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A SIMPLICIAL ALGORITHM FOR THE NONLINEAR STATIONARY POINT PROBLEM ON AN UNBOUNDED POLYHEDRON

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A Simplicial Algorithm for the Nonlinear Stationary Point Problem on an Unbounded Polyhedron

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ABSTRACT: A path following algorithm is proposed for finding a solution to the nonlinear stationary point problem on an unbounded, convex and pointed polyhedron. The algorithm can start at an arbitrary point of the polyhedron. The path to be followed by the algorithm is described as the path of zeros of some piecewise continuously differentiable function defined on an appropriate subdivided manifold. This manifold is induced by a generalized primal-dual pair of subdivided manifolds. The path of zeros can be approximately followed by dividing the polyhedron into simplices and replacing the original function by its piecewise linear approximation with respect to this subdivision. The piecewise linear path of this function can be generated by alternating replacement steps and linear programming pivot steps. We also state a condition under which the path of zeros converges to a solution and we describe how the algorithm operates when the problem is linear or when the polyhedron is the Cartesian product of a polytope and an unbounded polyhedron.

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1. Introduction

Let K be a convex polyhedron in R^n. We assume that K is unbounded and pointed, i.e., K has at least one vertex, and that K is represented by the set $\{ x \in R^n \mid A^t x \leq b \}$, where A is an $n \times m$ matrix and b an n-vector. Further, let f be a continuously differentiable function from K to R^n. Then the (nonlinear) stationary point problem for f on K is to find a point x in K such that

$$(z - x)^t f(x) \geq 0$$

for any point z in K. We call x a stationary point of f on K. If the function f is affine on K we call the problem the linear stationary point problem. The stationary point problem on an unbounded convex polyhedron is frequently met in mathematical programming, for example to find a Karush-Kuhn-Tucker point for an optimization problem with linear constraints.

To solve the nonlinear stationary point problem on K we propose a path-following algorithm. Such an algorithm traces the set of zeros of a piecewise continuously differentiable function g defined from an $(n+1)$-dimensional subdivided manifold to R^n. In case the zero vector is a regular value of the function g there exists a path of zeros initiating from an arbitrarily chosen point in K. The $(n+1)$-dimensional subdivided manifold is induced by a generalized primal-dual pair of subdivided manifolds, where the primal sets are determined by the faces of K and the dual sets are determined by the normal cones of these faces. A primal-dual pair of subdivided manifolds is a basic framework in path-following techniques for finding fixed points or solving stationary point problems, see for example [DY], [K1], [KY], [TY], [Y1], [Y2], and [Y3].

The path S of zeros of the function can be approximately followed by a simplicial algorithm. This algorithm subdivides first the set K into simplices in some appropriate way and replaces the function f by its piecewise linear approximation $\bar{f}$ with respect to this triangulation. For this function the path of zeros of g becomes piecewise linear and can therefore be followed by making alternating replacement steps and linear programming pivot steps for a sequence of adjacent simplices of varying dimension.

Since the set K is unbounded, the path S may diverge to infinity. We state a simple condition on the function under which the path S is bounded and therefore leads from the starting point to a solution of the problem. We also describe how the algorithm should be adapted in case K is the Cartesian product of a polytope and an unbounded, convex polyhedron and under which conditions the path S is bounded for this case. We conclude the paper with a short description of the algorithm when the function is affine on K. The convergence condition for this problem is related to the well-known condition of copositive plus in case of the linear complementarity problem.

This paper is a generalization of path-following techniques introduced earlier for solving stationary point problems. In [Y1] such a method has been proposed for the linear stationary point problem on a polytope, i.e., on a bounded polyhedron. In [TY] the nonlinear stationary point problem on a polytope was treated. Finally, in [DY] a path-following algorithm for the linear stationary point problem on a polyhedral cone was introduced.

This paper is organized as follows. Section 2 briefly reviews a basic theorem for path-following algorithms and extends the concept of a primal-dual pair of subdivided manifolds. In section 3 we describe the generalized pair of primal-dual subdivided manifolds which will underlie the algorithm. Section 4 defines the path of zeros from an arbitrary point and leading to either infinity or a solution. We describe how this path approximately can be followed by a simplicial algorithm. In section 5 we state a convergence condition guaranteeing that the path is bounded. Finally, section 6 and section 7 discuss the cases when K is the product of a polytope and a convex polyhedron and when f is affine on K, respectively.

2. Generalization of the Primal-Dual Pair of Subdivided Manifolds

We shall review briefly a basic theorem for path following algorithms and extend the concept of the primal-dual pair of subdivided manifolds introduced by Kojima and Yamamoto [KY].

We call an l-dimensional convex polyhedron a cell or an l-cell. When a cell X is a face (see for example [SW]) of a cell Y, we write $X \preceq Y$. We denote $X \prec Y$ when X is a proper face of Y. Particularly when an $(l-1)$-cell X is a face of an l-cell Y, we call X a facet of Y and denote it by $X \triangleleft Y$.

A collection $\mathcal{L}$ of cells of the same dimension, say l, is called an l-dimensional subdivided manifold if it satisfies the following conditions:

(1) any two cells of $\mathcal{L}$ intersect in a common face unless the intersection is empty,

(2) any facet of a cell of $\mathcal{L}$ lies in at most two cells of $\mathcal{L}$,

(3) each point of cells of $\mathcal{L}$ has a neighborhood which intersects finitely many cells of $\mathcal{L}$.

The last condition is referred to as local finiteness. We denote the collection of all faces of cells of $\mathcal{L}$ by $\bar{\mathcal{L}}$, i.e.,

$$\bar{\mathcal{L}} = \{ X \mid X \text{ is a face of some cell of } \mathcal{L} \},$$

and the union of all cells of $\mathcal{L}$ by $|\mathcal{L}|$, i.e.,

$$|\mathcal{L}| = \bigcup [X \mid X \text{ is a cell of } \mathcal{L}].$$

It is noteworthy that $\bar{\mathcal{L}}$ consists of cells of various dimensions. By the second and most crucial condition each $(l-1)$-cell of $\bar{\mathcal{L}}$ lies in either one or two l-cells of $\mathcal{L}$. We refer to the collection of those $(l-1)$-cells lying in exactly one l-cell of $\mathcal{L}$ as the boundary of $\mathcal{L}$ and denote it by $\partial\mathcal{L}$. A continuous mapping g from $|\mathcal{L}|$ into R^n is piecewise continuously differentiable (abbreviated by PC^1) on $\mathcal{L}$ if the restriction of g to each cell of $\mathcal{L}$ has a continuously differentiable extension. We denote the Jacobian matrix of g at point x of any cell C of $\mathcal{L}$ by $Dg(x;C)$. A point $c \in R^n$ is a regular value of the PC^1 mapping $g : |\mathcal{L}| \to R^n$ if

$$x \in B \preceq C \in \mathcal{L} \quad \text{and} \quad g(x) = c \quad \text{imply} \quad \dim\{ Dg(x;C)y \mid y \in B \} = n.$$

We now state one of the basic theorems for a path following algorithm [K].

THEOREM 2.1. Let $\mathcal{L}$ be an $(n+1)$-dimensional subdivided manifold in R^n and $g : |\mathcal{L}| \to R^n$ be a PC^1 mapping. Suppose $c \in R^n$ is a regular value of g and $g^{-1}(c) \neq \emptyset$. Then $g^{-1}(c)$ is a disjoint union of paths and loops, where a path is a 1-dimensional subdivided manifold homeomorphic to one of the intervals $(0,1)$, $(0,1]$ and $[0,1]$ and a loop is a 1-dimensional subdivided manifold homeomorphic to the 1-dimensional sphere. Furthermore they satisfy the following conditions:

(1) $g^{-1}(c) \cap X$ is either empty or a 1-manifold for each $X \in \mathcal{L}$,

(2) a loop of $g^{-1}(c)$ does not intersect $|\partial\mathcal{L}|$,

(3) if a path S of $g^{-1}(c)$ is compact, ∂S consists of two distinct points in $|\partial\mathcal{L}|$.

We first generalize the primal-dual pair of subdivided manifolds proposed in [KY]. In [KY] the dual operator relating a pair of subdivided manifolds was assumed to satisfy several conditions including one-to-one. We will here relax these conditions. Let $\mathcal{P}$ and $\mathcal{D}$ be subdivided manifolds. A dual operator, say d, is defined on $\bar{\mathcal{P}}$ and assigns to each cell of $\bar{\mathcal{P}}$ either the empty set or a cell Y of $\bar{\mathcal{D}}$ such that for some fixed positive integer l, called the degree,

$$dim X + dim Y = l$$

holds. We denote the image of $X \in \bar{\mathcal{P}}$ under the operator d by X^d. When a pair of subdivided manifolds $\mathcal{P}$ and $\mathcal{D}$ is linked by such an operator d, we call the triplet $(\mathcal{P}, \mathcal{D}, d)$ a generalized primal-dual pair of subdivided manifolds, $GPDM$ in short. We allow a dual operator to assign the same cell of $\bar{\mathcal{D}}$ to more than one cell of $\bar{\mathcal{P}}$, that is, to be a non-injective dual operator. Letting

$$\mathcal{L} = \{\, X \times X^d \mid X \in \bar{\mathcal{P}}, X^d \neq \emptyset \,\}, \tag{2.1}$$

the conditions required for $\mathcal{L}$ to be a subdivided manifold are given in the next lemma.

LEMMA 2.2. Suppose $(\mathcal{P}, \mathcal{D}, d)$ is a $GPDM$ with degree l. Let $\mathcal{L}$ be defined by (2.1). Then $\mathcal{L}$ is an l-dimensional subdivided manifold if and only if for any $(l$-$1)$-cell $X \times Y$ of $\bar{\mathcal{L}}$:

(1) there are at most two cells Z of $\bar{\mathcal{P}}$ such that

$$X \triangleleft Z \quad \text{and} \quad Z^d = Y, \tag{2.2}$$

(2) if $Y \lhd X^d$ then there is at most one cell Z of $\bar{\mathcal{P}}$ satisfying (2.2).

Proof. Among the three conditions of a subdivided manifold the second one is crucial and the others will be seen straightforward. Note that an $(l-1)$-cell $X \times Y$ of $\bar{\mathcal{L}}$ is a facet of an l-cell $Z \times Z^d$ of $\mathcal{L}$ if and only if either

$$X = Z \quad \text{and} \quad Y \lhd Z^d \tag{2.3}$$

or

$$X \lhd Z \quad \text{and} \quad Y = Z^d \tag{2.4}$$

holds. By the first condition there are at most two cells $Z \times Z^d$ of $\mathcal{L}$ satisfying (2.4). Furthermore the second condition means that there is at most one such cell if a cell $Z \times Z^d = X \times X^d$ satisfying (2.3) exists. Therefore we have shown that $X \times Y$ lies in at most two l-cells of $\mathcal{L}$.
The "only if" part is also readily seen by the same argument. //

The following lemma characterizes the cells constituting the boundary $\partial\mathcal{L}$ of $\mathcal{L}$.

LEMMA 2.3. An $(l-1)$-cell $X \times Y$ of $\bar{\mathcal{L}}$ belongs to the boundary if and only if the following conditions hold:

(1) if $Y \lhd X^d$, then there is no cell Z of $\bar{\mathcal{P}}$ satisfying (2.2),

(2) if $Y \ntriangleleft X^d$ then there is exactly one cell Z of $\bar{\mathcal{P}}$ satisfying (2.2).

3. Construction of a GPDM

As an underlying set we consider a pointed convex polyhedron in R^n, i.e., a convex polyhedron with vertices, defined by $K = \{ x \in R^n \mid A^t x \leq b \}$ with A an $n \times m$ matrix and b an m-vector. In what follows we shall present a subdivision of the polyhedron K and construct a $GPDM$ having this subdivision as the primal subdivided manifold.

It is well-known that K can be decomposed into a polytope and a polyhedral cone C containing the directions of all rays in K, and that C is given by $C = \{ x \mid A^t x \leq 0 \}$ (see for example [SW]). Since K is pointed, the cone C of rays is also pointed, namely $C \cap (-C) = \{0\}$. Indeed, suppose that $r \in C \cap (-C)$ and consider two points $v + r$ and $v - r$ for an arbitrarily chosen vertex v of K. Since r and $-r \in C$, both of these two points lie in K. If $r \neq 0$, then the point v would be a middle point of these points, which contradicts the fact that v is a vertex.

Let w be an arbitrary point of K. In the algorithm proposed below for solving the stationary point problem on K the point w will be the starting point. For some strictly positive m-vector γ let $h = -A\gamma$ and let $H^0 = \{ x \mid h^t x = h_0 \}$ be a hyperplane for some positive h_0. We can see that if h_0 is sufficiently large this hyperplane intersects every unbounded face of K while the negative halfspace $H^- = \{ x \mid h^t x \leq h_0 \}$ contains all vertices of K as well as w and hence also all bounded faces of K in its interior. To see this, let r be a nonzero vector of C. Since C is pointed, $A^t r \neq 0$. More precisely, $A^t r \leq 0$ and $(a_i)^t r < 0$ for at least one column a_i of A. Then by the definition of h,

$$h^t r = -\gamma^t A^t r > 0. \tag{3.1}$$

Therefore, when h_0 is large enough for the interior of H^- to contain all vertices of K and w, the hyperplane H^0 intersects every unbounded face of K.

Now we introduce several notations. Let $H^+ = \{ x \mid h^t x \geq h_0 \}$ be the positive halfspace of H^0. For any face F of K, let

$$F^- = \{ x \mid x \in F \cap H^- \}$$
$$F^0 = \{ x \mid x \in F \cap H^0 \}$$

and

$$F^+ = \{ x \mid x \in F \cap H^+ \}.$$

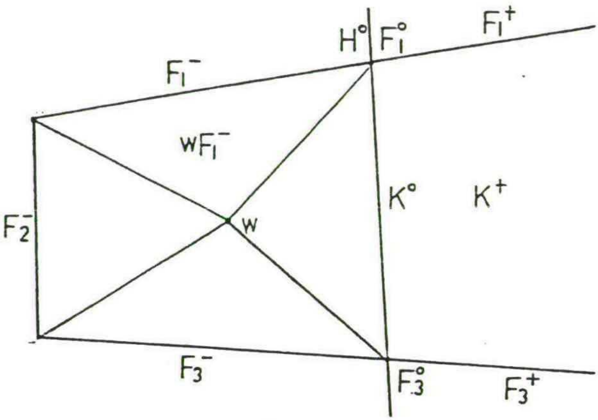

Figure. 3.1

Note that if some face F is entirely included in H^- then $F^- = F$. For an arbitrary subset G of K, we denote the convex hull of G and w by wG. Let

$$\mathcal{P} = \{ wF^- \mid w \notin F \lhd K \} \cup \{ wK^0 \} \cup \{ K^+ \}. \tag{3.2}$$

It can be easily proved that $\mathcal{P}$ is a subdivided manifold with the same dimension as K. Moreover the collection $\bar{\mathcal{P}}$ is equal to

$$\begin{aligned} \bar{\mathcal{P}} = & \{ wF^- \mid w \notin F \prec K \} \\ & \cup \{ wF^0 \mid F \text{ is an unbounded face of } K \} \\ & \cup \{ F^+ \mid F \text{ is an unbounded face of } K \} \\ & \cup \{ F^0 \mid F \text{ is an unbounded face of } K \} \\ & \cup \{ F^- \mid w \notin F \prec K \} \\ & \cup \{w\} \end{aligned} \tag{3.3}$$

and

$$|\mathcal{P}| = K. \tag{3.4}$$

An example is illustrated in Figure 3.1.

To make the dual subdivided manifold $\mathcal{D}$, we subdivide R^n in almost the same way as in [DY]. The normal cone at $x \in K$ to K is defined to be

$$N(x, K) = \{ y \mid y^t(z - x) \leq 0 \quad \text{for any} \quad z \in K \}. \tag{3.5}$$

It is the cone of all outward normal vectors at x to K. It is readily seen that normal cones are identical at any relative interior point of a face F of K. Therefore we denote it by F^*. Letting

$$I(F) = \{ \, i \mid (a_i)^t x = b_i \quad \text{for any} \quad x \in F \, \},$$

then F^* is equal to

$$F^* = \{ \, y \mid y = \sum_{i \in I(F)} \mu_i a_i, \; \mu_i \geq 0 \quad \text{for each} \quad i \in I(F) \, \}.$$

The dual subdivided manifold $\mathcal{D}$ is defined to be

$$\begin{aligned} \mathcal{D} = \{ \, &\{v\}^* \mid v \in U(K) \, \} \\ &\cup \{ \, F^* + < h > \mid F \quad \text{is an extreme ray of} \quad K \, \}, \end{aligned} \tag{3.6}$$

where $U(K)$ is the set of vertices of K and

$$< h > = \{ \, y \mid y = \alpha h \quad \text{for some} \quad \alpha \geq 0 \, \}, \tag{3.7}$$

being the ray in the direction h. Then $\mathcal{D}$ is obviously an n-dimensional subdivided manifold,

$$\begin{aligned} \bar{\mathcal{D}} = \{ \, &F^* \mid F \preceq K \, \} \\ &\cup \{ \, F^* + < h > \mid F \quad \text{is an unbounded face of} \quad K \, \} \end{aligned} \tag{3.8}$$

and

$$|\mathcal{D}| = R^n. \tag{3.9}$$

For constructing a $GPDM$ it remains to define an operator d linking the subdivided manifolds $\mathcal{P}$ and $\mathcal{D}$. Let

$$\begin{array}{ll} (wF^-)^d = F^* & \text{if } w \notin F \prec K \\ (wF^0)^d = F^* + < h > & \text{if } F \text{ is an unbounded face of } K \\ (F^+)^d = F^* + < h > & \text{if } F \text{ is an unbounded face of } K \\ (F^-)^d = \emptyset & \text{if } w \notin F \prec K \\ (F^0)^d = \emptyset & \text{if } F \text{ is an unbounded face of } K \\ (\{w\})^d = \emptyset \, . & \end{array} \tag{3.10}$$

Then the dimensions of a cell X in $\mathcal{P}$ and its dual cell X^d in $\mathcal{D}$ sum up to $n+1$ if X^d is nonempty, that is the $GPDM(\mathcal{P}, \mathcal{D}, d)$ constructed above has degree $n+1$. Let $\mathcal{M}$ be the collection of $(n+1)$-dimensional subdivided manifolds defined by (2.1) for this $GPDM(\mathcal{P}, \mathcal{D}, d)$. We shall show that $\mathcal{M}$ is an $(n+1)$-dimensional subdivided manifold by demonstrating that the $GPDM(\mathcal{P}, \mathcal{D}, d)$ satisfies the conditions of Lemma 2.2.

LEMMA 3.1. For any n-cell $X \times Y$ of $\bar{\mathcal{M}}$ derived from (3.3), (3.8) and (3.10) we have

(1) there are at most two cells Z of $\bar{\mathcal{P}}$ satisfying (2.2),

(2) if $Y \lhd X^d$, then there is at most one cell Z of $\bar{\mathcal{P}}$ satisfying (2.2).

Proof. From the definition of the dual operator d it follows that if at least two cells of $\bar{\mathcal{P}}$ are mapped to an identical cell they must be equal to wF^0 and F^+ for some unbounded face F of K. This means that condition (1) of Lemma 2.2 is satisfied. Next suppose that there are two different cells Z_1 and Z_2 in $\bar{\mathcal{P}}$ satisfying (2.2). Then $Z_1 = wF^0$ and $Z_2 = F^+$ for some unbounded face F of K. Since X is a facet of both Z_1 and Z_2, X must be F^0 and hence $X^d = \emptyset$. This proves that the second condition of Lemma 2.2 is also satisfied. //

Thus we have seen that $\mathcal{M}$ is an $(n+1)$-dimensional subdivided manifold as an immediate consequence of Lemma 2.2. By applying Lemma 2.3 to the $GPDM(\mathcal{P}, \mathcal{D}, d)$ considered here we obtain the following.

LEMMA 3.2. The boundary of M is equal to

$$\begin{aligned}
\partial\mathcal{M} = \{\ & \{w\} \times F^* \mid w \notin F \prec K,\ \dim F = 0 \ \} \\
\cup \{\ & \{w\} \times (F^* + < h >) \mid F \text{ is an extreme ray of } K \ \} \\
\cup \{\ & F^+ \times F^* \mid F \text{ is an unbounded face of } K \ \} \\
\cup \{\ & F^- \times F^* \mid w \notin F \prec K \ \} \\
\cup \{\ & wF^0 \times F^* \mid w \in F,\ F \text{ is an unbounded face of } K \ \} \\
\cup \{\ & wF^- \times G^* \mid w \in G,\ w \notin F \lhd G,\ G \preceq K \ \}
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
|\partial\mathcal{M}| = & \left(\bigcup [\{w\} \times \{v\}^* \mid v \text{ is a vertex of } K,\ v \neq w\ ]\right) \\
& \cup \left(\bigcup [\{w\} \times (F^* + < h >) \mid F \text{ is an extreme ray of } K\ ]\right) \\
& \cup \left(\bigcup [\ F \times F^* \mid \{w\} \neq F \preceq K\ ]\right).
\end{aligned} \tag{3.12}$$

Note that

$$|\partial\mathcal{M}| \supset (\ \{w\} \times (R^n \setminus \{w\}^*)) \cup \left(\bigcup [\ F \times F^* \mid \{w\} \neq F \preceq K\ ]\right).$$

4. Path Following Technique

Let $\mathcal{M}$ be the $(n+1)$-dimension subdivided manifold obtained from the $GPDM(\mathcal{P}, \mathcal{D}, d)$ as described in the previous section and let f be a continuously differentiable function from K to R^n. To find a stationary point of f on K, we consider the system

$$g(x,y) \equiv f(x) + y = 0, \ (x,y) \in |\mathcal{M}|. \tag{4.1}$$

If $0 \in R^n$ is a regular value of the mapping g, then from applying Theorem 2.1 to system (4.1) we obtain that $g^{-1}(0)$ consists of disjoint paths and loops. Suppose the starting point w is not a stationary point of f on K then we see from Lemma 3.2 that $(w, -f(w)) \in g^{-1}(0) \cap |\partial\mathcal{M}|$. Consequently, the connected component of $g^{-1}(0)$ containing $(w, -f(w))$ is a path. In the following we denote this path by S. Also according to Theorem 2.1, if the path S is bounded, then it will provide a distinct end point (x,y) in $|\partial\mathcal{M}|$. Since (x,y) satisfies the system of equations (4.1), $y = -f(x)$. If $x = w, (x,y)$ would coincide with $(w, -f(w))$. Therefore, according to (3.12), $(x,y) = (x, -f(x))$ lies in $F \times F^*$ for some face F of K and x is a stationary point of f on K.

To follow the path S in $|\mathcal{M}|$, we subdivide K into simplices such that each cell F in $\bar{\mathcal{P}}$ is triangulated. An appropriate simplicial subdivision of K is obtained by first triangulating the set K^- as described in [TY]. Notice that the starting point w is a vertex of this triangulation. In order to triangulate K^+, note that K^+ is the union of $K^0 + < h >$ and $F^+ + < h >$ over all unbounded facets F of K. The subset $K^0 + < h >$ can be triangulated in exactly the same way as wK^0 and each subset $F^+ + < h >$ in a similar way as wF^- by using projections of $w + h$ on the faces of F^+ instead of projections of w on the faces of F^-, as illustrated in Figure 4.1.

Let $\bar{f}$ be the piecewise linear approximation of f with repect to the triangulation. Taking $\bar{f}$ instead of f in (4.1), the path T of solutions to (4.1) originating at $(w, -f(w))$ is piecewise linear and can therefore be followed by making pivoting steps in subsequent systems of linear equations. For ease of description we restrict ourselves to a polyhedron K for which none of the inequalities $(a_i)^t x \leq b_i$ is redundant and each vertex is an end point of exactly n 1-faces of K. Now let (x,y) be a point on the path T. Then in some t-cell X of $\bar{\mathcal{P}}$ there is a simplex σ with vertices $w^1, \ldots, w^{t+1}$ such that x lies in σ and $\bar{f}(x)$ in X^d. Hence, there exist nonnegative numbers λ_i^*, $i = 1, \ldots, t+1$,

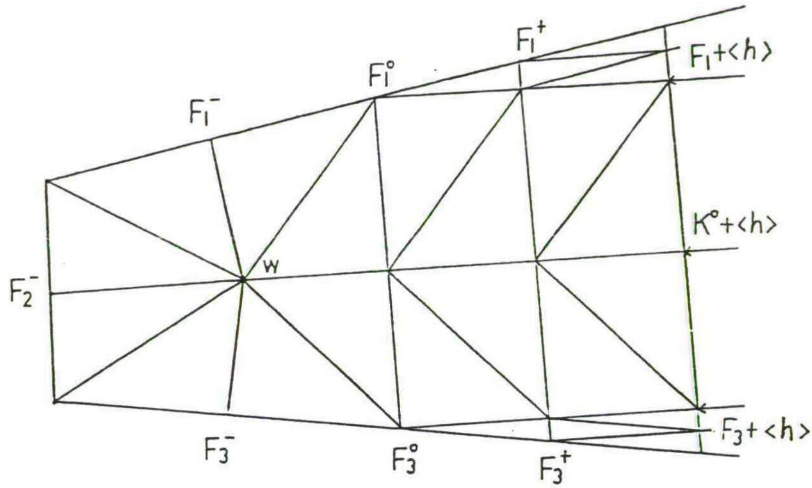

Figure. 4.1

such that $x = \sum_i \lambda_i^* w^i$ and $\sum_i \lambda_i^* = 1$. Moreover, if $X = wF^-$ there exist nonnegative numbers μ_k^*, $k \in I(F)$, such that $y = \sum_{k\in I(F)} \mu_k^* a^k$, and if $X = wF^0$ or $X = F^+$, there exist nonnegative numbers μ_k^*, $k \in I(F)$, and α^* such that $y = \sum_{k\in I(F)} \mu_k^* a^k + \alpha^* h$. Notice that $I(F)$ consists of $n+1-t$ indices because F is determined by this number of equalities. In case not all vertices of K are determined by n 1-faces of K we refer to [TY]. Since (x, y) is a solution of (4.1) with $\bar{f}$ instead of f and $\bar{f}(x) = \sum_i \lambda_i^* f(w^i)$, it follows that λ_i^*, $i = 1, \ldots, t+1$, μ_k^*, $k \in I(F)$, is a nonnegative solution to the system of linear equations

$$\sum_{i=1}^{t+1} \lambda_i \begin{bmatrix} f(w^i) \\ 1 \end{bmatrix} + \sum_{k\in I(F)} \mu_k \begin{bmatrix} a^k \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.2}$$

if $X = wF^-$, and that λ_i^*, $i = 1, \ldots, t+1$, μ_k^*, $k \in I(F)$, α^* is a nonnegative solution to the system of linear equations

$$\sum_{i=1}^{t+1} \lambda_i \begin{bmatrix} f(w^i) \\ 1 \end{bmatrix} + \sum_{k\in I(F)} \mu_k \begin{bmatrix} a^k \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} h \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.3}$$

if $X = wF^0$ or $X = F^+$. The system (4.2) or (4.3) has a line segment of solutions corresponding to a line segment of points $x = \sum_i \lambda_i w^i$ in σ, assuming nondegeneracy. At an end point of solutions one of the variables is equal to zero. When $\lambda_i = 0$ for some $i \in \{1, \ldots, t+1\}$, x lies in the facet τ opposite the vertex w^i of σ. This facet lies either in the boundary $\mathrm{bd}X$ of X or is a facet of just one other t-simplex $\bar{\sigma}$ in the cell X with vertices w^h, $h \neq i$, and $\bar{w}^i \neq w^i$. Then in the latter case to continue the path T in $\bar{\sigma}$, a pivoting step is made with $(f(\bar{w}^i)^t, 1)^t$. Suppose τ lies in $\mathrm{bd}X$ and $X = wF^-$. Then x is a stationary point of $\bar{f}$ on K if τ lies in F^-. Otherwise, either τ lies in wG^- with G a facet of F or τ lies in wF^0. In the first case the path T can be continued in wG^- by pivoting $((a^k)^t, 0)^t$ into (4.2), where k is the unique index in $I(G)$ not in $I(F)$. In the latter case the path T can be continued in wF^0 by pivoting $(h^t, 0)^t$ into (4.2). Now, suppose τ lies in $\mathrm{bd}X$ and $X = wF^0$ or $X = F^+$. Then when $X = wF^0$, τ lies either in wG^0 for some facet G of F or in F^0, and when $X = F^+$, τ lies either in G^+ for some facet G of F or in F^0. When τ is in wG^0 or G^+, the path T can be continued by pivoting $((a^k)^t, 0)^t$ into (4.3), where k is the unique index in $I(G)$ not in $I(F)$. When τ lies in F^0, τ is the facet of a unique t-simplex $\bar{\sigma}$ in F^+ if $X = wF^0$ and in wF^0 if $X = F^+$, and the path T can be continued in $\bar{\sigma}$ by making a pivoting step with $(f(\bar{w})^t, 1)^t$ in (4.3), where $\bar{w}$ is the vertex of $\bar{\sigma}$ opposite τ.

We now consider the case that at an end point of solutions of (4.2) or (4.3) we have that $\mu_k = 0$ for some $k \in I(F)$. Let G be the unique face of K such that $I(G) = I(F) \setminus \{k\}$. Then with $x = \sum_i \lambda_i w^i$ we have that $f(x) = \sum_i \lambda_i f(w^i) = -\sum_{k \in I(G)} \mu_k a^k \in G^*$. First, suppose that $X = wF^-$. If $w \in G$ then also $x \in G$ since F is a facet of G. Therefore, $w \in G$ implies $(x, -f(x)) \in G \times G^*$, and hence x is a stationary point of $\bar{f}$ on K. In case $w \notin G$ or if $X = wF^0$ or F^+, then σ is a facet of a unique $(t+1)$-dimensional simplex $\bar{\sigma}$ in wG^-, respectively wG^0 or G^+, and the path can be continued in $\bar{\sigma}$ by making a pivoting step with $(f(\bar{w})^t, 1)^t$ in (4.2) or (4.3), where $\bar{w}$ is the vertex of $\bar{\sigma}$ opposite σ.

Finally, we consider the case that in (4.3) at an end point $\alpha = 0$. If $X = wF^0$ and $w \notin F$, then σ is a facet of a unique $(t+1)$-simplex $\bar{\sigma}$ in wF^- and the path can be continued in wF^- by making a pivoting step with $(f(\bar{w})^t, 1)^t$ in (4.3), where $\bar{w}$ is the vertex of $\bar{\sigma}$ opposite σ. If $X = wF^0$ and $w \in F$ or if $X = F^+$, then $x = \sum_i \lambda_i w^i \in F$ and $\bar{f}(x) = \sum_i \lambda_i f(w^i) =$

$-\sum_{k\in I(F)} \mu_k a^k \in F^*$, so that x is a stationary point of $\bar{f}$ on K .

This completes the description of how to follow approximately the path S by making alternating pivoting and replacement steps for a sequence of adjacent simplices of varying dimension. This sequence starts at w and terminates with a simplex containing a stationary point $\bar{x}$ of $\bar{f}$ on K . This point $\bar{x}$ is an approximate stationary point of f on K . To improve the accuracy of the approximation if necessary we can take a new trangulation of K with the point $\bar{x}$ as the new starting point w and having a finer mesh size and apply the same procedure.

5. Convergence Condition

In this section we state a condition under which the path S is bounded and therefore leads from w to a stationary point of f on K .

LEMMA 5.1. Let (x,y) be a solution of the system

$$g(x,y)=0,\ (x,y)\in F^{+}\times(F^{*}+<h>). \tag{5.1}$$

If x is not a stationary point, then

$$r^t y > 0,$$

for any nonzero vector r in the cone C satisfying $(a_i)^t r = 0$ for all $i \in I(F)$.
Proof. The point y in $F^* + <h>$ is equal to $B\mu + \alpha h$ for some vector $\mu \geq 0$ and number $\alpha \geq 0$, where B denotes the submatrix of A consisting of the column vectors a_i for $i \in I(F)$. Since x is not a stationary point, $\alpha > 0$. Then

$$r^t y = r^t(B\mu + \alpha h) = (B^t r)^t \mu + \alpha h^t r = \alpha h^t r > 0$$

by the choice of h .   //

CONDITION 5.2. There is a set $U \subset R^n$ such that $U \cap K$ is bounded and for each point $x \in K \setminus U$ there is a nonzero vector $\tilde{r}$ in $C \cap \{\, r \in R^n \mid (a_i)^t r = 0$ whenever $(a_i)^t x = b_i \,\}$ satisfying

$$\tilde{r}^t f(x) \geq 0.$$

LEMMA 5.3. Under Condition 5.2 the path S does not diverge.

Proof. Suppose the contrary. Then there is a solution $(\tilde{x}, \tilde{y})$ of the system (4.1) such that $\tilde{x} \in F^+ \setminus U$ for some face F, since the continuity of the function f requires the x-component to diverge. Therefore by Lemma 5.1 and Condition 5.2 we see that $\tilde{r}^t(f(\tilde{x}) + \tilde{y}) > 0$ for some vector $\tilde{r}$, which contradicts that $(\tilde{x}, \tilde{y})$ is a solution of (4.1). //

6. Stationary Point Problems on a Cartesian Product of a Polytope and a Polyhedron

We consider a stationary point problem defined on the Cartesian product of a polytope and a polyhedron. The product is again a polyhedron and the theory of section 3 could still be applied to this case. However, it will be quite useful to consider it separately because a lot of problems are defined on such product sets. Let

$$K_1 = \{ \ x_1 \in R^{n_1} \mid A_1^t x_1 \leq b_1 \ \}$$

be a nonempty polytope and let

$$K_2 = \{ \ x_2 \in R^{n_2} \mid A_2^t x_2 \leq b_2 \ \}$$

be a nonempty, convex, unbounded and pointed polyhedron, with A_i an $n_i \times m_i$ matrix and b_i an m_i-vector for $i = 1, 2$. We consider the stationary point problem for a continuous function f from $K_1 \times K_2$ to $R^{n_1} \times R^{n_2}$. We denote $f(x)$ by $f(x) = \big(f_1(x_1, x_2), f_2(x_1, x_2)\big)$ and call $(x_1, x_2) \in K_1 \times K_2$ a stationary point of f on $K_1 \times K_2$ if

$$(z_1 - x_1)^t f_1(x_1, x_2) + (z_2 - x_2)^t f_2(x_1, x_2) \geq 0$$

for any point $(z_1, z_2) \in K_1 \times K_2$.

In the same way as in the preceding sections we will construct a $GPDM$ by introducing an artificial hyperplane and corresponding half spaces defined by

$$H^\pi = \{ \ (x_1, x_2) \in R^{n_1 + n_2} \mid h_2^t x_2 \ \rho \ h_0 \ \},$$

where π is $-$, 0 and $+$ when ρ is $\leq$, $=$ and $\geq$, respectively, where $h_2 = -A_2 \gamma$ for some fixed positive vector γ, and $h_0 > 0$ is chosen such that the interior of the half space H^- contains all vertices of $K_1 \times K_2$ as well as the starting point $w = (w_1, w_2)$. Note that $h_2^t r_2 > 0$ for any nonzero vector r_2 in the set

$$C_2 = \{\ r_2 \in R^{n_2}\ |\ A_2^t r_2 \leq 0\ \}$$

of directions of rays of K_2, which we have seen is a pointed cone. It is clear that a face F of $K_1 \times K_2$ is itself a Cartesian product of faces of K_1 and K_2, which we will denote by F_1 and F_2, respectively. Let

$$H_2^\pi = \{\ x_2 \in R^{n_2}\ |\ h_2^t x_2\ \rho\ h_0\ \},$$

where π is $-$, 0 and $+$ when ρ is $\leq$, $=$ and $\geq$, respectively. We define

$$F_2^\pi = F_2 \cap H_2^\pi \quad \text{for} \quad \pi = -,\ 0 \text{ and } +.$$

Then

$$F^\pi = F_1 \times F_2^\pi \quad \text{for} \quad \pi = -,\ 0 \text{ and } +.$$

It is also clear that

$$F^* = F_1^* \times F_2^*,$$

where F_1^* and F_2^* are defined with respect to $K_1 \subset R^{n_1}$ and $K_2 \subset R^{n_2}$, respectively. Thus with the dual operator d defined as follows we obtain a *GPDM*:

$$\begin{array}{ll}
\left(w(F_1 \times F_2^-)\right)^d = F_1^* \times F_2^* & \text{if } w \notin F_1 \times F_2 \prec K \\
\left(w(F_1 \times F_2^0)\right)^d = F_1^* \times (F_2^* + < h_2 >) & \text{if } F_2 \text{ is an unbounded face of } K_2 \\
(F_1 \times F_2^+)^d = F_1^* \times (F_2^* + < h_2 >) & \text{if } F_2 \text{ is an unbounded face of } K_2 \\
(F_1 \times F_2^-)^d = \emptyset & \text{if } w \notin F_1 \times F_2 \prec K \\
(F_1 \times F_2^0)^d = \emptyset & \text{if } F_2 \text{ is an unbounded face of } K_2 \\
(\{w\})^d = \emptyset\,. &
\end{array}$$

The collection $\mathcal{M}$ of cells, each cell being the Cartesian product of a primal cell and its dual, is clearly a subdivided $(n_1 + n_2 + 1)$-manifold with boundary $\partial\mathcal{M}$ containing $\{(w_1, w_2)\} \times \left(R^{n_1} \times R^{n_2} \setminus \{(w_1, w_2)\}^*\right)$ and $(F_1 \times F_2) \times (F_1^* \times F_2^*)$ for all faces F_1 of K_1 and F_2 of K_2. Therefore in case that the starting point $w = (w_1, w_2)$ is not a stationary point, the point $(x_1, x_2, y_1, y_2) = (w_1, w_2, -f_1(w_1, w_2), -f_2(w_1, w_2))$ lies in the boundary $|\partial\mathcal{M}|$ of $|\mathcal{M}|$ and under the regular value assumption there is a path leading from it to either a stationary point or to infinity. Thus in exactly the same way as in the preceding sections the problem is now reduced to tracing the path S of solutions to the system

$$(f_1(x_1,x_2), f_2(x_1,x_2)) + (y_1,y_2) = 0,\ (x_1,x_2,y_1,y_2) \in |\mathcal{M}|.$$

The remarkable feature of this path is shown in the following lemma, where

$$S_x = \{\ (x_1,x_2) \mid (x_1,x_2,y_1,y_2) \in S \quad \text{for some} \quad (y_1,y_2) \in R^{n_1+n_2}\ \}.$$

<u>LEMMA 6.1</u>. If $(\tilde{x}_1,\tilde{x}_2) \in S_x \cap H^+$, then $\tilde{x}_1$ is a solution of the stationary point problem $(f_1(x_1,\tilde{x}_2),\ K_1)$, i.e., $\tilde{x}_1^t f_1(\tilde{x}_1,\tilde{x}_2) \le x_1^t f_1(\tilde{x}_1,\tilde{x}_2)$ for all $x_1 \in K_1$.

<u>Proof.</u> Since $(\tilde{x}_1,\tilde{x}_2) \in H^+$, it is in $F_1 \times F_2^+$ for some face F_1 of K_1 and some unbounded face F_2 of K_2. By the construction of the $GPDM$

$$(-f_1(\tilde{x}_1,\tilde{x}_2), -f_2(\tilde{x}_1,\tilde{x}_2)) \in F_1^* \times (F_2^* + < h_2 >).$$

This means that $\tilde{x}_1$ is a solution of the problem $(f_1(x_1,\tilde{x}_2),\ K_1)$. //

<u>LEMMA 6.2</u>. Let $(\tilde{x}_1,\tilde{x}_2,\tilde{y}_1,\tilde{y}_2)$ be a point of S. Suppose that $(\tilde{x}_1,\tilde{x}_2)$ is not a stationary point and $\tilde{x}_2$ lies in K_2^+. Then

$$r_2^t \tilde{y}_2 > 0$$

for any nonzero vector r_2 in the cone C_2 such that $(a_{2i})^t r_2 = 0$ whenever $(a_{2i})^t \tilde{x}_2 = b_{2i}$, where a_{2i} is the i^{th} column of A_2 and b_{2i} is the i^{th} component of b_2.

<u>Proof.</u> Let B_2 be the submatrix of A_2 consisting of the columns a_{2i} such that $(a_{2i})^t \tilde{x}_2 = b_{2i}$. Then $\tilde{y}_2 = B_2\mu + \alpha h_2$ for some vector $\mu \ge 0$ and number $\alpha \ge 0$. Since $(\tilde{x}_1,\tilde{x}_2)$ is in H^+ and is not a stationary point, we have $\alpha > 0$ by Lemma 6.1. Therefore

$$r_2^t \tilde{y}_2 = r_2^t(B_2\mu + \alpha h_2) = \alpha h_2^t r_2 > 0. \quad //$$

<u>CONDITION 6.3</u>. There is a set $U_2 \subset R^{n_2}$ such that $U_2 \cap K_2$ is bounded and for each point $\tilde{x}_2 \in K_2 \setminus U_2$ one of the following conditions holds:

(a) $(f_1(x_1,\tilde{x}_2),\ K_1)$ has no stationary points,

(b) for each point $x_1 \in K_1$ there is a nonzero vector $\tilde{r}_2$ in $C_2 \cap \{\ r_2 \in R^{n_2} \mid (a_{2i})^t r_2 = 0 \ \text{ if } \ (a_{2i})^t \tilde{x}_2 = b_{2i}\ \}$ such that $\tilde{r}_2^t f_2(x_1,\tilde{x}_2) \ge 0$. //

THEOREM 6.4. Under Condition 6.3 the path S will not diverge.

Proof. Suppose the contrary. Then there is a point $(\tilde{x}_1, \tilde{x}_2) \in S_x \cap H^+$ such that $\tilde{x}_2 \notin U_2$. By Lemma 6.1, $\tilde{x}_1$ is a solution of $\left(f_1(x_1, \tilde{x}_2),\ K_1\right)$. Therefore condition (b) must be satisfied at this point, so that for some nonzero vector $\tilde{r}_2$ in $C_2 \cap \{\, r_2 \in R^{n_2} \mid (a_{2i})^t r_2 = 0 \quad \text{if} \quad (a_{2i})^t \tilde{x}_2 = b_{2i} \,\}$ we must have

$$\tilde{r}_2^t f_2(\tilde{x}_1, \tilde{x}_2) \geq 0.$$

On the other hand we have seen in Lemma 6.2 that

$$\tilde{r}_2^t f_2(\tilde{x}_1, \tilde{x}_2) = \tilde{r}_2^t(-\tilde{y}_2) < 0.$$

This is a contradiction. //

7. Linear Stationary Point Problems

In this section we consider a special but important case where the function f from K to R^n is an affine function, i.e., $f(x) = Qx + q$, where Q is an $n \times n$ matrix and q is an n-vector. For simplicity of notations we confine ourselves to the linear stationary point problem defined on a polyhedron instead of the product of a polytope and a polyhedron. Like for comlementary pivoting algorithms for solving a linear complementarity problem we show that if the matrix Q is copositive plus on the polyhedral cone C and the problem has a stationary point, the path does not go to infinity and consequently leads to one of the solutions.

DEFINITION 7.1. The matrix Q is copositive plus on C if

(a) $r^t Q r \geq 0$ for any $r \in C$,

(b) $(Q + Q)^t r = 0$ if $r \in C$ and $r^t Q r = 0$.

LEMMA 7.2. There exists no point $x \in K$ such that $Qx + q = -A\mu$ for some vector $\mu \geq 0$ if and only if there is a $(v, u) \in R^n \times R^m$ such that $v \in C$, $Q^t v = Au$, $b^t u + q^t v < 0$ and $u \geq 0$.

Proof. There exists no point x in K satisfying $Qx + q = -A\mu$ for $\mu \geq 0$ if and only if the system

$$\begin{aligned} & A^t(x_1 - x_2) \leq b \\ & Q(x_1 - x_2) + q = -A\mu \\ & x_1, x_2, \mu \geq 0 \end{aligned} \tag{7.1}$$

is not solvable. By Farkas' Alternative Theorem, we have an equivalent statement to (7.1): the following system

$$\begin{aligned} &Q^t v - Au = 0 \\ &A^t v \le 0 \\ &u \ge 0 \\ &b^t u + q^t v < 0 \end{aligned} \tag{7.2}$$

is solvable. This means the existence of a point v in C such that $Q^t v = Au$ and $b^t u + q^t v < 0$ for some $u \ge 0$. //

<u>LEMMA 7.3</u>. Let Q be copositive plus on C. If the path S is unbounded and does not contain a point which provides a stationary point, then the stationary point problem has no solutions.

<u>Proof.</u> Suppose S is unbounded, then there are $(x, y) \in S$ and $(\bar{x}, \bar{y}) \neq 0$ such that $(x, y) + \beta(\bar{x}, \bar{y}) \in S$ for any $\beta \ge 0$. Then

$$\bar{y} + Q\bar{x} = 0. \tag{7.3}$$

Moreover as β increases, $(x, y) + \beta(\bar{x}, \bar{y})$ will be entirely contained in a cell $F^+ \times (F^* + < h >)$ for some face $F \preceq K$. Here note that $\bar{x} \neq 0$ because the contrary would yield $(\bar{x}, \bar{y}) = 0$. Then we have

$$\begin{aligned} &x \in F^+ \\ &y = y\prime + \lambda h \text{ for some } y\prime \in F^* \text{ and some } \lambda \ge 0, \end{aligned}$$

and

$$\begin{aligned} &\bar{x} \in C \cap \{\, r \in R \mid (a_i)^t r = 0 \text{ whenever } (a_i)^t x = b_i \,\} \\ &\bar{y} = \bar{y}\prime + \mu h \text{ for some } \bar{y}\prime \in F^* \text{ and some } \mu \ge 0. \end{aligned}$$

Therefore we have

$$\bar{x}^t Q \bar{x} = \bar{x}^t(-\bar{y}) = \bar{x}^t(-\bar{y}\prime - \mu h) = -\mu \bar{x}^t h.$$

Suppose $\mu > 0$. By the choice of h and since $\bar{x} \in C$, we have $\mu \bar{x}^t h > 0$, which contradicts that Q is copositive plus on C. Therefore $\mu = 0$ and $\bar{x}^t Q \bar{x} = 0$. If $\lambda = 0$, then $y = y\prime \in F^*$. This means that the point x is a stationary point. Since we have assumed that S does not contain such a point, we see that $\lambda > 0$. Since $\bar{x}^t Q\bar{x} = 0$ implies $(Q + Q^t)\bar{x} = 0$, we have

$$Q^t\bar{x} = -Q\bar{x} = \bar{y} = \bar{y}\prime + \mu h = \bar{y}\prime \in F^*.$$

In other words there is some vector u satisfying

$$\begin{aligned} &Q^t\bar{x} = Au \\ &u_i \geq 0 \quad \text{for } i \in I(F) \\ &u_i = 0 \quad \text{for } i \notin I(F). \end{aligned} \tag{7.4}$$

We also have

$$\bar{x}^t(-y) = \bar{x}^t(Qx + q) = x^t Q^t \bar{x} + q^t \bar{x} = x^t(-Q\bar{x}) + q^t\bar{x} = x^t\bar{y} + q^t\bar{x}. \tag{7.5}$$

On the other hand, since $\bar{x} \in C$ and $y\prime \in F^*$,

$$\bar{x}^t(-y) = \bar{x}^t(-y\prime - \lambda h) = -\bar{x}^t y\prime - \lambda\bar{x}^t h = -\lambda\bar{x}^t h < 0. \tag{7.6}$$

From (7.5) and (7.6) we have $x^t\bar{y} + q^t\bar{x} < 0$. Since $x \in F^+$, we also have that $A^t x + s = b$ for some slack variable vector s satisfying

$$\begin{aligned} &s_i \geq 0 \quad \text{for } i \notin I(F) \\ &s_i = 0 \quad \text{for } i \in I(F). \end{aligned}$$

Then

$$\begin{aligned} b^t u + q^t\bar{x} &< b^t u - x^t\bar{y} = (A^t x + s)^t u - x^t(-Q\bar{x}) \\ &= x^t Au + s^t u - x^t(Q^t\bar{x}) = x^t(Au - Q^t\bar{x}) + s^t u = 0. \end{aligned} \tag{7.7}$$

From (7.4) and (7.7) and Lemma 7.2, we conclude that there are no stationary points. //

The algorithm for tracing the piecewise linear path S, being linear on any cell of $\mathcal{M}$, is quite similar to that proposed in Yamamoto [Y1] for solving linear stationary point problems on polytopes. We will only give an outline here. Suppose we are at a point (x, y) on the path, i.e.,

$$Qx + q + y = 0, \ (x, y) \in X \times X^d, \tag{7.8}$$

for some cell $X \times X^d$ of $\mathcal{M}$. By the decomposition theorem of a polyhedron each point of a polyhedron is a sum of two points: a convex combination of vertices of the polyhedron and a nonnegative combination of directions of extreme rays. Let U and R be the sets of vertices and extreme rays of X, respectively. Then a point $x \in X$ is written as

$$\begin{aligned} & x = \sum_{u \in U} \lambda_u u + \sum_{r \in R} \alpha_r r \\ & \sum_{u \in U} \lambda_u = 1 \\ & \lambda_u \geq 0, \ \alpha_r \geq 0. \end{aligned}$$

On the other hand X^d is the cone generated by coefficient vectors a_i of binding constraints of the face corresponding to X. Then a point $y \in X^d$ is written as

$$y = \sum \mu_i a_i, \ \mu_i \geq 0.$$

Therefore (7.8) has a solution if and only if the system

$$\begin{aligned} & \sum \lambda_u \begin{bmatrix} Qu \\ 1 \end{bmatrix} + \sum \alpha_r \begin{bmatrix} Qr \\ 0 \end{bmatrix} + \sum \mu_i \begin{bmatrix} a_i \\ 0 \end{bmatrix} = \begin{bmatrix} -q \\ 1 \end{bmatrix} \\ & \lambda_u \geq 0, \ \mu_i \geq 0, \ \mu_i \geq 0 \end{aligned} \tag{7.9}$$

has a solution (λ, α, μ). It should be noted here that a vertex of X is either the starting point w or a vertex of some face of K^- corresponding to X and that a ray of X is also a ray of some face of K. More precisely,

$$\begin{array}{lll} U = \{w\} \cup \{ \text{ vertices of } F^- \}, & R = \emptyset & \text{when } X = wF^- \\ U = \{w\} \cup \{ \text{ vertices of } F^0 \}, & R = \emptyset & \text{when } X = wF^0 \\ U = \{ \text{ vertices of } F^0 \}, & R = \{ \text{ rays of } F \} & \text{when } X = F^+ . \end{array}$$

In every case a vertex or a ray can be generated in need if we keep the index set of binding constraints, including $H^0 = \{ x \in R^n \mid h^t x = h_0 \}$, determining the face F in storage.

Suppose we are at an end point of the line segment or half line of the path within $X \times X^d$. Since the path is linear within $X \times X^d$ an appropriate choice of the objective function $c_x x + c_y y$ makes the current end point the unique maximal solution of the linear program:

$$\begin{array}{ll}\text{max.} & c_x x + c_y y,\\ \text{s.t.} & x = \sum \lambda_u u + \sum \alpha_r r\\ & y = \sum \mu_i a_i\\ & \sum \lambda_u \begin{bmatrix} Qu \\ 1 \end{bmatrix} + \sum \alpha_r \begin{bmatrix} Qr \\ 0 \end{bmatrix} + \sum \mu_i \begin{bmatrix} a_i \\ 0 \end{bmatrix} = \begin{bmatrix} -q \\ 1 \end{bmatrix}\\ & \lambda_u \geq 0,\ \alpha_r \geq 0,\ \mu_i \geq 0.\end{array}$$

In fact, the outward normal vector at any point (x, y) to $X \times X^d$ may serve as (c_x, c_y). Then the other end point, when the path within $X \times X^d$ is a line segment, or the diverging direction, when it is a half line, can be found by solving the following linear minimization program:

$$\begin{array}{ll}\text{min.} & c_x x + c_y y,\\ \text{s.t.} & x = \sum \lambda_u u + \sum \alpha_r r\\ & y = \sum \mu_i a_i\\ & \sum \lambda_u \begin{bmatrix} Qu \\ 1 \end{bmatrix} + \sum \alpha_r \begin{bmatrix} Qr \\ 0 \end{bmatrix} + \sum \mu_i \begin{bmatrix} a_i \\ 0 \end{bmatrix} = \begin{bmatrix} -q \\ 1 \end{bmatrix}\\ & \lambda_u \geq 0,\ \alpha_r \geq 0,\ \mu_i \geq 0.\end{array}$$

From this we see that this problem is a typical application of the Dantzig-Wolfe decomposition principle for large structured linear programs. By solving a sequence of these problems we can trace the path and finally after a finite number of iterations we meet with an end point of the path or find that the path goes to infinity.

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