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Abstract

In this paper we introduce a variable dimension simplicial algorithm on a cartesian product of unit simplices to find a zero point of a continuous function with special structure. The special structure of the function allows us to perform the linear programming pivot steps of the algorithm in a small system of equations. Moreover, a specific simplicial subdivision of the simplotope underlies the algorithm. The path of points generated by the algorithm approximately follows a piecewise smooth path in the simplotope. The latter path can be interpreted as being generated by an adjustment process. We discuss two applications, an international trade economy and an economy with increasing returns to scale. In both applications the zero points of the function induce equilibria in the economies.

Keywords : simplotope, simplicial subdivision, economic equilibria, linear programming pivot step

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0. Introduction

Let S^ℓ denote the ℓ-dimensional unit simplex in $R^{\ell+1}$, i.e., $S^\ell = \{x \in R_+^{\ell+1} | \Sigma_{j=1}^{\ell+1} x_j = 1\}$, and let S be the cartesian product of $m+1$ times S^ℓ. An element or point in S will be denoted by $x = (x_0, x_1, ..., x_m)$, with $x_j = (x_{j1}, ..., x_{j(\ell+1)})^T$ an element or point in S^ℓ, $j = 0, 1, ..., m$. A component x_{jk} of x in S, $k = 1, ..., \ell$, $j = 1, ..., m$, is called the k-th component of x_j or the (j,k)-th component of x. For all j, the set $I(j)$ denotes the index set $\{(j,1), ..., (j, \ell+1)\}$, and I denotes the union of $I(j)$ over all $j = 0, 1, ..., m$.

Let there be given a continuous function f from S into $R^{m(\ell+1)}$ satisfying, for all $x \in S$ and $j = 0, 1, ..., m$,

$$\Sigma_{k=1}^{\ell+1} f_{jk}(x) = 0,$$

and

$$f_{jk}(x) > 0 \qquad \text{if } x_{jk} = 0.$$

Moreover, for $j = 1, ..., m$, $f_j(x)$ depends only on x_0 and x_j, so that we may write $f_j(x) = f_j(x_0, x_j)$. The problem is to find a zero point of f in S, i.e., a point x^* in S such that $f(x^*) = 0$. As will be discussed in Section 5, this kind of zero point problems on S with special structure of the function f arises in several economic applications, such as an international trade model (see van der Laan [1985]) or a general equilibrium model with increasing returns to scale production (see Kamiya [1988]).

In this paper we describe an efficient simplicial variable dimension algorithm on S for finding a zero point of f in S. Under some regularity and nondegeneracy conditions, the algorithm approximately follows a piecewise smooth path, P, in S. This path connects an arbitrarily chosen point x^0 in S with a zero point of f and is approximately followed by tracing from x^0 a piecewise linear path, $\bar{P}$, in S. The latter path can be generated by performing linear programming pivot steps and making replacement steps in a simplicial subdivision of S. By utilizing the special structure of f and taking a specific triangulation of S, the linear programming pivot steps can be made very efficiently while pivoting is typically performed in just one small $(\ell+2) \times (\ell+2)$ matrix instead of in an $((m+1)(\ell+1)+1) \times ((m+1)(\ell+1)+1)$ matrix.

Initially, along the path P at the starting point x^0 the components x_{jk} of x for which

$f_{jk}(x)$ is negative are relatively decreased while for all j the components x_{jk} of x_j for which $f_{jk}(x)$ is positive are relatively increased. For simplicity we assume that x^0 lies in the relative interior of S, i.e., $x^0_{jk} > 0$ for all $(j,k) \in I$. In general, along the path P the components x_{jk} of x for which $f_{jk}(x)$ is negative are all relatively equal to each other and smaller than all other components of x, whereas for all j the components x_{jk} of x_j for which $f_{jk}(x)$ is positive are all relatively equal to each other and larger than the other components of x_j. In applications the path P can be interpreted in this way as the path generated by some adjustment process on S.

This paper is organized as follows. In Section 1, we give a formal description of the path P followed approximately by the algorithm. Section 2 presents the simplicial subdivision of S which will underlie the algorithm. In Section 3, the steps of the algorithm are discussed in case of sublinearity. Sublinearity occurs when for all $j = 1, ..., m$, $f_j(\hat{w}^q) = f_j(w^{q-1}) - f_j(w^q) + f_j(w^{q+1})$, in case for some q, $2 \le q \le t$, in a simplex with vertices $w^1, ..., w^q, ..., w^{t+1}$ the vertex w^q is replaced by $\hat{w}^q$. In case of sublinearity the pivot step itself becomes very simple and no pivoting is needed. In Section 4 the steps of the algorithm are given in case sublinearity does not occur. Finally, in Section 5 the applications are given.

1. Description of the Path

In order to describe formally the path P to be followed by the algorithm, let us consider the set B of points x in S for which for all $(j,k) \in I$,

$$\frac{x_{jk}}{x^0_{jk}} = min_{(i,h)\in I} \frac{x_{ih}}{x^0_{ih}} \quad \text{if} \quad f_{jk}(x) < 0$$

and (1.1)

$$\frac{x_{jk}}{x^0_{jk}} = max_{h=1,\dots,\ell+1} \frac{x_{jh}}{x^0_{jh}} \quad \text{if} \quad f_{jk}(x) > 0.$$

Clearly, the point x^0 satisfies (1.1) with the minimum and all the maxima equal to one. Notice that the minimum is taken over all indices (in I) and that each maximum is taken separately for all $j = 0, 1, ..., m$. Furthermore, for each j either $f_j(x) = 0$ or for some $1 \le h \ne k \le \ell + 1$, $f_{jh}(x) < 0$ and $f_{jk}(x) > 0$. So, let $s \in R^{m(\ell+1)}$ be a sign vector, i.e., for all $(j,k) \in I$ we have $s_{jk} \in \{-1, 0, +1\}$. We call a sign vector s feasible if for at least one j both $s_j \not\leq 0$ and $s_j \not\geq 0$ and for all other j, $s_j = 0$. For a (feasible) sign vector s we define

$$I_j^+(s) = \{(j,k) \in I(j) \mid s_{jk} = +1\}, \ I^+(s) = \cup_{j=1}^m I_j^+(s),$$

$$I_j^0(s) = \{(j,k) \in I(j) \mid s_{jk} = 0\}, \ I^0(s) = \cup_{j=1}^m I_j^0(s),$$

and

$$I_j^-(s) = \{(j,k) \in I(j) \mid s_{jk} = -1\}, \ I^-(s) = \cup_{j=1}^m I_j^-(s).$$

For a feasible sign vector s, let the subsets $A(s)$ and $C(s)$ be defined by

$$A(s) = \{x \in S \mid \frac{x_{jk}}{x_{jk}^0} = min_{(i,h)\in I} \frac{x_{ih}}{x_{ih}^0} \ \text{ if } (j,k) \in I^-(s), \text{ and}$$

$$\frac{x_{jk}}{x_{jk}^0} = max_{h=1,\ldots,\ell+1} \frac{x_{jh}}{x_{jh}^0} \ \text{ if } (j,k) \in I^+(s)\},$$

and

$$C(s) = Cl\{x \in S \mid \text{sgn} f(x) = s\},$$

where Cl denotes the closure and sgn is taken componentwise. Clearly, a point $x \in S$ satisfies (1.1) if and only if for some feasible sign vector s the point x lies in the intersection $B(s)$ of $A(s)$ and $C(s)$, i.e., B is the union of $B(s)$ over all feasible sign vectors s. Assuming regularity and differentiability of f, each nonempty set B(s) is a smooth 1-manifold, consisting of a disjoint set of smooth loops and paths. Each path has two end points, each end point x lying in the boundary of $A(s)$ or having $f_{jk}(x) = 0$ for some $(j,k) \notin I^0(s)$. Assuming nondegeneracy, an end point of a path in $B(s)$ is either the starting point x^0, or a zero point of f, or an end point of exactly one other path in some $B(s')$ where s' differs from s in only one or two components. Moreover, the starting point x^0 is an end point of just one path in some $B(s^0)$, where s^0=sgn $f(x^0)$, whereas each zero point of f is an end point of also just one path in some $B(s)$, where s is uniquely determined by the set $A(s)$, in which the zero point lies. Hence, the union B of the sets $B(s)$ over all feasible s consists, under certain conditions, of piecewise smooth disjoint loops and paths with two end points, one end point being x^0 and all others being zero points of f. The path P is then the piecewise smooth path in B which connects x^0 with a zero point of f. All other paths, if any, connect two different zero points of f. Notice that an end point of a path in $A(s)$ cannot lie in the boundary of S since for a point x in the intersection of $A(s)$ and the boundary of S we have that for all $(j,k) \in I^-(s)$, $x_{jk} = 0$, implying that $f_{jk}(x)$ is positive and not negative for these indices.

As described below the path P can approximately be followed by a simplicial variable dimension algorithm. This algorithm traces from x^0 a piecewise linear path $\bar{P}$ of points x satisfying (1.1), with f replaced by a piecewise linear approximation, $\bar{f}$, with respect to some subdivision of S into simplices. This simplicial subdivision is such that it subdivides each subset $A(s)$ into t-dimensional simplices, where t is the dimension of $A(s)$. For a point x in some t-dimensional simplex in $A(s)$ with vertices $w^1,..., w^{t+1}$, i.e., $x = \Sigma_{i=1}^{t+1}\lambda_i w^i$ for some unique nonnegative numbers λ_i summing up to 1, the piecewise linear approximation of f at x is defined by

$$\bar{f}(x) = \Sigma_{i=1}^{t+1}\lambda_i f(w^i).$$

The piecewise linear path $\bar{P}$ connects x^0 with a zero point of $\bar{f}$ and can be generated by efficient linear programming pivot steps, in order to trace a linear piece of the path in a simplex, and by replacement steps, in order to move from one simplex to an adjacent simplex containing the next (linear) piece of $\bar{P}$.

2. The Simplicial Subdivision of S

The simplicial subdivision which will underlie the algorithm in tracing the piecewise linear path $\bar{P}$ in S is such that for any feasible sign vector s it triangulates the set $A(s)$. This triangulation of $A(s)$ consists of a collection of t-dimensional simplices, where t is the dimension of $A(s)$ equal to

$$t = |I^0(s)| - |\{j|s_j = 0\}| + 1,$$

with $|\cdot|$ denoting the number of elements. In order to describe the triangulation of $A(s)$, whose union over all feasible s will give the desired triangulations of S, we first subdivide $A(s)$ into t-dimensional subsets. For $j = 0, 1, ..., m$, let T_j be a subset of $I_j^0(s)$ such that $T_j = I_j^0(s)$ if $s_j \neq 0$, and $T_j = I_j^0(s) \setminus \{(j, k_0^j)\}$ for some index $(j, k_0^j) \in I(j)$ if $s_j = 0$. Clearly, the union T of T_j over all j consists of $t-1$ elements. Next, for all j let

$$\gamma_j(T_j) = ((j, k_1^j), ..., (j, k_{t(j)}^j))$$

be a permutation of the $t(j)$ elements of T_j, and let

$$\gamma(T) = (\gamma_0(T_0), ..., \gamma_m(T_m)).$$

Further, for all j let the index set $Z_j^+(s)$ be defined by $Z_j^+(s) = I_j^+(s)$ if $s_j \neq 0$ and $Z_j^+(s) = \{(j, k_0^j)\}$ if $s_j = 0$, and let $Z^+(s)$ be the union of $Z_j^+(s)$ over all j. Then for any subset of indices L of I we define the projection $r(L)$ of x^0 by $r_j(L) = x_j^0$ if $L \cap I(j) = \emptyset$ and if $L \cap I(j) \neq \emptyset$

$$\begin{aligned} r_{jk}(L) &= x_{jk}^0 / \Sigma_{(j,h)\in L} x_{jh}^0 \quad && \text{for } (j,k) \in L, \\ &= 0 && \text{for } (j,k) \notin L, \end{aligned}$$

i.e., $r(L)$ is the relative projection of the starting point x^0 on the face $S(L)$ of S defined by

$$S(L) = \{x \in S \mid x_{jk} = 0 \text{ if } L \cap I(j) \neq \emptyset \text{ and } (j,k) \notin L\}.$$

Now the t-dimensional subset $A(s, \gamma(T))$ of $A(s)$ is defined by

$$A(s,\gamma(T)) = \{x \in S \mid x = x^0 + \alpha(0)q(0) + \Sigma_{(j,k)\in T}\alpha(j,k)q(j,k),$$

$$\text{with for all } j, \; 0 \leq \alpha(j, k_{t(j)}^j) \leq ... \leq \alpha(j, k_1^j) \leq \alpha(0) \leq 1\},$$

where $q(0) = r(Z^+(s)) - x^0$ and for $(j,k) \in T$

$$q(j,k_i^j) = r(Z^+(s) \cup \{(j,k_1^j), ..., (j,k_i^j)\}) - r(Z^+(s) \cup \{(j,k_1^j), ..., (j,k_{i-1}^j)\}).$$

Clearly, the collection of $A(s,\gamma(T))$ over all permutation vectors $\gamma(T)$ and sets T is a subdivision of $A(s)$. The simplicial subdivision of S is now obtained by subdividing each set $A(s,\gamma(T))$ into t-dimensional simplices.

Let n be a positive integer, then n^{-1} will be the grid size of the triangulation. Each subset $A(s,\gamma(T))$ is triangulated by the collection $G^n(s,\gamma(T))$ of t-simplices σ with vertices $w^1, ..., w^{t+1}$ in S such that

1) $w^1 = x^0 + a(0)n^{-1}q(0) + \Sigma_{(i,h)\in T} a(i,h)n^{-1}q(i,h)$ for certain integers $a(0)$ and $a(i,h)$, $(i,h) \in T$, such that for all j, $0 \leq a(j,k_{t(j)}^j) \leq ... \leq a(j,k_1^j) \leq a(0) \leq n-1$;

2) $w^{i+1} = w^i + n^{-1}q(\pi_i)$, $i = 1, ..., t$, with $\pi = (\pi_1, ..., \pi_t)$ a permutation of the element 0 and the $t-1$ elements of T such that $q > q'$ if $a(\pi_q) = a(\pi_{q'})$ and one of the

following conditions holds:

i) $\pi_{q'} = 0$ and $\pi_q = (j, k_1^j)$ for some j;

ii) $\pi_{q'} = (j, k_{i-1}^j)$ and $\pi_q = (j, k_i^j)$ for some i, $2 \le i \le t(j)$ and some j.

The union $G^n(s)$ of $G^n(s, \gamma(T))$ over all permutation vectors $\gamma(T)$ and sets T is a triangulation of $A(s)$. Finally, the union G^n of $G^n(s)$ over all feasible sign vectors s induces a triangulation of S with grid size n^{-1}.

3. The Algorithm on S in Case of Sublinearity

In S, the algorithm follows the piecewise linear path from x^0 of points x satisfying (1.1) with f replaced by its piecewise linear approximation $\bar{f}$ with respect to G^n. Each point x on this path lies in $A(s)$ for some feasible sign vector s with $s_0 = \text{sgn } \bar{f}_0(x)$ and $s_j = \text{sgn } \bar{f}_j(x_0, x_j)$, $j = 1, ..., m$. So, for some $\gamma(T)$, let $\sigma(w^1, ..., w^{t+1})$ be a t-simplex in $A(s, \gamma(T))$ containing x, then there exist unique nonnegative numbers λ_i, $i = 1, ..., t+1$, with $\Sigma_{i=1}^{t+1} \lambda_i = 1$ such that $x = \Sigma_{i=1}^{t+1} \lambda_i w^i$ and hence $\bar{f}_0(x) = \Sigma_{i=1}^{t+1} \lambda_i f_0(w^i)$ and $\bar{f}_j(x_0, x_j) = \Sigma_{i=1}^{t+1} \lambda_i f_j(w_0^i, w_j^i)$ for $j = 1, ..., m$. Since $s_0 = \text{sgn } \bar{f}_0(x)$ and $s_j = \text{sgn } \bar{f}_j(x_0, x_j)$, $j = 1, ..., m$, there also exist nonnegative numbers μ_{jh} such that $\bar{f}_{jh}(x) = \mu_{jh} s_{jh}$ for all $(j, h) \in I$. Hence λ_i, $i = 1, ..., t+1$, and μ_{jh}, $(j, h) \notin I^0(s)$, are a solution to the system of $(m+1)(\ell+1)+1$ equations defined by

$$\Sigma_{i=1}^{t+1} \lambda_i \begin{bmatrix} f_0(w^i) \\ f_1(w_0^i, w_1^i) \\ \vdots \\ f_m(w_0^i, w_m^i) \\ 1 \end{bmatrix} - \Sigma_{(j,h) \notin I^0(s)} \mu_{jh} s_{jh} \begin{bmatrix} e(j,h) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (3.1)$$

where the (j, h)-th unit vector $e(j, h)$ in $R^{(m+1)(\ell+1)}$ is defined by $e_{jh}(j, h) = 1$ and 0 elsewhere. When, for some $j = 1, ..., m$, $s_j = 0$ holds, we obtain that $\Sigma_{h=1}^{\ell+1} f_{jh}(w_0^i, w_j^i) = 0$ for all i, and hence the rank of the j-th part of the matrix in (3.1) is not full, i.e., $rank\, [f_j(w_0^1, w_j^1) \cdots f_j(w_0^{t+1}, w_j^{t+1})] = \ell$. Similarly, when $s_0 = 0$, we also have $rank\, [f_0(w^1) \cdots f_0(w^{t+1})] = \ell$. Therefore, if for some $j = 0, ..., m$, $s_j = 0$ holds, we delete one of the equations in the j-th part of the system (3.1), e.g., the k_ℓ^j-th equation. So, for a given feasible sign vector s and a permutation $\gamma(T)$, let $\bar{f}_j(x_0, x_j) = f_j(x_0, x_j)$ if $s_j \neq 0$, $j = 1, ..., m$, and

$\bar{f}_0(x) = f_0(x)$ if $s_0 \neq 0$, and let

$$\bar{f}_j(x_0, x_j) = (f_{j1}(x_0, x_j), \ldots, f_{j(k^j_\ell - 1)}(x_0, x_j), f_{j(k^j_\ell + 1)}(x_0, x_j), \ldots, f_{j(\ell+1)}(x_0, x_j))^T$$

if $s_j = 0$, $j = 1, \ldots, m$, and

$$\bar{f}_0(x) = (f_{01}(x), \ldots, f_{0(k^0_\ell - 1)}(x), f_{0(k^0_\ell + 1)}(x), \ldots, f_{0(\ell+1)}(x))^T$$

if $s_0 = 0$. Now the variables λ_i, $i = 1, \ldots, t+1$, and μ_{jh}, $(j,h) \notin I^0(s)$, are a solution to the system of $(m+1)\ell + |\{j | s_j \neq 0\}| + 1$ equations

$$\Sigma_{i=1}^{t+1} \lambda_i \begin{bmatrix} \bar{f}_0(w^i) \\ \bar{f}_1(w^i_0, w^i_1) \\ \vdots \\ \bar{f}_m(w^i_0, w^i_m) \\ 1 \end{bmatrix} - \Sigma_{(j,h) \notin I^0(s)} \mu_{jh} s_{jh} \begin{bmatrix} e(j,h) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad (3.2)$$

where each $e(j,h)$ is of appropriate length. System (3.2) has just one variable more than there are equations. So, assuming nondegeneracy, system (3.2) has a line segment of solutions, if any, corresponding to a line segment of points x in σ with $x = \Sigma_{i=1}^{t+1} \lambda_i w^i$ and satisfying (1.1) with f replaced by $\bar{f}$. At each of the two end points of the solution set of (3.2), exactly one of the variables λ_i, $i = 1, \ldots, t+1$, and μ_{jh}, $(j,h) \notin I^0(s)$, is equal to zero. The set of solutions to (3.2) can be generated by making a linear programming pivot step with one of the variables being zero at an end point. After the pivot step, one of the other variables has become zero, yielding the other end point of solutions. The variable which is zero at this end point determines the adjacent simplex in which the next line segment of points satisfying (1.1) with respect to $\bar{f}$ lies.

In order to perform an efficient pivot step in (3.2), we make use, in case $s_0 \neq 0$, of $(u+1)$-vectors $d(k)$, $k = 1, \ldots, \ell+2$, where $u = (m+1)(\ell+1) - |\{i | s_i = 0\}|$, an $(\ell+2) \times (\ell+2)$ matrix P^{-1}, and, for $j = 1, \ldots, m$, an $(\ell+1) \times (\ell+1)$ matrix D_j^{-1} if $s_j \neq 0$ and an $\ell \times \ell$ matrix D_j^{-1} if $s_j = 0$. If $s_0 = 0$, we will make use of $(u+1)$-vectors $d(k)$, $k = 1, \ldots, \ell+1$, an $(\ell+1) \times (\ell+1)$ matrix P^{-1}, and, for $j = 1, \ldots, m$, matrices D_j^{-1}. From these vectors and matrices the solution at a new end point is calculated. The pivoting steps need only to be made in P^{-1} and sometimes also in one or two of the D_j^{-1}'s. We only describe here the case $s_0 \neq 0$ (the case $s_0 = 0$ is the same except that the dimension of the $d(k)$'s and P^{-1} is one less). If by a linear programming step the variable λ_q for some $1 \leq q \leq t+1$ becomes 0, then the point x with $x = \Sigma_{i=1}^{t+1}\lambda_i w^i$ lies in the facet τ of σ opposite to the vertex w^q. If the facet τ does not lie in the boundary of $A(s,\gamma(T))$, then there is exactly one t-simplex $\hat{\sigma}(w^1,...,w^{q-1},\hat{w}^q,w^{q+1},...,w^{t+1})$ in $G^*(s,\gamma(T))$ sharing the facet τ with $\sigma(w^1,...,w^{t+1})$. The parameters of $\hat{\sigma}$ are obtained from Table 1, where a becomes $a+e(\pi_1)$ $(a-e(\pi_t))$ means $a(\pi_1)$ becomes $a(\pi_1)+1$ ($a(\pi_t)$ becomes $a(\pi_t)-1$) and the other a's do not change.

	w^1 becomes	π becomes	a becomes
$q=1$	$w^1+n^{-1}q(\pi_1)$	$(\pi_2,...,\pi_t,\pi_1)$	$a+e(\pi_1)$
$1<q<t+1$	w^1	$(\pi_1,...,\pi_q,\pi_{q-1},...,\pi_t)$	a
$q=t+1$	$w^1-n^{-1}q(\pi_t)$	$(\pi_t,\pi_1,...,\pi_{t-1})$	$a-e(\pi_t)$

Table 1. Replacement Step with Vertex w^q

In particular, when $2 \leq q \leq t$ and $\pi_{q-1}=(j,k)$ and $\pi_q=(j',k')$ for some (j,k) and $(j',k') \in T$ with $j,j' \neq 0$ and $j \neq j'$, then τ does not lie in the boundary of $A(s,\gamma(T))$. In this case, we can exploit sublinearity in order to make an l.p. pivot step in (3.2) with the variable corresponding to the new vertex $\hat{w}^q$ of $\hat{\sigma}$ if at the other end point of the solution set of (3.2) with respect to σ also one of the λ_i's, say $\lambda_{q'}$, was equal to zero. Sublinearrity is due to the fact that

$$\ell^-(\hat{w}^q) = \ell^-(w^{q+1}) + \ell^-(w^{q-1}) - \ell^-(w^q),$$

where, for $x \in S$, $\ell(x) = (\ell_0(x), \ell^-(x))$ with $\ell_0(x) = \bar{f}_0(x)$ and $\ell_j(x) = \bar{f}_j(x_0,x_j)$, $j = 1,...,m$. In Section 4, the cases are discussed when there is no sublinearity. Let $(\pi_{t+1},...,\pi_u)$ be a permutation of the elements not in $I^0(s)$. Then for $k=1,...,\ell+2$, the $(u+1)$-vector $d(k)$ satisfies $d_q(k)=0$ and

$$\Sigma_{i=1\ i\neq q}^{t+1} d_i(k)\begin{pmatrix}\ell^-(w^i)\\1\end{pmatrix} - \Sigma_{h=t+2}^{u+1} d_h(k)s_{\pi_{h-1}}\begin{pmatrix}e^-(\pi_{h-1})\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

when $k=\ell+2$ or $(0,k)\in T$, and $d(k)=e(k'+1)$ with k' such that $\pi_{k'}=(0,k)$ when $(0,k)\notin T$ and $k<\ell+2$. For $k=1,...,\ell+2$, the k-th column of the matrix P is equal to

$$p(k) = \begin{bmatrix}1\\b(k)\end{bmatrix} \text{ if } (0,k)\in T \text{ or } k=\ell+2,$$

where $b(k) = \Sigma_{i=1}^{t+1} d_i(k)\ell_0(w^i) - \Sigma_{h=t+2}^{n+1} d_h(k) s_{\pi_{h-1}} e_0(\pi_{h-1})$, and

$$p(k) = \begin{bmatrix} 0 \\ e_0(0,k) \end{bmatrix} \text{ if } (0,k) \notin T \text{ and } k < \ell + 2.$$

Lemma 3.1: Let $\xi \in R^{\ell+2}$ be equal to $P^{-1}e(1)$, where $e(1) = (1, 0, ..., 0)^T \in R^{\ell+2}$. Then $\beta = [d(1) \cdots d(\ell+2)]\xi$ solves the system (3.2) with $\lambda_q = 0$.

Let $\Pi_j = \{i \mid \pi_i = (j,k) \in T\}$ for $j = 0, ..., m$, and let i_0 be the index for which $\pi_{i_0} = 0$. Then the system (3.2) is equivalent with the system of equations

$$\lambda'_{i_0}(\ell(w^{i_0+1}) - \ell(w^{i_0})) + \Sigma_{i\in\Pi_0}\lambda'_i(\ell(w^{i+1}) - \ell(w^i)) - \Sigma_{(0,h)\notin I_0^0(s)}\mu_{0h}s_{0h}e(0,h)$$

$$+\Sigma_{j=1}^{m}\left[\Sigma_{i\in\Pi_j}\lambda'_i\begin{pmatrix} \ell_0(w^{i+1}) - \ell_0(w^i) \\ 0 \\ \vdots \\ 0 \\ \ell_j(w^{i+1}) - \ell_j(w^i) \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \Sigma_{(j,h)\notin I_j^0(s)}\mu_{jh}s_{jh}e(j,h)\right] = -\ell(w^1), \quad (3.3)$$

with $1 \geq \lambda'_1 \geq \cdots \geq \lambda'_t \geq 0$ and $\mu_{jh} \geq 0$ for all $(j,h) \notin I^0(s)$, where $\lambda'_i = \Sigma_{j=i+1}^{t+1}\lambda_j$, $i = 1, ..., t+1$. For $j = 1, ..., m$ the columns of D_j are the vectors $\ell_j(w^{i+1}) - \ell_j(w^i)$ for $i \in \Pi_j$ and the unit vectors $e_j(j,h)$, $(j,h) \notin I_j^0(s)$. If for some j the columns of D_j are not mutually independent, we replace some of the columns by vectors $\ell_j(w^{i+1}) - \ell_j(w^i)$ with $i \in \Pi_0$ or $i = i_0$ in order to make each D_j nonsingular. Such a choice is always possible because the matrix of column vectors in system (3.2) is assumed to have full rank. The matrices D_j^{-1} are needed in the nonsublinear cases to perform an l.p. pivoting step.

In the rest of this section we descibe how to perform a pivoting step in the sublinear case. So, let $2 \leq q \leq t$ and $\pi_{q-1} = (j,k)$ and $\pi_q = (j',k')$ for some (j,k) and $(j',k') \in T$ with $j \neq j'$ and $j, j' \neq 0$, and suppose that $\lambda_{q'}$ is 0 at the other end point of the solution set to (3.2). For the facet τ opposite w^q, let $d(h)$, $h = 1, ..., \ell+2$, be as given above. Let τ' be the facet of σ opposite to $w^{q'}$ and let $d'(h)$, $h = 1, ..., \ell+2$, be these vectors corresponding to τ'. Next, let h^* be such that $d'(h^*) \neq d(h)$ for all $1 \leq h \leq \ell+2$, and let R be the $(u+1) \times (u+1)$ matrix defined by

$$R = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & 1 & & & \\ & & & -1 & & & \\ & & & 1 & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

with the -1 in the q-th column. Further let L_σ^- be the matrix defined by

$$L_\sigma^- = \begin{bmatrix} \ell^-(w^1) & \dots & \ell^-(w^{t+1}) & -s_{\pi_{t+1}} e^-(\pi_{t+1}) & \dots & -s_{\pi_u} e^-(\pi_u) \\ 1 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}.$$

By the sublinearity, $L_{\hat\sigma}^- = L_\sigma^- R$ and $L_\sigma^- RRd'(h^*) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Now let $d(0) = Rd'(h^*)$, so that $L_{\hat\sigma}^- d(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then we define $\zeta = d(0) - [d(1) \cdots d(\ell+2)]P^{-1}p(0)$, where $p(0) = \begin{pmatrix} 1 \\ b(0) \end{pmatrix}$ with

$$b(0) = \Sigma_{i=1}^{t+1} d_i(0)\ell_0(w^i) - \Sigma_{h=t+2}^{s+1} d_h(0) s_{\pi_{h-1}} e_0(\pi_{h-1}).$$

Clearly, $L_{\hat\sigma}\zeta = 0$, where

$$L_{\hat\sigma} = \begin{bmatrix} \ell(w^1) & \dots & \ell(\hat w^q) & \dots & \ell(w^{t+1}) & -s_{\pi_{t+1}} e(\pi_{t+1}) & \dots & -s_{\pi_u} e(\pi_u) \\ 1 & \dots & 1 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}.$$

With the vector ζ a ratio-test now can be performed. Suppose that λ_{q*} becomes equal to zero then the new $\hat d(h)$'s and $\hat P^{-1}$ which correspond to the facet τ^* of $\hat\sigma$ opposite to the vertex w^{q*} are calculated as follows. At the end of this section, we refer to the case that μ_{jk} becomes zero for some $(j,k) \notin I^0(s)$. First choose an $d(\hat h)$ such that $d_{q*}(\hat h) \neq 0$ and $d_{q*}(0)$, then for all $h = 1, ..., \ell+2$, $\hat d(h)$ is taken equal to $d(h) - c_h(d(0) - d(\hat h))$, where

$$c_h = \frac{d_{q*}(h)}{d_{q*}(0) - d_{q*}(\hat h)}.$$

We remark that if, for some h with $(0,h) \in T$ or $h = t+2$, $max_{i=1,...,t+1}|\hat d_i(h)|$ becomes too large, then we can take

$$\delta \hat d_i(h) + (1-\delta)\hat\lambda_i, \qquad i = 1, ..., t+1$$

for some small $\delta > 0$, where $\hat\lambda_i$ is the new solution value of λ_i.

For $h = 1, \ldots, \ell + 2$, $\hat{p}(h)$ is then equal to $p(h) - c_h(p(0) - p(\hat{h}))$. Therefore $\hat{P}^{-1}$ can be obtained from the matrix P^{-1} as follows. Let Q be the $(\ell + 3) \times (\ell + 2)$ matrix defined by

$$Q = \begin{bmatrix} 0 \\ P^{-1} \end{bmatrix},$$

let $\psi = \begin{pmatrix} 1 \\ -P^{-1}p(0) \end{pmatrix}$, and for $h = 1, \ldots, \ell + 2$, let the $(\ell + 3) \times (\ell + 3)$ matrix C_h be defined by

$$C_h = \begin{bmatrix} 1 & & -c_h & & \\ & \ddots & & & \\ & & 1 & & \\ & & c_h & \ddots & \\ & & & & 1 \end{bmatrix},$$

where $-c_h$ is in the first row and h-th column and c_h in the $\hat{h}$-th row and the h-th column. Next, let $\hat{Q} = C_{\hat{h}}^{-1}\Pi_{h \neq \hat{h}} C_h^{-1} Q$, and $\hat{\psi} = C_{\hat{h}}^{-1}\Pi_{h \neq \hat{h}} C_h^{-1} \psi$, so that we have $[p(0),\ \hat{P}]\hat{Q} = I$ and $[p(0),\ \hat{P}]\hat{\psi} = 0$. Consequently, the h-th row of $\hat{P}^{-1}$ is equal to the $(h+1)$-th row of the matrix $\hat{Q} - \hat{\psi}\hat{Q}_1/\hat{\psi}_1$, where $\hat{Q}_1$ is the first row of $\hat{Q}$. This completes the description of a linear programming pivot step in case of sublinearity and in case after the ratio-test with ζ some λ_q becomes zero.

Finally, if after the ratio-test with the vector ζ some $\mu_{jk}, (j,k) \notin I^0(s)$, becomes zero, then we cannot utilize sublinearity to compute the new $d(h)$'s and P^{-1}. In this case, we first construct a vector $d(0)$ as in the case (a)-(i) in Section 4. Then we compute the new $d(h)$'s and P^{-1} corresponding to σ as discussed at the end of Section 4.

4. The Algorithm on S in Case of Non-Sublinearity

In this section we describe the steps of the algorithm when sublinearity cannot be utilized or does not occur. In each case we describe how to calculate a vector $d(0)$. Unless, through the ratio-test, for some $(j,k) \notin I^0(s)$, μ_{jk} becomes zero, we can obtain $\hat{d}(h), h = 1, \ldots, \ell + 2$, and $\hat{P}^{-1}$ for the next step from $d(0)$ in the same way as described in Section 3. The case that μ_{jk} for some $(j,k) \notin I^0(s)$ becomes zero is discussed at the end of this section. In general only a pivot step in one of the matrices D_j^{-1} is needed in each case.

First, we consider the case for which, after a pivot step in (3.2) for some σ in $A(s, \gamma(T))$, $\lambda_q = 0$ and the facet τ opposite w^q does not lie in the boundary of $A(s, \gamma(T))$. This induces the subcases:

(a) $2 \leq q \leq t$

(i) $\pi_{q-1} = (j,k)$ and $\pi_q = (j',k')$ with $j, j' \neq 0$ and $j \neq j'$, and at the other end point of the solution set of (3.2) not $\lambda_{q^*} = 0$ for some $q^* \neq q$

(ii) $\pi_{q-1} = (j,k)$ and $\pi_q = (j,k')$ with $j \neq 0$

(iii) $\pi_{q-1} = (j,k)$ with $j \neq 0$, and $\pi_q = (0,k')$ or 0

(iv) $\pi_{q-1} = (0,k')$ or 0, and $\pi_q = (j,k)$ with $j \neq 0$

(v) $\pi_{q-1} = (0,k)$ or 0, and $\pi_q = (0,k')$ or 0;

(b) $q = 1$

(i) $\pi_1 = (j,k)$, $j \neq 0$

(ii) $\pi_1 = (0,k)$ or 0;

(c) $q = t+1$

(i) $\pi_{t+1} = (j,k)$, $j \neq 0$

(ii) $\pi_{t+1} = (0,k)$ or 0.

In all these subcases, there is exactly one t-simplex $\hat{\sigma}$ in $G^n(s,\gamma(T))$ adjacent to σ sharing τ. The parameters of $\hat{\sigma}$ can be obtained from Table 1. Let $\hat{w}^q$ be the vertex of $\hat{\sigma}$ opposite to τ. In order to calculate $\ell^-(\hat{w}^q)$, in general just one $\ell_j(\hat{w}^q)$ needs to be evaluated. In case (a)-(i), let D_j^{-k} be the matrix obtained from D_j by deleting the column corresponding to $i = q-1$. Let B_h be the matrix defined by

$$B_h = [\ell_h(w^{i_0+1}) - \ell_h(w^{i_0}), \quad \ell_h(w^{i+1}) - \ell_h(w^i), i \in \Pi_0], \; h = 1, ..., m.$$

We choose any column of B_j, say B_{jk^*}, to make the matrix $[B_{jk^*}, D_j^{-k}]$ nonsingular. Let $\bar{D}_{j'}^{-1}$ be the matrix obtained from $D_{j'}^{-1}$ by replacing $\ell_{j'}(w^{q+1}) - \ell_{j'}(w^q)$ by $\ell_{j'}(w^{q+1})$. In order to determine $d(0)$, let the vector $c = (c_0, c_1, ..., c_m)$ with $c_0 \in R^{t(0)+1}$ if $s_0 \neq 0$ ($c_0 \in R^{\ell+1}$ if $s_0 = 0$), $c_j \in R^{\ell}$ if $s_j \neq 0$ ($c_j \in R^{\ell-1}$ if $s_j = 0$), $c_i \in R^{\ell+1}$ if $s_i \neq 0$ ($c_i \in R^{\ell}$ if $s_i = 0$), be defined by $c_{0h} = 1$ for $h \neq k^*$,

$$c_{j'} = \bar{D}_{j'}^{-1}(\ell_{j'}(\hat{w}^q) - \ell_{j'}(w^1) - B_{j'}c_0),$$

$$(c_{0k^*}, (c_j)^T)^T = [B_{jk^*}, D_j^{-k}]^{-1}(\ell_j(\hat{w}^q) - \ell_j(w^1) - B_j^{-k^*}c_0^{-k^*}), \quad (4.1)$$

where $B_j^{-k^*}$ is the submatrix of B_j by deleting its k^*-th column and $c_0^{-k^*}$ is the subvector of c_0 by deleting its k^*-th component, and for $i \neq 0, j, j'$,

$$c_i = D_i^{-1}(\ell_i(\hat{w}^q) - \ell_i(w^1) - B_i c_0 - c_{j'k'}\ell_i(w^{q+1})). \qquad (4.2)$$

After some subtractions and permutations and having added zeros, we easily obtain from the vector c a vector $d \in R^{n+1}$ such that

$$L_{\bar{s}}^- d = \begin{pmatrix} \ell^-(\hat{w}^q) \\ 1 \end{pmatrix}.$$

Then $d(0)$ can be chosen to be equal to

$$d(0) = d(1) + d - e(q).$$

With $d(0)$ a ratio-test is performed and the new $d(h)$'s and P^{-1} can be obtained as described in Section 3. To obtain the new D_j^{-1}, one pivoting step is made in $[B_{jk^*}, D_j^{-k}]^{-1}$ to replacing B_{jk^*} by $\ell_j(w^{q+1}) - \ell_j(\hat{w}^q)$. The new $D_{j'}^{-1}$ is obtained by making one pivoting step, replacing $\ell_{j'}(w^{q+1})$ by $\ell_{j'}(\hat{w}^q) - \ell_{j'}(w^{q-1})$.

In case (a)-(ii), let D_j^{-k} be the matrix obtained from D_j by deleting the column corresponding to $i = q - 1$ and by replacing $\ell_j(w^{q+1}) - \ell_j(w^q)$ by $\ell_j(w^{q+1})$. We choose any column of B_j, say B_{jk^*}, to make the matrix $[B_{jk^*}, D_j^{-k}]$ nonsingular. Let the vector $c = (c_0, c_1, ..., c_m)$ be defined as in case (a)-(i) except that (4.2) now holds for all $i \neq 0, j$, and in (4.2) $c_{j'k'}$ is replaced by $c_{jk'}$. After the ratio-test, in order to obtain the new D_j^{-1} two pivoting steps are made in $[B_{jk^*}, D_j^{-k}]$ to replacing B_{jk^*} by $\ell_j(w^{q+1}) - \ell_j(\hat{w}^q)$ and $\ell_j(w^{q+1})$ by $\ell_j(\hat{w}^q) - \ell_j(w^{q-1})$.

In case (a)-(iii), let B_{hk^*} be the column of B_h corresponding to $i = q$ and replace it by $\ell_h(w^{q+1})$ for all $h \neq 0$. Let D_j^{-k} be the submatrix of D_j by deleting the column corresponding to $i = q - 1$. Now the vector c is defined as in the previous subcase except that in (4.2) the term with $c_{jk'}$ is missing. After the ratio-test, one pivoting step is made in $[B_{jk^*}, D_j^{-k}]^{-1}$ to replace B_{jk^*} by $\ell_j(w^{q+1}) - \ell_j(\hat{w}^q)$ whereas, for all i, B_{ik^*} becomes equal to $\ell_i(\hat{w}^q) - \ell_i(w^{q-1})$.

In case (a)-(iv), let the k^*-th column of each B_h correspond to $i = q - 1$ and let $\bar{D}_j^{-1}$ be the matrix obtained from D_j^{-1} by replacing $\ell_j(w^{q+1}) - \ell_j(w^q)$ by $\ell_j(w^{q+1})$. Now the vector

$c = (c_0, c_1, ..., c_m)$ with $c_0 \in R^{t(0)+1}$ if $s_0 \neq 0$ ($c_0 \in R^{\ell+1}$ if $s_0 = 0$), and for $i \neq 0$, $c_i \in R^{\ell+1}$ if $s_i \neq 0$ ($c_i \in R^{\ell}$ if $s_i = 0$), is defined by $c_{0k^*} = 0$, $c_{0h} = 1$ for $1 \leq h \leq t(0)+1$ and $h \neq k^*$,

$$c_j = \bar{D}_j^{-1}(\ell_j(\hat{w}^q) - \ell_j(w^1) - B_j c_0),$$

and for $i \neq 0, j$,

$$c_i = D_i^{-1}(\ell_i(\hat{w}^q) - \ell_i(w^1) - B_i c_0 - c_{jk}\ell_i(w^{q+1})).$$

After the ratio-test, a pivoting step is made in $\bar{D}_j$ to replace $\ell_j(w^{q+1})$ by $\ell_j(\hat{w}^q) - \ell_j(w^{q-1})$ and $\ell_i(w^{q+1}) - \ell_i(\hat{w}^q)$ replaces $\ell_i(w^q) - \ell_i(w^{q-1})$ in each B_i.

In case (a)-(v), let the k^*-th column of each B_h correspond to $i = q-1$ and let $\bar{B}_h$ be the matrix obtained from B_h by replacing the column corresponding to $i = q$ by $\ell_h(w^{q+1})$, for all $h \neq 0$. Now the vector $c = (c_0, ..., c_m)$ is defined by $c_{0k^*} = 0$, $c_{0h} = 1$ for $1 \leq h \leq t(0)+1$ and $h \neq k^*$, and for $i \neq 0$,

$$c_i = D_i^{-1}(\ell_i(\hat{w}^q) - \ell_i(w^1) - \bar{B}_i c_0). \tag{4.3}$$

After the ratio-test, $\ell_i(w^{q+1})$ becomes $\ell_i(\hat{w}^q) - \ell_i(w^{q-1})$ and $\ell_i(w^q) - \ell_i(w^{q-1})$ becomes $\ell_i(w^{q+1}) - \ell_i(\hat{w}^q)$ in each $\bar{B}_i$.

In case (b)-(i), let D_j^{-k} be the submatrix of D_j by deleting the column corresponding to $i = 1$, and let B_{jk^*} be a column of B_j making $[B_{jk^*}, D_j^{-k}]$ nonsingular. The vector c is obtained as in case (a)-(ii) except that in (4.1) and (4.2) w^1 is replaced by w^2 and in (4.2) the term with $c_{jk'}$ is missing. After the ratio-test a pivoting step is performed to replacing B_{jk^*} by $\ell_j(\hat{w}^1) - \ell_j(w^{t+1})$ whereas $\ell(w^1)$ becomes $\ell(w^2)$. In case (b)-(ii), the vector c is obtained as in case (a)-(v) except that w^1 is replaced by w^2. After the ratio-test, in each $\bar{B}_i$, $\ell_i(w^2) - \ell_i(w^1)$ becomes $\ell_i(\hat{w}^1) - \ell_i(w^{t+1})$, and $\ell(w^1)$ becomes $\ell(w^2)$.

In case (c)-(i), let D_j^{-k} be the submatrix of D_j by deleting the column corresponding to $i = t$ and let B_{jk^*} be a column of B_j such that $[B_{jk^*}, D_j^{-k}]$ is nonsingular. Then the vector $c = (c_0, c_1, ..., c_m)$ is defined by $c_{0h} = 1$ for $1 \leq h \leq t(0)+1$ and $h \neq k^*$,

$$(c_{0k^*}, (c_j)^T)^T = [B_{jk^*}, D_j^{-k}]^{-1}(\ell_j(\hat{w}^{t+1}) - B_j^{-k^*} c_0^{-k^*}),$$

and for all other i,

$$c_i = D_i^{-1}(\ell_i(\hat{w}^{t+1}) - B_i c_0).$$

After the ratio-test, one pivoting step is made in $[B_{jk^*}, D_j^{-k}]^{-1}$ to replacing B_{jk^*} by $\ell_j(w^1) - \ell_j(\hat{w}^{t+1})$ whereas $\ell(w^1)$ becomes $\ell(\hat{w}^{t+1})$. In case (c)-(ii), let the k^*-th column of each B_h correspond to $i = t$. Then the vector $c = (c_0, c_1, ..., c_m)$ is defined by $c_{0k^*} = 0$, $c_{0h} = 1$ for $1 \leq h \leq t(0)$ and $h \neq k^*$, and for $i \neq 0$,

$$c_i = D_i^{-1}(\ell_i(\hat{w}^{t+1}) - B_i c_0).$$

After the ratio-test, in each B_i, $\ell_i(w^{t+1}) - \ell_i(w^t)$ becomes $\ell_i(w^1) - \ell_i(\hat{w}^{t+1})$, and $\ell(w^1)$ becomes $\ell(\hat{w}^{t+1})$.

In the next lemma, we describe when a facet opposite a vertex of a simplex in $G^n(s, \gamma(T))$ lies on the boundary.

Lemma 4.1: Let $\sigma(w^1, ..., w^{t+1})$ be a t-simplex in $G^n(s, \gamma(T))$, then the facet τ opposite to the vertex w^q lies in the boundary of $A(s, \gamma(T))$ if and only if $t = 1$ and $\tau = \{x^0\}$ or if one of the following cases holds:

i) $q = 1$, $\pi_1 = 0$, and $a(\pi_1) = n - 1$

ii) $2 \leq q \leq t$, $\pi_{q-1} = 0$, $\pi_q = (j, k_1^j)$ for some j, and $a(\pi_{q-1}) = a(\pi_q)$

iii) $2 \leq q \leq t$, $\pi_{q-1} = (j, k_{i-1}^j)$, $\pi_q = (j, k_i^j)$ for some $2 \leq i \leq t(j)$, and $a(\pi_{q-1}) = a(\pi_q)$

iv) $q = t + 1$, $\pi_t = (j, k_{t(j)}^j)$ for some j, and $a(\pi_t) = 0$.

In case i), the facet τ lies in the face $S(I^0(s) \cup I^+(s))$ of S, i.e., at any end point x in τ, for all $(j, k) \in I^-(s)$, we must have $x_{jk} = 0$ and therefore $\bar{f}_{jk}(x) > 0$. For a point x on $\bar{P}$ we have according to (3.1) that for all $(j, k) \in I^-(s)$, $\bar{f}_{jk}(x) = s_{jk}\mu_{jk} \leq 0$. Consequently, case i) cannot occur in the algorithm. The other cases can occur if $\lambda_q = 0$ in (3.1) and we now describe how the algorithm then continues.

In case ii), if $j \neq 0$ and $s_j = 0$, then the facet τ is also a facet of a t-simplex in $A(s, \gamma(T'))$, where $T' = T \cup \{(j, k_0^j)\} \setminus \{(j, k_1^j)\}$ and $\gamma_j(T_j') = ((j, k_0^j), (j, k_2^j), ..., (j, k_\ell^j))$, while π becomes $(\pi_1, ..., \pi_{q-1}, (j, k_0^j), \pi_{q+1}, ..., \pi_t)$. The new vertex $\hat{w}^q$ is therefore equal to $w^{q-1} + n^{-1}q(0)$. Let $\bar{D}_j$ be the matrix obtained from D_j by replacing the column corresponding to $i = q$ by $\ell_j(w^{q+1})$. The pivot step to follow the path in σ is now the same as above for the case (a)-(iv) except that after the ratio-test the pivoting is made in $\bar{D}_j^{-1}$ to replacing $\ell_j(w^{q+1})$ by $\ell_j(w^{q+1}) - \ell_j(\hat{w}^q)$ whereas $\ell_i(\hat{w}^q) - \ell_i(w^{q-1})$ becomes the first column of $B_i, i = 1, ..., m$.

In case ii), if $j \neq 0$ and $s_j \neq 0$, then the facet τ lies in the boundary of $A(s)$ and is a $(t-1)$-simplex in $A(s', \gamma(T'))$, where $s'_{jk_1^j} = +1, s'_{ih} = s_{ih}$ for all $(i,h) \neq (j, k_1^j)$, $T' = T \backslash \{(j, k_1^j)\}$, and $\gamma_j(T') = \{(j, k_2^j), ..., (j, k_{t(j)}^j)\}$, while π becomes $(\pi_1, ..., \pi_{q-1}, \pi_{q+1}, ..., \pi_t)$. Let $\bar{D}_j$ be the same as just above, then $c = (c_0, ..., c_m)$ is defined by $c_{01} = 0$, $c_{0h} = 1$ for $h = 2, ..., t(0)+1$,

$$c_j = \bar{D}_j^{-1}(e_j(j, k_1^j) - \ell_j(w^1) - B_j c_0),$$

and for $i \neq 0, j$,

$$c_i = D_i^{-1}(-\ell_i(w^1) - B_i c_0).$$

After the ratio-test, in $\bar{D}_j^{-1}$, $\ell_j(w^{q+1})$ is replaced by $e_j(j, k_1^j)$, and the first column of each B_i becomes $\ell_i(w^{q+1}) - \ell_i(w^{q-1})$.

In case ii), if $j = 0$ and $s_0 = 0$, then the facet τ is also a facet of a t-simplex in $A(s, \gamma(T'))$, where $T' = T \cup \{(0, k_0^0)\} \setminus \{(0, k_1^0)\}$ and $\gamma_0(T'_0) = ((0, k_0^0), (0, k_2^0), ..., (0, k_\ell^0))$, while π becomes $(\pi_1, ..., \pi_{q-1}, (0, k_0^0), \pi_{q+1}, ..., \pi_t)$. The new vertex $\hat{w}^q$ is therefore equal to $w^{q-1} + n^{-1}q(0)$. Let $\bar{B}_h$ be the matrix obtained from B_h by replacing the column corresponding to $i = q$ by $\ell_h(w^{q+1})$, then $c = (c_0, ..., c_m)$ is defined by $c_{01} = 0$, $c_{0h} = 1$ for $h = 2, ..., t(0)+1$, and for $i \neq 0$

$$c_i = D_i^{-1}(\ell_i(\hat{w}^q) - \ell_i(w^1) - \bar{B}_i c_0).$$

After the ratio-test, in each $\bar{B}_i$, $\ell_i(w^{q+1})$ is replaced by $\ell_i(w^{q+1}) - \ell_i(\hat{w}^q)$ and the first column by $\ell_i(\hat{w}^q) - \ell_i(w^{q-1})$.

In case ii), if $j = 0$ and $s_0 \neq 0$, then the facet τ lies on the boundary of $A(s)$ and is a $(t-1)$-simplex in $A(s', \gamma(T'))$, where $s'_{0k_1^0} = +1$, $s'_{ih} = s_{ih}$ for all $(i,h) \neq (0, k_1^0)$, $T' = T \backslash \{(0, k_1^0)\}$, $\gamma_0(T'_0) = \{(0, k_2^0), ..., (0, k_{t(0)}^0)\}$, while π becomes $(\pi_1, ..., \pi_{q-1}, \pi_{q+1}, ..., \pi_t)$. Let each $\bar{B}_h$ be the matrix as defined just above. Then $(c_0, ..., c_m)$ is defined by $c_{01} = 0$, $c_{0h} = 1$ for $h = 2, ..., t(0)+1$, and for $i \neq 0$,

$$c_i = D_i^{-1}(-\ell_i(w^1) - \bar{B}_i c_0).$$

After the ratio-test, in each $\bar{B}_i$, the column $\ell_i(w^{q+1})$ is deleted and $\ell_i(w^{q+1}) - \ell_i(w^{q-1})$ becomes the first column.

In case iii), the facet τ is also a facet of a t-simplex in $A(s, \gamma'(T))$, where $\gamma'_j(T_j) = ((j, k_1^j), ..., (j, k_i^j), (j, k_{i-1}^j), ..., (j, k_{t(j)}^j))$ and $\gamma'_h(T_h) = \gamma_h(T_h)$ for $h \neq j$, while π becomes

$(\pi_1, ..., \pi_q, \pi_{q-1}, ..., \pi_t)$. The new vertex $\hat{w}^q$ is therefore equal to $w^{q-1} + n^{-1}q(j, k_i^j)$. The pivoting step is identical to the subcases of case (a) above when $\pi_{q-1} = (j,k)$ and $\pi_q = (j,k')$ for some j, except that after the ratio-test, when $j \neq 0$, two pivoting steps are made in $[B_{jk^*}, D_j^{-k}]^{-1}$ in order to replace B_{jk^*} by $\ell_j(\hat{w}^q) - \ell_j(w^{q-1})$ and $\ell_j(w^{q+1})$ by $\ell_j(w^{q+1}) - \ell_j(\hat{w}^q)$, whereas, when $j = 0$, in each $\bar{B}_h$, $\ell_h(w^{q+1})$ becomes $\ell_h(w^{q+1}) - \ell_h(\hat{w}^q)$ and $\ell_h(w^q) - \ell_h(w^{q-1})$ becomes $\ell_h(\hat{w}^q) - \ell_h(w^{q-1})$. When $s_j = 0$, $j \neq 0$, and $i = \ell$, then an extra row-pivoting step has to be performed in D_j^{-1} and B_j has to be adapted in order to exchange the row corresponding to the index $(j, k_{\ell-1}^j)$ with the one corresponding to (j, k_ℓ^j). When $s_j = 0$, $j = 0$, and $i = \ell$, then the $d(k)$'s and P^{-1} must be adapted in order to exchange the row and in P^{-1} also the column corresponding to the index $(0, k_{\ell-1}^0)$ with the ones corresponding to $(0, k_\ell^0)$.

In case iv), the facet τ is in the boundary of $A(s)$. If $s_j = 0$ and $j \neq 0$ then τ is a $(t-1)$-simplex in $A(s', \gamma(T'))$, where $s'_{jk_0^j} = +1$, $s'_{jk_\ell^j} = -1$, $s'_{ih} = s_{ih}$ for all $(i,h) \neq (j, k_0^j)$ and (j, k_ℓ^j), $T' = T \backslash \{(j, k_\ell^j)\}$, and $\gamma_j(T'_j) = ((j, k_1^j), ..., (j, k_{\ell-1}^j))$, whereas π becomes $(\pi_1, ..., \pi_{t-1})$. Let D_j^{-k} be the submatrix of D_j by deleting the column corresponding to $i = t$ and let B_{jk^*} be a column of B_j making $[B_{jk^*}, D_j^{-k}]$ nonsingular. Now the vector $c = (c_0, ..., c_m)$ with $c_j \in R^{\ell-1}$ is defined by $c_{0h} = 1$ for $h \neq k^*$,

$$(c_{0k^*}, (c_j)^T)^T = [B_{jk^*}, D_j^{-k}]^{-1}[e_j(j, k_0^j) - \ell_j(w^1) - B_j c_0],$$

and for $i \neq 0, j$,

$$c_i = D_i^{-1}(-\ell_i(w^1) - B_i c_0).$$

After the ratio-test, we first make a pivoting step in $[B_{jk^*},\ D_j^{-k}]^{-1}$ to replacing B_{jk^*} by $e_j(j, k_0^j)$. Then we add to each $d(h)$ the component and to B_j and D_j^{-1} the row corresponding to the index (j, k_ℓ^j) and we add $-e_j(j, k_\ell^j)$ to D_j^{-1}.

In case iv), if $s_j \neq 0$ and $j \neq 0$ then τ is a $(t-1)$-simplex in $A(s', \gamma(T'))$, where $s'_{jk_{t(j)}^j} = -1$, $s'_{ih} = s_{ih}$ for all $(i,h) \neq (j, k_{t(j)}^j)$, $T' = T \backslash \{(j, k_{t(j)}^j)\}$, and $\gamma_j(T'_j) = ((j, k_1^j), ..., (j, k_{t(j)-1}^j))$, whereas π becomes $(\pi_1, ..., \pi_{t-1})$. Now the vector $c = (c_0, ..., c_m)$ with $c_j \in R^\ell$ is defined by $c_{0h} = 1$ for $h \neq k^*$,

$$(c_{0k^*}, (c_j)^T)^T = [B_{jk^*}, D_j^{-k}](-e_j(j, k_{t(j)}^j) - \ell_j(w^1) - B_j c_o),$$

where $B_{jk\cdot}$ and D_j^{-k} are as defined in the previous case, and for $i \neq 0, j$,

$$c_i = D_i^{-1}(-\ell_i(w^1) - B_i c_0).$$

After the ratio-test, one pivoting step is made in $[B_{jk\cdot}, D_j^{-k}]^{-1}$ in order to replace $B_{jk\cdot}$ by $-e_j(j, k^j_{t(j)})$.

In case iv), if $j = 0$ and $s_0 = 0$ then the facet τ is a $(t-1)$-simplex in $A(s', \gamma(T'))$, where $s'_{0k_0^0} = +1$, $s'_{0k_\ell^0} = -1$, $s'_{ih} = s_{ih}$ for $(i,h) \neq (0, k_0^0)$ and $(0, k_\ell^0)$, $T' = T \backslash \{(0, k_\ell^0)\}$, and $\gamma_0(T_0') = ((0, k_1^0), \ldots, (0, k_{\ell-1}^0))$, whereas π becomes $(\pi_1, \ldots, \pi_{t-1})$. Now the vector $c = (c_0, \ldots, c_m)$ with $c_0 \in R^\ell$ is defined by $c_{0h} = 1$ for all h and for $i \neq 0$,

$$c_i = D_i^{-1}(-\ell_i(w^1) - B_i c_0).$$

After the ratio-test, the $d(h)$'s and P^{-1} must be adapted by adding rows and to P^{-1} also a column corresponding to the index $(0, k_\ell^0)$. If $j = 0$ and $s_0 \neq 0$ then the facet τ is a $(t-1)$-simplex in $A(s', \gamma(T'))$, where $s'_{0k^0_{t(0)}} = -1, s'_{ih} = s_{ih}$ for $(i,h) \neq (0, k^0_{t(0)})$, $T' = T \backslash \{(0, k^0_{t(0)}\}$, and $\gamma_0(T_0') = ((0, k_1^0), \ldots, (0, k^0_{t(0)-1}))$, whereas π becomes $(\pi_1, \ldots, \pi_{t-1})$. The vector $c = (c_0, \ldots, c_m)$ is defined as above for the case $s_0 = 0$. Note that after the ratio-test, no changes need to be made. This concludes the case of Lemma 4.1.

Next, suppose that by a pivoting step with respect to a t-simplex $\sigma(w^1, \ldots, w^{t+1})$ in $A(s, \gamma(T))$ a variable μ_{jk}, for some $(j,k) \notin I^0(s)$, becomes zero. Let $(I_j^0(s))^c$ be the complement of $I_j^0(s)$ in $I(j)$. If $|(I_j^0(s))^c| > 2$ then σ is a facet of just one $(t+1)$-simplex σ' in $A(s', \gamma(T'))$, where $s'_{jk} = 0$, $s'_{ih} = s_{ih}$ for all $(i,h) \neq (j,k)$, $T' = T \cup \{(j,k)\}$, $\gamma_j(T_j') = ((j,k), \gamma_j(T_j))$ if $s_{jk} = +1$, and $\gamma_j(T_j') = (\gamma_j(T_j), (j,k))$ if $s_{jk} = -1$, whereas π becomes $(\pi_1, \ldots, \pi_{i_0}, (j,k), \pi_{i_0+1}, \ldots, \pi_t)$ if $s_{jk} = +1$ and $(\pi_1, \ldots, \pi_t, (j,k))$ if $s_{jk} = -1$. If $j \neq 0$ then we replace $e_j(j,k)$ in D_j^{-1} by a column $B_{jk\cdot}$ of B_j to make $[B_{jk\cdot}, D_j^{-k}]$ nonsingular. Now c is defined as above for the case (a)-(ii) except that the new vertex is $\hat{w}^{i_0+2}$ if $s_{jk} = +1$ and $\hat{w}^{t+2}$ if $s_{jk} = -1$ and that in (4.2) the term $c_{jk'}\ell_i(w^{q+1})$ is missing. In order to perform the ratio-test, we compute the new $d(h)$'s and P^{-1} corresponding to σ as follows. Let $\bar{L}^-_{\sigma'}$ be the matrix obtained from $L^-_{\sigma'}$ by deleting the (i_0+2)-th column if $s_{jk} = +1$ and the $(t+2)$-th column if $s_{jk} = -1$. We first construct a vector $\nu \in R^n$ such that

$$\bar{L}^-_{\sigma'}\nu = \begin{pmatrix} -s_{jk}e^-(j,k) \\ 0 \end{pmatrix}$$

using the same method as above for the case (a)-(iv) or (a)-(v) if $s_{jk} = +1$ except that we replace $\ell_j(\hat{w}^q)$ and $\ell_i(\hat{w}^q)$ by $e_j(j,k)$ and 0, respectively, and for the case (c)-(i) or (c)-(ii) if $s_{jk} = -1$ except that we replace $\ell_j(\hat{w}^q)$ and $\ell_i(\hat{w}^q)$ by $-e_j(j,k)$ and 0, respectively. Let $\bar{\nu} = (\nu_1, ..., \nu_{i_0+1}, 0, \nu_{i_0+2}, ..., \nu_n)$ if $s_{jk} = +1$ and $\bar{\nu} = (\nu_1, ..., \nu_{t+1}, 0, \nu_{t+2}, ..., \nu_n)$ if $s_{jk} = -1$. Further, for each h, let $\bar{d}(h) = (d_1(h), ..., d_{i_0+1}(h), d_p(h), d_{i_0+2}(h), ..., d_{p-1}(h), d_{p+1}(h), ..., d_{n+1}(h))$ if $s_{jk} = +1$ and $\bar{d}(h) = (d_1(h), ..., d_{t+1}(h), d_p(h), d_{t+2}(h), ..., d_{p-1}(h), d_{p+1}(h), ..., d_{n+1}(h))$ if $s_{jk} = -1$, where $\pi_{p-1} = (j,k)$. We define for all h,

$$\hat{d}(h) = \bar{d}(h) + \bar{d}_{i_0+2}(h)(\bar{\nu} - e(i_0+2)) \quad \text{if} \quad s_{jk} = +1,$$

and

$$\hat{d}(h) = \bar{d}(h) + \bar{d}_{t+2}(h)(\bar{\nu} - e(t+2)) \quad \text{if} \quad s_{jk} = -1,$$

where $e(i_0+2)$ and $e(t+2)$ are the (i_0+2)-th unit vector and $(t+2)$-th unit vector, respectively. It is easy to check that $L^{-}_{\sigma'}\hat{d}(h) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and that $\hat{d}_{i_0+2}(h) = 0$ if $s_{jk} = +1$ and $\hat{d}_{t+2}(h) = 0$ if $s_{jk} = -1$, for all h. The $\hat{P}^{-1}$ for the next step can be obtained from the $\hat{d}(h)$'s in the same way as described in Section 3. After the ratio-test, we replace $B_{jk\cdot}$ by $\ell_j(\hat{w}^{t+2}) - \ell_j(w^{t+1})$ if $s_{jk} = +1$ and by $\ell_j(w^{i_0+2}) - \ell_j(\hat{w}^{i_0+1})$ if $s_{jk} = -1$. In the latter case, the first column of each B_i becomes $\ell_i(\hat{w}^{i_0+1}) - \ell_i(w^{i_0})$. If $j = 0$, the vector $c = (c_0, ..., c_m)$ is defined by $c_{0h} = 1$ for all h and for $i \neq 0$,

$$c_i = D_i^{-1}(\ell(\hat{w}^{t+2}) - \ell_i(w^1) - B_i c_0) \quad \text{if} \quad s_{jk} = +1$$

and

$$c_i = D_i^{-1}(\ell(\hat{w}^{i_0+1}) - \ell_i(w^1) - B_i c_0) \quad \text{if} \quad s_{jk} = -1.$$

In order to perform the ratio-test, we calculate the $\hat{d}(h)$'s and $\hat{P}^{-1}$ corresponding to σ as follows. We set $\hat{d}(h) = d(h)$ for all $h \neq k$ and $\hat{d}(k) = d(0)$. From the $\hat{d}(h)$'s, we can easily obtain $\hat{P}^{-1}$. After the ratio-test, each B_i is extended with the column $\ell_i(\hat{w}^{t+2}) - \ell_i(w^{t+1})$ if $s_{jk} = +1$ and with the column $\ell_i(w^{i_0+2}) - \ell_i(\hat{w}^{i_0+1})$ if $s_{jk} = -1$. In the latter case the first column of each B_i becomes $\ell_i(\hat{w}^{i_0+1}) - \ell_i(w^{i_0})$. Notice that in all these cases only $\ell_0(\hat{w}^{t+1})$ and $\ell_j(\hat{w}^{t+1})$ or $\ell_0(\hat{w}^{i_0+1})$ and $\ell_j(\hat{w}^{i_0+1})$ need to be evaluated if $j \neq 0$.

If $|(I_j^0(s))^c| = 2$ and $|(I^0(s))^c| = 2$ then the corresponding point $x = \Sigma\lambda_i w^i$ is a zero point of $\bar{f}$ and therefore an approximate zero of f, where $(I^0(s))^c$ is the complement of $I^0(s)$ in I. If $|I_j^0(s)| = 2$, but $|I^0(s)| > 2$, then $I_j^+(s)$ as well as $I_j^-(s)$ consists of one element, say (j,k) and (j,h), respectively, and both μ_{jk} and μ_{jh} become zero simultaneously. Then σ is a facet of just one $(t+1)$-simplex σ' in $A(s', \gamma(T'))$, where $s'_j = 0$, $s'_i = s_i$ for $i \neq j$, $T' = T \cup \{(j,h)\}$, $\gamma_j(T'_j) = (\gamma_j(T_j), (j,h))$, whereas π becomes $(\pi_1, ..., \pi_t, (j,h))$. If $j \neq 0$, then first we delete from each $d(i)$ the component corresponding to the index (j,h) and from B_j and D_j^{-1} the row and from D_j^{-1} also the column corresponding to the index (j,h). Next, in D_j^{-1}, we replace $e_j(j,k)$ by a column B_{jk^*} of B_j to make $[B_{jk^*}, D_j^{-k}]$ nonsingular. Now the vector c is defined as above for the case (a)-(ii) except that the new vertex is equal to $\hat{w}^{t+2} = w^{t+1} + n^{-1}q(j,k)$ and in (4.2) the term $c_{jk'}\ell_i(w^{q+1})$ is missing. The new $d(h)$'s and P^{-1} corresponding to σ can be obtained as in the previous case for $j \neq 0$. After the ratio-test, we replace B_{jk^*} by $\ell_j(\hat{w}^{t+2}) - \ell_j(w^{t+1})$. If $j = 0$, then first we delete from each $d(i)$ the component and from P^{-1} the row and column corresponding to the index $(0,h)$. Now the vector $c = (c_0, ..., c_m)$ with $c_0 \in R^\ell$ is defined by $c_{0h} = 1$ for all h and for $i \neq 0$,

$$c_i = D_i^{-1}(\ell_i(\hat{w}^{t+2}) - \ell_i(w^1) - B_i c_0).$$

The new $d(h)$'s and P^{-1} corresponding to σ can be obtained as in the previous case for $j = 0$. After the ratio-test, each B_i is extended with the column $\ell_i(\hat{w}^{t+2}) - \ell_i(w^{t+1})$. This concludes the steps of the algorithm in case of non-sublinearity.

Finally, we describe the initialization of the algorithm. At the starting point x^0, let $s^0 = \text{sgn } f(x^0)$, then

$$L_{\sigma(x^0)} = \begin{bmatrix} f(x^0) & -s_{01}^0 e(0,1) & \dots & -s_{m(\ell+1)}^0 e(m, \ell+1) \\ 1 & 0 & \dots & 0 \end{bmatrix}.$$

Hence we can take $d(h) = e(h+1)$ for $h = 1, ..., \ell+1$, and $d(\ell+2) = (1, 0, ..., 0, s_{11}^0 f_{11}(x^0), ..., s_{m(\ell+1)}^0 f_{m(\ell+1)}(x^0)) \in R^{(m+1)(\ell+1)+1}$. From these $d(h)$'s, we easily obtain the initial P^{-1}. All the D_j's are initially $(\ell+1) \times (\ell+1)$ identity matrices and the B_i's do not exist at x^0. The algorithm starts by making an l.p. pivoting step with $(f(x^1)^T, 1)^T$ in (3.2), where $x^1 = x^0 + n^{-1}q(0)$. Note that because of the nondegeneracy assumption s^0 does not contain any zero and hence there is no other sign vector s with $|I^0(s)| = 0$ such that, with respect to the unique simplex in $A(s)$ having x^0 as a vertex, x^0 corresponds to a basic solution of (3.2). Following the steps described above, the algorithm terminates within a finite number of steps with an $x^* \in S$ for which $\bar{f}(x^*) = 0$. If x^* is not accurate enough to yield a zero point of f, the algorithm can be restarted at x^* with a smaller grid size n^{-1}.

5. Applications

In this section we describe how the algorithm on S presented in this paper can be used to compute equilibria in an international trade model or in an economy with increasing returns to scale.

First of all we remark that the unit simplices in the cross product S need not to be of the same dimension. The steps of the algorithms are the same in case S is the cartesian product of $m+1$ unit simplices $S^{\ell(j)}$, $j = 0, 1, \ldots, m$, where $\ell(j)$ is the dimension of the j-th simplex in the product, i.e., $S = S^{\ell(0)} \times \cdots \times S^{\ell(m)}$. The only difference is that for each j the number ℓ should be replaced by $\ell(j)$.

In an international trade model there are m different countries. In country j, $j = 1, \ldots, m$, there are $\ell(j)$ domestic goods which are traded only inside country j. Moreover, there are $\ell(0)+1$ nondomestic goods to be traded all over the world. Given a price vector $p \in R_+^{\ell(0)+1} \times \Pi_{j=1}^m R_+^{\ell(j)}$, $p \neq 0$, with for $j \neq 0$, p_{jk} the price of domestic commodity k in country j, $1 \leq k \leq \ell(j)$, and with p_{0k} the price of common good k, $1 \leq k \leq \ell(0)+1$, let $z^j(p)$ be the total excess demand vector of country j at p. Clearly, for $j \neq 0$, $z_i^j(p) = 0$ when $i \neq j$, while $z_j^j(p)$ and $z_0^j(p)$ only depend on p_0 and p_j, i.e., $z_i^j(p) = z_i^j(p_0, p_j)$ for $i = 0, j$. The equilibrium problem is to find a price vector p^* such that $\Sigma_{j=0}^m z^j(p^*) = 0$, i.e.,

$i)\ z_j^j(p_0^*, p_j^*) = 0, \quad j = 1, \ldots, m$

and

$ii) \Sigma_{j=1}^m z_0^j(p_0^*, p_j^*) = 0.$

As shown in van der Laan [1985], this problem can be formulated as a zero point problem on S. Let $x \in S$, then we define price vectors $q^j = (q_0^j, q_j^j) \in R_+^{\ell(0)+1} \times R_+^{\ell(j)}$ in country j by $q_{0k}^j = x_{j(\ell(j)+1)} x_{0k}$, $k = 1, \ldots, \ell(0)+1$, and $q_{jk}^j = x_{jk}$, $k = 1, \ldots, \ell(j)$. Then the function $f : S \to \Pi_{j=0}^m R^{\ell(j)+1}$ is defined by

$$f_0(x) = (m+1)^{-1} \Sigma_{j=1}^m z_0^j(q^j)$$

and for $j = 1, ..., m$,

$$f_{jk}(x) = z^j_{jk}(q^j), \qquad k = 1, ..., \ell(j),$$
$$= \Sigma_{h=1}^{\ell(0)+1} x_{0h} z^j_{0h}(q^j), \qquad k = \ell(j) + 1.$$

A zero point x^* of f in S yields an equilibrium price vector p^*, where $p^*_0 = x^*_0$ and, for $j = 1, ..., m$, $p^*_{jk} = x^*_{jk}/x^*_{j(\ell(j)+1)}$, $k = 1, ..., \ell(j)$. Clearly, the function f is such that $f_j(x)$ depends only on x_0 and x_j, $j = 1, ..., m$. Moreover, $f_{jk}(x) > 0$ if $x_{jk} = 0$, for all $(j, k) \in I$, which follows from the monotonicity of the preference relations of the consumers, i.e., the excess demand for a domestic or common commodity is positive if its price tends to zero. Notice that $x_{j(\ell(j)+1)}$ relates the price level of the domestic goods to the price level of the common goods and that $f_{j(\ell(j)+1)}(x_0, x_j)$ is the deficit of country j, being positive (or infinite) if the domestic price level of the common goods tends to zero. So, the function f satisfies all the properties of Section 1, except that we do not have that $\Sigma_k f_{jk}(x) = 0$, for all j. However, because of Walras' law, i.e., every country will spend all its income, we have that $\Sigma_k x_{jk} f_{jk}(x) = 0$, $j = 1, ..., m$. Under this property, there also exists in general a piecewise smooth path P from x^0 to a zero point of f of points x satisfying (1.1), except that in case $f_j(x) \leq 0$, x_j satisfies

$$\frac{x_{jk}}{x^0_{jk}} = min_{h=1,...,\ell(j)+1} \frac{x_{jh}}{x^0_{jh}} \qquad \text{if} \quad f_{jk}(x) < 0.$$

The path P can be followed piecewise linearly as described in the previous sections, making use of the special structure of f. Because for each index $j \neq 0$ Walras' law holds instead of the sum of the components equals zero, we must allow for sign vectors s for which for some but not all j, $s_j \geq 0$ and $s_j \neq 0$, or $s_j \leq 0$ and $s_j \neq 0$, instead of $s_j = 0$. For a simplicial algorithm on S for functions satisfying just Walras' law but without (using) the special structure of f we refer to Doup, van den Elzen, and Talman [1987] and Doup [1988]. The replacement steps in the simplicial subdivision are the same as for the latter algorithm, whereas in each step the computation of $d(0)$ and the new $d(k)$'s, P^{-1}, and D_j^{-1}'s is identical as described in the Sections 3 and 4.

In case of an economy with increasing returns to scale there are given m firms and $\ell + 1$ commodities. Each firm j is represented by a production set Y_j and a pricing rule

$\varphi_j : bdY_j \rightarrow S^\ell$. Also given is a demand function $d : S^\ell \times Y \rightarrow R^{\ell+1}$, where $Y = \Pi_{j=1}^m Y_j$, satisfying $p^T d(p,y) = p^T(\Sigma_{j=1}^m y_j + w)$ for all $p \in S^\ell$ and $y \in Y$ (Walras' law) if $\varphi_j(y_j) = p$ for all j, with $w \in R^{\ell+1}$ the vector of the total intial endowment. We call $(p^*, y^*) \in S^\ell \times Y$ an equilibrium if at (p^*, y^*) all firms set the same price vector and produce efficiently and if demand is equal supply, i.e.,

$i)$ $\varphi_j(y_j^*) = p^*$ and $y_j^* \in bdY_j, j = 1, ..., m$

and

$ii)$ $d(p^*, y^*) = \Sigma_{j=1}^m y_j^* + w$.

Under certain reasonable assumptions on the φ_j's, Y_j's, and d, the set of efficient attainable production plans in Y_j can be identified as S^ℓ and there is a price vector p^0 and a point $\delta^0 \in S^{\ell \times m}$ such that

$$(\varphi_j(\delta_j) - p^0)(\delta_j - \delta_j^0) > 0 \text{ if } \delta_j \in bdS^\ell \tag{5.1}$$

and

$$p^0 z(p,\delta) > 0 \quad \text{if} \quad \delta \in bdS^{\ell \times m} \quad \text{and} \quad \varphi_j(\delta_j) = p, \text{ for all } j \neq 0, \tag{5.2}$$

where φ_j is now redefined on S^ℓ and z is the net excess demand function. Moreover, we assume desirability of goods, i.e., $z_i(p,\delta) > 0$ whenever $\varphi_j(\delta_j) = p$ for all j and $p_i = 0, i = 1, ..., \ell+1$. With $x = (p,\delta) \in S \equiv S^\ell \times S^{\ell \times m}$, we define the function f from S to $R^{(\ell+1)\times(m+1)}$ by $f_0(x) = z(p,\delta)$ and for $j = 1, ..., m$,

$$f_j(x) = p - \varphi_j(\delta_j).$$

A zero point of f obviously induces an equilibrium vector. Clearly, for $j = 1, ..., m$, we have $\Sigma_{k=1}^{\ell+1} f_{jk}(x) = 0$ and $f_j(x)$ depends only on x_0 and x_j. Because of (5.1) there in general exists a piecewise smooth path P_1 of points $(p^0, \delta) \in \{p^0\} \times S^{\ell \times m}$ satisfying (1.1) for $j = 1, ..., m$, and connecting (p^0, δ^0) with a point (p^0, δ^1). At δ^1 we have that $\varphi_j(\delta_j^1) = p^0$ for all $j \neq 0$, i.e., all firms set the same price p^0. The path P_1 can be followed approximately as described in the previous sections except that f_0 is ignored, i.e., the matrix P^{-1} and the $d(h)$'s are not present at all and Π_0 is empty. Continuing in (p^0, δ^1), there now exists a path P_2 starting in (p^0, δ^1) of points $x = (p,\delta) \in S$ such that $\varphi_j(\delta_j) = p$ for all $j \neq 0$ while x_0 and $f_0(x)$ satisfy (1.1). The path P_2 can be followed as described before with $s_j = 0$ for all $j \neq 0$. Because of Walras' law, property (5.2), and the desirability condition, the path P_2 cannot meet the boundary of S and has therefore another end point. This end point is either a zero of f and yields then an equilibrium or it is a point (p^0, δ^2) with $\varphi_j(\delta_j^2) = p^0$ for all $j \neq 0$. In the latter case (p^0, δ^2) is also an endpoint of a path P_3 in $\{p^0\} \times S^{\ell \times m}$, satisfying again (1.1) for $j = 1, ..., m$. The other end point of P_3, (p^0, δ^3), is again an end point of a path P_4 in S satisfying (1.1) with $f_j(x_0, x_j) = 0$ for $j \neq 0$, etc. In this way, there exists a piecewise smooth path P in S connecting $x^0 = (p^0, \delta^0)$ with a zero point of f. The path consists of several pieces. Each piece lies either in $\{p^0\} \times S^{\ell \times m}$ and satisfies (1.1) for $j = 1, ..., m$ or it lies in S and satisfies (1.1) with $f_j(x) = 0$, $j = 1, ..., m$. Each piece can be followed as described in the Sections 3 and 4, making use of the structure of the function f. Notice that the algorithm terminates at an $\bar{x} \in S$ where $\bar{f}_0(\bar{x}) \leq 0$ or ≥ 0 and $\bar{f}_j(\bar{x}_0, \bar{x}_j) = 0$, $j = 1, ..., m$.

The piecewise smooth path P in S from x^0 to a zero point of f can be interpreted in both applications just described as the path followed by a price or quantity adjustment process. For example, in the case of the international trade economy, along the path, initially the prices of the commodities for which there is excess demand are relatively increased and those of the commodities for which there is excess supply are relatively decreased. Moreover, the relative price level for a country is initially increased (decreased) if the trade balance deficit is positive (negative). As soon as there is equilibrium on a market or balance, its corresponding price or price level is adjusted in order to keep it in equilibrium. When the price (level) becomes later on relatively too low or too high the market or balance becomes again in disequilibrium. The latter step prevents the process from cycling or hitting the boundary (of S). A similar interpretation holds for the adjustment along the path in case of an economy with increasing returns to scale production.

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