Z'X)^{-1'}$, with Z the matrix of instrumental variables, consisting of the constant vector and the observation vector for $\log \tilde{Q}_t$.

These estimates yielded the following results:

S.L.S.	$\log \tilde{K}_t^1/\tilde{L}_t$		$\log \tilde{K}_t^2/L_t$		$\log \tilde{K}_t^3/\tilde{L}_t$		$\log \tilde{K}_t^{u_1}/\tilde{L}_t$		$\log \tilde{K}_t^{u_2}/\tilde{L}_t$		$\log \tilde{K}_t^{u_3}/\tilde{L}_t$	
	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$
ct	5.13(5.19)	0.01(0.03)	5.18(5.28)	0.01(0.05)	5.24(5.50)	0.01(0.11)	5.08(5.12)	0.01(0.03)	5.13(5.19)	0.01(0.04)	5.19(5.28)	0.01(0.05)
$\log \frac{\tilde{w}_t}{\tilde{r}_t}$	0.67(0.45)	0.05(0.10)	0.56(0.28)	0.05(0.10)	0.43(0.14)	0.05(0.08)	0.74(0.56)	0.06(0.10)	0.63(0.40)	0.06(0.10)	0.50(0.24)	0.06(0.09)
	DW = 0.82 $\bar{\rho}$ = 0.70		DW = 0.61 $\bar{\rho}$ = 0.83		DW = 0.40 $\bar{\rho}$ = 0.95		DW = 0.86 $\bar{\rho}$ = 0.66		DW = 0.65 $\bar{\rho}$ = 0.76		DW = 0.44 $\bar{\rho}$ = 0.86	
	$\bar{R}^2$= 0.89 I =0.008 (DW = 1.09)		$\bar{R}^2$= 0.86 I =0.008 (DW = 0.83)		$\bar{R}^2$= 0.77 I = 0.008 (DW = 0.82)		$\bar{R}^2$= 0.90 I =0.009 (DW = 1.28)		$\bar{R}^2$= 0.87 I =0.009 (DW = 0.99)		$\bar{R}^2$= 0.81 I =0.009 (DW = 0.96)	
I.V.												
ct	5.13	0.01	5.18	0.01	5.24	0.01	5.08	0.01	5.13	0.01	5.19	0.01
$\log \frac{\tilde{w}_t}{\tilde{r}_t}$	0.73	0.06	0.61	0.06	0.48	0.06	0.81	0.06	0.69	0.06	0.56	0.06
	DW = 0.89 $\bar{\rho}$ = 0.65		DW = 0.67 $\bar{\rho}$ = 0.76		DW = 0.44 $\bar{\rho}$ = 0.88		DW = 0.94 $\bar{\rho}$ = 0.59		DW = 0.73 $\bar{\rho}$ = 0.68		DW = 0.50 $\bar{\rho}$ = 0.79	
	$\bar{R}^2$= 0.89 I =0.008		$\bar{R}^2$= 0.85 I =0.008		$\bar{R}^2$= 0.76 I =0.008		$\bar{R}^2$= 0.89 I =0.009		$\bar{R}^2$= 0.86 I =0.009		$\bar{R}^2$= 0.80 I =0.009	

Table 3.2 Simple Least squares and Instrumental Variables Estimation of (1.64)

All estimations show an elasticity of factor substitution significantly below unity. The mean value fluctuates around 0.60. It is higher for utilized than for available capital stock. Autocorrelation correction considerably decreases the value of σ, while by computing instrumental variable consistent estimates, σ slightly increases w.r.t.S.L.S., with a mean value of 0.61 for available capital stock and 0.69 for utilized capital stock.

It strikes however that the value of σ is heavily dependent upon the hypothesis made about the initial capital - output ratio: if the initial C.O.R. increases, σ strongly decreases (an increase of about 20% in the initial C.O.R. corresponds, in our case, to a decrease of more than 30% in σ). Therefore, it might be interesting to compute σ without taking account of a capital variable. This may be done by estimating the first equation of (1.63). Although the occuring error term, together with an additional error term for imperfections in profit maximization, is not independent of the explanatory variable $\overset{\circ}{Q}$, some idea might be obtained be applying S.L.S. (although being neither unbiased, nor consistent) on the relative first differences transformation:

$$(3.22)\quad \frac{Q_t}{L_t} = 0.08(-0.17) + 0.59(0.62)\frac{w_t}{p_t} + 0.37(0.36)Q_t \quad DW= 2.46 \quad \bar{\rho}=-0.20$$

$$(\hat{\sigma}_{\hat{\beta}_i})\quad (0.89)(0.67)\quad (0.12)(0.07)\quad (0.16)(0.13)\quad \bar{R}^2= 0.65 \quad I= 0.26$$

$$(DW = 1.85)$$

Although there is no multicollinearity present in (3.22), the value of the constant term (certainly for the autocorrelation correction), standing for $\frac{\lambda(1-\sigma)}{\nu}$, seems to be underestimated (perhaps also due to simultaneous equation bias). But the value of σ (0.60) confirms the previous findings, and although it is neither unbiased nor consistent, there is much evidence for saying that the average elasticity of factor substitution for postwar Belgian non-farm, non-residential industry amounts about 0.60 (interval 0.50 - 0.70). Combined with the significant positive coefficient for Q_t, (3.22) implies that there should be increasing returns to scale.

This might be verified from a direct estimation of C.E.S. function (1.62) either by non-linear least squares (Davidon, Hartley, Marquardt) or, assuming that $\log u_t = \varepsilon_t$ be normally distributed with mean zero and finite variance, by non-linear maximum likelihood methods (Scoring). This problem will be thoroughly discussed in a following paper, where both single equation M.L. and F.I.M.L. estimates for the non-linear equations involved in model (1.75) will be presented. In this paper, only S.L.S.- estimates of a linearized C.E.S. function and of the derived demand equations in the anticipated or expected profit maximization model (1.75) will be discussed irrespective of some possibly occurring specification error involved in the linearization of (1.62) itself and of inconsistent estimates due to simultaneous equation bias. The only purpose is to get some preliminary idea about the value of various parameters.

Following Kmenta [12], we may rewrite C.E.S. function (1.62) as:

$$(3.23) \quad \log \tilde{Q}_t = \log A + \lambda t - \frac{\nu}{\rho} f(\rho) + \varepsilon_t \quad \text{with}$$

$$(3.24) \quad f(\rho) = \log \left[\delta \tilde{K}_t^{-\rho} + (1 - \delta) \tilde{L}_t^{-\rho} \right]$$

and expand $f(\rho)$ in a Taylor series around $\rho = 0$ (C.D.-case):

$$(3.25) \quad f(\rho) = -\rho[\delta \log \tilde{K}_t + (1 - \delta) \log \tilde{L}_t] + \tfrac{1}{2}\rho^2 \delta(1 - \delta)(\log \tilde{K}_t - \mathrm{Log}\tilde{L}_t)^2$$

$$- \frac{1}{6} \rho^3\delta(1-\delta)(1-2\delta) (\log \tilde{K}_t - \log \tilde{L}_t)^3 + \text{higher order terms}$$

Considering terms only up to the 2nd order and substituting into C.E.S.-function (3.23):

$$(3.26) \quad \log \tilde{Q}_t = \log A + \lambda t + \nu\delta\log \tilde{K}_t + \nu(1-\delta) \log \tilde{L}_t - \frac{\nu\rho\delta\,(1-\delta)}{2}$$

$$(\log \tilde{K}_t - \log \tilde{L}_t)^2 + \varepsilon_t' \qquad \text{with } \varepsilon_t' = \varepsilon_t + \text{neglected}$$

higher order terms in (3.25), and differentiating (3.26):

$$(3.27) \quad Q_t = \lambda + \nu\delta K_t + \nu(1-\delta)L_t - \frac{\nu\rho\delta(1-\delta)}{2} \Delta \left(\log \frac{\tilde{K}_t}{\tilde{L}_t}\right)^2 + \Delta\, \varepsilon_t'$$

If we assume that $\Delta\, \varepsilon_t' = \eta_t$ is stochastically distributed with mean zero and constant variance we may apply S.L.S. upon (3.27). The results are presented in table 3.3.

A. Available Capital Stock

Q_t	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	F_i	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	F_i	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	F_i
ct	1.68(2.64)	1.21(1.59)			2.07(2.81)	1.01(1.44)			2.40(3.08)	0.84(1.29)		
L_t	0.65(0.88)	0.37(0.29)	57(57)	7.63	0.65(0.93)	0.33(0.26)	55(59)	4.06	0.69(0.99)	0.29(0.24)	57(61)	1.42
K_t^1	0.46(0.44)	0.43(0.44)	34(24)	6.61								
K_t^2					0.41(0.42)	0.37(0.41)	32(24)	5.53				
K_t^3									0.34(0.38)	0.31(0.38)	29(24)	3.50
$\Delta(\log \frac{\tilde{K}_t^j}{\tilde{L}_t})^2$	-0.02(-0.05)	0.07(0.06)	9 (19)	9.55	-0.03(-0.05)	0.07(0.06)	13(17)	6.24	-0.03(-0.05)	0.06(0.06)	14(15)	3.00
	DW = 1.43		$\bar{\bar{\rho}}$ = 0.42		DW = 1.39		$\bar{\bar{\rho}}$ = 0.43		DW = 1.36		$\bar{\bar{\rho}}$ = 0.44	
	$\bar{R}^2$ = 0.29		I = 0.39		$\bar{R}^2$ = 0.29		I = 0.39		$\bar{R}^2$ = 0.28		I = 0.39	
	DET = 0.42		χ_3^2= 8.90		DET = 0.52		χ_3^2= 11.98		DET = 0.68		χ_3^2= 18.43	
	(DW = 1.84)				(DW = 1.88)				(DW = 1.90)			

B. Utilized Capital Stock

Q_t	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	F_i	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	F_i	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	F_i
ct	2.32(2.72)	1.08(1.56)			2.57(2.99)	0.86(1.46)			2.74(3.22)	0.73(1.34)		
L_t	0.50(0.66)	0.55(0.41)	44(39)	24.45	0.61(0.76)	0.45(0.34)	60(46)	13.24	0.66(0.83)	0.38(0.30)	67(50)	7.01
$K_t^{u_1}$	0.37(0.57)	0.60(0.53)	39(40)	45.69								
$K_t^{u_2}$					0.24(0.49)	0.46(0.45)	30(37)	27.89				
$K_t^{u_3}$									0.18(0.44)	0.35(0.40)	25(36)	15.38
$\Delta(\log \frac{\tilde{K}_t^{u_j}}{\tilde{L}_t})^2$	-0.04(-0.08)	0.12(0.11)	16(21)	20.41	-0.03(-0.07)	0.11(0.09)	10(17)	12.66	-0.02(-0.06)	0.10(0.08)	8(14)	6.71
	DW = 1.28		$\bar{\bar{\rho}}$ = 0.48		DW = 1.27		$\bar{\bar{\rho}}$ = 0.49		DW = 1.26		$\bar{\bar{\rho}}$ = 0.49	
	$\bar{R}^2$ = 0.25		I = 0.40		$\bar{R}^2$ = 0.24		I = 0.40		$\bar{R}^2$ = 0.24		I = 0.40	
	DET = 0.15		χ_3^2= 2.61		DET = 0.22		χ_3^2= 4.05		DET = 0.33		χ_3^2= 6.57	
	(DW = 1.84)				(DW = 1.88)				(DW = 1.93)			

Table 3.3 Linearized C.E.S. functions for available and utilized capital stock.

Since the effect of multicollinearity seems to be strongly aggravated by the introduction, w.r.t. the C.D.-function, of the second order variable, also the determinants of the correlation matrices of the explanatory variables and a χ^2- test on its departure from a zero value (perfect dependence):

(3.28) $\chi^2_m = -[T - 1 - \frac{1}{6}(2n + 5)] \quad \log \left| 1 - |R| \right|$, with $m = \frac{1}{2}n(n-1)$ the degrees of freedom of the χ^2 -distributed value (3.28) and n the number of explanatory variables, the constant vector excluded, are given in table 3.3.

Compared with the χ^2 critical values for 3 degrees of freedom, being 7.82 at a 95% confidence interval and 11.34 at a 99% confidence interval, we may conclude that significant overall multicollinearity is present for the C.E.S. functions with capital as utilized capital. At the 1%- "inconfidence" level, overall multicollinearity seems also to be present in the first C.E.S. function with the capital input expressed as available capital stock. As indicated by the F-values in those equations, intercorrelation is principally relevant between the capital-input variable and the quadratic variable (critical F-values for 2 and 16 degrees of freedom are 3.63 at a 95% confidence level and 6.23 at a 99% confidence level). Due to this multicollinearity, one has to be careful with the interpretation of the parameter estimates of C.E.S. function (3.27), particularly w.r.t. the estimates involving a capital services variable.

Nevertheless, estimates of ν, δ, ρ and σ will be derived from the estimation of the vector $\beta = (\beta_o, \beta_1, \beta_2, \beta_3)' =$ $[\lambda, \nu\delta, \nu(1-\delta), -\frac{1}{2}\nu\rho\delta(1-\delta)]'$, while the corresponding (asymptotic) variance-covariance matrix may, in general, be computed from the approximation formula for functions of random variables. So, if

$\hat{\theta}_i = f_i(\hat{\beta}_o, \hat{\beta}_1, \ldots\ldots, \hat{\beta}_h) \qquad (h \leq 3),$

$$(3.29) \quad \mathrm{var}\,(\hat{\theta}_i) = \sum_{k=o}^{h} \left(\frac{\partial f_i}{\partial \beta_k}\right)^2_{\beta_k = \hat{\beta}_k} \mathrm{var}\,(\hat{\beta}_k) +$$

$$+ \sum_{\substack{k=o \\ k\neq \ell}}^{h} \sum_{\ell=o}^{h} \left(\frac{\partial f_i}{\partial \beta_k}\right)_{\beta_k = \hat{\beta}_k} \left(\frac{\partial f_i}{\partial \beta_\ell}\right)_{\beta_\ell = \hat{\beta}_\ell} \mathrm{cov}\,(\hat{\beta}_k\ \hat{\beta}_\ell) \text{ and}$$

$$(3.30) \quad \mathrm{cov}\,(\hat{\theta}_i\ \hat{\theta}_j) = \sum_{k=o}^{h} \left(\frac{\partial f_i}{\partial \beta_k}\right)_{\beta_k = \hat{\beta}_k} \left(\frac{\partial f_j}{\partial \beta_k}\right)_{\beta_k = \hat{\beta}_k} \mathrm{var}\,(\hat{\beta}_k) +$$

$$+ \sum_{\substack{k=o \\ k\neq \ell}}^{h} \sum_{\ell=o}^{h} \left(\frac{\partial f_i}{\partial \beta_k}\right)_{\beta_k = \hat{\beta}_k} \left(\frac{\partial f_j}{\partial \beta_\ell}\right)_{\beta_\ell = \hat{\beta}_\ell} \mathrm{cov}\,(\hat{\beta}_k \hat{\beta}_\ell)\ (\star),$$

with i,j = 1, 2, 3, 4 and $\hat{\theta} = (\hat{\nu}, \hat{\delta}, \hat{\rho}, \hat{\sigma})'$,

or the (asymptotic) standard errors for the $\hat{\theta}_i$ - elements are easily computed from (3.29), utilizing the definition of the $\hat{\beta}$- vector, as the nonnegative roots:

$$(3.31) \quad \text{S.E. } (\hat{\nu}) = \text{S.E. } (\hat{\beta}_1 + \hat{\beta}_2) = [\mathrm{var}\,(\hat{\beta}_1) + \mathrm{var}\,(\hat{\beta}_2) + 2\,\mathrm{cov}\,(\hat{\beta}_1\hat{\beta}_2)]^{\frac{1}{2}},$$

$$(3.32) \quad \text{S.E. } (\hat{\delta}) = \text{S.E. } \left(\frac{\hat{\beta}_1}{\hat{\beta}_1 + \hat{\beta}_2}\right) = \frac{1}{(\hat{\beta}_1 + \hat{\beta}_2)^2} [\hat{\beta}_2^2\ \mathrm{var}\,(\hat{\beta}_1) +$$

$$+ \hat{\beta}_1^2\ \mathrm{var}\,(\hat{\beta}_2) - 2\,\hat{\beta}_1\hat{\beta}_2 \mathrm{cov}\,(\hat{\beta}_1\hat{\beta}_2)]^{\frac{1}{2}},$$

($\star$) No confusion should be made between the various notations. Indeed, for clarity, we denoted here, in contradiction to the previous tables, the variances by var $(\hat{\beta}_i)$ and var $(\hat{\theta}_i)$ in stead of $\sigma^2_{\hat{\beta}_i}$ and $\sigma^2_{\hat{\theta}_i}$.

σ is reserved for the elasticity of substitution and the standard error $\hat{\sigma}_{\hat{\beta}_i}$ will be denoted by S.E. (β_i).

$$(3.33) \quad \text{S.E.}\ (\hat{\rho}) = \text{S.E.}\ \left[\frac{-2\hat{\beta}_3(\hat{\beta}_1 + \hat{\beta}_2)}{\hat{\beta}_1\ \hat{\beta}_2}\right] = 2\left[\frac{\hat{\beta}_3^2}{\hat{\beta}_1^4}\ \text{var}\ (\hat{\beta}_1) + \frac{\hat{\beta}_3^2}{\hat{\beta}_2^4}\ \text{var}\ (\hat{\beta}_2) + \right.$$

$$+ \frac{(\hat{\beta}_1 + \hat{\beta}_2)^2}{(\hat{\beta}_1\hat{\beta}_2)^2}\ \text{var}\ (\hat{\beta}_3) + \frac{2\hat{\beta}_3^2}{(\hat{\beta}_1\hat{\beta}_2)^2}\ \text{cov}\ (\hat{\beta}_1\hat{\beta}_2)$$

$$\left. - \frac{2\hat{\beta}_3(\hat{\beta}_1 + \hat{\beta}_2)}{\hat{\beta}_1^3\ \hat{\beta}_2}\ \text{cov}\ (\hat{\beta}_1\hat{\beta}_3) - \frac{2\hat{\beta}_3(\hat{\beta}_1 + \hat{\beta}_2)}{\hat{\beta}_2^3\ \hat{\beta}_1}\ \text{cov}\ (\hat{\beta}_2\hat{\beta}_3)\right]^{\frac{1}{2}}$$

and

$$(3.34) \quad \text{S.E.}\ (\hat{\sigma}) = \text{S.E.}\ \left(\frac{1}{1 + \hat{\rho}}\right) = \frac{1}{(1+\hat{\rho})^2}\ \text{S.E.}\ (\hat{\rho}).$$

The S.L.S. and autocorrelation corrected parameter values for λ, ν, δ, ρ and σ, together with their asymptotic standard errors (based upon the estimated variance- covariance matrix of the S.L.S. estimators) are gathered, for all six C.E.S. equations involved in table 3.3, in the following table:

	1		2		3		4		5		6	
	Est.	S.E.	Est.	S.E.	Est.	S.E.	Est.	S.E.	Est.	S.E.	Est.	S.E.
$\hat{\lambda}$	1.68(2.64)	1.21	2.07(2.81)	1.01	2.40(3.08)	0.84	2.32(2.72)	1.08	2.57(2.99)	0.86	2.74(3.22)	0.73
$\hat{\nu}$	1.11(1.32)	0.37	1.06(1.35)	0.35	1.03(1.37)	0.34	0.87(1.23)	0.30	0.85(1.25)	0.30	0.84(1.27)	0.29
$\hat{\delta}$	0.59(0.67)	0.33	0.62(0.69)	0.30	0.67(0.72)	0.25	0.57(0.54)	0.64	0.72(0.61)	0.51	0.78(0.65)	0.40
$\hat{\rho}$	[illegible](0.34)	0.51	0.23(0.35)	0.51	0.29(0.36)	0.50	0.42(0.52)	0.99	0.30(0.47)	0.99	0.29(0.42)	1.10
$\hat{\sigma}$	[illegible](0.75)	0.39	0.81(0.74)	0.34	0.78(0.74)	0.30	0.70(0.66)	0.49	0.77(0.68)	0.59	0.78(0.70)	0.67

Table 3.4. Coefficient estimators and (asymptotic) standard errors for C.E.S. parameters.

As was also indicated for the Cobb-Douglas models, there seems to be no significant departure from constant returns to scale. Point estimations based on available capital stock models indicate increasing returns while S.L.S.- estimates based on utilized capital point to decreasing returns, which is however corrected when account has been taken of the significant autocorrelation embodied in S.L.S.- estimation.

Due to the large standard errors of the estimated substitution parameter $\hat{\rho}$, there is even no significant departure from the unitary elasticity of substitution hypothesis, although all point estimates indicate values of σ below unity. Compared with the pretty reliable estimates (irrespective of possible simultaneous equation bias) of the equations in table 3.2. and equation (3.22), the estimates for σ, in table 3.4, seem to be a little exaggerated due to the multicollinearity involved.

A remarkable fact is that $\hat{\sigma}$ seems to increase if the initial capital-output ratio also increases and if the capital input variable is expressed as utilized capital. This again is probably a consequence of multicollinearity, being particularly strong for the models with utilized capital stock, so that estimates of σ for these, and likely also for the available capital stock models, are preferrably obtained from table 3.2. rather than from table 3.4.
Only the disembodied technical progress parameter (and as stated above, also the returns to scale parameter) seems to be rather reliably estimated (in the neighbourhood of 2% - 3% per year).

The simple least squares estimates (together with autocorrelation corrections) for the derived demand equations of C.E.S. model (1.75), transformed to relative first differences to avoid multicollinearity between t and log $\tilde{Q}_t$, are given in the following table:

	L_t		K_t^1		K_t^2		K_t^3		K_t^{u1}		K_t^{u2}		K_t^{u3}	
	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$
ct	-0.67(-0.61)	0.63(0.57)	2.66([illegible])	0.61(1.0[illegible])	2.22(4.39)	0.69(1.40)	1.72(5.14)	0.77(2.02)	1.94([illegible].99)	0.81(0.69)	1.50(2.79)	0.88(0.75)	1.01(2.57)	0.96(0.81)
Q_t	0.44(0.49)	0.15(0.12)	0.24(0.02)	0.14(0.08)	0.25(0.02)	0.16(0.07)	0.26(0.02)	0.18(0.07)	0.44(0.32)	0.19(0.12)	0.45(0.3[illegible])	0.20(0.11)	0.46(0.30)	0.22(0.11)
	DW = 1.27	$\bar{\rho}$ = 0.29	DW = 0.66	$\bar{\rho}$ = 0.79	DW = 0.53	$\bar{\rho}$ = 0.86	DW = 0.42	$\bar{\rho}$ = 0.91	DW = 0.60	$\bar{\rho}$ = 0.50	DW = 0.68	$\bar{\rho}$ = 0.56	DW = 0.58	$\bar{\rho}$ = 0.61
	$\bar{R}^2$ = 0.36	I = 0.69	$\bar{R}^2$ = 0.15	I = 0.34	$\bar{R}^2$ = 0.08	I = 0.41	$\bar{R}^2$ = 0.11	I = 0.51	$\bar{R}^2$ = 0.20	I = 0.42	$\bar{R}^2$ = 0.18	I = 0.48	$\bar{R}^2$ = 0.16	I = 0.57
	(DW = 1.51)		(DW = 1.55)		(DW = 1.62)		(DW = 1.67)		(DW = 1.49)		(DW = 1.48)		(DW = 1.48)	

<u>Table 3.5.</u> S.L.S. and corrected S.L.S. estimates for the C.E.S. derived demand equations (1.75)

The small negative constant in the derived demand equation for labour, standing for $-\dfrac{\lambda\rho}{\nu(\ell_1+\rho)}$, w.r.t. to the high, but positive (!), value for $-\dfrac{\lambda\rho}{\nu(\ell_2+\rho)}$, confirms once more that ℓ_1 is much larger than ℓ_2(smaller price elasticity of labour than for capital: see direct and C.D. estimates). This is also verified from the estimated coefficient for the relative first differences of production, the theoretical contents of which are

$\dfrac{\frac{\rho}{\nu}+\ell_o}{\ell_1+\rho}$ viz $\dfrac{\frac{\rho}{\nu}+\ell_o}{\ell_2+\rho}$. But the tangential S.L.S. coefficient for Q_t is about the same for labour and utilized capital input, which is explained by the decreasing returns to scale for the C.E.S. model with the latter capital variable (see table 3.4 and also lower constant values for "utilized capital model" than for "available capital model" in table 3.5).

It is also remarked from table 3.5 that the hypotheses about the initial capital output ratio have a strong influence upon the disembodied technical progress parameter (see decrease of the constant values) but in contradiction to the C.D. models (see eq. (3.13)-(3.19)), not so very much on the parameters ℓ_o and ℓ_1(together with ρ and ν). This might be a consequence of the erroneous specification regarding the elasticity of substitution there.

The only assumption still to test regarding this elasticity of substitution is its constancy over the sample period. This might be done in the way described in section I. C. and appendix B. In the subsequent paper announced, a more penetrating analysis about estimation, also inclusive homothetic production function models, will be given.

A preliminary test about the variability of σ during that sample period will now be given in the form of S.L.S. and corrected S.L.S. estimates of the profit maximizing or cost minimizing relationship (B 37), expressed in relative first differences.

A. Available Capital Stock.

Q_t/L_t	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i
w_t/P_t	0.75(0.76)	0.09(0.08)	88(92)	0.76(0.77)	0.09(0.07)	88(93)	0.78(0.78)	0.08(0.06)	90(94)
K_t^1/L_t	0.17(0.10)	0.18(0.17)	12(8)						
K_t^2/L_t				0.16(0.09)	0.19(0.17)	12(7)			
K_t^3/L_t							0.13(0.08)	0.19(0.17)	10(6)
	DW = 2.40	$\bar{\bar{\rho}}$ = - 0.17		DW = 2.35	$\bar{\bar{\rho}}$ = - 0.15		DW = 2.29	$\bar{\bar{\rho}}$ = - 0.13	
	$\bar{R}^2$ = 0.53	I = 0.31		$\bar{R}^2$ = 0.52	I = 0.32		$\bar{R}^2$ = 0.52	I = 0.32	
	(DW = 2.05)			(DW = 2.07)			(DW = 2.09)		

B. Utilized Capital Stock

	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i	$\hat{\beta}_i$	$\hat{\sigma}_{\hat{\beta}_i}$	e_i
w_t/P_t	0.77(0.73)	0.11(0.09)	93(90)	0.78(0.74)	0.10(0.08)	95(89)	0.80(0.75)	0.09(0.08)	98(89)
K_t^{u1}/L_t	0.11(0.18)	0.23(0.19)	7(10)						
K_t^{u2}/L_t				0.08(0.17)	0.23(0.20)	5(11)			
K_t^{u3}/L_t							0.03(0.16)	0.23(0.20)	2(11)
	DW = 2.25	$\bar{\bar{\rho}}$ = - 0.13		DW = 2.21	$\bar{\bar{\rho}}$ = - 0.12		DW = 2.16	$\bar{\bar{\rho}}$ = -0.09	
	$\bar{R}^2$ = 0.51	I = 0.32		$\bar{R}^2$ = 0.51	I = 0.32		$\bar{R}^2$ = 0.50	I = 0.32	
	(DW = 2.19)			(DW = 2.23)			(DW = 2.26)		

Table 3.6 Test for the "indirect approach" of V.E.S.-functions (B 37).

This experiment reveals that, according to the indirect approach for V.E.S. functions, the elasticity of factor substitution is not significantly varying since not any coefficient of the capital-labour ratio seems to differ significantly from zero at a 95% confidence level, provided of course that the inconsistency, often implied in the S.L.S. estimation of (B 37), does not play a too considerable role. Since all F-values for multicollinearity (see (3.11)) were below unity, this certainly represents no problem at all. It is clear, however, that a more thorough analysis is required. This should presumably be performed from a "direct approach" where it is assumed that the elasticity of substitution is some function of the capital intensity (see appendix B 2.1), while a suitable general framework would be the class of homothetic production models. More about this for a next opportunity.

Appendix A Derivation of the Class of C.E.S. Production Functions.

In this appendix, function (1.4) will be derived from definition (7) of the elasticity of substitution σ and it will be shown that three very known production functions can be derived from it as special cases (★):

ρ	-1	0	∞
σ	∞	1	0
C.E.S.	Linear Production Function	Cobb-Douglas	Leontief (fixed proportions)

Considering the general two-factor production function supposed to exist at a certain period:

(A 1) $Q = F\ (L,K)$, where

Q can be kept constant, say at a level $Q^{\star}$, so that K can be expressed as a function of L alone (without $Q^{\star}$ since it is kept constant on an isoquant level):

(A 2) $K = f\ (L)$, where the marginal rate of substitution is:

(A 3) $R = \frac{\partial K}{\partial L} = f'\ (L)$ and the elasticity of substitution

(A 4) $$\sigma = \frac{d(K/L)/K/L}{d\ f'(L)/f'(L)} = f'(L).\ \frac{L}{K}\left[\frac{L(dK/df'(L)) - K(dL/df'(L))}{L^2}\right]$$

$$= f'(L).\ \frac{L}{K}\ \frac{L(f'(L)/f''(L)) - K(1/f''(L))}{L^2}$$ with

(★) The derivation is based on Appendix A of [4].

(A 5) $f''(L) = \frac{d^2f}{dL^2} = \frac{d^2K}{dL^2} = f'(L). \frac{df'(L)}{dK}$

so that a non-linear second order differential equation is obtained:

(A 6) $\sigma LKf''(L) = L[f'(L)]^2 - Kf'(L)$, where solution will be the equation of an isoquant containing σ as a parameter and two arbitrary integration constants.

From the solution of (A6), three types of production functions, all of them are spectial cases of a C.E.S.-production function, will be derived.

a) <u>σ = 0</u> <u>Fixed Proportion or Leontief Production Function</u> (input-output analysis)

If σ = 0, from (A6): either

(A 7) $f'(L) = 0 \rightarrow K = f(L) = k$ (integration constant) or

(A 8) $Lf'(L) - K = 0 \rightarrow \frac{dK}{dL} = \frac{K}{L} \rightarrow \log K + \log k_1 = \log L + \log k_2$ so that

$K = \frac{k_2}{k_1} L$ (k_1, k_2 are integration constants) , which says

that, if the input of one factor increases, the input of the other factor must also increase so as to hold output constant. Consequently, (A 8) does not have to be considered since it implies a homogeneous production function of degree zero.

Function (A 7), being the Leontief production isoquant, shows that, given the output Q, the input of K is uniquely determined. An occurring change in factor prices has no effect at all on the capital-labour ratio (★) so that the factors combine in a fixed

(★) If there exists perfect competition on the factor markets or if the price elasticities of supply for factors are equal (so $\ell_1 = \ell_2$), the marginal rate of substitution is equal to the ratio of factor prices under profit maximization or cost minimization so that the elasticity of substitution becomes:

(7 bis) $\sigma = \frac{d \log K/L}{d \log w/r}$

proportion and substitutability of one factor for another is impossible.

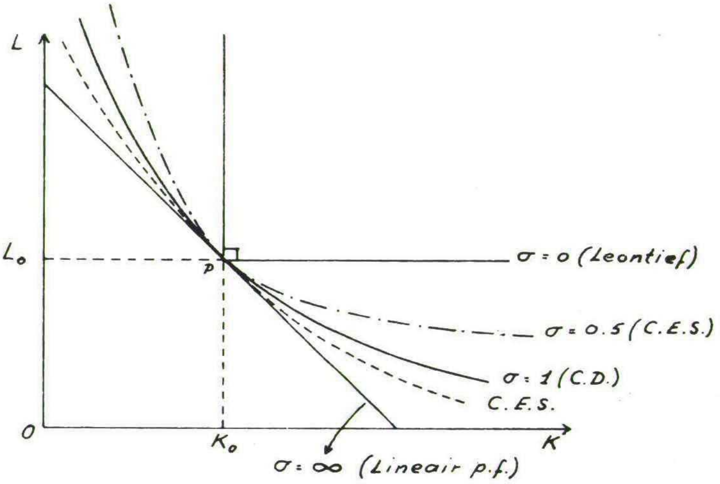

Figure 1 CES-class of isoquants (*)

So, if $\sigma = 0$, the isoquants can be written as:

(A 9) $L = L_o$ for $K \geq K_o$ (excess capital)

$K = K_o$ for $L \geq L_o$ (excess labour)

which are right-angled curves where the profit maximizing economy produces only at the corner point P.

(*) The isoquants in figure 1 are symmetric because capital intensity is supposed to be equal to labour intensity ($\delta = 0.5$). For $\delta \neq 0.5$, each isoquant with $\sigma \neq 0$ and ∞ will be asymmetric w.r.t. left upper and right lower parts.

If $\sigma > 0$, (A 6) has to be solved, which can be easily performed as:

Substitute $L = e^h$ with $\frac{dh}{dL} = e^{-h}$, $\frac{d^2h}{dL^2} = -e^{-2h}$ and

(A 10) $\frac{d^2K}{dL^2} = f''(L)e^{-2h} - f'(L)e^{-2h}$, so that substituting in (A 6)

and eliminating e^{-h}:

(A 11) $\sigma e^h K \left[f''(L)\, e^{-2h} - f'(L)e^{-2h} \right] = e^{-h} [f'(L)]^2 - K [f'(L)]\, e^{-h}$

and substituting for e^{-h}:

(A 12) $\sigma K f''(L) + K f'(L)(1 - \sigma) - [f'(L)]^2 = 0$, with derivatives

taken with respect to h.

Since h does not occur explicitly in (A 12), we can reduce the 2nd order differential equation by setting $f'(L) = \frac{dK}{dh} = s$, so that (A 12) becomes:

(A 13) $\sigma K \frac{ds}{dh} + Ks(1 - \sigma) - s^2 = 0$ and because

(A 14) $\frac{df'(L)}{dh} = \frac{ds}{dK}\,\frac{dK}{dh} = \frac{ds}{dK}\, s$, (A 13) becomes

(A 15) $\sigma K \frac{ds}{dK} + K(1 - \sigma) - s = 0$

Utilizing $K^{-1/\sigma}$ as the integration factor, the solution of the first order differential equation is:

(A 16) $s = \frac{dK}{dh} = K + cK^{1/\sigma}$ (c = integration constant).

b) <u>$\sigma = 1$ Cobb- Douglas Production Function</u>

If $\sigma = 1$, the first order differential equation (A 16) can be solved as follows:

(A 17) $\frac{dK}{K} = (1 + c)dh \rightarrow \log K = (1 + c)h + c_1$ (c = constant) or

$K = k\, e^{(1 + c)h}$ with $k = e^{c_1}$ and since $L = e^h$, we find

(A 18) $K = k\, L^{(1 + c)}$ as the equation of the constant production line.

Deriving a homogeneous production function of degree ν, we may write:

(A 19) $Q = F(L,K) = F(L^{(1 + c)}/K) = F(z)$ $(z = \frac{1}{k})$ and by

Euler's theorem:

(A 20) $$\nu Q = L\frac{\partial Q}{\partial L} + K\frac{\partial Q}{\partial K} = (1 + c)\;\frac{L^{(1 + c)}}{K}\;\frac{dQ}{dz} - \frac{L^{(1 + c)}}{K}\;\frac{dQ}{dz}$$

$$= cz\frac{dQ}{dz}$$, which has as solution:

(A 21) $Q = k(L^{(1 + c)}/K)^{\nu/c}$ (k = integration constant) which is equal to the Cobb- Douglas function:

(A 22) $Q = kL^{\frac{\nu}{c} + \nu}K^{-\frac{\nu}{c}} = kL^{\alpha}K^{\beta}$ with $\alpha = \frac{\nu + \nu c}{c}$, $\beta = -\frac{\nu}{c}$ and $\alpha + \beta = \nu$ (*)

c) <u>σ = any constant value:</u> <u>CES - Production Function</u>

Rewriting the first order differential equation (A 16) and utilizing:

(A 24) $\frac{dK}{dL} = \frac{dK}{dh}\cdot\frac{dh}{dL} = \frac{dK}{dh}\,e^{-h}$ or $\frac{dK}{dh} = e^h\frac{dK}{dL} = L\frac{dK}{dL}$, we find:

(*) Note that C.D.-function (1.1) is obtained if $A_t = Ae^{\lambda t}$ is substituted for k in (A 22). Note also that the unitary elasticity of substitution of (1.1) can directly be derived from the marginal products (1.2) and application of the definition (7) of σ:

(A 23) $$\sigma = \frac{d\log K_t/L_t}{d\log \partial K_t/\partial L_t} = \frac{d\log K_t/L_t}{d(\log \alpha/\beta + \log K_t/L_t)} = \frac{d\log K_t/L_t}{d\log K_t/L_t} = 1.$$

(A 25) $L \frac{dK}{dL} = K + cK^{1/\sigma}$ or $\frac{dK}{K(1 + cK^{\frac{1}{\sigma} - 1})} = \frac{dL}{L}$

The implicit solution of this Bernouilli differential equation amounts to:

(A 26) $K^{1 - \frac{1}{\sigma}} - kL^{1 - \frac{1}{\sigma}} = z$ (both k and z are integration constants)

according to the produce for the C.D.-function (see A 19), a homogeneous C.E.S.-function of degree ν can be written from (A 26) as:

(A 27) $Q = F(K^{(1 - \frac{1}{\sigma})} - kL^{(1 - \frac{1}{\sigma})}) = F(z)$ and since $\rho = \frac{1 - \sigma}{\sigma}$

(A 28) $Q = F(K^{-\rho} - kL^{-\rho})$

Applying Euler's theorem for homogeneous production functions of degree ν on (A 28):

(A 29) $\nu Q = L \frac{\partial Q}{\partial L} + k \frac{\partial Q}{\partial K} = \frac{\partial Q}{\partial z} \rho (kL^{-\rho} - K^{-\rho})$ with solution

(A 30) $Q = c_1^{1/\rho} z^{-\nu/\rho} = c_1^{1/\rho} (K^{-\rho} - kL^{-\rho})^{-\nu/\rho}$ (c_1 = integration constant) which can be written as:

(A 31) $Q = k_1 [\delta K^{-\rho} + (1 - \delta)L^{-\rho}]^{-\nu/\rho}$ putting $c_1^{-\frac{1}{\nu}} = \delta k_1^{-\rho}$ and

$c_1^{-\frac{1}{\nu}} k = -(1 - \delta)k_1^{-\rho}$, which ensures that $0 < \delta < 1$

since the arbitrary constant is negative (slope of a convex isoquant z is negative), so that (A 31) is exactly the previously defined C.E.S.-function (1.4) with $k_1 = Ae^{\lambda t}$.

d) <u>if $\sigma \to \infty$:</u> <u>Linear Production Function.</u>

From the definition of isoquant (A 26) it is clear that it tends to a straight line if σ tends to infinity (see also figure 1); in the limit:

(A 32) $z^{\star} = K - kL$

Transforming to the production function:

(A 33) $Q = F(z^{\star})$ and applying Euler's theorem once more, we find:

(A 34) $Q = k_1[\,\delta K + (1 - \delta)L]^{\nu}$ so that a linear isoquant is always attained, but a linear production function only is if there are constant returns to scale ($\nu = 1$), at least when production is expressed in original dimension(★).

Finishing this appendix, it might be clear to stress that all previous production functions show constant elasticity of substitution, or changes in relative factor inputs and prices do not alter the elasticity. The value of the elasticity is determined by the underlying technology and changes in the underlying technology effect variations on the elasticity for every level of the factor inputs and prices. So, the constancy of the elasticity refers to its invariance with respect to changes in relative factor supplies and <u>not</u> to transformations of the underlying technology.

(★) Indeed, by convenient "normalization" of the production data, a linear production function is always obtained, also if $\nu \neq 1$.

Appendix B Production Functions related to C.E.S.: changing returns to scale and variable elasticity of substitution

1. C.E.S.- function with changing returns to scale.

From (A 26) and (A 31), the equation of a C.E.S. isoquant can be written as:

(B 1) $$z = (1 - \delta)\ L^{-\rho} + \delta K^{-\rho}$$

where z is an arbitrary constant (different from z in A 26), δ the capital intensity parameter and $\rho = \frac{1}{\sigma} - 1$ the substitution parameter.

Supposing now that returns to scale of the general production function(A 1) are a function of some changing variable(s), say a (decreasing) function h of total output, then Euler's theorem on homogeneous functions may be rewritten as:

(B 2) $$Q\ h(Q) = L\frac{\partial Q}{\partial L} + K\frac{\partial Q}{\partial K}$$ and since ρ (so σ) and k must take the same values for each isoquant, z in fact numbers these isoquants so that Q can be written as a function of z alone:

(B 3) $$h(Q) = \frac{dQ}{dz}\left[\frac{\partial z}{\partial L}\cdot\frac{L}{Q} + \frac{\partial z}{\partial K}\cdot\frac{K}{Q}\right]$$ or

(B 4) $$h(Q) = -\rho\frac{dQ}{dz}\left[(1 - \delta)\ \frac{L}{Q}^{-\rho} + \delta\ \frac{K}{Q}^{-\rho}\right] = -\rho\frac{dQ}{dz}\frac{z}{Q}$$

For a specified form of h(Q), the production function is derived from the solution of differential equation (B 4).
Quite generally, h(Q) may be represented by a polynomial, say of degree n, in Q (*):

(B 5) $$h(Q) = a_o + a_1 Q + a_2 Q^2 + \ldots + a_n Q^n$$

(*) Other functions for h(Q), depending on a known and constant returns to scale parameter of the underlying production isoquant, will be discussed in the next appendix. Note also that for n = o, $h(Q) = a_o = \nu$ and solution of (B 4) yields the usual C.E.S.-production function (1.4).

Substituting (B 5) into (B 4), we have to integrate both sides of

(B 6) $\dfrac{dQ}{h(Q).Q} = -\dfrac{1}{\rho}\ \dfrac{dz}{z}$, which involves an expression in all n (real or complex) roots of polynomial (B 5). Since many arbitrary functions can be conveniently approximated by a quadratic:

(B 7) $h(Q) = a_o + a_1Q + a_2Q^2$ with real nonzero roots $(a_1^2 - 4a_0a_2 > 0)$, differential equation (B 6) may be solved by partial fraction with:

(B 8) $\dfrac{1}{(a_o + a_1Q + a_2Q^2)} = \dfrac{1}{a_2(Q - \alpha_1)(Q - \alpha_2)}$ with α_1 and α_2:

(B 9) $\alpha_1, \alpha_2 = \dfrac{-a_1 \pm \sqrt{a^2{}_1 - 4a_oa_2}}{2a_2}$ so that

(B 10) $\dfrac{1}{a_2}\left(\dfrac{\log Q}{\alpha_1\ \alpha_2} + \dfrac{\log\ |\ Q - \alpha_1|}{\alpha_1\ (\alpha_1 - \alpha_2)} + \dfrac{\log\ |\ Q - \alpha_2|}{\alpha_2\ (\alpha_2 - \alpha_1)}\right) = -\dfrac{1}{\rho}\log z + c$

with c an arbitrary integration constant.

Since $\alpha_1\ \alpha_2 = \dfrac{a_o}{a_2} = \dfrac{\nu}{a_2}$, (B 10), using (B 1), can be rewritten as:

(B 11) $Q|Q - \alpha_1|^{\beta_1}\ |Q - \alpha_2|^{\beta_2} = e^c\left[(1 - \delta)L^{-\rho} + \delta K^{-\rho}\right]^{-\nu/\rho}$ so that, with

$$Q|Q - \alpha_1|^{\beta_1}\ |Q - \alpha_2|^{\beta_2} = f(Q) = Q^D$$

(B 12) $e^c = Ae^{\lambda t}$

$\beta_1 = \dfrac{\alpha_2}{\alpha_1 - \alpha_2}$ and $\beta_2 = \dfrac{\alpha_1}{\alpha_2 - \alpha_1}$, production function (1.8) emerges.

The term $|Q - \alpha_1|^{\beta_1}\ |Q - \alpha_2|^{\beta_2}$ acts as a deflator if the actual returns to scale imply a higher level of output than that implied by unchanging returns to scale and in the opposite case, it acts as an inflator.

2. V.E.S.- production functions.

In this appendix, only the variability of the elasticity of substitution of homogeneous production functions of degree one will be analysed. The discussion of more general V.E.S.-production functions will be deferred to the next appendix. In general, it may be noticed that, if the capital-labour ratio varies due to changes in the factor price ratio, it is possible that the elasticity of substitution will vary (i.e. if the marginal rate of substitution does not vary in a same proportion). Two approaches are clear then:

(i) either σ is supposed to depend, in a certain way, on the capital intensity or

(ii) a theory of production is introduced (say profit maximization or cost minimization) where the output-labour ratio is not only depending upon the relative wage ratio (see 1.63) but also upon the capital-labour ratio.

It is obvious that both approaches, the "direct" and the "indirect" one, are based upon empirical relevance.

B 2.1 σ is an explicit function of K/L (direct approach) (*)

To derive the explicit production function in this case of varying σ, we have to solve the differential equation involved by definition (7) for the first degree homogeneous production function:

(B 13) $$Q = F(L,K) = L\ F\left(1, \frac{K}{L}\right) = L\ f(k) \quad \text{or}$$

(B 14) $$q = f(k) \quad \text{with} \quad q = \frac{Q}{L} \quad \text{and} \quad k = \frac{K}{L}, \text{ the capital intensity.}$$

The marginal products with respect to labour and capital are:

(B 15) $$\frac{\partial Q}{\partial L} = f(k) + L\,\frac{df(k)}{dk}.\,\frac{\partial k}{\partial L} = f(k) - \frac{K}{L}\ f'(k) = q - kq' \quad \text{and}$$

(B 16) $$\frac{\partial Q}{\partial K} = L\,\frac{d\ f(k)}{dk}\,\frac{\partial k}{\partial K} = f'(k) = q' \qquad \text{with } q' = \frac{dq}{dk} \quad ,$$

(*) The idea of this appendix is based upon the work of R.Sato [18].

or the marginal rate of substitution is equal to:

(B 17) $R = \frac{\partial Q/\partial L}{\partial Q/\partial K} = \frac{q}{q'} - k$, and the elasticity of substitution

(according def. (7)):

(B 18) $\sigma = \frac{dk/k}{dR/R} = \sigma\ (k)$ so that the (2nd order) differential equation

(B 19) $\frac{dR}{R} = \frac{dk}{k\sigma(k)}$ with $R = \frac{q}{q'} - k$

has to be solved. This yields (in 2 steps):

(B 20) $R = c \exp \int \frac{d \log k}{\sigma(k)} = \frac{q}{q'} - k = \frac{q\ dk}{dq} - k$ and

(B 21) $q = c_1 \exp \int \frac{dk}{k + c \exp \int \frac{d \log k}{\sigma(k)}}$, where c and c_1 are arbitrary

positive constants of integration.
To work out the integral term in the r.h.s. of (B 21), we have to specify the function $\sigma(k)$. Empirical relecance has shown(*) that, for most economies, σ will first increase to a certain level (above unity) if the capital labour ratio increases, and then will decrease (until a value below unity) if capital accumulates faster than labour. This dependence implies a parabolic function:

(B 22) $\sigma(k) = a_o + a_1 k + a_2 k^2$, where a_2 is the form parameter denoting nonlinearity of (B 22) (**)

(*) See e.g. Wise,J. and Y. Yeh : "Econometric Techniques for analyzing Wage and Productivity Differentials with application to Manufacturing Industries in U.S.A., India and Japan", presented at the 1965 Econometric Society Meeting, Chicago.

(**) In view of the integrations involved in (B 21), it is already clear a priori that explicit forms for $\sigma(k)$ other than polynomials are quite difficult to handle. If $\sigma(k)$ is e.g. an exponential function of type $k^{1/a}$, the integrand on the r.h.s. of (B 21) becomes $\frac{dk}{k + c \exp(-ak^{-1/a})}$ which seems a very difficult term to integrate.
In the case of polynomials, no exponential terms appear in the integrand and exponential and log terms are nicely "compensating" each other.

Substitution in (B 19) and partial fraction yields:

(B 23) $\frac{dR}{R} = \frac{dk}{k(a_o + a_1k - a_2k^2)} = \frac{dk}{a_2k(k - \alpha_1)(k - \alpha_2)}$ under the

condition that

(B 24) $a^2{}_1 > 4a_oa_2$ (α_1, α_2 are the positive real roots of B 22)

Since $\alpha_1\alpha_2 = \frac{a_o}{a_2}$, we obtain:

(B 25) $R = Ck^{1/a_o} |k - \alpha_1|^{\frac{\beta_1}{a_o}} |k - \alpha_2|^{\frac{\beta_2}{a_o}}$ with $c > o$, $\beta_1 = \frac{\alpha_2}{\alpha_1 - \alpha_2}$ and

$\beta_2 = \frac{\alpha_1}{\alpha_2 - \alpha_1}$

Taking accout of (B 17), we finally get (see B 21)

(B 26) $q = c_1 \exp \int \frac{dk}{k + ck^{1/a_o}|k - \alpha_1|^{\frac{\beta_1}{a_o}}|k - \alpha_2|^{\frac{\beta_2}{a_o}}}$

Even if a_o is set equal to one, the term on the r.h.s. of (B 26) is an almost hopeless task to integrate, unless β_1, β_2 would be very simple numbers, which obviously implies a too strong assumption.

Therefore, it will be assumed a priori that the parabolic has in fact a very flat top such that a_2 of (B 22) may be somewhat neglected. The basis of this reasoning is that the elasticity of substitution only starts decreasing when the capital stock is being accumulated at an exceptional high rate. So, (B 23) becomes:

(B 27) $\frac{dR}{R} = \frac{dk}{k(a_o + a_1k)} = \frac{dk}{a_ok} - \frac{a_1}{a_o} \frac{dk}{(a_o + a_1k)}$ with solution:

(B 28) $R = \frac{q}{q'} - k = c \left(\frac{k}{a_o - a_1k}\right)^{1/a_o}$ so that the production function

(B 26) becomes:

(B 29) $$q = c_1 \exp \int \frac{dk}{k + c\left(\frac{k}{a_o + a_1 k}\right)^{1/a_o}} \qquad (c_1,\ c > 0)$$

To perform the integration on the r.h.s. of (B 29) analytically, we have to assume simple values for a_o:

(i) say for $a_o = 1$

Then the integrand of (B 29) is simply rewritten by partial fraction as:

(B 30) $$\frac{dk}{k + c\left(\frac{k}{1 + a_1 k}\right)} = \frac{(1 + a_1 k)dk}{k(1 + a_1 k + c)} = \frac{dk}{(1 + c)k} + \frac{c a_1 dk}{(1+c)(1+a_1 k+c)} \quad \text{or}$$

(B 31) $$q = c_1 \exp \left(\frac{1}{1 + c} \log k + \frac{c}{1 + c} \log (1 + a_1 k + c) + c_2\right)$$

$$= c_3 k^{\frac{1}{1+c}} (1 + a_1 k + c)^{\frac{c}{1+c}}$$ or for the original production data:

(B 32) $$Q = c_3 K^{\frac{1}{1+c}} [(1 + c) L + a_1 K]^{\frac{c}{1+c}}$$

$$= c_3 K^{\frac{1}{1+c}} \left(L + \frac{a_1}{1 + c} K\right)^{\frac{c}{1+c}}$$ which is precisely the V.E.S. function (1.10).

(ii) say for $a_o = \frac{1}{2}$.

Then rewriting integrand (B 29):

(B 33) $$\frac{dk}{k + c\left(\frac{k}{\frac{1}{2} + a_1 k}\right)^2} = \frac{(\frac{1}{2} + a_1 k)^2}{k(\frac{1}{2} + a_1 k) + ck^2} + \frac{\frac{1}{4} + a_1 k + a^2{}_1 k^2}{k(\frac{1}{4} + (a_1+c)k+a_1^2\, k^2)} \quad \text{and}$$

since $(a_1 + c)^2 > a^2{}_1$ is always satisfied, we may apply partial fraction to solve the differential equation. Consequently, (B 26) becomes:

(B 34) $$q = c_2 k \left| k-\alpha_1 \right|^{\frac{1}{a^2{}_1}\left(\frac{1}{4\alpha_1(\alpha_1-\alpha_2)} + \frac{a_1}{(\alpha_1-\alpha_2)} + \frac{a^2{}_1\alpha_1}{(\alpha_1-\alpha_2)}\right)} \cdot \left| k-\alpha_2 \right|^{\frac{1}{a^2{}_1}\left(\frac{1}{4\alpha_2(\alpha_2-1)} + \frac{a_1}{\alpha_2-\alpha_1} + \frac{a^2{}_1\alpha_2}{\alpha_2-\alpha_1}\right)}$$

with

$$\text{(B 35)} \quad \alpha_1, \alpha_2 = \frac{-(a_1 + c) \pm \sqrt{(a_1 + c)^2 - a^2{}_1}}{2a^2{}_1}$$

B 2.2 Empirical V.E.S. production function based on profit maximization or cost minimization ("indirect" approach)(*)

Under profit maximization or cost minimization, the marginal product of labour is equal to:

(B 36) $\frac{\partial Q}{\partial L} = \frac{w\ell_1}{p\ell_o}$,so that, because of (B 15), the assumed relationship between log q, log $\frac{w\ell_1}{p\ell_o}$ and k may be written as:

$$\text{(B 37)} \quad \log q = \log a + b \log (q - kq') + g \log k \text{ or}$$

$$\text{(B 38)} \quad q = a\left(q - k\frac{dq}{dk}\right)^b k^g = a\left(q^2\frac{dz}{dk}\right)^b k^g \text{ with } z = \frac{k}{q}.$$

Rewriting the differential equation (B 38) in terms of z and k as:

$$\text{(B 39)} \quad k^{(1-g-2b)/b}\, dk = a^{1/b}\, z^{(1-2b)/b}\, dz \quad \text{and integrating:}$$

$$\text{(B 40)} \quad \frac{k^{(1-g-b)/b}}{(1-g-b)/b} = a^{1/b} \frac{z^{(1-b)/b}}{(1-b)/b} + c \quad \text{(c = arbitrary constant of}$$

integration).

Substituting for $z = \frac{k}{q}$, the final V.E.S.-production function is obtained:

$$\text{(B 41)} \quad q = \left[\frac{(1-b)}{(1-g-b)a^{1/b}}\, k^{-g/b} - \frac{c(1-b)}{b\, a^{1/b}}\, k^{-\left(\frac{1-b}{b}\right)}\right]^{-\frac{b}{1-b}}$$

$$= (\alpha\, k^{-m\rho} + \beta\, k^{-\rho})^{-\frac{1}{\rho}} \quad \text{with}$$

(*) The V.E.S. production function treaded here is that of Liu and Hildebrand. Ser Nerlove [17].

$$\rho = \frac{1-b}{b} \;;\; m = \frac{g}{1-b} \;;\; \alpha = \frac{(1-b)}{(1-g-b)a^{1/b}} \quad \text{and}$$

$$\beta = \frac{-c(1-b)}{b\ a^{1/b}}$$

, which is equal to the constant returns to scale C.E.S.-function if m = o (i.e. if g = o) (*) .

To obtain the final expression for the elasticity of substitution, the marginal rate of substitution formula of (B 17) may be utilized so that:

(B 44) $$\frac{d \log R}{dk} = \frac{-kq''}{q-kq'} - \frac{q''}{q'} = - \frac{qq''}{q'(q-kq')} \text{ with } q' = \frac{dq}{dk} \text{ and } q'' = \frac{d^2q}{dk^2}$$

and the elasticity of substitution becomes:

(B 45) $$\sigma = -\frac{q'(q-kq')}{kq\ q''}$$ where, for V.E.S.-function (B 41),

(B 46) $$q' = q^{1+\rho} \left[\alpha m k^{-(1+m\rho)} + \beta k^{-(1+\rho)}\right]$$ (**) and

(B 50) $$q'' = \rho m \frac{w\ell_1}{p\ell_o} k^{-2} - (1+\rho) \frac{q'}{kq} \frac{w\ell_1}{p\ell_o}$$ such that:

(B 51) $$\sigma = \frac{1}{(1+\rho) - \rho m \frac{q}{kq'}}$$ where use has been made of (B 48).

(*) The V.E.S.-production function, expressed in original production units, is directly obtained as:

(B 42) $$Q = \left[\alpha L^{-\rho} (\tfrac{K}{L})^{-m\rho} + \beta K^{-\rho}\right]^{-\frac{1}{\rho}}$$ and setting $\alpha = (1-\delta)A_t^{-\rho}$, $\beta = \delta A_t^{-\rho}$ and providing a time subscript:

(B 43) $$Q_t = A_t \left[(1-\delta)(\tfrac{K_t}{L_t})^{-m\rho} L_t^{-\rho} + \delta K_t^{-\rho}\right]^{-\frac{1}{\rho}}$$, which is precisely equation (1.11) will $A_t = Ae^{\lambda t}$.

(**) The contents of (B 37) may be formalized by raising (B 41) to power $-\rho$ and multiplying with $q^{1+\rho}$:

(B 47) $$q = q^{1+\rho} (\alpha k^{-m\rho} + \beta k^{-\rho})$$ and from (B 15),(B 36) and (B 46):

(B 48) $$q - kq' = \frac{w\ell_1}{p\ell_o} = q^{1+\rho} \alpha(1-m)\ k^{-m\rho} \quad \text{or}$$

(B 49) $$\log q = -\frac{1}{1+\rho} \log \alpha(1-m) + \frac{1}{1+\rho} \log \frac{w\ell_1}{p\ell_o} + \frac{m\rho}{1+\rho} \log k.$$

Since

(B 52) $$\frac{q}{kq'} = \frac{\frac{w\ell_1}{p\ell_o}}{kq'} + 1 = \frac{\frac{w\ell_1}{p\ell_o}}{\frac{K}{L}\frac{r\ell_2}{p\ell_o}} + 1 = \frac{Lw\ell_1}{Qp\ell_o} \Big/ \frac{Kr\ell_2}{Qp\ell_o} + 1$$ and if $\ell_1 = \ell_2$:

(B 53) $$\frac{q}{kq'} = \frac{s_L + s_K}{s_K} = \frac{1}{s_K}$$, with s_K the share of capital, or

(B 54) $$\sigma = \frac{1}{(1+\rho) - \frac{\rho m}{s_K}}$$

Appendix C Homothetic Production Function. (*)

In general, it may be said that a certain neo-classical production function $F(L,K)$ of an arbitrary degree of homogeneity ν and with constant or variable elasticity of substitution may be monotonically transformed to a neo-classical production function as (1.12) which is homothetic and has a changing returns to scale parameter, varying according to a preassigned relationship to the output level. Such transformed production functions are also called "generalized" production functions (see[22]) and are characterized by the property that all expansion paths are along straight lines through the origin.

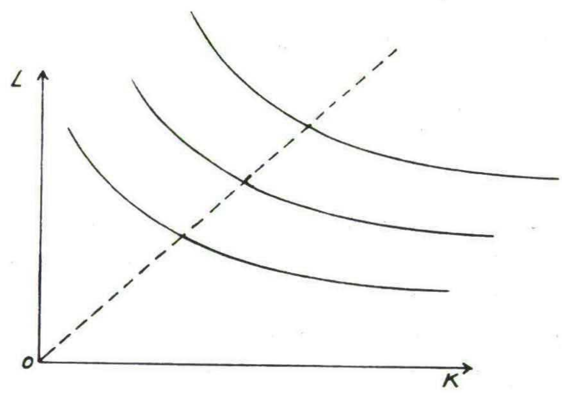

Figure 2 Homothetic Production isoquants.

So, for a returns to scale function $h(Q)$, we may apply the generalized Euler's theorem for homothetic functions:

(*) This appendix is based upon the work of Clemhout [7] and Zellner-Revankar [22] .

(C 1) $h(Q).\ Q = L \frac{\partial Q}{\partial L} + K \frac{\partial Q}{\partial K}$ with positive marginal products:

(C 2) $\frac{\partial Q}{\partial L} = \frac{dG}{dF} \frac{\partial F}{\partial L} > o$

$\frac{\partial Q}{\partial K} = \frac{dG}{dF} \frac{\partial F}{\partial K} > o$,

since $o \leq F(L,K) < o$ or $\frac{dG}{dF} > o$ (monotonic transf.), $\frac{\partial F}{\partial L}, \frac{\partial F}{\partial K} > o$,

or

(C 3) $h(Q)\ Q = \frac{dG}{dF} (L \frac{\partial F}{\partial L} + K \frac{\partial F}{\partial K}) = \frac{dG}{dF} \nu F$ by application of Euler's theorem on the homogeneous production function F of degree ν.

Rewriting (C 3):

(C 4) $\frac{dF}{\nu F} = \frac{dQ}{h(Q).Q}$, we notice that its general solution is given by production function (1.12).

Since the elasticity of substitution of G (or Q) is the same as that associated with F(because both functions have the same marginal rate of substitution: quotient of both marginal products in (C 2)), σ is constant for all isoquants along a ray but not necessarily constant along one isoquant (because F may be a V.E.S. function). So, the isoquants only differ in scale but not in shape so that they are parallel to each other (figure 2). It is the curvature of the production surface which indicates the type of returns to scale. This curvature can be formalized in various ways. In general it may be said that returns to scale diminish if the output level is increased.

A. Zellner and N. Revankar [22] investigate three expressions for h(Q):

(i)

(C 5) $h(Q) = \nu(1 - \frac{Q}{c})$ with $o \leq Q < c$ or substituting in (C 4):

(C 6) $\frac{dF}{F} = \frac{dQ}{Q(1-\frac{Q}{c})} = \frac{dQ}{Q} - \frac{d(c - Q)}{c - Q}$ or solving, we find the homothetic production function:

(C 7) $Q = (c - Q)\,Fc_1 = \frac{cFc_1}{1 + Fc_1}$ or for the arbitrary positive constant $c_1 = 1$:

(C 8) $Q = \frac{cF}{1+F} = \frac{c}{F^{-1}+1} = \frac{c}{e^{-\log F}+1}$ so that Q and log F are interrelated by a logistic function.

(ii)

(C 9) $h(Q) = \nu\left[\alpha + \frac{\beta}{F}\right]$ with $F = G^{-1}(Q)$, $\alpha, \beta > o$ or substituting into (C 4) and solving:

(C 10) $\frac{dQ}{Q} = \frac{dF(\alpha + \beta F^{-1})}{F} = \frac{\alpha dF}{F} + \frac{\beta dF}{F^2}$ or

(C 11) $Q = c_1\, F^{\alpha}\, e^{-\frac{\beta}{F}}$, with c_1 an arbitrary positive constant

(iii)

(C 12) $h(Q) = \nu + a\left(\frac{b-Q}{b+Q}\right)$ with $o \leq a < \nu$.

Again substituting into (C 4):

(C 13) $\frac{dF}{F\nu} = \frac{dQ}{Q\left[\nu + a\left(\frac{b-Q}{b+Q}\right)\right]} = \frac{dQ}{(\nu+a)Q} + \frac{1}{b}\,\frac{(a-\nu)}{(a+\nu)}\,\frac{dQ}{\left[\nu + a\left(\frac{b-Q}{b+Q}\right)\right]}$ and solving:

(C 14) $Q\,\{(1+c)\,b + (1-c)\,Q\}^{\frac{2c}{1-c}} = c_1\;F^{1+c}$ with c_1 a positive constant of integration and $c = \frac{a}{\nu}$.

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