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The Open-Loop Discounted Linear Quadratic Differential Game for Regular Higher Order Index Descriptor Systems

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Abstract: In this paper we consider the discounted linear quadratic differential game for descriptor systems that have an index larger than one. We derive both necessary and sufficient conditions for existence of an open-loop Nash (OLN) equilibrium. In a small macro-economic stabilization game we illustrate that the corresponding optimal response is generically cyclic.

Keywords: linear quadratic differential games; open-loop information structure, descriptor systems.

Jel-codes: C61, C72, C73.

1 Introduction

Dynamic game theory brings together three features that are key to many situations in economy, ecology, and elsewhere: optimizing behavior, presence of multiple agents, and enduring consequences of decisions. For that reason this framework is often used to analyze various policy problems in these areas (see e.g. [2], [9] and [19]).
In applications one often encounters however systems described by a set of ordinary differential equations subject to some algebraic constraints. These systems are known as descriptor systems.
Differential games for descriptor systems were, e.g., already studied by [20] and [21]. More recently [5] and [6] studied for index one systems the linear quadratic differential game. In [5] the open-loop information case is studied, whereas in [6] the case that players use linear state feedback controls is dealt with.
In this paper we take a first step to solve differential games for descriptor systems which have an index that is higher than one. We consider the problem of two players who like to optimize their performance given by a usual quadratic cost function depending both on the state and control variables in which both variables are discounted. The underlying system is described by a set of differential and algebraic equations and we assume that the system is regular, that is, consistent initial states yield a unique state trajectory. The index of the system is $k > 1$. It is well-known that in that case the state trajectory includes $(k-1)^{th}$ order derivatives of the applied input. For that reason we consider the $(k-1)^{th}$ order derivative of the applied input as our control instrument.
We assume that the information structure of the game is of the open-loop type. That is, both

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players only know the initial state and structure of the system, and the set of admissible inputs, $\mathcal{U}$, are functions of time which are $k-1$ times differentiable.

Linear quadratic control problems play an important role in applications. Particularly in economics usually the cost of the players is discounted to stress the short-term cost. Since the derivatives of the input function naturally appear in the state trajectory of descriptor systems it seems natural to include these terms in the cost function of the players too. This motivates the problem setting that will be formulated in the next section in detail. The linear quadratic control problem subject to descriptor systems has been considered in the literature by various authors. The approach taken here, to introduce additional states and to consider the $(k-1)^{th}$ derivative of the input function as the control instrument, was considered e.g. in [18]. More references on the regulator problem for descriptor systems can be found, e.g., in [16], [12], [22] or [6] . Like many approaches for solving the linear quadratic control problem for descriptor systems, in this paper we solve the corresponding game problem using the Weierstrass canonical form of the pencil $\lambda E - A$ (see (1)). Using the corresponding state transformation, it is possible to reduce the problem to a reduced order standard game problem. Using the theory for affine linear quadratic differential games as documented in [3] and [4] it is possible then to solve the game for both a finite and infinite planning horizon.

The outline of the paper is as follows. The next section formalizes the problem statement and summarizes some basic properties about descriptor systems. In section three we present the main results for the finite planning horizon, whereas section four contains those about the infinite planning horizon. In section five we illustrate some of the theory by a simple example from macro-economics. The example demonstrates in particular that cyclic behaviour may arise within this framework. Finally section six concludes.

2 Preliminaries

Consider the differential algebraic equation

$$\bar{E}\dot{x}(t) = \bar{A}x(t) + f(t),\ x(0) = x_0, \tag{DAE}$$

and the associated matrix pencil

$$\lambda\bar{E} - \bar{A}. \tag{1}$$

From, e.g., [1] we recall the following results. System (DAE) and (1) are said to be *regular* if the characteristic polynomial $\det(\lambda\bar{E} - \bar{A})$ is not identically zero. If the pencil (1) is not regular, then the system (DAE) is under-determined in the sense that consistent initial conditions do not uniquely determine solutions (see [7]). If the pencil (1) is regular, then the roots of the characteristic polynomial are the finite eigenvalues of the pencil. If $\bar{E}$ is singular, the pencil is said to have infinite eigenvalues which may be identified as the zero eigenvalues of the inverse pencil $\bar{E} - \lambda\bar{A}$. From [7] we recall the so-called Weierstrass canonical form.

Theorem 2.1 *If (1) is regular, then there exist nonsingular matrices X and Y such that*

$$Y^T\bar{E}X = \begin{bmatrix} I_n & 0 \\ 0 & N \end{bmatrix} \text{ and } Y^T\bar{A}X = \begin{bmatrix} J & 0 \\ 0 & I_r \end{bmatrix}, \tag{2}$$

where $J \in \mathbb{R}^{n\times n}$ is a matrix in Jordan form whose elements are the finite eigenvalues, $I_k \in \mathbb{R}^{k\times k}$ is the identity matrix and $N \in \mathbb{R}^{k\times k}$ is a nilpotent matrix also in Jordan form. J and N are unique up to permutation of Jordan blocks. □

If (1) is regular the solutions of (DAE) take the form

$$x(t) = X_1 x_1(t) + X_2 x_2(t) \tag{3}$$

where with $X = [X_1\ X_2]$, $Y = [Y_1^T\ Y_2^T]$, $X_1,\ Y_1^T \in \mathbb{R}^{(n+r)\times n}$, $X_2,\ Y_2^T \in \mathbb{R}^{(n+r)\times r}$ and

$$\begin{aligned} x_1(t) &= e^{Jt}x_1(0) + \int_0^t e^{J(t-s)}Y_1 f(s)ds; \qquad (4)\\ & \quad x_1(0) = [I_n\ 0]X^{-1}x_0 \\ x_2(t) &= -\sum_{i=0}^{k-1} N^i Y_2 \frac{d^i}{dt^i} f(t), \qquad (5) \end{aligned}$$

under the consistency condition that $x_2(0) = 0$ (see e.g. [10]). Here k is the degree of nilpotency of N. That is the integer k for which $N^k = 0$ and $N^{k-1} \neq 0$. The *index* of the pencil (1) and of the descriptor system (DAE) is the degree k of nilpotency of N. If E is nonsingular, we define the index to be zero.

From the above formulae it is obvious that the solution $x(t)$ will not contain derivatives of the function f if and only if $k \leq 1$. In that case the solution $x(t)$ is called *impulse free*. In general, the solution $x(t)$ involves derivatives of order $k-1$ of the forcing function f if (DAE) has index k. To verify whether (DAE) has an index of at most one see, e.g., [11].

In this paper we assume that the dynamics of the game is described by

$$\bar{E}\dot{x}(t) = \bar{A}x(t) + \bar{B}_1 u_1(t) + \bar{B}_2 u_2(t),\ x(0) = x_0, t > 0. \tag{6}$$

where $\bar{E},\ \bar{A} \in \mathbb{R}^{(n+r)\times(n+r)}$, $\text{rank}(\bar{E}) = n$, $\bar{B}_i \in \mathbb{R}^{(n+r)\times m_i}$, $u_i \in \mathbb{R}^{m_i}$ is the input by which player i can manipulate the system and x_0 is assumed to be a consistent initial state (so $x_0 = x(0_+)$). That is, x_0 is such that the system $\bar{E}\dot{x}(t) = \bar{A}x(t),\ x(0) = x_0$, has a unique solution for $t > 0$. Assuming that system (6) has index $k > 1$ let $u_j^{(i)} := \frac{d^i u_j(t)}{dt^i}, i = 1, \cdots, k-2$, $v_j(t) := \frac{d^{(k-1)}u_j(t)}{dt^{(k-1)}}$ and

$$x^{e^T}(t) := [x^T(t)\ u_1^T(t)\ \cdots u_1^{(k-2)^T}\ u_2^T(t)\ \cdots\ u_2^{(k-2)^T}\ v_1^T\ v_2^T]. \tag{7}$$

We consider then the next quadratic cost functional J_i for player i:

$$\int_0^T e^{-\theta t}\{[x^{e^T}(t)\bar{M}_i x^e(t)\}dt + e^{-\theta T}x^{e^T}(T)\bar{Q}_{iT}x^e(T). \tag{8}$$

Here all matrices are constant in time, both $\bar{M}_i$ and $\bar{Q}_{iT}$ are symmetric and $\theta > 0$ is the discount factor.

The problem addressed in this paper is to find the OLN equilibria for the game (6,8) as defined below.

Definition 2.2 *Assume (6) is regular and has index $k > 1$. Let x_0 be a consistent initial state and $u^i(0)$, $i = 1, \cdots, k-2$ be given. Furthermore, let $\mathcal{U}$ denote the set of functions which are $k-1$ times differentiable with $u^{(k-1)}$ piecewise continuous. Then $(u_1^*, u_2^*) \in \mathcal{U}$ is an open-loop Nash (OLN) equilibrium if for every (u_1, u_2^*) and $(u_1^*, u_2) \in \mathcal{U}$, $J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*)$ and $J_2(u_1^*, u_2^*) \leq J_1(u_1^*, u_2)$.* □

With

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} := X^{-1}x, \text{ with } x_1 \in \mathbb{R}^n \text{ and } x_2 \in \mathbb{R}^r \tag{9}$$

we have that for any consistent initial state the system dynamics (6) can be rewritten as (see (3-5)):

$$\begin{aligned} \dot{x}_1(t) &= Jx_1(t) + Y_1\bar{B}_1u_1(t) + Y_1\bar{B}_2u_2(t),\ x_1(0) = [I\ 0]X^{-1}x_0 & (10)\\ x_2(t) &= -\sum_{i=0}^{k-1} N^iY_2(\bar{B}_1u_1^{(i)}(t) + \bar{B}_2u_2^{(i)}(t)) & (11)\\ &= -\sum_{i=0}^{k-2} N^iY_2(\bar{B}_1u_1^{(i)}(t) + \bar{B}_2u_2^{(i)}(t)) - N^{k-1}Y_2\bar{B}_1v_1(t) - N^{k-1}Y_2\bar{B}_2v_2(t). & (12) \end{aligned}$$

Next introduce the discounted state and control vector

$$x_z^T := e^{-\frac{1}{2}\theta t}[x_1^T(t)\ u_1^T(t)\ \cdots u_1^{(k-2)^T}(t)\ u_2^T(t)\ \cdots\ u_2^{(k-2)^T}(t)] \text{ and } w_i(t) := e^{-\frac{1}{2}\theta t}v_i^T(t) \tag{13}$$

$$\text{together with } z^T(t) := [x_z^T(t)\ w_1^T(t)\ w_2^T(t)]. \tag{14}$$

Then, with $m := m_1 + m_2$,

$$\begin{aligned} P_i &:= [Y_2\bar{B}_i\ NY_2\bar{B}_i\ \cdots\ N^{k-2}Y_2\bar{B}_i],\ i = 1,2, \text{ and} & (15)\\ Z_1 &:= [I_n\ 0_{n\times km}];\ Z_2 := -[0_{r\times n}\ P_1\ P_2\ N^{k-1}Y_2\bar{B}_1\ N^{k-1}Y_2\bar{B}_2] & (16) \end{aligned}$$

we have that

$$e^{-\frac{1}{2}\theta t}x(t) = X\begin{bmatrix} e^{-\frac{1}{2}\theta t}x_1(t) \\ e^{-\frac{1}{2}\theta t}x_2(t) \end{bmatrix} = X\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} z(t) =: L_1z(t). \tag{17}$$

Furthermore, with $E_2 := [0_{km\times n}\ I_{km}]$, we have that

$$e^{-\frac{1}{2}\theta t}x^e(t) = \begin{bmatrix} L_1 \\ E_2 \end{bmatrix} z(t) =: Lz(t). \tag{18}$$

Next, let $A_i := [Y_1\bar{B}_i\ 0_{n\times(k-2)m_i}],\ i = 1,2$; and with $I \in \mathbb{R}^{m_i\times m_i}$

$$D_i := \begin{bmatrix} -\frac{1}{2}\theta I & I & 0 & \cdots & 0 \\ 0 & & & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & & I \\ 0 & & \cdots & 0 & -\frac{1}{2}\theta I \end{bmatrix} \in \mathbb{R}^{(k-1)m_i\times(k-1)m_i} \text{ and } B_{z_i} := \begin{bmatrix} 0 \\ \\ \vdots \\ 0 \\ I \end{bmatrix} \in \mathbb{R}^{(k-1)m_i\times m_i}.$$

Using this, it is obvious then that the game (6,8) has a set of OLN equilibrium actions $(u_1(.), u_2(.))$ if and only if $(v_1(.), v_2(.))$ are OLN equilibrium actions for the game defined by

$$\begin{aligned} \dot{x}_z(t) &= \begin{bmatrix} J - \frac{1}{2}\theta I & A_1 & A_2 \\ 0 & D_1 & 0 \\ 0 & 0 & D_2 \end{bmatrix} x_z(t) + \begin{bmatrix} 0 \\ B_{z_1} \\ 0 \end{bmatrix} w_1(t) + \begin{bmatrix} 0 \\ 0 \\ B_{z_2} \end{bmatrix} w_2(t) \\ &=: Ax_z(t) + B_1w_1(t) + B_2w_2(t), & (19) \end{aligned}$$

with $x_z^T(0) := x_{z0} = [(([I\ 0]X^{-1}x_0)^T\ u_1^T(0)\ \cdots u_1^{(k-2)^T}(0)\ u_2^T(0)\ \cdots\ u_2^{(k-2)^T}(0)]$ is such that (10,11) holds and

$$J_i \;=\; \int_0^T \{[z^T(t)L^T\bar{M}_iLz(t)\}dt + z^T(T)L^T\bar{Q}_{iT}Lz(T). \tag{20}$$

To avoid the inclusion of controls in the scrap value, we make the standard assumption (see also [16]) that

$$L^T\bar{Q}_{iT}L = \begin{bmatrix} Q_{iT} & 0 \\ 0 & 0 \end{bmatrix},\ i = 1,2, \text{where } Q_{iT} \in \mathbb{R}^{n+(k-1)m\times n+(k-1)m}.$$

Moreover, let

$$L^T\bar{M}_iL =: M_i =: \begin{bmatrix} Q_i & V_i & W_i \\ V_i^T & R_{1i} & N_i \\ W_i^T & N_i^T & R_{2i} \end{bmatrix},\ \text{where } Q_i \in \mathbb{R}^{n+(k-1)m\times n+(k-1)m},\ R_{ji} \in \mathbb{R}^{m_j\times m_j}. \tag{21}$$

Then, (20) can be rewritten as

$$J_i = \int_0^T \{[z^T(t)M_iz(t)\}dt + x_z^T(T)Q_{iT}x_z(T). \tag{22}$$

3 The finite planning horizon

In this section we consider the game (6,8) under the assumption that T is finite. As shown in the previous section the open-loop Nash equilibria are found by determining the open-loop Nash equilibria of the game defined by (19) and (22). Assuming that $R_{ii} > 0,\ i = 1,2$ and matrix G (see the Appendix for the introduced notation, in particular for matrix $\tilde{M}$ (35)) is invertible the solution for the last-mentioned game is well-known. From e.g. [3] we recall the next result.

Theorem 3.1 *Assume that the two Riccati differential equations*

$$\begin{aligned} \dot{K}_1(t) &= -A^TK_1(t) - K_1(t)A + (K_1(t)B_1 + V_1)R_{11}^{-1}(B_1^TK_1(t) + V_1^T) - Q_1,\ K_1(T) = Q_{1T}, \quad (23) \\ \dot{K}_2(t) &= -A^TK_2(t) - K_2(t)A + (K_2(t)B_2 + W_2)R_{22}^{-1}(B_2^TK_2(t) + W_2^T) - Q_2,\ K_2(T) = Q_{2T}, \quad (24) \end{aligned}$$

have a symmetric solution $K_i(.)$ on $[0,T],\ i = 1,2.$

1. *Then (19,22) has an OLN for every initial state if and only if matrix*

$$\tilde{H}(T) = [I\ 0\ 0]e^{-\tilde{M}T}\begin{bmatrix} I \\ Q_{1T} \\ Q_{2T} \end{bmatrix}$$

is invertible.

2. *Assume that the nonsymmetric Riccati differential equation*

$$\dot{P}(t) = -\tilde{A}_2^T P(t) - P(t)\tilde{A} + P(t)BG^{-1}\tilde{B}^T P(t) - \tilde{Q};\ P^T(T) = [Q_{1T}^T,\ Q_{2T}^T] \tag{25}$$

has a solution P on $[0,T]$.
Then (19,22) has a unique OLN for every initial state. Moreover, the equilibrium actions are

$$\begin{bmatrix} w_1^*(t) \\ w_2^*(t) \end{bmatrix} = -G^{-1}(H + \tilde{B}^T P(t))\tilde{\Phi}(t,0)x_{z0},$$

where $\tilde{\Phi}(t,0)$ is the solution of the transition equation

$$\dot{\tilde{\Phi}}(t,0) = (A - BG^{-1}(H + \tilde{B}^T P(t)))\tilde{\Phi}(t,0);\ \tilde{\Phi}(0,0) = I.$$

□

Assumptions (23,25) imply that for both players the optimal control problem that arises if the action of his opponent is known is solvable. In case item 1 in the above theorem holds, but item 2 does not apply, then the corresponding equilibrium actions can be determined by solving a linear two-point boundary value problem. The reader should then proceed similarly as in e.g. [5, Theorem 5].

Corollary 3.2 *Assume that the two Riccati differential equations (23) have a symmetric solution $K_i(t)$ on $[0,T]$ and the nonsymmetric Riccati differential equation (25) has a solution $P(t)$ on $[0,T]$. Then (6,8) has a unique OLN for every initial consistent state x_0 and $u^{(i)}(0)$, $i = 1, \cdots, k-2$. Moreover, the equilibrium actions are*

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = e^{\frac{1}{2}\theta t} \begin{bmatrix} 0_{m_1 \times n} & I_{m_1} & 0_{m_1 \times (k-2)m + m_2} \\ 0_{m_2 \times n + (k-2)m_1} & I_{m_2} & 0_{m_2 \times (k-2)m_2} \end{bmatrix} x_z(t),$$

where $x_z(t)$ solves

$$\dot{x}_z(t) = (A - BG^{-1}(H + \tilde{B}^T P(t))x_z(t),\ x_z(0) = x_{z0}.$$

Moreover, the corresponding equilibrium state trajectory equals (see (17))

$$x^*(t) = L_1[e^{\frac{1}{2}\theta t} x_z^T(t)\ v_1^T(t)\ v_2^T(t)]^T.$$

□

4 The infinite planning horizon

In this section we assume that the cost functional player $i = 1, 2$, likes to minimize is:

$$\lim_{T \to \infty} J_i(x_0, u_1(0), \cdots, u_1^{(k-2)}(0), u_2(0), \cdots, u_2^{(k-2)}(0), u_1, u_2, T), \tag{26}$$

where

$$J_i(x_0, u_1(0), \cdots, u_1^{(k-2)}(0), u_2(0), \cdots, u_2^{(k-2)}(0), u_1, u_2, T) = \int_0^T e^{-\theta t}\{x^{e^T}(t)\bar{M}_i x^e(t)\}dt.$$

subject to (6).
We assume that the matrix pairs $(\bar{E}, \bar{A}, \bar{B}_i)$, $i = 1, 2$, are finite dynamics stabilizable. That is [1] $\text{rank}([\lambda\bar{E} - \bar{A}, \bar{B}_i]) = n + r, \forall\lambda \in \mathbb{C}_0^+$.
Following the analysis of section 2 it can be easily shown that the game (6,26) has an OLN if and only if the game defined by

$$\lim_{T\to\infty} \int_0^T \{z^T(t) M_i z(t)\} dt \tag{27}$$

subject to (19) has an OLN. Furthermore, $(\bar{E}, \bar{A}, \bar{B}_i)$ is finite dynamics stabilizable if and only if (A, B_i) is stabilizable. So under this assumption, in principle, each player is capable to stabilize the system (6) on his own. This property is a prerequisite to derive the main results below.
We assume that the players choose control functions belonging to the set of functions of time which are $k-1$ times differentiable and which are such that the state of the closed-loop system converges to zero, $\mathcal{U}_s$. Notice that the assumption that the players use simultaneously stabilizing controls implies that stabilization of the system is a common objective of both players (see e.g. [3] for a discussion).
In the rest of the paper the symmetric algebraic Riccati equations

$$\begin{aligned} 0 &= -A^T K_1 - K_1 A + (K_1 B_1 + V_1) R_{11}^{-1} (B_1^T K_1 + V_1^T) - Q_1, \\ 0 &= -A^T K_2 - K_2 A + (K_2 B_2 + W_2) R_{22}^{-1} (B_2^T K_2 + W_2^T) - Q_2. \end{aligned} \tag{28}$$

and the asymmetric algebraic Riccati equation

$$0 = \widetilde{A}_2^T P + P\widetilde{A} - PBG^{-1}\widetilde{B}^T P + \widetilde{Q} \tag{29}$$

or, equivalently,

$$0 = A_2^T P + PJ - (PB + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}) G^{-1} (\widetilde{B}^T P + H) + Q$$

play a crucial role. Let $\sigma(X)$ denote the spectrum of a matrix X.

Definition 4.1 A solution $P \in \mathbb{R}^{2n\times n}$ of the algebraic Riccati equation (29) is called

a. *stabilizing*, if $\sigma(\tilde{A} - BG^{-1}\tilde{B}^T P) \subset \mathbb{C}^-$;

b. *left-right stabilizing*[2] (LRS) if

i. it is a stabilizing solution, and
ii. $\sigma(-\tilde{A}_2^T + PBG^{-1}\tilde{B}^T) \subset \mathbb{C}_0^+$; □

The next lemma summarizes the relationship between the LRS solution of (29) and the stable graph subspace (or disconjugate subspace, see [8]) of matrix $\tilde{M}$ introduced in (35). A proof of it can be found in [4] and [14]. One way to calculate the (left-right) stabilizing solutions of (29) is by determining the invariant subspaces of matrix M. Details on this issue can be found, e.g., in [3].

[1] $\mathbb{C}^- = \{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) < 0\}$; $\mathbb{C}_0^+ = \{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) \geq 0\}$.
[2] In [3] such a solution is called strongly stabilizing.

Lemma 4.2

1. *The algebraic Riccati equation (29) has a LRS solution P if and only if matrix $\tilde{M}$ has an n-dimensional stable graph subspace and $\tilde{M}$ has $2n$ eigenvalues (counting algebraic multiplicities) in $\mathbb{C}_0^+$.*

2. *If the algebraic Riccati equation (29) has a LRS solution, then it is unique.* □

From [4] we recall the following two main results.

Theorem 4.3 *Assume that*

1. *the set of coupled algebraic Riccati equations (29) has a set of stabilizing solutions P_i, $i = 1, 2$; and*

2. *the two algebraic Riccati equations (28) have a stabilizing solution $K_i(.)$, $i = 1, 2$.*

Then the linear quadratic differential game (19,27) has an OLN for every initial state.
Moreover, with $F := -G^{-1}(H + \tilde{B}^T P)$, one set of equilibrium actions is given by:

$$\begin{bmatrix} w_1^*(t) \\ w_2^*(t) \end{bmatrix} = F\tilde{\Phi}(t,0)x_z(0), \tag{30}$$

where $\tilde{\Phi}(t,0)$ is the solution of the transition equation

$$\dot{\tilde{\Phi}}(t,0) = (A + BF)\tilde{\Phi}(t,0);\ \tilde{\Phi}(0,0) = I.$$

The costs by using the actions (30) for the players are $x_z^T(0)\bar{C}_i x_z(0)$, $i = 1, 2$, where, with $A_{cl} := A + BF$, $\bar{C}_i$ is the unique solution of the Lyapunov equation

$$[I,\ F^T]M_i[I,\ F^T]^T + A_{cl}^T\bar{C}_i + \bar{C}_i A_{cl} = 0. \tag{31}$$

□

Notice that in case the set of algebraic Riccati equations (29) has more than one set of stabilizing solutions, there exists more than one open-loop Nash equilibrium. Matrix M has then a stable subspace of dimension larger than n. In that case, generically, for every initial state there will exist an infinite number of open-loop Nash equilibria.

The next Theorem 4.4 gives conditions under which there exists a unique OLN. Moreover, it shows that in case there is a unique equilibrium the corresponding actions are obtained by those described in Theorem 4.3.

Theorem 4.4 *Consider the differential game (19,27).*
This game has a unique open-loop Nash equilibrium for every initial state if and only if

1. *The set of coupled algebraic Riccati equations (29) has a LRS solution, and*

2. *the two algebraic Riccati equations (28) have a stabilizing solution.*

Moreover, in case this game has a unique equilibrium, the unique equilibrium actions are given by (30). □

Similarly like we did for the finite-planning horizon case one can reformulate the above results also in terms of the original system. Corollary 4.5 below states such a result if there is a unique equilibrium. In that case the corresponding equilibrium strategies can also be synthesized as a feedback strategy.

Corollary 4.5 *Assume that the two Riccati equations (28) have a symmetric stabilizing solution K_i and the nonsymmetric Riccati equation (29) has a LRS solution P.*
Then (6,8) has a unique OLN for every initial consistent state x_0 and $u^{(i)}(0)$, $i = 1, \cdots, k-2$. Moreover, the equilibrium actions are

$$\begin{bmatrix} u_1^*(t) \\ u_2^*(t) \end{bmatrix} = e^{\frac{1}{2}\theta t} \begin{bmatrix} 0_{m_1 \times n} & I_{m_1} & 0_{m_1 \times (k-2)m+m_2} \\ 0_{m_2 \times n+(k-2)m_1} & I_{m_2} & 0_{m_2 \times (k-2)m_2} \end{bmatrix} x_z(t),$$

where $x_z(t)$ solves

$$\dot{x}_z(t) = (A - BG^{-1}(H + \tilde{B}^T P))x_z(t), \; x_z(0) = x_{z0}.$$

Moreover, the corresponding equilibrium state trajectory equals (see (17))

$$x^*(t) = L_1[e^{\frac{1}{2}\theta t} x_z^T(t) \; v_1^T(t) \; v_2^T(t)]^T.$$

The corresponding costs are as in Theorem 4.3. □

Remark 4.6 Notice that the open-loop strategies are independent of the cost player i attaches to the actual control instruments used by player j (i.e. they are independent of R_{ij}, $i \neq j$. So in this case player i will ignore the effects of the $(k-1)^{th}$ order derivative of the control u_j, used by player j.

5 An Example

In this section we consider a simple macro-economic stabilization problem. Assume that a monetary and fiscal authority like to stabilize some key macro-economic variables, i.e. the real interest rate, r, inflation, $\dot{p}$, and the output gap, y, after a shock has occurred. The system is described by the following equations:

$$\begin{aligned} r(t) &= i(t) - \dot{p}(t) && (32) \\ \dot{y}(t) &= -\alpha(i(t) - \dot{p}(t)) + \beta f(t) && (33) \\ m(t) - p(t) &= \gamma y(t) - \delta i(t) && (34) \end{aligned}$$

Here $p(t)$ is the price level, $i(t)$ denotes the nominal interest rate, $m(t)$ is the money supply and $f(t)$ the fiscal policy. The first two instruments, the nominal interest rate and money supply, are determined by the monetary authority of the country, whereas the level of the third instrument, the fiscal policy, is set by the government. Here, equation (32) models the real interest rate, (33) is a

simple growth equation of the output gap and (34) models asset market equilibrium (see e.g. [17]). Assume that an initial shock in the real interest rate, price level and output gap has occurred, all equal to one.
Introducing as the state variable $x(t) := [r(t)\ p(t)\ y(t)]^T$, $u_1(t) := [i(t)\ m(t)]^T$ and $u_2(t) := f(t)$ the model can be rewritten as (6), where

$$\bar{E} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\alpha & 1 \\ 0 & 0 & 0 \end{bmatrix},\ \bar{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & -\gamma \end{bmatrix},\ \bar{B}_1 = \begin{bmatrix} 1 & 0 \\ -\alpha & 0 \\ \delta & 1 \end{bmatrix},\ \bar{B}_2 = \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix},\ \text{and } x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

It is easily verified that this system is regular if and only if $\mu := 1 + \alpha\gamma \neq 0$. With

$$Y^T := \begin{bmatrix} 0 & 1 & 0 \\ \mu & \gamma & -1 \\ 0 & 0 & -1 \end{bmatrix};\ X := \frac{1}{\mu}\begin{bmatrix} 0 & -1 & 1 \\ -\gamma & 0 & 1 \\ 1 & 0 & \alpha \end{bmatrix},$$

$k = 2;\ n = 1$ and $r = 2$, we have

$$Y^T\bar{E}X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } Y^T\bar{A}X := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So this model has index two.
Elementary calculations show that our model (32-34) can be rewritten into the form (19) with

$$x_z(t) := e^{-\frac{1}{2}\theta t}[y(t)\ i(t)\ m(t)\ f(t)]^T;\ w_1(t) := e^{-\frac{1}{2}\theta t}[\dot{i}(t)\ \dot{m}(t)]^T;\ w_2(t) := e^{-\frac{1}{2}\theta t}\dot{f}(t);\ J = 0;$$
$$A_1 = [-\alpha\ 0];\ A_2 = \beta; D_1 = \begin{bmatrix} \frac{-1}{2}\theta & 0 \\ 0 & \frac{-1}{2}\theta \end{bmatrix};\ D_2 = \frac{-1}{2}\theta;\ B_{z_1} = I_2 \text{ and } B_{z_2} = 1.$$

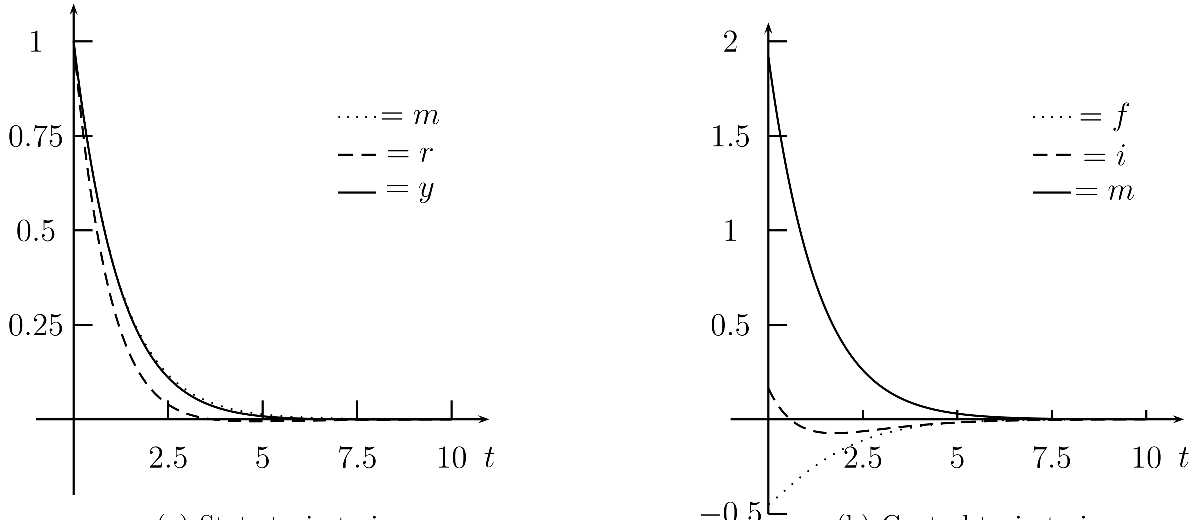
Assuming that the cost functional matrices are of the form $\bar{M}_i = [\text{diag}(\rho_{ij})],\ i = 1, 2$, where $\rho_{17},\ \rho_{18}$ and ρ_{29} are positive, we get from (21) that with

$$L = \begin{bmatrix} L_1 \\ E_2 \end{bmatrix},\ \text{where } L_1 = \frac{1}{\mu}\begin{bmatrix} 0 & 1 & 0 & \beta\gamma & -\delta & -1 & 0 \\ -\gamma & \delta & 1 & 0 & 0 & 0 & 0 \\ 1 & \alpha\delta & \alpha & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } E_2 = [0_{1\times 6}\ I_6],$$

$M_i = \frac{1}{\mu^2}*$

$$\begin{bmatrix} \gamma^2\rho_{i2} + \rho_{i3} & \delta\nu_i & \nu_i & 0 & 0 & 0 & 0 \\ \delta\nu_i & \rho_{i1} + \delta^2\tau_i + \mu^2\rho_{i4} & \delta\tau_i & \gamma\beta\rho_{i1} & -\delta\rho_{i1} & -\rho_{i1} & 0 \\ \nu_i & \delta\tau_i & \tau_i + \mu^2\rho_{i5} & 0 & 0 & 0 & 0 \\ 0 & \gamma\beta\rho_{i1} & 0 & \gamma^2\beta^2\rho_{i1} + \mu^2\rho_{i6} & -\gamma\beta\delta\rho_{i1} & -\gamma\beta\rho_{i1} & 0 \\ 0 & -\delta\rho_{i1} & 0 & -\gamma\beta\delta\rho_{i1} & \delta^2\rho_{i1} + \mu^2\rho_{i7} & \delta\rho_{i1} & 0 \\ 0 & -\rho_{i1} & 0 & -\gamma\beta\rho_{i1} & \delta\rho_{i1} & \rho_{i1} + \mu^2\rho_{i8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu^2\rho_{i9} \end{bmatrix}.$$

Here $\tau_i := \rho_{i2} + \alpha^2\rho_{i3}$ and $\nu_i := -\gamma\rho_{i2} + \alpha\rho_{i3}$.
In the Appendix we calculated the resulting matrix $\tilde{M}$. Choosing next $\alpha = \delta = \frac{1}{2};\ \gamma = 1;\ \beta = \frac{3}{4};$



(a) State trajectories. (b) Control trajectories.

Figure 1: Example with $\gamma = 1$, $\beta = \frac{3}{4}$, $\alpha = \delta = \frac{1}{2}$, $\theta = 0.15$ and $x(0) = [1\ 1\ 1]^T$.

$\theta = 0.15$ (see e.g. [17]); $M_1 = \mathrm{diag}(2;2;1;2;2;1;3;3;2)$ and $M_2 = \mathrm{diag}(1;1;2;1;1;2;2;2;3)$ we obtain for the infinite planning horizon problem that there is a unique Nash equilibrium. The LRS solution $P =: [P_1; P_2]$, corresponding with these parameters, of the algebraic Riccati equation (29) is

$$P_1 = \begin{bmatrix} 2.2983 & -1.1452 & -0.5815 & 1.1654 \\ -1.1767 & 3.8712 & 0.9343 & -0.2142 \\ -0.5435 & 0.9236 & 3.0723 & 0.1003 \\ 1.5569 & -0.2957 & 0.0148 & 2.6569 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 2.5603 & -0.8946 & 0.0753 & 1.4173 \\ -1.2775 & 2.2990 & 0.3377 & -0.8147 \\ 0.4338 & 0.3008 & 1.7192 & 0.4675 \\ 1.3597 & -0.3636 & 0.2017 & 3.3466 \end{bmatrix}.$$

This yields the next feedback gain, F, and closed-loop matrix, $A + BF$ (see Theorem 4.3):

$$\begin{bmatrix} 0.3515 & -1.0793 & -0.1839 & 0.1522 \\ 0.0996 & 0.1144 & -0.7690 & 0.1282 \\ -0.4532 & 0.1212 & -0.0672 & -1.1155 \end{bmatrix} \text{ and } \begin{bmatrix} -0.075 & -0.5 & 0 & 0.75 \\ 0.3515 & -1.1543 & -0.1839 & 0.1522 \\ 0.0996 & 0.1144 & -0.8440 & 0.1282 \\ -0.4532 & 0.1212 & -0.0672 & -1.1905 \end{bmatrix},$$

respectively.

The initial state of the transformed system (9), corresponding with the initial state $x(0) = [1;1;1]$, is $[x_1(0)\ x_2(0)] = [0.5\ 0.5\ 2]$. To determine a consistent initial state for system (19) we assume that the control used at time $t = 0$ is Fx_{z0} and that (since our problem setting concerns a perturbation problem) the initial control $[u_1(0)\ u_2(0)]$ is chosen such that its norm[3] is as small as possible. Some elementary calculations show that the initial control $[u_1(0)\ u_2(0)]$ (satisfying (12)) which is consistent with $x(0) = [1;1;1]$ and whose norm is as small as possible is $[0.1639\ 1.9181\ -0.4530]$. In Figures 1 and 2 we plotted the corresponding equilibrium state and control trajectories, respectively.

It is easily verified that the closed-loop system has complex eigenvalues. That implies that the closed-loop response shows oscillations. Figure 1 shows the optimal control policy for fiscal policy and nominal interest rate.

From (36) it is clear that if the weight attached by the government for changing his fiscal policy is not

[3]For simplicity reasons we choose all weights in the norm here the same, an assumption which of cause can be simply adapted.

too small the eigenvalues of matrix $\tilde{M}$ are close to $\frac{1}{2}\theta$ (4×) and those of matrix $C = \begin{bmatrix} C_1 & -C_2 \\ -C_4 & -C_1^T \end{bmatrix}$. Since C is a Hamiltonian matrix its eigenvalues are symmetrically distributed w.r.t. the imaginary axis. Furthermore it can be shown that C has no eigenvalues on the imaginary axis ((C_1, C_2) is stabilizable, (C_4, C_1) is detectable and $C_4 > 0$ (see e.g. [3, Proposition 5.15])). So matrix $\tilde{M}$ has exactly 4 stable eigenvalues in this case. Consequently, the game has generically a unique equilibrium. Moreover it can be shown (by a detailed analysis of its characteristic polynomial) that C has a complex eigenvalue if the discount factor is not too large (if this discount factor becomes large the eigenvalues are approximately $\pm\frac{1}{2}\theta$ (4×)). This shows that the cyclic behaviour of the closed-loop system we found with the chosen set of parameters in this example is not a coincidence, but holds in general (for a reasonable choice of parameters).

6 Concluding Remarks

In this paper we considered the linear-quadratic differential game for descriptor systems which have an index k, larger than one. Since the state trajectory in that case is a function of up to the $(k-1)^{th}$ order derivatives of the applied control, we considered here cost functions which take this dependency into account. By actually penalizing the $(k-1)^{th}$ derivatives of the input function, in fact this $(k-1)^{th}$ order derivative can be viewed as the control instrument and one obtains a regular linear-quadratic differential game. Using the standard results on the regular linear-quadratic differential games we derived then both necessary and sufficient conditions for OLN equilibria in this game.
We considered both a finite and infinite planning horizon. For the infinite horizon the standard literature on linear-quadratic differential games requires that the system should be stabilizable by all players individually. For that reason we considered in the general set-up a cost function where future cost are discounted. For the finite planning horizon this assumption can be dropped.
The above results can be generalized straightforwardly to the N-player case. Furthermore, since Q_i are assumed to be indefinite, the obtained results can be directly used to (re)derive properties for the zero-sum game. Notice, moreover, that if the discount factor θ is "large enough" the infinite horizon game has generically a unique OLN equilibrium.
We illustrated the theory by a simple macro-economic stabilization problem. The example shows that the optimal response by the players gives rise to oscillatory behaviour of the closed-loop system under fairly generally accepted choices for the set of model parameters. A phenomenon one often observes in economics. This raises the question whether this kind of response is typical for this type of control problems. A more detailed analysis of this phenomenon is planned for the future.
Obviously there are still many open problems to be solved. In particular, we did not worry about numerical aspects. In applications it is well-known that for higher-order index systems this is a serious issue. From that perspective it might be worthwhile to consider index reduction algorithms that have been developed in the literature (see e.g. [15], [13]) to reduce the system to an index one system first and next use the results from [5] to calculate the OLN equilibria. Maybe such an approach would also facilitate to analyse the undiscounted case for an infinite planning horizon. Furthermore, all of these problems can be analyzed also under different information structures.

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Appendix Notation

The next shorthand notation will be used.
$S_i := B_i R_{ii}^{-1} B_i^T$;

$$G := \left[\begin{array}{c} [0\ I\ 0]\, M_1 \\ [0\ 0\ I]\, M_2 \end{array} \right] \left[\begin{array}{cc} 0 & 0 \\ I & 0 \\ 0 & I \end{array} \right] = \left[\begin{array}{cc} R_{11} & N_1 \\ N_2^T & R_{22} \end{array} \right]$$

$$\begin{aligned}
A_2 &:= \mathrm{diag}\{A, A\};\ B := [B_1,\ B_2];\ \tilde{B}^T := \mathrm{diag}\{B_1^T, B_2^T\};\ \tilde{B}_1^T := \left[\begin{array}{c} B_1^T \\ 0 \end{array} \right];\ \tilde{B}_2^T := \left[\begin{array}{c} 0 \\ B_2^T \end{array} \right]; \\
H_i &:= [I\ 0\ 0] M_i \left[\begin{array}{cc} 0 & 0 \\ I & 0 \\ 0 & I \end{array} \right] = [V_i,\ W_i],\ i = 1, 2;\ H := \left[\begin{array}{c} [0\ I\ 0]\, M_1 \\ [0\ 0\ I]\, M_2 \end{array} \right] \left[\begin{array}{c} I \\ 0 \\ 0 \end{array} \right] = \left[\begin{array}{c} V_1^T \\ W_2^T \end{array} \right]; \\
\tilde{A} &:= A - BG^{-1}H;\ \tilde{S}_i := BG^{-1}\tilde{B}_i^T;\ \tilde{Q}_i := Q_i - H_i G^{-1} H;\ \tilde{A}_2^T := A_2^T - \left[\begin{array}{c} H_1 \\ H_2 \end{array} \right] G^{-1} \tilde{B}^T \text{ and}
\end{aligned}$$

$$\tilde{M} := \left[\begin{array}{cc} \tilde{A} & -\tilde{S} \\ -\tilde{Q} & -\tilde{A}_2^T \end{array} \right], \text{ where } \tilde{S} := [\tilde{S}_1,\ \tilde{S}_2],\ \tilde{Q} := \left[\begin{array}{c} \tilde{Q}_1 \\ \tilde{Q}_2 \end{array} \right]. \tag{35}$$

Matrix $\tilde{M}$ for the example

Following the notation of the Appendix above one can construct matrix $\tilde{M}$ for the example. After some lengthy elementary calculations we get that with $\theta_1 := \frac{1}{2}\theta$ and $det := \delta^2 \rho_{11}\rho_{18} + \rho_{11}\rho_{17} + \mu^2 \rho_{17}\rho_{18}$

$$\tilde{M} = \left[\begin{array}{ccc} C_1 & -C_2 & -C_3 \\ -C_4 & -C_1^T & 0_{4\times 4} \\ -C_5 & -C_6 & C_7 \end{array} \right], \tag{36}$$

where

$$C_1 = -\theta_1 I_4 + \begin{bmatrix} 0 & -\alpha & 0 & \beta \\ 0 & c_{11} & 0 & \gamma\beta c_{11} \\ 0 & c_{12} & 0 & \gamma\beta c_{12} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ with } c_{11} = \frac{\delta\rho_{11}\rho_{18}}{det},\ c_{12} = \frac{\rho_{11}\rho_{17}}{det};$$

$$C_2 = \frac{1}{det}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{11}+\mu^2\rho_{18} & -\delta\rho_{11} & 0 \\ 0 & -\delta\rho_{11} & \delta^2\rho_{11}+\mu^2\rho_{17} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0;\ C_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\rho_{29}} \end{bmatrix};$$

$$C_4 = \frac{1}{\mu^2}\begin{bmatrix} \gamma^2\rho_{12}+\rho_{13} & \delta\nu_1 & \nu_1 & 0 \\ \delta\nu_1 & c_{41} & \delta\tau_1 & c_{42} \\ \nu_1 & \delta\tau_1 & \tau_1+\mu^2\rho_{15} & 0 \\ 0 & c_{42} & 0 & c_{43} \end{bmatrix} > 0, \text{ with}$$

$$c_{41} = \mu^2\rho_{14} + \delta^2\tau_1 + \mu^2\rho_{18}c_{12},\ c_{42} = \mu^2\gamma\beta\rho_{18}c_{12},\ c_{43} = \mu^2(\rho_{16} + \gamma^2\beta^2\rho_{18}c_{12});$$

$$C_5 = \frac{1}{\mu^2}\begin{bmatrix} \gamma^2\rho_{22}+\rho_{23} & \delta\nu_2 & \nu_2 & 0 \\ \delta\nu_2 & c_{51} & \delta\tau_2 & \frac{\rho_{21}}{\rho_{11}}c_{42} \\ \nu_2 & \delta\tau_2 & \tau_2+\mu^2\rho_{25} & 0 \\ 0 & \frac{\rho_{21}}{\rho_{11}}c_{42} & 0 & c_{53} \end{bmatrix} > 0, \text{ with}$$

$$c_{51} = \mu^2\rho_{24} + \delta^2\tau_2 + \mu^2\rho_{18}\frac{\rho_{21}}{\rho_{11}}c_{12},\ c_{53} = \mu^2(\rho_{26} + \gamma^2\beta^2\rho_{18}\frac{\rho_{21}}{\rho_{11}}c_{12});$$

$$C_6 = \frac{1}{det}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \delta\rho_{21}\rho_{18} & \rho_{21}\rho_{17} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \beta\gamma\delta\rho_{21}\rho_{18} & \beta\gamma\rho_{21}\rho_{17} & 0 \end{bmatrix};\ C_7 = \theta_1 I_4 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\beta & 0 & 0 & 0 \end{bmatrix}.$$

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