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On the sum of Laplacian eigenvalues of graphs

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Abstract

Let k be a natural number and let G be a graph with at least k vertices. A.E. Brouwer conjectured that the sum of the k largest Laplacian eigenvalues of G is at most $e(G)+\binom{k+1}{2}$, where $e(G)$ is the number of edges of G. We prove this conjecture for $k=2$. We also show that if G is a tree, then the sum of the k largest Laplacian eigenvalues of G is at most $e(G)+2k-1$.

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1 Introduction

Let G be a simple graph with the vertex set $V(G)=\{v_1,\ldots,v_n\}$. The *degree* of a vertex $v\in V(G)$, denoted by $d(v)$, is the number of neighbors of v. The *Laplacian matrix* of G is the $n\times n$ matrix $\mathcal{L}(G)=[\ell_{ij}]$ that records the vertex degrees $d(v_1),\ldots,d(v_n)$ on its diagonal and for any $i\neq j$, $1\leqslant i,j\leqslant n$, $\ell_{ij}=-1$ if v_i and v_j are adjacent and $\ell_{ij}=0$, otherwise. It is well-known that $\mathcal{L}(G)$ is positive semi-definite and so its eigenvalues are nonnegative real numbers. The eigenvalues of $\mathcal{L}(G)$ are called the *Laplacian eigenvalues* of G and are denoted by $\mu_1(G)\geqslant\mu_2(G)\geqslant\cdots\geqslant\mu_n(G)$. Note that each row sum of $\mathcal{L}(G)$ is 0 and therefore, $\mu_n(G)=0$.

In this paper, we investigate the sum $\mathcal{S}_k(G)=\sum_{i=1}^{k}\mu_i(G)$ for $1\leqslant k\leqslant n$. We denote the edge set of G by $E(G)$ and we let $e(G)=|E(G)|$. In [2], A.E. Brouwer has conjectured the following.

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Conjecture 1 *Let G be a graph with n vertices. Then $\mathcal{S}_k(G) \leqslant e(G) + \binom{k+1}{2}$ for $k = 1, \ldots, n$.*

In [2], Brouwer points out that, by the use of computer, he has checked Conjecture 1 for all graphs with at most 10 vertices. For $k = 1$, the conjecture follows from the well-known inequality $\mu_1(G) \leqslant |V(G)|$ (see [6, p. 281]). Here, we prove Conjecture 1 for $k = 2$. We also show that $\mathcal{S}_k(T) \leqslant e(T) + 2k - 1$ for any tree T and any $1 \leqslant k \leqslant n$ from which the conjecture follows for trees.

Some results and conjectures related to $\mathcal{S}_k(G)$ can be found in the literature. First we state the Grone-Merris conjecture [7]. Let $d_i^{\mathsf{T}} = |\{v \in V(G) \,|\, d(v) \geqslant i\}|$ for $i = 1, \ldots, n$. The numbers $d_1^{\mathsf{T}} \geqslant d_2^{\mathsf{T}} \geqslant \cdots \geqslant d_n^{\mathsf{T}}$ are called the *conjugate degrees* of G. The Grone-Merris conjecture asserts that $\mathcal{S}_k(G) \leqslant \sum_{i=1}^{k} d_i^{\mathsf{T}}$ for $k = 1, \ldots, n$. This inequality for $k = 1$ is immediate from $\mu_1(G) \leqslant |V(G)|$ and the equality obviously occurs for $k = n-1, n$. Moreover, the conjecture has been proved whenever $k = 2$ [3, Theorem 7.1] or G is a tree [9]. Next, we note that the upper bound

$$\mathcal{S}_k(G) \leqslant \frac{2mk + \sqrt{mk(n-k-1)(n^2-n-2m)}}{n-1},$$

is obtained in [10], where $1 \leqslant k < n$ and $m = e(G)$.

2 Notation and Preliminaries

We first present some notation and definitions. For a subset X of $V(G)$, $\mathcal{N}(X)$ denotes the set of vertices which have at least one neighbor in X. An *independent set* in G is a subset Y of $V(G)$ such that no two distinct vertices in Y are adjacent. Two distinct edges of G are called *independent* if they have no common endpoint. A set of pairwise independent edges in G is called a *matching*. The maximum size of a matching in G is known as the *matching number* of G, denoted by $m(G)$. For two graphs G_1 and G_2, the *union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and whose edge set is $E(G_1) \cup E(G_2)$. If $V(G_1) \cap V(G_2) = \varnothing$, then the union of G_1 and G_2 is denoted by $G_1 + G_2$. We denote the complete graph, star and path with n vertices by K_n, S_n and P_n, respectively. The complete bipartite graph with the part sizes m and n is denoted by $K_{m,n}$.

Brouwer has checked Conjecture 1 for all graphs with at most 10 vertices. For our purpose we only need the following statement.

Lemma 1 [2] *For any graph G with at most 8 vertices, $\mathcal{S}_2(G) \leqslant e(G) + 3$.*

We next state some lemmas and theorems which will be used in the subsequent sections.

Lemma 2 *Let n be a natural number.*

(i) *The Laplacian eigenvalues of K_n are n with multiplicity $n-1$, and 0.*

(ii) *The Laplacian eigenvalues of S_n are n, 1 with multiplicity $n-2$, and 0.*

The following lemma gives an affirmative answer to Conjecture 1 for $k = 1$.

Lemma 3 [6, p. 281] *If G is a graph with n vertices, then $\mu_1(G) \leqslant n$.*

Theorem 1 [6, p. 291] *Let G be a graph with n vertices and let G' be a graph obtained from G by inserting a new edge into G. Then the Laplacian eigenvalues of G and G' interlace, that is,*

$$\mu_1(G') \geqslant \mu_1(G) \geqslant \cdots \geqslant \mu_n(G') = \mu_n(G) = 0.$$

Theorem 2 [8] *Let G be a graph. Then $\mu_1(G) \leqslant \max\{d(v) + m(v) \mid v \in V(G)\}$, where $m(v)$ is the average of the degrees of the vertices of G adjacent to the vertex v.*

Theorem 3 [1] *Let G be a graph with n vertices and vertex degrees $d_1 \geqslant \cdots \geqslant d_n$. If G is not $K_s + (n-s)K_1$, then $\mu_s(G) \geqslant d_s - s + 2$ for $1 \leqslant s \leqslant n$.*

The following theorem from matrix theory plays a key role in our proofs. We denote the eigenvalues of a symmetric matrix M by $\lambda_1(M) \geqslant \cdots \geqslant \lambda_n(M)$.

Theorem 4 [4] (see also [5]) *Let A and B be two real symmetric matrices of size n. Then for any $1 \leqslant k \leqslant n$,*

$$\sum_{i=1}^{k} \lambda_i(A+B) \leqslant \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B).$$

An immediate consequence of Theorem 4 is the following corollary which will be used frequently.

Corollary 1 *Let $G_1, \ldots, G_r$ be some edge disjoint graphs. Then $\mathcal{S}_k(G_1 \cup \cdots \cup G_r) \leqslant \sum_{i=1}^{r} \mathcal{S}_k(G_i)$ for any k.*

The following Lemma asserts that to prove Conjecture 1 for $k = 2$, it suffices to consider connected graphs.

Lemma 4 *Let G be a graph. Then either $\mathcal{S}_2(G) = \mathcal{S}_2(H)$ for a connected component H of G or $\mathcal{S}_2(G) \leqslant e(G) + 2$.*

Proof. If the first statement does not hold, then G has two connected components H_1 and H_2 such that $\mu_1(G) = \mu_1(H_1)$ and $\mu_2(G) = \mu_1(H_2)$. By Lemma 3, we have $\mu_1(H_i) \leqslant |V(H_i)| \leqslant e(H_i) + 1$ for $i = 1, 2$. Therefore, $\mathcal{S}_2(G) \leqslant (e(H_1) + 1) + (e(H_2) + 1) \leqslant e(G) + 2$. □

The next lemma is the key to our approach. It gives a sufficient condition for the truth of Conjecture 1 with $k = 2$, that holds for almost all graphs.

Lemma 5 *If G is a graph with a subgraph H for which $\mathcal{S}_2(H) \leqslant e(H)$, then $\mathcal{S}_2(G) \leqslant e(G) + 3$.*

Proof. Assume that G is a counterexample with a minimum possible number of edges. By Corollary 1, we have $e(G) + 3 < \mathcal{S}_2(G) \leqslant \mathcal{S}_2(H) + \mathcal{S}_2(G - H)$. This implies that $\mathcal{S}_2(G - H) > e(G - H) + 3$, which contradicts the minimality of $e(G)$. □

Lemma 6 *Let G be a graph with n vertices. Suppose that there exist two non-adjacent vertices $u, v \in V(G)$ such that $\mu_k(G) \geqslant d(u) + d(v) + 2$ for some integer k, $1 \leqslant k \leqslant n$. If G' is the graph obtained from G by inserting edge $e = \{u, v\}$ into G, then $\mathcal{S}_k(G') \leqslant \mathcal{S}_k(G) + 1$.*

Proof. For $i = 1, \ldots, n$, define $\epsilon_i = \mu_i(G') - \mu_i(G)$. By Theorem 1, $\epsilon_i \geqslant 0$ for any i. Let $d_1 \geqslant \cdots \geqslant d_n$ and $d'_1 \geqslant \cdots \geqslant d'_n$ be vertex degrees of G and G', respectively. Recall that for any graph Γ, considering the trace of the matrix $\mathcal{L}(\Gamma)^2$, we have

$$\sum_{i=1}^{|V(\Gamma)|} \mu_i(\Gamma)^2 = \sum_{v \in V(\Gamma)} d(v)^2 + 2e(\Gamma).$$

Applying this fact, we have

$$\begin{aligned}\sum_{i=1}^{n} \mu_i(G')^2 &= \sum_{i=1}^{n} d_i'^2 + 2e(G') \\ &= \sum_{i=1}^{n} d_i^2 + 2e(G) + 2d(u) + 2d(v) + 4 \\ &= \sum_{i=1}^{n} \mu_i(G)^2 + 2\big(d(u) + d(v) + 2\big).\end{aligned}$$

This yields that

$$\begin{aligned}2\mu_k(G) \sum_{i=1}^{k} \epsilon_i &\leqslant \sum_{i=1}^{k} 2\epsilon_i \mu_i(G) \\ &\leqslant \sum_{i=1}^{n} \mu_i(G')^2 - \sum_{i=1}^{n} \mu_i(G)^2 \\ &= 2\big(d(u) + d(v) + 2\big).\end{aligned}$$

Since $\mu_k(G) \geqslant d(u) + d(v) + 2$, $\mathcal{S}_k(G') - \mathcal{S}_k(G) = \sum_{i=1}^{k} \epsilon_i \leqslant 1$ and the assertion follows. □

3 Trees and threshold graphs

In the following, we obtain an upper bound for the sum of the k largest Laplacian eigenvalues of a tree which implies Conjecture 1 for trees.

Theorem 5 *Let T be a tree with n vertices. Then $\mathcal{S}_k(T) \leqslant e(T) + 2k - 1$ for $1 \leqslant k \leqslant n$.*

Proof. We prove the assertion by induction on $|V(T)|$. If T is a star, then by Lemma 2(ii), $\mathcal{S}_k(T) = n+k-1$ for $1 \leqslant k < n$, and we are done. Thus assume that T is not a star. Then T has an edge whose removing leaves a forest F consisting of two trees T_1 and T_2, both having at least one edge. Suppose that k_i of the k largest eigenvalues of F comes from the Laplacian spectrum of T_i for $i = 1, 2$, where $k_1 + k_2 = k$. If one of k_i, say k_2, is zero, then by $|V(T_2)| \geqslant 2$, Corollary 1, and the induction hypothesis, we conclude that $\mathcal{S}_k(T) = \mathcal{S}_k(F \cup K_2) \leqslant \mathcal{S}_{k_1}(T_1) + \mathcal{S}_k(K_2) \leqslant (e(T_1) + 2k_1 - 1) + 2 \leqslant n + 2k - 2 = e(T) + 2k - 1$. Otherwise, using Corollary 1 and the induction hypothesis, we have $\mathcal{S}_k(T) = \mathcal{S}_k(T_1 \cup T_2 \cup K_2) \leqslant \mathcal{S}_{k_1}(T_1) + \mathcal{S}_{k_2}(T_2) + \mathcal{S}_k(K_2) \leqslant (e(T_1) + 2k_1 - 1) + (e(T_2) + 2k_2 - 1) + 2 = e(T) + 2k - 1$. This completes the proof. □

A *threshold graph* is a graph obtained from K_1 by a sequence of operations of the form (i) adding an isolated vertex or (ii) taking the complement. It is clear that adding isolated vertices to a graph only increases the multiplicity of the Laplacian eigenvalue 0. This observation and the next theorem shows that Conjecture 1 is valid for threshold graphs.

Theorem 6 *Let G be a graph with n vertices and $1 \leqslant k \leqslant n-2$. If $S_k(G) \leqslant e(G) + \binom{k+1}{2}$, then $S_{n-k-1}(\overline{G}) \leqslant e(\overline{G}) + \binom{n-k}{2}$, where $\overline{G}$ is the complement of G.*

Proof. From [6, p. 280], we have $\mu_i(\overline{G}) = n - \mu_{n-i}(G)$ for $i = 1, \ldots, n-1$. Therefore,

$$\begin{aligned}
S_{n-k-1}(\overline{G}) &= n(n-k-1) - \big(\mu_{k+1}(G) + \cdots + \mu_{n-1}(G)\big) \\
&= n(n-k-1) - 2e(G) + \big(\mu_1(G) + \cdots + \mu_k(G)\big) \\
&= n(n-k-1) - \binom{n}{2} + e(\overline{G}) + \big(\mu_1(G) + \cdots + \mu_k(G)\big) - e(G) \\
&\leqslant e(\overline{G}) + n(n-k-1) - \binom{n}{2} + \binom{k+1}{2} \\
&= e(\overline{G}) + \binom{n-k}{2},
\end{aligned}$$

as desired. □

4 The case $k = 2$

In this section, we prove Conjecture 1 for $k = 2$. First we establish the conjecture for graphs with matching number at most three and then we conclude the assertion using Lemma 5.

Lemma 7 *Let G be a graph with $m(G) = 1$. Then $\mathcal{S}_2(G) \leqslant e(G) + 3$.*

Proof. Let $n = |V(G)|$. Since $m(G) = 1$, it is easily checked that either $G = S_m + (n-m)K_1$ for some m, $1 \leqslant m \leqslant n$ or $G = K_3 + (n-3)K_1$. By Lemma 2, the assertion holds. □

We say that a connected graph has the form $\triangle$ if it has a subgraph H isomorphic to K_3 such that every edge is incident with some vertex of H.

Lemma 8 *Let G be a graph of the form $\triangle$. Then $\mathcal{S}_2(G) \leqslant e(G)+3$.*

Proof. Let $n = |V(G)|$ and $d_1^{\mathsf{T}} \geqslant \cdots \geqslant d_n^{\mathsf{T}}$ be the conjugate degrees of G. If t is the number of vertices of degree 1 in G, then it is not hard to see that $2(n-t-3) \leqslant e(G)-t-3$. This implies that $d_2^{\mathsf{T}} = n-t \leqslant e(G)-n+3$. Since $d_1^{\mathsf{T}} = n$, $d_1^{\mathsf{T}} + d_2^{\mathsf{T}} \leqslant e(G)+3$. By [3, Theorem 7.1], the Grone-Merris conjecture is true for $k=2$. Therefore, $\mathcal{S}_2(G) \leqslant d_1^{\mathsf{T}} + d_2^{\mathsf{T}} \leqslant e(G)+3$. □

Lemma 9 *Let $n \geqslant 3$ and let G be a connected spanning subgraph of $K_{2,n-2}$. Then $\mathcal{S}_2(G) \leqslant e(G)+3$.*

Proof. Assume that $\{\{v,w\},B\}$ is the partition of $V(G)$. For simplicity, we write $\mu_i(G) = \mu_i$ for $1 \leqslant i \leqslant n$. Let $d_1 \geqslant \cdots \geqslant d_n$ be the vertex degrees of G and let r and s be the number of vertices of degree 1 and 2 in B, respectively. By Theorem 5, we can assume that G is not a tree. Hence $s \geqslant 2$ and the degrees $d_1, d_2 \geqslant 2$ are the degrees of v and w. It is easily seen that s rows of $2I - \mathcal{L}(G)$ are identical and therefore the multiplicity of 2 as an eigenvalue of $\mathcal{L}(G)$ is at least $s-1$. Similarly, the multiplicity of 1 as eigenvalues of $\mathcal{L}(G)$ is at least $r-2$. If $\mu_2 \leqslant 2$, then Lemma 3 implies that $\mu_1 + \mu_2 \leqslant n+2 < e(G)+3$. Hence we may assume that $\mu_2 > 2$ and so $\mu_1 \geqslant \mu_2 \geqslant \mu_a \geqslant \mu_b \geqslant \mu_n = 0$ are the five remaining eigenvalues. By $\mathrm{trace}(\mathcal{L}(G)) = \sum_{i=1}^n \mu_i = \sum_{i=1}^n d_i$, we have $\mu_1 + \mu_2 + \mu_a + \mu_b \leqslant d_1 + d_2 + 4$. Finally, by the interlacing theorem [6, p. 193] for the $(n-2)\times(n-2)$ submatrix $D = \mathrm{diag}(1,\ldots,1,2,\ldots,2)$ of $\mathcal{L}(G)$, we find that $\mu_a \geqslant \mu_{n-2} \geqslant \lambda_{n-2}(D) \geqslant 1$. Hence $\mu_1 + \mu_2 \leqslant d_1 + d_2 + 4 - \mu_a - \mu_b \leqslant d_1 + d_2 + 3 = e(G)+3$. □

Lemma 10 *Let G be a graph with $m(G)=2$. Then $\mathcal{S}_2(G) \leqslant e(G)+3$.*

Proof. By Lemmas 1 and 4, we may assume that G is a connected graph with at least 7 vertices. First suppose that G has a subgraph $H = K_3$ with $V(H) = \{u,v,w\}$. If every edge of G has at least one endpoint in $V(H)$, then by Lemma 8, we are done. Hence assume that there exists an edge $e = \{a,b\}$ whose endpoints are in $V(G)\setminus V(H)$. Let $M = V(G)\setminus\{a,b,u,v,w\}$. Since $m(G)=2$, there are no edges between $V(H)$ and M. Since $|M| \geqslant 2$, it is easily seen that all vertices in M are adjacent to one of the endpoints of e, say a. Hence there are no edges between b and $V(H)$. Now by ignoring the edges between a and $V(H)$, we find a subgraph K of G which is a disjoint union of K_3 and a star with the center a. Since the graph $L = G - E(K)$ is a star, Corollary 1 yields that $\mathcal{S}_2(G) \leqslant \mathcal{S}_2(K) + \mathcal{S}_2(L) \leqslant (e(K)+1) + (e(L)+2) = e(G)+3$, as required.

Next assume that G has no K_3 as a subgraph. Suppose that $e_1 = \{a_1,b_1\}$ and $e_2 = \{a_2,b_2\}$ are two independent edges in G. Since G contains no $3K_2$ and K_3 as subgraphs, $M = V(G)\setminus\{a_1,b_1,a_2,b_2\}$ is an independent set and at least one of the two endpoints of e_i has no neighborhood in M for $i=1,2$. Assume those endpoints to be b_1 and b_2. If b_1 and b_2

are adjacent, then $|M| \geqslant 2$ yields that all vertices in M are adjacent to only one of the two vertices a_1 and a_2, say a_1. This implies that G is a bipartite graph with the vertex set partition $\{\{a_1, b_2\}, V(G) \setminus \{a_1, b_2\}\}$ and so Lemma 9 yields the assertion. Now assume that b_1 and b_2 are not adjacent. If a_1 and a_2 are adjacent, then G is a tree and we are done by Theorem 5. Otherwise, G is a bipartite graph with the vertex set partition $\{\{a_1, a_2\}, V(G) \setminus \{a_1, a_2\}\}$ and using Lemma 9, the proof is complete. □

Lemma 11 *Let G be a graph with $m(G) = 3$. Then $\mathcal{S}_2(G) \leqslant e(G) + 3$.*

Proof. By Lemmas 1 and 4, we may assume that G is a connected graph with at least 9 vertices. Using Lemma 5, we may suppose that G has no subgraph H with $\mathcal{S}_2(H) \leqslant e(H)$. In particular, Lemma 2 implies that G has no subgraph $3S_3$. Suppose that G has a subgraph $K = K_3 + 2K_2$. Let $x \in V(G) \setminus V(K)$. Since $m(G) = 3$, the vertex x is not incident with the subgraph K_3 of K and so G has a subgraph $H = K_3 + S_3 + K_2$. Now by Lemma 2, we have $\mathcal{S}_2(H) = e(H)$ and therefore G has no subgraph $K_3 + 2K_2$.

Let $e_1 = \{a_1, b_1\}, e_2 = \{a_2, b_2\}$ and $e_3 = \{a_3, b_3\}$ be three independent edges in G. Since $m(G) = 3$, $M = V(G) \setminus V(\{e_1, e_2, e_3\})$ is an independent set. Since G has no $4K_2$ and $K_3 + 2K_2$ as subgraphs, either $\mathcal{N}(a_i) \cap M = \varnothing$ or $\mathcal{N}(b_i) \cap M = \varnothing$, for $i = 1, 2, 3$. With no loss of generality, we may assume that $\mathcal{N}(M) \subseteq \{a_1, a_2, a_3\}$. We consider the following three cases.

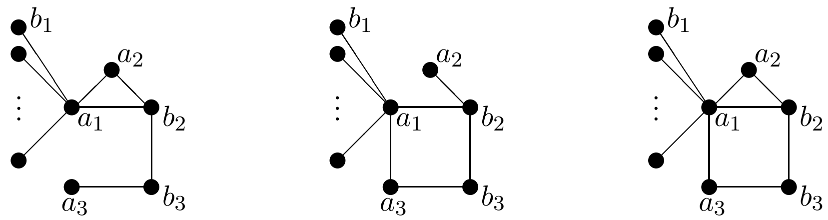
Case 1. $|\mathcal{N}(M)| = 3$. We have $\mathcal{N}(M) = \{a_1, a_2, a_3\}$. Since G has no $3S_3$, the bipartite subgraph $G - \{b_1, b_2, b_3\}$ has no perfect matching. By Hall's Theorem, there exists a subset of $\{a_1, a_2, a_3\}$ with 2 elements, say $\{a_2, a_3\}$, such that $|\mathcal{N}(\{a_2, a_3\}) \cap M| = 1$. This means that there exists exactly one vertex $y \in M$ which is adjacent to both a_2 and a_3. If $d(b_1) \geqslant 2$, then we clearly find a subgraph isomorphic to $3S_3$ in G, a contradiction. Therefore, $d(b_1) = 1$. Suppose that H is the star with center a_1 and $V(H) \subseteq \{a_1, a_2, a_3, b_2, b_3, y\}$. Then $G - E(H)$ is a disjoint union of a star S with center a_1 and a graph K containing P_5 with the vertex set $\{a_2, a_3, b_2, b_3, y\}$. Using Theorem 2, we have $\mu_1(P_5) \leqslant 4$ and by Lemma 2, we obtain that $\mu_1(K) \leqslant e(K)$. This yields that $\mathcal{S}_2(G - E(H)) \leqslant \mu_1(S) + \mu_1(K) \leqslant e(G - E(H)) + 1$. Thus $\mathcal{S}_2(G) \leqslant \mathcal{S}_2(H) + \mathcal{S}_2(G - E(H)) \leqslant e(G) + 3$, as desired.

Case 2. $|\mathcal{N}(M)| = 2$. Without loss of generality, assume that $\mathcal{N}(M) = \{a_1, a_2\}$. Since $m(G) = 3$, b_1 is not adjacent to b_2. If b_1 is adjacent to a_3 or b_3, then changing the role of e_1, e_2, e_3 by three independent edges $\{a_1, z\}$, e_2, e_3 for some vertex $z \in M \cap \mathcal{N}(a_1)$, we have Case 1. Therefore, we may assume that b_1, and similarly b_2, is adjacent to none of the vertices a_3 and b_3. Let H be the induced subgraph on $\{a_1, a_2, a_3, b_3\}$.

First assume that H has a subgraph $L = K_3$. If $\{a_1, a_2\}$ is an edge of L, then clearly any edge of G is incident with L and by Lemma 8, there is nothing to prove. Now assume that exactly one of the two vertices a_1 and a_2, say a_1, is a vertex in L. Let K be the disjoint union of L and the induced subgraph of G on $\{a_2, b_2\} \cup (\mathcal{N}(a_2) \cap M)$ which is a star with at least three vertices. Note that $G - E(K)$ is a star or a disjoint union of two stars. Now, by Lemma 2 and Corollary 1, $\mathcal{S}_2(G) \leqslant \mathcal{S}_2(K) + \mathcal{S}_2(G - E(K)) = (e(K) + 1) + (e(G - E(K)) + 2) = e(G) + 3$, as required.

Next suppose that H has no K_3 as a subgraph. Let $t = d(a_3) + d(b_3)$. We have $t = 3, 4$. It is not hard to see that $G - e_3$ contains two disjoint stars S_t with centers a_1 and a_2. Therefore, by Theorem 1, $\mu_2(G - e_3) \geqslant \mu_2(2S_t) = t$. Using Lemmas 6 and 10, we find that $\mathcal{S}_2(G) \leqslant \mathcal{S}_2(G - e_3) + 1 \leqslant (e(G - e_3) + 3) + 1 = e(G) + 3$, as required.

Case 3. $|\mathcal{N}(M)| = 1$. Without loss of generality, assume that $\mathcal{N}(M) = \{a_1\}$. If $d(b_1) \geqslant 2$, then we clearly find three independent edges e'_1, e'_2, e'_3 in G such that the set $M' = V(G) \setminus V(\{e'_1, e'_2, e'_3\})$ is an independent set and $|\mathcal{N}(M')| \geqslant 2$ which is dealt with as the previous cases. Hence we assume that $d(b_1) = 1$. Suppose that H is the star with center a_1 and the vertex set $V(H) \subseteq \{a_1, a_2, a_3, b_2, b_3\}$. Then $G - E(H)$ is a disjoint union of a star S with center a_1 and a graph L containing $2K_2$ with $V(L) = \{a_2, a_3, b_2, b_3\}$. First assume that $L \neq P_4$. Using Lemma 2(i) and Lemma 3, we have $\mu_1(L) \leqslant e(L)$. This yields that $\mathcal{S}_2(G - E(H)) \leqslant \mu_1(S) + \mu_1(L) \leqslant e(G - E(H)) + 1$. Thus $\mathcal{S}_2(G) \leqslant \mathcal{S}_2(H) + \mathcal{S}_2(G - E(H)) \leqslant e(G) + 3$, as desired. Next assume that $L = P_4$. With no loss of generality, suppose that L is the path $a_2 — b_2 — b_3 — a_3$. If $|\mathcal{N}(a_1) \cap L| = 1$, then G is a tree and the assertion follows from Theorem 5. If a_1 is adjacent to both b_2 and b_3, then by Lemma 8, there is nothing to prove. Suppose that a_1 is adjacent to none of b_2 and b_3. If we let K be the disjoint union of the star $G - V(L)$ and the edges $\{a_2, b_2\}$ and $\{a_3, b_3\}$, then the graph $G - E(K)$ is a disjoint union of a star with the center a_1 and the edge $\{b_2, b_3\}$. Now, by Lemma 2 and Corollary 1, we have $\mathcal{S}_2(G) \leqslant \mathcal{S}_2(K) + \mathcal{S}_2(G - E(K)) \leqslant (e(K) + 1) + (e(G - E(K)) + 2) = e(G) + 3$. If none of the above cases occurs, then G is one of the following forms:



G_1 G_2 G_3

If $G = G_1$, then by Theorem 3, we have $\mu_2(G) \geqslant 3$. Since $d(a_3)+d(b_3) = 3$, applying Lemma 6 for the graph $G-e_3$ and using Lemma 10, we find that $\mathcal{S}_2(G) \leqslant \mathcal{S}_2(G-e_3)+1 \leqslant (e(G-e_3)+3)+1 = e(G) + 3$, as required. Hence assume that $G = G_2$ or $G = G_3$. First suppose that $\mu_2(G) \geqslant 4$. Since $d(a_3) + d(b_3) = 4$, applying Lemma 6 for the graph $G - e_3$ and using Lemma 10, the result follows. Now suppose that $\mu_2(G) < 4$. By Theorem 2, we have $\mu_1(G_2) \leqslant |V(G_2)| - 1 = e(G_2) - 1$ and by Lemma 3, $\mu_1(G_3) \leqslant |V(G_3)| = e(G_3) - 1$. Therefore, $\mathcal{S}_2(G) < (e(G) - 1) + 4 = e(G) + 3$. This completes the proof. □

We now present the main theorem of the paper.

Theorem 7 *Let G be a graph with at least two vertices. Then $\mathcal{S}_2(G) \leqslant e(G) + 3$.*

Proof. Using Lemmas 7, 10 and 11, we may assume that G has a subgraph $H = 4K_2$, which satisfies $\mathcal{S}_2(H) = e(H)$. So the result follows by Lemma 5. □

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