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On semidefinite programming relaxations of the traveling salesman problem

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Abstract

We consider a new semidefinite programming (SDP) relaxation of the symmetric traveling salesman problem (TSP), that may be obtained via an SDP relaxation of the more general quadratic assignment problem (QAP). We show that the new relaxation dominates the one in the paper:

[D. Cvetković, M. Cangalović and V. Kovačević-Vujčić. Semidefinite Programming Methods for the Symmetric Traveling Salesman Problem. In *Proceedings of the 7th International IPCO Conference on Integer Programming and Combinatorial Optimization*, 1999, 126–136, Springer-Verlag, London, UK.]

Unlike the bound of Cvetković et al., the new SDP bound is not dominated by the Held-Karp linear programming bound, or vice versa.

Keywords: traveling salesman problem, semidefinite programming, quadratic assignment problem, association schemes

AMS classification: 90C22, 20Cxx, 70-08

JEL code: C61

1 Introduction

The quadratic assignment problem (QAP) may be stated in the following form:

$$\min_{X \in \Pi_n} trace(AXBX^T) \tag{1}$$

where A and B are given symmetric $n \times n$ matrices, and Π_n is the set of $n \times n$ permutation matrices.

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It is well-known that the QAP contains the symmetric traveling salesman problem (TSP) as a special case. To show this, we denote the complete graph on n vertices with edge lengths (weights) $D_{ij} = D_{ji} > 0$ $(i \neq j)$, by $K_n(D)$, where D is called the matrix of edge lengths (weights). The TSP is to find a Hamiltonian circuit of minimum length in $K_n(D)$. The n vertices are often called *cities*, and the Hamiltonian circuit of minimum length the *optimal tour*.

To see that TSP is a special case of QAP, let C_1 denote the adjacency matrix of $\mathcal{C}_n$ (the standard circuit on n vertices):

$$C_1 := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ 0 & & & & 0 & 1 \\ 1 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Now the TSP problem is obtained from the QAP problem (1) by setting $A = \frac{1}{2}D$ and $B = C_1$. To see this, note that every Hamiltonian circuit in a complete graph has adjacency matrix XC_1X^T for some $X \in \Pi_n$. Thus we may concisely state the TSP as

$$TSP_{opt} := \min_{X \in \Pi_n} trace \left(\frac{1}{2} DXC_1X^T \right). \tag{2}$$

The symmetric TSP is NP-hard in the strong sense [20], and therefore so is the more general QAP. In the special case where the distance function of the TSP instance satisfies the triangle inequality (metric TSP), there is a celebrated 3/2-approximation algorithm due to Christofides [9]. It is a long-standing (since 1975) open problem to improve on the 3/2 constant, since the strongest negative result is that a (1+1/219)-approximation algorithm is not possible, unless P=NP [21].

In the case when the distances are Euclidean in fixed dimension (the so-called planar or geometric TSP), the problem allows a polynomial-time approximation scheme [1]. A recent survey of the TSP is given by Schrijver [24], Chapter 58.

Main results and outline of this paper

In this paper we will consider semidefinite programming (SDP) relaxations of the TSP. We will introduce a new SDP relaxation of TSP in Section 2, that is motivated by the theory of association schemes. Subsequently, we will show in Section 3 that the new SDP relaxation coincides with the SDP relaxation for QAP introduced in [28] when applied to the QAP reformulation of TSP in (2). Then we will show in Section 4 that the new SDP relaxation dominates the relaxation due to Cvetković et al. [5]. The relaxation of Cvetković et al. is known to be dominated by the Held-Karp linear programming bound [6, 15], but we show in Section 5 that the new SDP bound is not dominated by the Held-Karp bound (or vice versa).

Notation

The space of $p \times q$ real matrices is denoted by $\mathbb{R}^{p\times q}$, the space of $k \times k$ symmetric matrices is denoted by $\mathcal{S}_k$, and the space of $k \times k$ symmetric positive semidefinite matrices by $\mathcal{S}_k^+$. We will sometimes also use the notation $X \succeq 0$ instead of $X \in \mathcal{S}_k^+$, if the order of the matrix is clear from the context. By $diag(X)$ we mean the n-vector composed of the diagonal entries of $X \in \mathcal{S}_n$.

We use I_n to denote the identity matrix of order n. Similarly, J_n and e_n denote the $n \times n$ all-ones matrix and all ones n-vector respectively, and $0_{n\times n}$ is the zero matrix of order n. We will omit the subscript if the order is clear from the context.

The *Kronecker product* $A \otimes B$ of matrices $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{r\times s}$ is defined as the $pr \times qs$ matrix composed of pq blocks of size $r \times s$, with block ij given by $A_{ij}B$ $(i = 1, \ldots, p)$, $(j = 1, \ldots, q)$.

The Hadamard (component-wise) product of matrices A and B of the same size will be denoted by $A \circ B$.

2 A new SDP relaxation of TSP

In this section we show that the optimal value of the following semidefinite program provides a lower bound on the length TSP_{opt} of an optimal tour:

$$\left.\begin{array}{rll} \min & \frac{1}{2} trace\left(DX^{(1)}\right) & \\ \text{subject to} & & \\ & X^{(k)} \geq 0, & k = 1, \ldots, d \\ & \sum_{k=1}^{d} X^{(k)} = J - I, & \\ & I + \sum_{k=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) X^{(k)} \succeq 0, & i = 1, \ldots, d \\ & X^{(k)} \in \mathcal{S}^n, & k = 1, \ldots, d, \end{array}\right\} \quad (3)$$

where $d = \lfloor \frac{1}{2}n \rfloor$ is the diameter of $\mathcal{C}_n$.

Note that this problem involves nonnegative matrix variables $X^{(1)}, \ldots, X^{(d)}$ of order n. The matrix variables $X^{(k)}$ have an interesting interpretation in terms of *association schemes*.

Association schemes

We will give a brief overview of this topic; for an introduction to association schemes, see Chapter 12 in [10], and in the context of SDP, [11].

Definition 2.1 (Asssociation scheme)**.** *Assume that a given set of $n\times n$ matrices $B_0, \ldots, B_t$ has the following properties:*

(1) B_i is a $0-1$ matrix for all i and $B_0 = I$;

(2) $\sum_i B_i = J$;

(3) $B_i = B_{i^*}^T$ for some i^*;

(4) $B_iB_j = B_jB_i$ *for all* i, j*;*

(5) $B_iB_j \in span\{B_1, \ldots, B_t\}$.

Then we refer to $\{B_1, \ldots, B_t\}$ *as an association scheme. If the* B_i*'s are also symmetric, then we speak of a symmetric association scheme.*

Note that item (4) (commutativity) implies that the matrices $B_1, \ldots, B_t$ share a common set of eigenvectors, and therefore can be simultaneously diagonalized. Note also that an association scheme is a basis of a matrix-$*$ algebra (viewed as a vector space). Moreover, one clearly has

$$trace(B_iB_j^T) = 0 \text{ if } i \neq j.$$

Since the B_i's share a system of eigenvectors, there is a natural ordering of their eigenvalues with respect to any fixed ordering of the eigenvectors. Thus the last equality may be interpreted as:

$$\sum_k \lambda_k(B_i)\lambda_k(B_j) = 0 \text{ if } i \neq j, \tag{4}$$

where the $\lambda_k(B_i)$'s are the eigenvalues of B_i with respect to the fixed ordering.

The association scheme of particular interest to us arises as follows. Given a connected graph $G = (V, E)$ with diameter d, we define $|V| \times |V|$ matrices $A^{(k)}$ $(k = 1, \ldots, d)$ as follows:

$$A_{ij}^{(k)} = \begin{cases} 1 & \text{if } \mathrm{dist}(i,j) = k \\ 0 & \text{else,} \end{cases} \qquad (i, j \in V),$$

where $\mathrm{dist}(i, j)$ is the length of the shortest path from i to j.

Note that $A^{(1)}$ is simply the adjacency matrix of G. Moreover, one clearly has

$$I + \sum_{k=1}^{d} A^{(k)} = J.$$

It is well-known that, for $G = \mathcal{C}_n$, the matrices $A^{(k)}$ $(k = 1, \ldots, d \equiv \lfloor n/2 \rfloor)$ together with $A^{(0)} := I$ form an association scheme, since $\mathcal{C}_n$ is a distance regular graph.

It is shown in the Appendix to this paper, that for $G = \mathcal{C}_n$, the eigenvalues of the matrix $A^{(k)}$ are:

$$\lambda_m(A^{(k)}) = 2\cos(2\pi mk/n), \quad m = 0, \ldots, n-1,\ k = 1, \ldots, \lfloor (n-1)/2 \rfloor,$$

and, if n is even,

$$\lambda_{n/2}(A^{(k)}) = \cos(k\pi) = (-1)^k.$$

In particular, we have

$$\lambda_m(A^{(k)}) = \lambda_k(A^{(m)}) \qquad k, m = 1, \ldots, \lfloor (n-1)/2 \rfloor. \tag{5}$$

Also note that

$$\lambda_m(A^{(k)}) = \lambda_{n-m}(A^{(k)}), \quad k, m = 1, \ldots, \lfloor (n-1)/2 \rfloor, \tag{6}$$

so that each matrix $A^{(k)}$ $(k = 1, \ldots, d)$ has only $1 + \lfloor n/2 \rfloor$ distinct eigenvalues.

Verifying the SDP relaxation (3)

We now show that setting $X^{(k)} = A^{(k)}$ $(k = 1, \ldots, d)$ gives a feasible solution of (3). We only need to verify that

$$I + \sum_{k=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) A^{(k)} \succeq 0, \qquad i = 1, \ldots, d.$$

We will show this for odd n, the proof for even n being similar.

Since the $A^{(k)}$'s may be simultaneously diagonalized, the last LMI is the same as:

$$2 + \sum_{k=1}^{d} \lambda_k(A^{(i)})\lambda_j(A^{(k)}) \geq 0, \qquad i, j = 1, \ldots, d,$$

and by using (5) this becomes:

$$2 + \sum_{k=1}^{d} \lambda_k(A^{(i)})\lambda_k(A^{(j)}) \geq 0, \qquad i, j = 1, \ldots, d.$$

Since $\lambda_0(A^{(i)}) = 2$ $(i = 1, \ldots, d)$, and using (4), one can easily verify that the last inequality holds. Indeed, one has

$$\begin{aligned}
& 2 + \sum_{k=1}^{d} \lambda_k(A^{(i)})\lambda_k(A^{(j)}) \\
= \;& 2 + \frac{1}{2}\sum_{k=1}^{n-1} \lambda_k(A^{(i)})\lambda_k(A^{(j)}) \quad \text{(by (6))} \\
= \;& 2 - \frac{1}{2}\lambda_0(A^{(i)})\lambda_0(A^{(j)}) + \frac{1}{2}\sum_{k=0}^{n-1} \lambda_k(A^{(i)})\lambda_k(A^{(j)}) \\
= \;& \begin{cases} 2 - 2 + 0 = 0 & \text{if } (i \neq j), \text{ by (4)} \\ 2 - 2 + \frac{1}{2}\sum_{k=0}^{n-1} \left(\lambda_k(A^{(i)})\right)^2 \geq 0 & \text{if } (i = j). \end{cases}
\end{aligned}$$

Thus we have established the following result.

Theorem 2.1. *The optimal value of the SDP problem (3) provides a lower bound on the optimal value TSP_{opt} of the associated TSP instance.*

3 Relation of (3) to an SDP relaxation of QAP

An SDP relaxation of the QAP problem (1) was introduced in [28], and further studied for specially structured instances in [7].

When applied to the QAP reformulation of TSP in (2), this SDP relaxation takes the form:

$$\left.\begin{array}{rl} \min & \frac{1}{2} trace(C_1 \otimes D)Y \\ \text{subject to} & \\ & trace((I \otimes (J-I))Y + ((J-I) \otimes I)Y) = 0 \\ & trace(Y) - 2e^T y = -n \\ & \begin{pmatrix} 1 & y^T \\ y & Y \end{pmatrix} \succeq 0, \;\; Y \geq 0. \end{array}\right\} \tag{7}$$

It is easy to verify that this is indeed a relaxation of problem (2), by noting that setting $Y = \text{vec}(X)\text{vec}(X)^T$ and $y = diag(Y)$ gives a feasible solution if $X \in \Pi_n$.

In this section we will show that the optimal value of the SDP problem (7) actually equals the optimal value of the new SDP relaxation (3). The proof is via the technique of *symmetry reduction*.

Symmetry reduction of the SDP problem (7)

Consider the following form of a general semidefinite programming problem:

$$p^* := \min_{X \succeq 0, X \geq 0} \{ \; trace(A_0 X) \; : \; trace(A_k X) = b_k, \quad k = 1, \ldots, m \}, \tag{8}$$

where the A_i $(i = 0, \ldots, m)$ are given symmetric matrices.

If we view (7) as an SDP problem in the form (8), the data matrices of problem (7) are:

$$\begin{pmatrix} 0 & 0^T \\ 0 & \frac{1}{2} C_1 \otimes D \end{pmatrix}, \begin{pmatrix} 0 & 0^T \\ 0 & I \otimes (J-I) + (J-I) \otimes I, \end{pmatrix}, \begin{pmatrix} 0 & -e^T \\ -e & 2I \end{pmatrix}, \begin{pmatrix} 1 & 0^T \\ 0 & 0 \end{pmatrix}. \tag{9}$$

Definition 3.1. *We define the* automorphism group *of a matrix $Z \in \mathbb{R}^{k \times k}$ as*

$$aut(Z) = \{ P \in \Pi_k \; : \; PZP^T = Z \}.$$

Symmetry reduction of problem (8) is possible under the assumption that the multiplicative matrix group

$$\mathcal{G} := \bigcap_{i=0}^{m} \text{aut}(A_i)$$

is non-trivial. We call $\mathcal{G}$ the symmetry group of the SDP problem (8).

For the matrices (9), the group $\mathcal{G}$ is given by the matrices

$$\mathcal{G} := \left\{ \begin{pmatrix} 1 & 0^T \\ 0 & P \otimes I \end{pmatrix} \; : \; P \in \mathcal{D}_n \right\}, \tag{10}$$

where $\mathcal{D}_n$ is the (permutation matrix representation of) the dihedral group of order n, i.e. the automorphism group of $\mathcal{C}_n$.

The basic idea of symmetry reduction is given by the following result.

Theorem 3.1 (see e.g. [8]). *If X is a feasible (resp. optimal) solution of the SDP problem (8) with symmetry group $\mathcal{G}$, then*

$$\bar{X} := \frac{1}{|\mathcal{G}|} \sum_{P \in \mathcal{G}} P^T X P$$

is also a feasible (resp. optimal) solution of (8).

Thus there exist optimal solutions in the set

$$\mathcal{A}_{\mathcal{G}} := \left\{ \frac{1}{|\mathcal{G}|} \sum_{P \in \mathcal{G}} P^T X P \ : \ X \in \mathbb{R}^{n \times n} \right\}.$$

This set is called the centralizer ring (or commutant) of $\mathcal{G}$ and it is a matrix $*$-algebra. For the group defined in (10), it is straightforward to verify that the centralizer ring is given by:

$$\mathcal{A}_{\mathcal{G}} := \left\{ \begin{pmatrix} \alpha & x^T \\ y & C \otimes Z \end{pmatrix} \ \middle| \ \alpha \in \mathbb{R},\ C = C^T \text{ circulant},\ Z \in \mathbb{R}^{n \times n},\ x, y \in \mathbb{R}^{n^2} \right\} \tag{11}$$

where $x^T = [x_1 e^T \dots x_n e^T]$ and $y^T = [y_1 e^T \dots y_n e^T]$ for some scalars x_i and y_i $(i = 1, \dots, n)$, where $e \in \mathbb{R}^n$ is the all-ones vector, as before.

Thus we may restrict the feasible set of problem (7) to feasible solutions of the form (11).

If we divide y and Y in (7) into blocks:

$$y = \left(\left(y^{(1)} \right)^T \cdots \left(y^{(n)} \right)^T \right)^T,$$

and

$$Y = \begin{pmatrix} Y^{(11)} & \dots & Y^{(1n)} \\ \vdots & \ddots & \vdots \\ Y^{(n1)} & \dots & Y^{(nn)} \end{pmatrix},$$

where $y^{(i)} \in \mathbb{R}^n$ and $Y^{(ij)} = Y^{(ji)T} \in \mathbb{R}^{n \times n}$, then feasible solutions of (7) satisfy

$$\begin{pmatrix} 1 & \left(y^{(1)} \right)^T & \dots & \left(y^{(n)} \right)^T \\ y^{(1)} & Y^{(11)} & \dots & Y^{(1n)} \\ \vdots & \vdots & \ddots & \vdots \\ y^{(n)} & Y^{(n1)} & \dots & Y^{(nn)} \end{pmatrix} \succeq 0. \tag{12}$$

Feasible solutions have the following additional structure (see [28] and Theorem 3.1 in [7]):

- $Y^{(ii)}$ $(i = 1, \dots, n)$ is a diagonal matrix;
- $Y^{(ij)}$ $(i \neq j)$ is a matrix with zero diagonal;

- $trace(JY^{(ij)}) = 1$ $(i, j = 1, \ldots, n)$;
- $\sum_{i=1}^{n} Y^{(ij)} = e\left(y^{(j)}\right)^T \quad (j = 1, \ldots, n)$;
- $diag(Y) = y$.

Since $diag(Y) = y$ for feasible solutions, we have $y^{(i)} = diag(Y^{(ii)})$ $(i = 1, \ldots, n)$. Moreover, since we may also assume the structure (11), we have that

$$y^{(i)} = y_i e \quad (i = 1, \ldots, n),$$

for some scalar values y_i. This implies that the diagonal elements of $Y^{(ii)}$ all equal y_i. Since the diagonal elements of $Y^{(ii)}$ sum to 1, we have $y_i = 1/n$ and $diag(Y^{(ii)}) = (1/n)e$. Thus the condition:

$$\begin{pmatrix} 1 & y^T \\ y & Y \end{pmatrix} \succeq 0$$

reduces to

$$Y - \frac{1}{n^2} J \succeq 0$$

by the Shur complement theorem. This is equivalent to

$$(I \otimes Q^*)Y(I \otimes Q) - \frac{1}{n^2}(I \otimes Q^*)J(I \otimes Q) \succeq 0$$

where Q is the discrete Fourier transform matrix defined in (25) in the Appendix.

Using the properties of the Kronecker product and of Q we get

$$\begin{pmatrix} Q^*Y^{(11)}Q & \cdots & Q^*Y^{(1n)}Q \\ \vdots & \ddots & \vdots \\ Q^*Y^{(n1)}Q & \cdots & Q^*Y^{(nn)}Q \end{pmatrix} - J \otimes \begin{pmatrix} \frac{1}{n} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \succeq 0.$$

Recall that $Y^{(ii)} = \frac{1}{n} I$ and that we may assume $Y^{(ij)}$ $(i \neq j)$ to be symmetric circulant, say

$$Y^{(ij)} = \sum_{k=1}^{d} x_k^{(ij)} C_k, \quad (i \neq j),$$

where C_k $(k = 1, \ldots, d)$ forms a basis of the symmetric circulant matrices with zero diagonals (see the Appendix for the precise definition). Note that the nonnegativity of $Y^{(ij)}$ is equivalent to $x_k^{(ij)} \geq 0$ $(k = 1, \ldots, d)$. Since $trace(JY^{(ij)}) = 1$ one has

$$\sum_{k=1}^{d} x_k^{(ij)} = \frac{1}{2n} \quad (i \neq j).$$

Since $\sum_{i=1}^{n} Y^{(ij)} = e\left(y^{(j)}\right)^T = \frac{1}{n} J$, one also has

$$\sum_{k=1}^{d} \sum_{i=1}^{n} x_k^{(ij)} C_k = \frac{1}{n} J.$$

By the definition of the C_k's, this implies that

$$\sum_{i=1}^{n} x_k^{(ij)} = \begin{cases} \frac{1}{n} & \text{if } 1 \le k \le \lfloor (n-1)/2 \rfloor \\ \frac{1}{2n} & \text{if } k = n/2 \ (n \text{ even}). \end{cases} \tag{13}$$

Moreover,

$$Q^* Y^{(ij)} Q = \sum_{k=1}^{d} x_k^{(ij)} D_k, \quad (i \neq j),$$

where D_k is the diagonal matrix with the eigenvalues (26) of C_k on its diagonal.

Thus the LMI becomes

$$\begin{pmatrix} \frac{1}{n} I & \cdots & \sum_{k=1}^{d} x_k^{(1n)} D_k \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{d} x_k^{(1n)} D_k & \cdots & \frac{1}{n} I \end{pmatrix} - J \otimes \begin{pmatrix} \frac{1}{n} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \succeq 0. \tag{14}$$

The left hand side of this LMI is a block matrix with each block being a diagonal matrix. Thus this matrix has a chordal sparsity structure (n disjoint cliques of size n). We may now use the following lemma to obtain the system of LMI's (3).

Lemma 3.1 (cf. [14])**.** *Assume a $nt \times nt$ matrix has the block structure*

$$M := \begin{pmatrix} D^{(11)} & \cdots & D^{(1n)} \\ \vdots & \ddots & \vdots \\ D^{(n1)} & \cdots & D^{(nn)} \end{pmatrix},$$

where $D^{(ij)} \in \mathcal{S}_t$ are diagonal $(i, j = 1, \ldots, n)$. Then $M \succeq 0$ if and only if:

$$\begin{pmatrix} D_{ii}^{(11)} & \cdots & D_{ii}^{(1n)} \\ \vdots & \ddots & \vdots \\ D_{ii}^{(n1)} & \cdots & D_{ii}^{(nn)} \end{pmatrix} \succeq 0 \quad i = 1, \ldots, t.$$

Applying the lemma to the LMI (14), and setting

$$X_{ij}^{(k)} = 2n x_k^{(ij)}, \qquad k = 1, \ldots, \lfloor n/2 \rfloor, \tag{15}$$

yields the system of LMI's in (3).

Thus we have established the following result.

Theorem 3.2. *The optimal values of the semidefinite programs (3) and (7) are equal.*

4 Relation of (3) to an SDP relaxation of Cvetković et al.

We will now show that the new SDP relaxation (3) dominates an SDP relaxation (16) due to Cvetković et al. [5]. This latter relaxation is based on the fact that the spectrum of the Hamiltonian circuit $\mathcal{C}_n$ is known. In particular, the smallest eigenvalue of its Laplacian is zero and corresponds to the all ones eigenvector, while the second smallest eigenvalue equals $2 - 2\cos\left(\frac{2\pi}{n}\right)$.

The relaxation takes the form:

$$TSP_{opt} \geq \min \frac{1}{2} trace(DX)$$

subject to

$$\left.\begin{array}{rcl} Xe & = & 2e, \\ \mathrm{diag}(X) & = & 0, \\ 0 & \leq & X \leq J, \\ 2I - X + \left(2 - 2\cos\left(\frac{2\pi}{n}\right)\right)(J - I) & \succeq & 0. \end{array}\right\} \tag{16}$$

Note that the matrix variable X corresponds to the adjacency matrix of the minimal length Hamiltonian circuit.

Theorem 4.1. *The SDP relaxation (3) dominates the relaxation (16).*

Proof. Assume that given $X^{(k)} \in \mathcal{S}^n$ $(k = 1, \ldots, d)$ satisfy (3). Then, $\mathrm{diag}(X^{(1)}) = 0$, while (13) and (15) imply

$$X^{(k)}e = 2e \quad (k = 1, \ldots, \lfloor (n-1)/2 \rfloor),$$

and $X^{(n/2)}e = e$ if n is even. In particular, one has $X^{(1)}e = 2e$. It remains to show that

$$2I - X^{(1)} + \left(2 - 2\cos\left(\frac{2\pi}{n}\right)\right)(J - I) \succeq 0,$$

which is the same as showing that

$$2I - X^{(1)} + \left(2 - 2\cos\left(\frac{2\pi}{n}\right)\right)\sum_{k=1}^{d} X^{(k)} \succeq 0, \tag{17}$$

since

$$\sum_{k=1}^{d} X^{(k)} = J - I.$$

We will show that the LMI (17) may be obtained as a nonnegative aggregation of the LMI's

$$I + \sum_{k=1}^{d} X^{(k)} \succeq 0$$

and

$$I + \sum_{k=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) X^{(k)} \succeq 0 \quad (i = 1, \ldots, d).$$

The matrix of coefficients of these LMI's is a $(d+1) \times (d+1)$ matrix, say A, with entries:

$$A_{ij} = \cos\left(\frac{2\pi ij}{n}\right) \quad (i, j = 0, \ldots, d).$$

Since we may rewrite (17) as

$$2I + \left(1 - 2\cos\left(\frac{2\pi}{n}\right)\right) X^{(1)} + \left(2 - 2\cos\left(\frac{2\pi}{n}\right)\right) \sum_{k=2}^{d} X^{(k)} \succeq 0,$$

we need to show that the linear system $Ax = b$ has a nonnegative solution, where

$$b := \left[2, \left(1 - 2\cos\left(\frac{2\pi}{n}\right)\right), \left(2 - 2\cos\left(\frac{2\pi}{n}\right)\right), \ldots, \left(2 - 2\cos\left(\frac{2\pi}{n}\right)\right)\right]^T.$$

One may verify that, for n odd, the system $Ax = b$ has a (unique) solution given by

$$x_i = \frac{4}{n} \begin{cases} d\left(1 - \cos\left(\frac{2\pi}{n}\right)\right) & \text{if } i = 0 \\ \cos\left(\frac{2\pi}{n}\right) - \cos\left(\frac{2\pi i}{n}\right) & \text{for } i = 1, \ldots, d. \end{cases}$$

Note that x is nonnegative, as it should be. If n is even, the solution is

$$x_i = \frac{4}{n} \begin{cases} \frac{(n-1)}{2}\left(1 - \cos\left(\frac{2\pi}{n}\right)\right) & \text{if } i = 0 \\ \cos\left(\frac{2\pi}{n}\right) - \cos\left(\frac{2\pi i}{n}\right) & \text{for } i = 1, \ldots, d-1 \\ \frac{1}{2}\cos\left(\frac{2\pi}{n}\right) - \frac{1}{2}\cos\left(\frac{2\pi i}{n}\right) & \text{for } i = d. \end{cases}$$

□

In the section with numerical examples, we will present instances where the new SDP relaxation (3) is strictly better than (16).

5 Relation to the Held-Karp bound

One of the best-known linear programming (LP) relaxations of TSP is the LP with sub-tour elimination constraints:

$$TSP_{opt} \geq \min \frac{1}{2} trace(DX)$$

subject to

$$\left.\begin{array}{rcll} Xe & = & 2, & \\ \mathrm{diag}(X) & = & 0, & \\ 0 & \leq & X \leq J, & \\ \sum_{i \in \mathcal{I},\, j \notin \mathcal{I}} X_{ij} & \geq & 2 & \forall\, \emptyset \neq \mathcal{I} \subset \{1, \ldots, n\}. \end{array}\right\} \tag{18}$$

This LP relaxation dates back to 1954 and is due to Dantzig, Fulkerson and Johnson [6]. Its optimal value coincides with the LP bound of Held and Karp [15] (see e.g. Theorem 21.34 in [16]), and the optimal value of the LP is commonly known as the *Held-Karp bound*.

The last constraints are called *sub-tour elimination inequalities* and model the fact that $\mathcal{C}_n$ is 2-connected. Although there are exponentially many sub-tour elimination inequalities, it is well-known that the LP (18) may be solved in polynomial time using the ellipsoid method; see e.g. Schrijver [24], §58.5.

It was shown by Goemans and Rendl [12] that this LP relaxation dominates the SDP relaxation (16) by Cvetković et al. [5]. The next theorem shows that the LP relaxation (18) does not dominate the new SDP relaxation (3), or vice versa.

Theorem 5.1. *The LP sub-tour elimination relaxation (18) does not dominate the new SDP relaxation (3), or vice versa.*

Proof. Define the 8×8 symmetric matrix $\bar{X}$ as the weighted adjacency matrix of the graph shown in Figure 1.

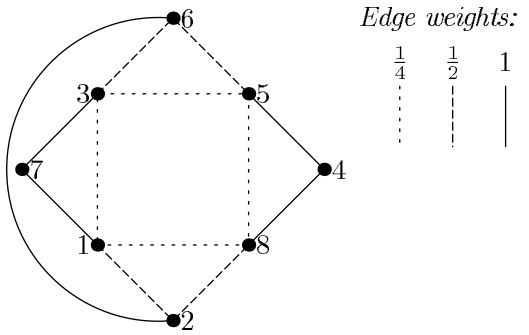

Figure 1: The weighted graph used in the proof of Theorem 5.1.

The matrix $\bar{X}$ satisfies the sub-tour elimination inequalities, since the minimum cut in the graph in Figure 1 has weight 2.

On the other hand, there does not exist a feasible solution of (3) that satisfies $X^{(1)} = \bar{X}$, as may be shown using SDP duality theory.

Conversely, in Section 7 we will provide examples where the optimal value of (18) is strictly greater than the optimal value of (3) (see e.g. the instances gr17, gr24 and bays24 there). □

6 An LMI cut via the number of spanning trees

In addition to the sub-tour elimination inequalities, there are several families of linear inequalities known for the TSP polytope; for a review, see Naddef [18] and Schrijver [24], Chapter 58.

Of particular interest to us is a valid nonlinear inequality that models the fact that $\mathcal{C}_n$ has n distinct spanning trees. To introduce the inequality we require a general form of the *matrix tree theorem*; see *e.g.* Theorem VI.29 in [26] for a proof.

Theorem 6.1 (Matrix tree theorem)**.** *Let a simple graph $G = (V, E)$ be given and associate with each edge $e \in E$ a real variable x_e. Define the (generalized) Laplacian of G with respect to x as the $|V| \times |V|$ matrix with entries*

$$L(G)(x)_{ij} := \begin{cases} \sum_{e\,:\,e \cap i \neq \emptyset} x_e & \text{if } i = j, \\ -x_e & \text{if } \{i, j\} = e, \\ 0 & \text{else.} \end{cases}$$

Now all principal minors of $L(G)(x)$ of order $|V| - 1$ equal:

$$\sum_T \prod_{e \in T} x_e, \tag{19}$$

where the sum is over all distinct spanning trees T of G.

In particular, if $L(G)(x)$ is the usual Laplacian of a given graph, then $x_e = 1$ for all edges e of the graph, and expression (19) evaluates to the number of spanning trees in the graph.

Thus if X corresponds to the approximation of the adjacency matrix of a minimum tour, then one may require that:

$$\det (2I - X)_{2:n,2:n} \geq n, \tag{20}$$

where $X_{2:n,2:n}$ denotes the principle submatrix of X obtained by deleting the first row and column.

The inequality (20) may be added to the above SDP relaxations (16) and (3) (with $X = X^{(1)}$), since the set

$$\{Z \succeq 0 \ : \ \det Z \geq n\}$$

is LMI representable; see e.g. Nemirovski [19], §3.2.

We know from numerical examples that (20) is not implied by the relaxation of Cvetković et al. (16), but do not know any examples where it is violated by a feasible $X^{(1)}$ of the new relaxation (3). Nevertheless, we have been unable to show that (20) (with $X = X^{(1)}$) is implied by (3).

7 Numerical examples

In Table 1 we give the lower bounds on some small TSPLIB[1] instances for the two SDP relaxations (3) and (16), as well as the LP relaxation with all sub-tour elimination constraints (18) (the Held-Karp bound). These instances

[1] `http://www.iwr.uni-heidelberg.de/groups/comopt/software/TSPLIB95/`

have integer data, and the optimal values of the relaxations were rounded up to obtain the bounds in the table.

The SDP problems were solved by the interior point software CSDP [2] using the Yalmip interface [17] and Matlab 6.5, running on a PC with two 2.1 GHz dual-core processors and 2GB of memory.

Problem	SDP bound (16)	SDP bound (3) (time)	LP bound (18)	TSP_{opt}
gr17	1810	2007 (39s)	2085	2085
gr21	2707	2707 (139s)	2707	2707
gr24	1230	1271 (1046s)	1272	1272
bays29	1948	2000 (2863s)	2014	2020

Table 1: Lower bounds on some small TSPLIB instances from various convex relaxations.

Note that the relaxation (3) can indeed be strictly better than (16), as is clear from the gr17, bays24 and bays29 instances. Also, since the LP relaxation (18) gives better bounds than (3) for all four instances, it is worth recalling that this will not happen in general, by Theorem 5.1.

The LMI cut from (20) was already satisfied by the optimal solutions of (16) and (3) for the four instances.

A second set of test problems was generated by considering all facet defining inequalities for the TSP polytope on 8 nodes; see [3] for a description of these inequalities, as well as the SMAPO project web site[2].

The facet defining inequalities are of the form $\frac{1}{2}trace(DX) \geq RHS$ where $D \in \mathcal{S}_n$ has nonnegative integer entries and RHS is an integer. From each inequality, we form a symmetric TSP instance with distance matrix D. Thus the optimal value of the TSP instance is the value RHS. In Table 2 we give the optimal values of the LP relaxation (18) (*i.e.* the Held-Karp bound), the SDP relaxation of Cvetković et al. (16), and the new SDP relaxation (3) for these instances, as well as the right-hand-side RHS of each inequality $\frac{1}{2}trace(DX) \geq RHS$. For $n = 8$, there are 24 classes of facet-defining inequalities. The members of each class are equal modulo a permutation of the nodes, and we need therefore only consider one representative per class. The first three classes of inequalities are sub-tour elimination inequalities.

The numbering of the instances in Table 2 coincides with the numbering of the classes of facet defining inequalities on the SMAPO project web site.

The new SDP bound (3) is only stronger than the Held-Karp bound (18) for the instances 16, 21 and 23 in Table 2, and for the instances 1, 5, 9, 12, 14, 17, 18, 19 and 22 the two bounds coincide. For the remaining 18 instances the Held-Karp bound is better than the SDP bound (3). However, if the bounds are rounded up, the SDP bound (3) is still better for the instances 16, 21 and 23, whereas the two (rounded) bounds are equal for all the other instances. Adding

[2] `http://www.iwr.uni-heidelberg.de/groups/comopt/software/SMAPO/tsp/`

Inequality	SDP bound (16)	SDP bound (3)	Held-Karp bound (18)	RHS
1	2	2	2	2
2	1.098	1.628	2	2
3	1.172	1.172	2	2
4	8.507	8.671	9	10
5	9	9	9	10
6	8.566	8.926	9	10
7	8.586	8.586	9	10
8	8.570	8.926	9	10
9	9	9	9	10
10	8.411	8.902	9	10
11	8.422	8.899	9	10
12	0	0	0	0
13	10.586	10.667	11	12
14	12	12	12	13
15	12.408	12.444	$12\frac{2}{3}$	14
16	14	14.078	14	16
17	16	16	16	18
18	16	16	16	18
19	16	16	16	18
20	15.185	15.926	16	18
21	18	18.025	18	20
22	20	20	20	22
23	23	23.033	23	26
24	34.586	34.739	35	38

Table 2: Results for instances on $n = 8$ cities, constructed from the facet defining inequalities.

the LMI cut from (20) did not change the optimal values of the SDP relaxations (16) or (3) for any of the instances.

For $n = 9$, there are 192 classes of facet defining inequalities of the TSP polytope [4]. Here the SDP bound (3) is better than the Held-Karp bound for 23 out of the 192 associated TSP instances. Similar to the $n = 8$ case, when rounding up, the rounded SDP bound remains better in all 23 cases and coincides with the rounded Held-Karp bound in all the remaining cases.

8 Concluding remarks

Wolsey [27] showed that the optimal value of the LP relaxation (18) is at least 2/3 the length of an optimal tour for metric TSP (see also [25]). An interesting question is whether a similar result may be proved for the new SDP relaxation (3).

Finally, the computational perspectives of the SDP relaxation (3) are somewhat limited due to its size. However, since it provides a new polynomial-time convex approximation of TSP with a rich mathematical structure, it is our hope that it may lead to a renewed interest in improving approximation results for

metric TSP.

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Appendix: Circulant matrices

Our discussion of circulant matrices is condensed from the review paper by Gray [13].

A circulant matrix has the form

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & & \\ & c_{n-1} & c_0 & c_1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & c_1 \\ c_1 & & \cdots & & c_{n-1} & c_0 \end{bmatrix}. \tag{21}$$

Thus the entries satisfy the relation

$$C_{ij} = c_{(j-i) \mod n}. \tag{22}$$

The matrix C has eigenvalues

$$\lambda_m(C) = c_0 + \sum_{k=1}^{n-1} c_k e^{-2\pi\sqrt{-1}mk/n}, \quad m = 0, \ldots, n-1.$$

If C is symmetric with n odd, this reduces to

$$\lambda_m(C) = c_0 + \sum_{k=1}^{(n-1)/2} 2c_k \cos(2\pi mk/n), \quad m = 0, \ldots, n-1, \tag{23}$$

and when n is even we have

$$\lambda_m(C) = c_0 + \sum_{k=1}^{n/2-1} 2c_k \cos(2\pi mk/n) + c_{n/2}\cos(m\pi), \quad m = 0, \ldots, n-1. \tag{24}$$

The circulant matrices form a commutative matrix $*$-algebra, as do the symmetric circulant matrices. In particular, all circulant matrices share a set of eigenvectors, given by the columns of the *discrete Fourier transform matrix*:

$$Q_{ij} := \frac{1}{\sqrt{n}} e^{-2\pi\sqrt{-1}ij/n}, \quad i, j = 0, \ldots, n-1. \tag{25}$$

One has $Q^*Q = I$, and Q^*CQ is a diagonal matrix for any circulant matrix C. Also note that $Q^*e = \sqrt{n}e$.

We may define a basis $C^{(0)}, \ldots, C^{\lfloor n/2 \rfloor}$ for the symmetric circulant matrices as follows: to obtain $C^{(i)}$ we set $c_i = c_{n-i} = 1$ in (21) and all other c_j's to zero. (We set $C_0 = 2I$ and also multiply $C_{n/2}$ by 2 if n is even).

By (23) and (24), the eigenvalues of these basis matrices are:

$$\lambda_m(C^{(k)}) = 2\cos(2\pi mk/n), \quad m = 0, \ldots, n-1,\ k = 0, \ldots, \lfloor n/2 \rfloor. \tag{26}$$

Also note that

$$\lambda_m(C^{(k)}) = \lambda_{n-m}(C^{(k)}), \quad m = 1, \ldots, \lfloor n/2 \rfloor,\ k = 0, \ldots, \lfloor n/2 \rfloor$$

so that each matrix $C^{(k)}$ has only $1 + \lfloor n/2 \rfloor$ distinct eigenvalues.

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