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SIMPLICAL ALGORITHMS FOR FINDING STATIONARY POINTS, A UNIFYING DESCRIPTION

by

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1. Introduction.

In the past decennium a large number of simplicial restart algorithms for finding a zero of a continuous function f from R^n into itself has been developed. Besides the well-known algorithm of Merrill [15], many variable dimension algorithms have been proposed. These methods allow for a start at an arbitrary point from which one among several directions (rays) is followed to leave this point. These rays define a collection of cones of variable dimension in which the search for an approximate fixed point takes place. In each cone movement occurs through simplicial pivoting. This class of methods has been initiated by Van der Laan and Talman [8, 9, 10, 12] and Reiser [16], who proposed algorithms with n+1 and 2n rays. Later on algorithms with 2^n rays (Wright [24]) and 3^n-1 rays (Kojima and Yamamoto [5,6]) have been proposed, whereas an algorithm with 2 rays has been developed by Saigal [17] and Yamamoto [25].

Several authors have given an interpretation of the variable dimension algorithm in one way or another. An interpretation as tracing zeroes of a homotopy function from an affine function on the zero-level to the function of interest on the one-level has been given by Todd [20,21], Todd and Wright [23], Wright [24] and Van der Laan and Talman [13]. In Van der Laan [7] and Talman [18], it has been shown that if an algorithm works in a t-dimensional cone, the function value lies in an (n+1-t)-dimensional "dual" set. Operating in cones of varying dimensions, a path of points is followed which terminates with a zero of f. This is closely related to the work of Kojima and Yamamoto [5,6], who developed a basic theory for designing variable dimension algorithms by introducing the concept of a primal-dual pair of subdivided manifolds. A similar unifying theory has been proposed by Freund [3].

In all references mentioned above it is necessary to be familiar with concepts like simplicial subdivision, piecewise linear approximation and complementary pivoting. Kojima [4] states that, we quote, "Understanding such technical terminologies .... often causes difficulty in finding out a basic idea of the vd algorithms . The author feels that this has been an obstacle to vd algorithms being popular though they have been attracting much attention" (pp. 200-201). To make the basic idea clear, Kojima describes the variable dimension algorithms in the framework of homotopy continuation methods (see for an overview Allgower and Georg [1]). In this paper we will follow this approach to give a general description for the limiting paths of the algorithms mentioned above. To do so we rewrite the zero point problem as the problem of finding a stationary point. Then we give a procedure to find such a point. This procedure is a generalization of the method given in Eaves [2] for finding a stationary point of an affine function, and generates a path of stationary points of the function restricted to an expanding convex set. Taking this set in an appropriate way, we will show that there is a close relation between the path of stationary points and the limiting paths of the simplicial restart algorithms.

This paper is organized as follows. In section 2 we develop the basic framework. In section 3 we discuss the $(n+1)$-ray algorithm, in section 4 the $2n$-, 2^n- and (3^n-1)-ray algorithms, and in section 5 Merrill's algorithm and the 2 ray algorithm. Finally in section 6 some concluding remarks are made.

2. Computing stationary points.

Let f be a continuous function from R^n into itself and suppose we want to find a zero point x^* of f. Clearly $f(x^*)=0$ if and only if x^* is a stationary point of f, i.e.

$$x^*f(x^*) \geq x\ f(x^*) \quad \text{for all } x\in R^n. \tag{2.1}$$

More general we have that x^* is a stationary point of f in $S\subset R^n$ if

$$x^*f(x^*) \geq x\ f(x^*) \quad \text{for all } x\in S. \tag{2.2}$$

Eaves [2] proposed a method to find such a stationary point of an affine function $f(x)=Cx+c$ on the polyhedron $A=\{x\in R^n|Ax\leq a\}$. To do so, he adapted Lemke's algorithm by introducing auxiliary constraints $B(t)=\{x\in R^n|Bx\leq b+te\}$, where $e=(1,...,1)^T$ and t varies over $[0,\infty)$. The m×n matrix B and the m-vector b are such that for t=0 some arbitrarily chosen starting point v is the unique solution to $Ax\leq a$ and $Bx\leq b$.

Now, let us consider the points $(x(t),t)$ such that $x(t)$ solves (2.2) for $S=A\cap B(t)$. Since $A\cap B(0)=\{v\}$, it follows that $(v,0)$ is a trivial solution. Moreover, with deleting the technical details, we have that the set of solution points $(x(t),t)$ is a disjoint union of curves (1-dimensional manifolds) in $R^n\times R_+$, assuming that some regularity conditions are satisfied. Since v is the unique solution for t=0, the path starting in $(v,0)$ cannot return in $R^n\times\{0\}$. Since $A\cap B(t)$ is bounded for any t and the path $(x(t),t)$ starting in $(v,0)$ cannot stay in a bounded set, t must go to infinity. Clearly, if $x(t)$ solves (2,2) for $S=A$ then $x(t)$ is a stationary point of f on A. When A is bounded we have that $A\cap B(t)=A$ for t sufficiently large and a stationary point can be found by following the path of points starting in $(v,0)$.

Zangwill and Garcia [26] have discussed the possibility of using path following procedures for establishing existence of equilibria. They apply such a procedure to an exchange economy model. To be more specific, letting $z(p)\in R^n$ be the vector of net demands given the price vector $p\in S^{n-1}=\{p\in R^n_+|\Sigma^n_{j=1}p_j=1\}$, a path of points $(p(t),t)$ is followed where $p(t)$ solves

$$\text{find } p^* \text{ such that } p^*z(p^*) \geq pz(p^*) \quad \text{for all } p\in S^{n-1}(t) \tag{2.3}$$

where $S^{n-1}(t)=\{p\in S^{n-1}|p\geq\hat{p}-(t+\varepsilon)e\}$, $\varepsilon>0$ very small, $0\leq t\leq 1$, and $\hat{p}$ some arbitrarily chosen initial price vector. So, $p(t)$ is a stationary point of $z: S^{n-1}\rightarrow R^n$ on $S^{n-1}(t)$. Let k be the unique index for which $z_j(\hat{p})$, $j=1,...,n$ is maximal.

Clearly, then for ε sufficiently small, in $S^{n-1}(0)$ there is a unique solution $\overset{\circ}{p}$, namely

$$\overset{\circ}{p}_k = \hat{p}_k+(n-1)\varepsilon \quad \text{and} \quad \overset{\circ}{p}_j = \hat{p}_j-\varepsilon \qquad j \neq k.$$

Following the path of solutions $(p(t),t)$ starting from $(\overset{\circ}{p},0)$ an equilibrium is reached at $t=1$.

Clearly, this approach is very close to the idea of Eaves. In fact, the only difference is that z is not necessarily an affine function and that $B(0)$ has been replaced by $S^{n-1}(0)$ on which (2.3) has a unique solution which can easily be found.

In Van der Laan and Talman [14] it is shown that the path of points $(p(t),t)$ is the limiting path of the simplicial variable dimension algorithm introduced in Van der Laan and Talman [8] if the labelling rule and the underlying triangulation are chosen in an appropriate way.

In this paper we will use Eaves' idea to give a unified framework of describing the limiting paths of simplicial restart algorithms to find a zero of a continuous function on R^n. Therefore we return to the problem (2.1). To solve this problem, let v be some arbitrarily chosen point and $B(0)$ some "arbitrarily small" n-dimensional convex compact subset of R^n containing zero in its interior. With arbitrarily small we mean that for $x \in B(0)$, $\|x\|_2 \leq \varepsilon$ for some arbitrarily small positive ε. Finally, let $B(t)=(1+t)B(0) = \{x \in R^n \mid (1+t)^{-1}x \in B(0)\}$. Now, let $(x(t),t)$ be a pair such that

$$x(t)f(x(t)) \geq xf(x(t)) \qquad \text{for all } x \in S(t), \tag{2.4}$$

where $S(t)=\{v\}+B(t)$. Then again we have that under some regularity conditions the set of points $(x(t),t)$ is a disjoint union of loops and paths. Since $B(0)$ is arbitrarily small, we may assume that for $t=0$, there is a unique solution $x(0)$. In fact, this will be the case for v almost everywhere. Then the curve starting in $(x(0),0)$ cannot return to the zero level. Furthermore it cannot be a loop. Since $B(t)$ is bounded for any fixed t, we must have that going along the path starting in $(x(0),0)$, t goes to infinity. Clearly, $x(t)$ solves (2.1) if $x(t) \in \text{int } S(t)$. So, following the path, a solution $x(t^*)$ to (2.1), and hence a zero of f has been found as soon as

$$x(t) = x(t^*) \in \text{int } S(t) \quad \text{for} \quad t>t^*.$$

Since $0 \in \text{int } B(0)$ we have that $B(t)$ and hence $S(t)$ expands to infinity, i.e. for each bounded set C we have that $C \subset S(t)$ for t sufficiently large. In the next theorem we prove that a condition originally due to Merrill [15] is sufficient for guaranteeing that $x(t)$ is bounded. Let $B(w,\mu)=\{x \in R^n \mid \|x-w\|_2 \le \mu\}$.

Theorem 2.1. Suppose there exist $w \in R^n$ and $\mu>0$ such that for all $x \notin B(w,\mu)$, f satisfies

$$f(x)(x-w) < 0.$$

Then the path $(x(t),t)$ starting in $(x(0),0)$ is bounded in x.

Proof. Let $\bar{t}$ be so large that $B(w,\mu) \subset \text{int } S(\bar{t})$. Now suppose that the path is not bounded. Then the projection of the path on R^n crosses bd $S(\bar{t})$ in some solution $x(\bar{t})$. Hence $x(\bar{t}) \notin B(w,\mu)$. Therefore

$$f(x(\bar{t}))(x(\bar{t})-w) < 0$$

However, since $x(\bar{t})$ solves (2.4) we also have that $x(\bar{t})f(x(\bar{t})) \ge xf(x(\bar{t}))$ for all $x \in S(t)$. Since $w \in S(\bar{t})$ a contradiction is obtained.

□

Theorem 2.1. says that if Merrill's condition is satisfied a zero of f will be reached following the path of points $(x(t),t)$ starting in $(x(0),0)$.

In the next section we will show that by choosing $B(0)$ in an appropriate way there is a close relation between the path of stationary points and the limiting path of the simplicial restart algorithms. For simplicity, in the following the path X denotes the projection on R^n of the path $(x(t),t)$ starting in $(x(0),0)$. We will see that the path X coincides with the limiting path except that the latter starts in v instead of $x(0)$. Recall however that $x(0)$ lies arbitrarily close to v.

Before treating some particular choices of $B(0)$ we first consider the general case. So, let $B(0)$ be some compact convex set containing 0 in its interior and let $x(t)$ be a solution to (2.4). Then either $x(t) \in \text{int } S(t)$, implying that $f(x(t))=0$ and hence $x(t)$ is a zero of f, or $x(t) \in \text{bd } S(t)$. In the latter case it follows from (2.4) that

$$S(t) \subset \underline{H}(f(x(t)),\ x(t)f(x(t))),$$

where for some $p\in R^n\setminus\{0\}$, $c\in R$, $H(p,c)=\{x\in R^n \mid px=c\}$ and $\underline{H}(p,c)$ is the half space below the hyperplane $H(p,c)$, i.e. $\underline{H}(p,c)=\{x\in R^n \mid px\leq c\}$. So, as illustrated in figure 1, $S(t)$ is in the half space below the hyperplane through $x(t)$ with normal $f(x(t))$.

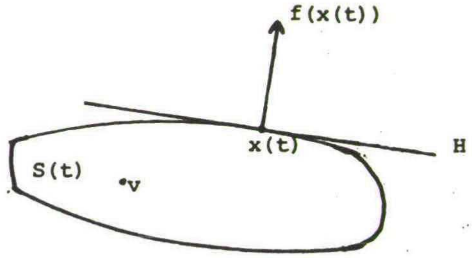

Figure 1. $S(t) \subset \underline{H}$ with H the hyperplane through $x(t)$ with normal $f(x(t))$.

For $B(0)$ being an n-simplex the path X is illustrated in figure 2. Observe that t does not increase monotonically moving along the path. For $t=t_1$, y is the unique solution to (2.4). However, for $t=t_2$, (2.4) has three solutions, namely x^1, x^2 and w^2, which are all lying on the path $x(t)$ starting in $x(0)$. In x^1 and x^2 we have that $H(f(x^i), x^i f(x^i))$ contains $\text{conv}(w^1,w^2)$. For w^2, we have that w^2 is the only point of $S(t_2)$ in $H(f(w^2), w^2 f(w^2))$. Going along the path, t increases until z^1 is reached, t decreases from z^1 till z^2 and increases again after z^2 has been passed. A zero is obtained at x^*.

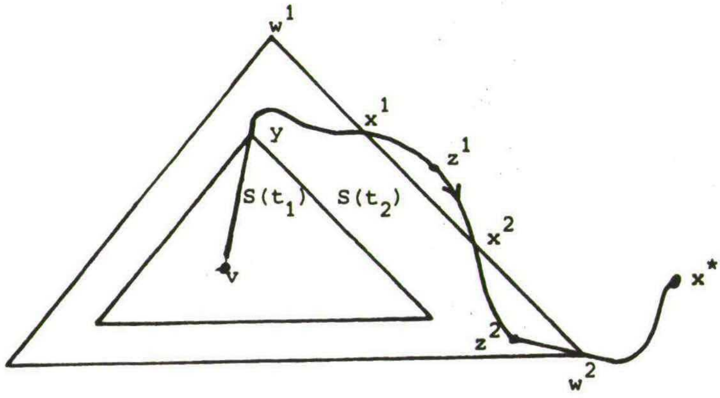

Figure 2. The path X for $B(0)$ being an n-simplex, n=2.

3. The (n+1)-ray variable dimension algorithm.

In this section we show that if $B(0)$ is chosen in an appropriate way, the path X is the limiting path of the (n+1)-ray variable dimension algorithm of Van der Laan and Talman [9,10], when using the UK-triangulation with U a nonsingular $n \times n$ matrix with diagonal elements equal to 1 and off-diagonal elements equal to $-\alpha^{-1}$, $\alpha > n-1$ (see [11]).

To do so, we first give the limiting path of the algorithm. Therefore, we define for $T \subset I_{n+1}$, the sets $A(T)$ by

$$A(T) = \{x \in R^n \mid x = v + \Sigma_{j \in T} \lambda_j u(j),\ \lambda_j \geq 0\},$$

where $u(j)$, $j=1,\dots,n$ is the j-th column of the matrix U and $u(n+1) = \Sigma_{i=1}^{n} -u(i)$. Moreover, let ℓ: $R^n \to R^n \times \{0\}$ be defined by $\ell(x) = (f(x)^T, 0)^T$. Finally, for $T \subset I_{n+1}$, we define the sets $C(T)$ and $H(T)$ by

$$C(T) = \{x \in R^n \mid \ell_k(x) = \max_j \ell_j(x),\quad k \in T\},$$

and $H(T) = A(T) \cap C(T)$. Observe that for T with $|T| \neq 0$, $n+1$, $A(T)$ is a $|T|$-dimensional set, whereas $C(T)$ is $(n+1-|T|)$-dimensional. So, under some regularity conditions, the set of points $x \in R^n$ with $x \in H = \cup_T H(T)$ is a collection of paths and loops with endpoints the set of points x in $\{H(T) \mid |T| = 0, n+1\}$.

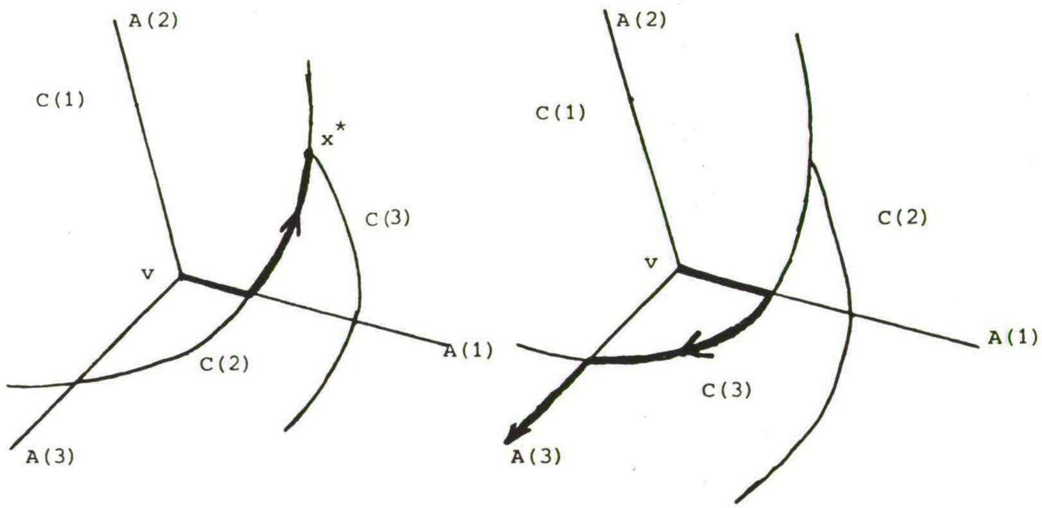

Figure 3. Two examples of the limiting path of the (n+1)-ray algorithm.

Clearly, when $|T|=0$, we have $T=\emptyset$ and hence by definition of $A(T)$, v is the unique endpoint for $T=\emptyset$. For all other endpoints x^* we have that $T=I_{n+1}$ and hence $x^*\epsilon C(I_{n+1})$, which implies that for all $k\epsilon I_n$, $f_k(x^*)=\ell_k(x^*)=\ell_{n+1}(x^*)=0$, i.e. x^* is a zero of f. Now, as shown in [7, page 103,4] the limiting path of the (n+1)-ray algorithm is the path of points of H starting in v. Clearly, as illustrated in figure 3, this path either goes to infinity or ends with a zero x^*.

Now define $w^i=u(i)$, $i=1,\dots,n$ and $w^{n+1}=(-\alpha^{-1},\dots,-\alpha^{-1})^T$ and let $\sigma(w^1,\dots,w^{n+1})$ be the convex hull of $w^1,\dots,w^{n+1}$. Then we will show that taking $B(0) = \varepsilon\sigma(w^1,\dots,w^{n+1})$ for some arbitrarily small positive ε, the path of points $x(t)$ which solves (2.4) starting in $x(0)$ coincides with the limiting path described above. Therefore, observe that $A(T)$ can be written as

$$A(T) = \{x\epsilon R^n \mid x=v+\lambda\tau(w^j,j\epsilon T),\ \lambda\geq 0\},$$

where $\tau(w^j,j\epsilon T)$ is the convex hull of the points εw^j, $j\epsilon T$. Furthermore, from the definition of $u(i)$ it follows that

$$B(0) = \{x\epsilon R^n \mid B^T x\leq b\},$$

where B is the $n\times(n+1)$ matrix $B = [e,-I]$ and b is the n-vector with $b_1=\varepsilon(1-(n-1)\alpha^{-1})$ and $b_j=\varepsilon\alpha^{-1}$, $j=1,\dots,n$. Observe that $b_1>0$ by the fact that $\alpha>n-1$. The set $B(0)$ is pictured in figure 4.

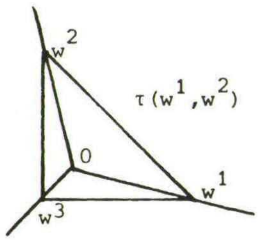

Figure 4. The set $B(0)$ for $n=2$, $\varepsilon=1$.

Let $(x(t),t)$ solve (2.4) and suppose $x(t)\not\in \text{int } S(t)$. Then for some $T\subset I_{n+1}$, $x(t)\epsilon\{v\}+(1+t)\tau(w^j,j\epsilon T)$ and hence $x(t)\epsilon A(T)$. Moreover we have that

$$(1+t)\tau(w^j,j\epsilon T) \subset H(f(x(t)),\ x(t)f(x(t))).$$

So, by considering the structure of B it follows immediately that

$$f_k(x(t)) = \max_j f_j(x(t)) \quad k \in T \quad n+1 \notin T$$

and

$$f_k(x(t)) = 0 = \max_j f_j(x(t)) \quad k \in T \backslash \{n+1\} \text{ if } n+1 \in T,$$

implying that $\ell_k(x(t)) = \max_j \ell_j(x(t))$, $k \in T$. So $x(t) \in C(T)$ and hence $x(t) \in H(T)$. Reversely if $T \neq I_{n+1}$, then $x \in H(T)$ implies that

$$xf(x) \geq \bar{x}f(x) \text{ for all } \bar{x} \in S(t),$$

with t such that $x \in \text{bd } S(t)$. Finally, $x(t) \in \text{int } S(t)$ iff $x(t) \in H(I_{n+1})$. So, $x(t)$ solves (2.4) iff for some T, $x(t) \in H(T)$. This shows that the limiting path of the (n+1)-ray algorithm coincides with the path X , with the remark that the latter starts in x(0) in stead of v. As a further result the next corollary follows immediately from theorem 2.1.

Corollary 3.1. Let f satisfy Merrill's condition. Then the limiting path of the (n+1)-ray algorithm with rays $u(i)$, $i=1,\ldots,n+1$ is bounded.

For the simplicial path this result has also been proved in [10]. There it is also shown that the proof does not hold for the K-triangulation. Here we come to the same conclusion. We obtain the K-triangulation by letting α go to infinity. Then w^{n+1} becomes 0, and b becomes $(\varepsilon, 0, \ldots, 0)^T$. Hence $0 \in \text{bd } B(0)$, implying that B(t) does not extend into the -e direction, so that not each bounded set C lies in B(t), for t large enough. This implies that for the K-triangulation the limiting path cannot be described by the path of solutions x(t) to (2.4) and hence corollary 3.1 does not follow. However, as has been observed by Todd (private communication), when using the K-triangulation, convergence is obtained by choosing an appropriate system of equations with which the algorithm starts. In the framework described above convergence is obtained by taking $B(0) = \varepsilon\sigma(w^1, \ldots, w^{n+1})$ with $w^i = e(i)$, $i=1,\ldots,n$, where $e(i)$ is the i-th unit column and $w^{n+1} = -e$. Then it can easily be shown that the path X is the limiting path of the (n+1)-ray algorithm using the K-triangulation if ℓ is redefined by $\ell(x) = (f(x)^T, -\Sigma_j f_j(x))^T$. Observe that this has been done by Talman [18] and Kojima and Yamamoto [6]. In this case, Merrill's condition is sufficient for convergence. The just defined ℓ is the limiting case for

$$\ell(x) = (f(x)^T, \frac{n-\alpha}{1+\alpha} \Sigma_j f_j(x))^T \tag{3.1}$$

when α goes to infinity. For the optimal triangulation with $\alpha = n + \sqrt{n+1}$ (see [11]) Todd [22] showed that (3.1) is optimal for using in zero point algorithms. In general, for (3.1) the limiting path of the (n+1)-ray algorithm is the path X with w^{n+1} replaced by $w^{n+1} = u(n+1)$.

4. The 2n, 2^n and (3^n-1)-ray algorithms.

In this section we consider some algorithms which can be seen as elements of a class of algorithms. In particular we show that the 2n-ray algorithm [10], [12] and [16] and the 2^n-ray algorithm [23] are extreme cases of a class of (3^n-1)-ray algorithms.

Firstly we give the set B(0) depending on n-1 parameters. By special choices of these parameters we get one of the above mentioned algorithms. Since the class of (3^n-1)-ray algorithms given by Kojima and Yamamoto [6] depends on one parameter only, the class of algorithms we will describe is considerably larger.

To characterize B(0), let $\alpha_1,\dots,\alpha_n$ be positive numbers such that

$$\alpha_1=1 \text{ and } \frac{i-1}{i}\alpha_{i-1} \le \alpha_i \le \frac{\alpha_{i-1}\alpha_{i-2}}{2\alpha_{i-2}-\alpha_{i-1}} \qquad (4.1)$$

with $\alpha_0=1$. Now, for a sign vector $s\in R^n$ with $s_i\in\{-1,0,1\}$, let $|s|$ be the number of nonzero elements. Then for $|s|\ge 1$, we define $w(s)\in R^n$ by

$$w_i(s) = s_i\ \alpha_{|s|} \quad , \qquad i=1,\dots,n.$$

Hence $\Sigma_i |w_i(s)|=|s|\alpha_{|s|}$. The set B(0) is now defined by

$$B(0) = \varepsilon \text{ conv } \{w(s) \,\big|\, |s|\ge 1\}.$$

The restrictions (4.1) yield that all vertices w(s) are in bd(B(0)). More precisely, in case of strict inequalities all vertices are extreme points.
For n=2, ε=1 and $\alpha_2=\frac{3}{4}$, B(0) is drawn in figure 5. For n=3, ε=1, and $\alpha_2=\frac{3}{4}$, $\alpha_3={}^{11}/20$ B(0) is drawn as far as B(0) lies in R^n_+. Observe that for $\alpha_2=\frac{3}{4}$, (4.1) implies $\frac{1}{2}\le\alpha_3\le{}^3/5$. For $\alpha_3=\frac{1}{2}$, D is the centerpoint of the 2-simplex ABC, whereas for $\alpha_3={}^3/5$

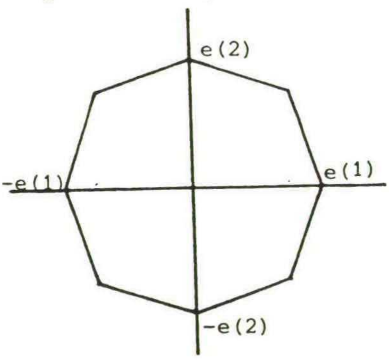

Figure 5a. n=2, ε=1, $\alpha_2=\frac{3}{4}$.

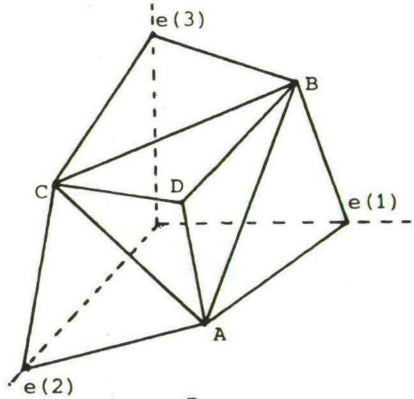

Figure 5b. $B(0)\cap R^n_+$, n=3, ε=1, $\alpha_2=\frac{3}{4}$, $\alpha_3={}^{11}/20$.

D is the intersection point of the three planes through $e(1)$, A, B; $e(2)$, A, C and $e(3)$, B, C.

Now for some $0<\gamma<{}^1/n$, set

$$\alpha_k = \frac{1-(n-1)\gamma}{1-(n-k)\gamma}, \quad k=1,\ldots,n. \tag{4.2}$$

Clearly, these α's satisfy (4.1) with $\alpha_2=\{1-(n-1)\gamma\}\{1-(n-2)\gamma\}^{-1}$ and the upper bound of (4.1) for α_k, $k=3,\ldots,n$. Then the path X starting in $x(0)$ is the limiting path of the (3^n-1)-ray algorithm given in [6] where γ is the parameter introduced by these authors on page 16. This equivalence follows from the fact that $x(t)$ solves (2.4) iff $y=f(x(t))/\Sigma_j|f_j(x(t))|$ satisfies the conditions in [6, page 17] for t, I, J chosen in an appropriate way. Instead of doing this technical exercise we will consider some interesting cases.

For $\gamma=1/n$ we have $\alpha_k = k^{-1}$, $k=1,\ldots,n$, implying that $B(0)$ becomes the n-dimensional octahedron $\{x\epsilon R^n|\ ||x||_1\le\varepsilon\}$. For s a sign vector, let

$$X(s) = \{x\epsilon R^n|x_i=0 \text{ if } s_i=0, \text{ and } x_is_i\ge 0 \text{ if } s_i\ne 0\}.$$

Then $x(t)$ solves (2.4) iff for some s, $x(t)\epsilon X(s)$ and

$$f_i(x(t)) = s_i \max_j|f_j(x(t))| \text{ if } s_i\ne 0.$$

and

$$|f_i(x(t))| < \max_j|f_j(x(t))| \text{ if } s_i=0.$$

which shows that the path X starting in $x(0)$ is the limiting path of the 2n-ray algorithm.

On the other hand for $\gamma=0$ we have $\alpha_k= 1$, $k=1,\ldots,n$, implying that $B(0)$ is the n-dimensional unit cube $\{x\epsilon R^n|\ ||x||_\infty\le\varepsilon\}$. This case is dual to the above mentioned case and it can easily be shown that the path X is the limiting path of the 2^n-ray algorithm. So, both the 2n and 2^n-ray algorithms are extreme cases of the class of (3^n-1)-ray algorithms. For $\gamma=1/n$ and 0 the paths X are illustrated in the figures 6 and 7.

By (4.2) the α's are related by the single parameter γ. However, given the restrictions (4.1), $\alpha_1,\ldots,\alpha_n$ can be chosen arbitrarily. An interesting case to investigate should be

$$\alpha_k = k^{-\frac{1}{2}} \quad k=1,\ldots,n,$$

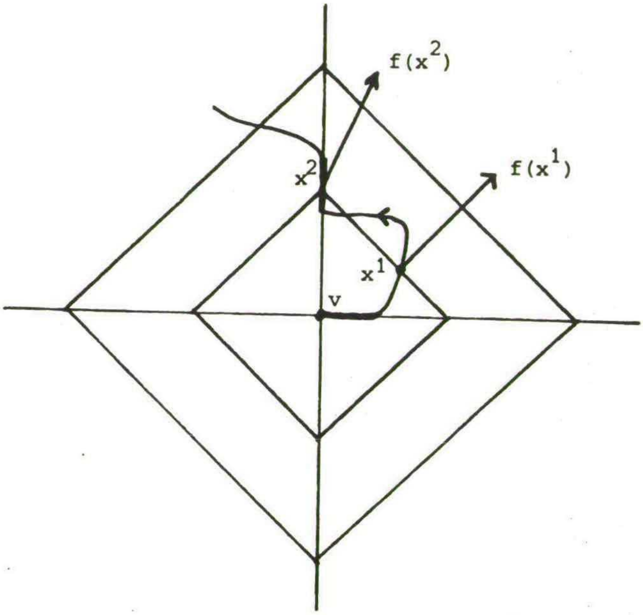

Figure 6. The 2n-ray algorithm, n=2, $B(0)=\{x\epsilon R^n |\ \|x\|_1 \le \varepsilon\}$.

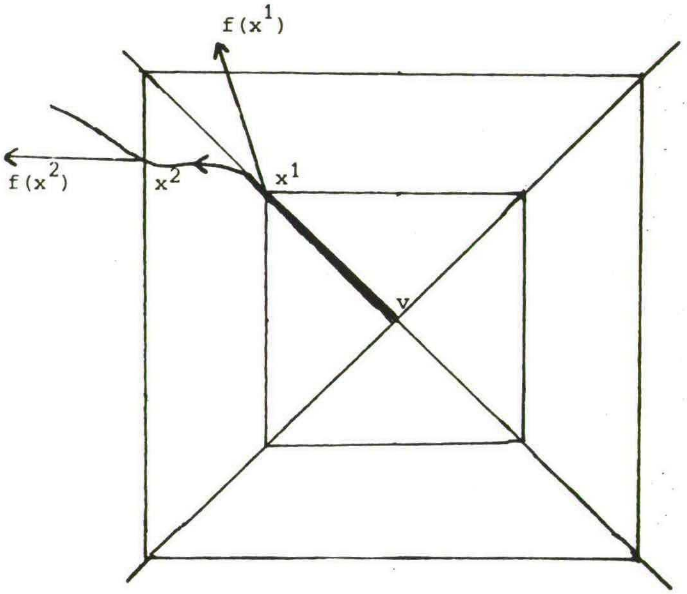

Figure 7. The 2^n-ray algorithm, n=2, $B(0)=\{x\epsilon R^n |\ \|x\|_\infty \le \varepsilon\}$.

which results in lying all points w(s) on the ball $\{x\epsilon R^n |\ \|x\|_2 = 1\}$ (see fig. 8). To conclude this section, observe that from theorem 2.1, it follows that Merrill's condition is sufficient to prove the convergence of the class of (3^n-1)-ray algorithms, including the 2n and 2^n-ray algorithms.

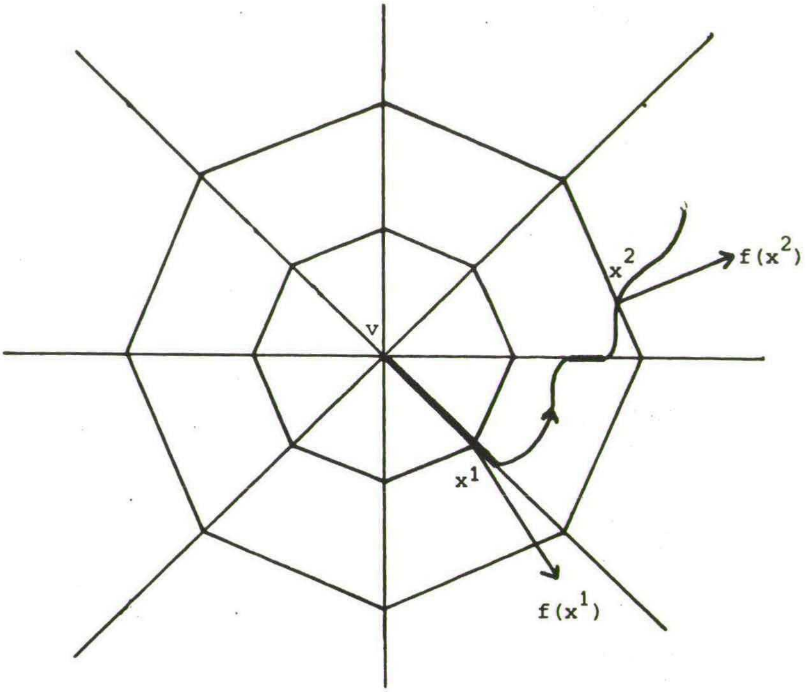

Figure 8. The (3^n-1)-ray algorithm, $n=2$, $\alpha_2 = 1/\sqrt{2}$.

5. Merrill's algorithm and the 2-ray algorithm.

To see how Merrill's algorithm fits in the framework developed in section 3, notice that this algorithm traces a path of zeroes to the homotopy function

$$h(x,\lambda) = \lambda f(x)+(1-\lambda)A(v-x), \qquad 0\leq\lambda\leq 1,$$

where A is some positive definite matrix. Hence, for a solution $(x(\lambda),\lambda)$ we have that

$$f(x(\lambda)) = \frac{1-\lambda}{\lambda} A(x(\lambda)-v).$$

So, for A the identity matrix $f(x(\lambda))$ and $x(\lambda)-v$ are parallel to each other, i.e. for each solution $(x(\lambda),\lambda)$ we have that $f(x(\lambda))$ is a positive multiple of $x(\lambda)-v$.

Since A is positive definite, there exists a matrix P such that $P^TP=A$. Now, for some small ε, define $B(0)=P^{-1}B_2$, where $B_2=\{x\in R^n \mid \|x\|_2\leq\varepsilon\}$ and $P^{-1}B_2=\{x\in R^n \mid Px\in B_2\}$. Then, it can easily be seen that for some positive α a point $(x(t),t)$ that solves (2.4) satisfies

$$f(x(t)) = \alpha A(x(t)-v)$$

and hence the path x(t) starting in x(0) is the limiting path of Merrill's algorithm. It follows again as a corollary that Merrill's condition is sufficient to prove convergence. For A=I, the path X is shown in figure 9.

Finally, let us consider the case that

$$A = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \gamma & & & & \cdot \\ \cdot & & \gamma^2 & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & \gamma^{n-1} \end{bmatrix}.$$

Then from the discussion above it follows easily that the path of points starting in x(0) becomes very close to the limiting path of the 2-ray algorithm of Saigal [17] and Yamamoto [25] if γ goes to zero. However, the limiting case $\gamma=0$ cannot be described by this theory, so that it does not follow that Merrill's condition is sufficient for the convergence of the 2-ray algorithm.

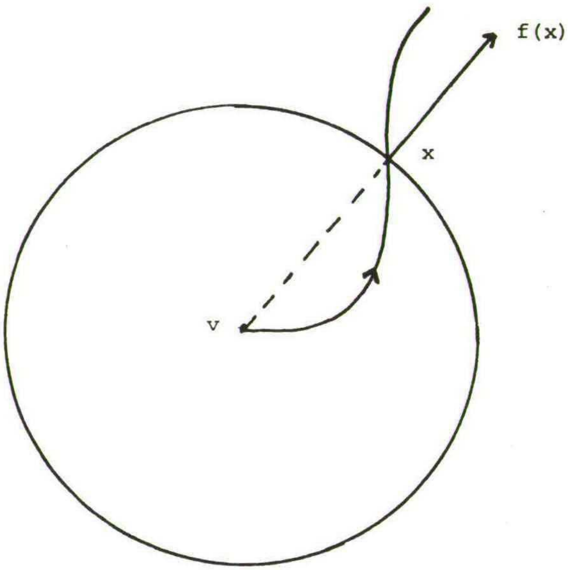

Figure 9. Merrill's algorithm, n=2, $B(0)=\{x \in R^n \mid \|x\|_2 \leq \varepsilon\}$.

6. Conclusions.

In this paper we showed that simplicial restart algorithms for finding zeroes can be seen as generating a path of stationary points with respect to a convex set expanding from the starting point. By specifying this set several restart algorithms fit in this framework. It should be noted that this approach can also be applied to the nonlinear complementarity problem. If in formula (2.4), $S(t)$ is replaced by $S(t)\cap R^n_+$, a path $(x(t),t)$ with $x(t)$ solving (2.4) can be generated. A solution to the NLCP $x\geq 0$, $f(x)\leq 0$ and $xf(x)=0$ is found as soon as $x(t)$ lies in $R^n_+ \cap \text{int } S(t)$. For the linear complementarity problem Talman and Van der Heyden [19] adapted the 2n-algorithm. Clearly the algorithm allows for an arbitrarily chosen starting point. The piecewise linear path coincides with the path of stationary points of f with $S(t)=\{x\in R^n_+ \mid \|x-v\|_1 \leq (1+t)\varepsilon\}$.

Considering Merrill's algorithm, we have that for $A=I$, the limiting path is obtained by taking an expanding ball with center the starting point v. For the variable dimension algorithms we take sets which can be seen as approximations to the ball. In case of the (n+1)-ray algorithm, $B(0)$ is an n-simplex. For the optimal UK-triangulation with $\alpha=n+\sqrt{n+1}$ (see [11]), we have that with $w^i=u(i)$, $i=1,\dots,n+1$, all vertices of the n-simplex $B(0)$ are on a ball with center the origin. Furthermore for the 2n-ray (2^n-ray) algorithm $B(0)$ is an n-octahedron (n-cube), i.e. $B(0)$ is a "2n-point approximation" (2^n-point) to the ball. Finally taking $\alpha_k=k^{-\frac{1}{2}}$, we have that for the (3^n-1)-ray algorithm all 3^n-1 vertices of $B(0)$ are on a ball. So, we can conclude that the limiting paths of the variable dimension algorithm become closer to that of Merrill's algorithm (with $A=I$) if the number of rays increases. In this view we can say that Merrill's algorithm is a "super-variable dimension" algorithm. In our opinion the mean path length decreases if the number of rays increases. On the other hand following the path by simplicial approximation we have less lower-dimensional pieces if the number of rays increases. In particular there are no lower-dimensional pieces for Merrill's algorithm. Further numerical experiments with the algorithms is necessary to obtain some insight in the number of rays which will be optimal. We should not be surprised if numerical experiments will show that for the 2n-ray algorithm the ratio between the length of path and the piece which can be followed by lower dimensional simplices is optimal.

Finally we remark that the approach given in this paper is closely related to the homotopy interpretation of Kojima and Yamamoto [6, page 6], see also [4]. In fact their homotopy function is obtained from the Kuhn-Tucker conditions for a solution to (2.4).

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SIMPLICIAL ALGORITHMS FOR FINDING STATIONARY POINTS, A UNIFYING DESCRIPTION

by

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ABSTRACT.

In this paper we consider the limiting paths of simplicial algorithms for finding a zero point. Therefore we rewrite the zero point problem as the problem of finding a stationary point. The latter problem can be solved by generating a path of stationary points of the function restricted to an expanding convex set. The limiting path of a simplicial algorithm to find a zero is obtained by choosing this set in an appropriate way. So, almost all simplicial algorithms fit in this basic framework. Using this framework it can be shown very easily that Merrill's condition is sufficient for convergence of the algorithms.

KEY WORDS: zero point, stationary point, simplicial algorithms, limiting path, Merrill's condition.

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