A discrete multivariate mean value theorem with applications

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Discrete Optimization

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Abstract

We establish a discrete multivariate mean value theorem for the class of positive maximum component sign preserving functions. A constructive and combinatorial proof is given based upon a simplicial algorithm and vector labeling. Moreover, we apply this theorem to a discrete nonlinear complementarity problem and an economic equilibrium problem with indivisibilities and show the existence of solutions in both problems under certain mild conditions.
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1. Introduction

Fixed (or zero) point theorems are fundamental tools for establishing the existence of solutions to nonlinear problems in various fields including mathematics, economics and engineering. The most well-known fixed point theorem is the Brouwer theorem, stating that any continuous function mapping from an *n*-dimensional unit ball to itself has a fixed point. Among many extensions, Kakutani's theorem generalizes the Brouwer theorem from a single-value function to point-to-set mappings and together with the Brouwer theorem has been widely used in the economic literature. Starting with Scarf (1967), simplicial algorithms have been developed, which can effectively find an approximate fixed point of any a priori given accuracy in a finite number of steps, and lead to constructive proofs to many fundamental fixed point theorems including Brouwer's and Kakutani's. More importantly, these algorithms can

be used to solve many practical problems that actually require the location of a solution. Efficient simplicial algorithms can be found in Eaves (1972), Eaves and Saigal (1972), Merrill (1972), Laan van der and Talman (1979, 1981), Reiser (1981), Saigal (1983), Freund (1984), and Yamamoto (1984) among others. For a background on the subject, one may consult with Allgower and Georg (1990), Todd (1976), and Yang (1999).

This paper is concerned with the existence of an integral solution to the system of nonlinear equations

$$f(x) = 0^n,$$

where 0^n is the *n*-vector of zeros, f is a nonlinear function from $\mathbb{Z}^n$ to $\mathbb{R}^n$, and $\mathbb{Z}^n$ is the set of integer vectors in the *n*-dimensional Euclidean space $\mathbb{R}^n$. An integral solution x^* is called a *discrete zero point* of f and the problem is called the *discrete zero point problem*. Obviously, this problem is equivalent to *the discrete fixed point problem* of the function $g(x) = f(x) + x$. While many fixed point theorems such as Brouwer's or Kakutani's concern continuous or upper semi-continuous mappings defined on a nonempty convex and compact set, the current problem concerns functions whose domain is a discrete set rather than a convex set and which do not possess any kind of continuity or upper semi-continuity. The major motivation of studying

the current discrete problem comes from the recent studies on exchange economies with indivisible goods and broadly speaking on models with discrete variables in the area of economics.

The study of the discrete zero point problem dates back to Tarski (1955). He shows that a weakly increasing function mapping from a finite lattice into itself has at least one fixed point. Somehow surprisingly, this result has long been the only prominent fixed point theorem having a discrete nature, in contrast to the rich literature on fixed point theorems of continuous nature. Recently, progress has been made toward relaxing the monotonicity assumption in Tarski's theorem, by Iimura et al. (2005), Danilov and Koshevoy (2004) and Yang (accepted for publication, 2004). They were all motivated by Iimura's (2003) discrete fixed point statement. In Iimura et al. (2005) a corrected version of Iimura's discrete fixed point theorem is established, while a similar theorem is given by Danilov and Koshevoy (2004). Both these papers deal with the class of so-called *direction preserving functions* owing to Iimura (2003), which need not be monotonic. In Yang (accepted for publication, 2004) a number of more general discrete fixed (and zero) point theorems are established, which contain the results of Iimura et al. (2005) and Danilov and Koshevoy (2004) as special cases. The existence theorems of Yang concern the class of so-called *locally gross direction preserving* mappings, which is substantially more general than the class of Iimura's direction preserving mappings. In addition to these existence results, Yang (accepted for publication) also studies discrete nonlinear complementarity problems and presents several sufficient conditions for the existence of solution for this class of problems. In Danilov and Koshevoy (accepted for publication) and Yang (2004) a class of more general but more complex functions is introduced for the existence of a discrete fixed point. All the results mentioned above are proved using the machinery of topology such as the Brouwer fixed point theorem or Borsuk-Ulam theorem. Such proofs are therefore nonconstructive and indirect. More recently, Laan van der et al. (accepted for publication, 2006, 2007) propose a constructive approach, namely, simplicial algorithms to find a discrete fixed (or zero) point of direction preserving functions and locally gross direction preserving functions under general conditions.

The objective of this paper is to establish a further general discrete zero or fixed point theorem based upon the class of so-called *positive maximum component sign preserving functions*. This class of functions generalizes substantially the class of direction preserving functions but differs from the class of locally gross direction preserving functions. Furthermore, when applied to an economic context, this class of functions admits an economically meaningful interpretation. Our discrete zero point theorem can be seen as a discrete analogue of the well-known multivariate mean value theorem for continuous functions (see Istratescu (1981) and Yang (1999)) and thus will be called *a discrete multivariate mean value theorem*. A constructive and combi-

natorial proof for this theorem will be given. The argument is based on the familiar idea of following a piecewise linear path of points in a triangulation. More precisely, we adapt the so-called $2n$-ray simplicial algorithm of Laan van der and Talman (1981) and Reiser (1981), to the current discrete setting. This algorithm was originally proposed to approximate a fixed point of a continuous function. In the current discrete setting, the algorithm will operate on an integral triangulation of $\mathbb{R}^n$ underlying the function f. Starting from any integral point in $\mathbb{Z}^n$, the algorithm generates a finite sequence of adjacent simplices of varying dimension and terminates in a finite number of steps with a simplex in which one of its vertices is a discrete zero point. As a result, this yields a constructive and combinatorial proof for our discrete multivariate mean value theorem. Furthermore, we discuss two applications, the discrete nonlinear complementarity problem and a discrete equilibrium existence problem with indivisibilities.

This paper is organized as follows. Section 2 presents basic concepts. Sections 3 establishes the discrete multi-variate mean value theorem. Section 4 discusses two applications in complementarity theory and economic theory. Section 5 concludes.

2. Basic concepts

We first give some general notation. For a given positive integer n, let N denote the set $\{1, 2, \ldots, n\}$. For $x, y \in \mathbb{R}^n$, $x \cdot y$ stands for the inner product of x and y. For $i \in N$, $e(i)$ denotes the ith unit vector of $\mathbb{R}^n$. Given a set $D \subseteq \mathbb{R}^n$, $Co(D)$ and $Bd(D)$ denote the convex hull of D and the relative boundary of D, respectively. For any x and y in $\mathbb{R}^n$, we say y is lexicographically greater than x, and denote it by $y \succeq x$, if the first nonzero component of $y - x$ is positive.

For any integer t, $0 \leqslant t \leqslant n$, the t-dimensional convex hull of $t + 1$ affinely independent points $x^1, \ldots, x^{t+1}$ in $\mathbb{R}^n$ is called a *t-simplex* or *simplex* and will be denoted by σ or $\sigma = < x^1, \ldots, x^{t+1} >$. The extreme points $x^1, \ldots, x^{t+1}$ of a t-simplex σ are called the *vertices* of σ. A k-simplex τ is called a *face* or k-face of a t-simplex σ if all vertices of τ are also vertices of σ. A k-face τ of a t-simplex σ is called a *facet* of σ if $k = t - 1$, i.e., if the number of vertices of τ is one less than the number of vertices of the simplex. Two integral points x and y in $\mathbb{Z}^n$ are said to be *cell-connected* if $\max_{h \in N} |x_h - y_h| \leqslant 1$, i.e., their distance is less than or equal to one according to the maximum norm. A simplex is said to be *integral* if all of its vertices are cell-connected and integral vectors.

Given an m-dimensional convex set D, a collection $\mathscr{T}$ of m-dimensional simplices is a *triangulation* or *simplicial subdivision* of the set D, if (i) D is the union of all simplices in $\mathscr{T}$, (ii) the intersection of any two simplices of $\mathscr{T}$ is either empty or a common face of both, and (iii) any neighborhood of any point in D only meets a finite number of simplices of $\mathscr{T}$. A facet of a simplex of $\mathscr{T}$ either lies on the boundary of D and is not a facet of any other simplex of

$\mathcal{T}$ or is a facet of precisely one other simplex of $\mathcal{T}$. A triangulation is called *integral* if all its simplices are integral. Two points x and y are *simplicially connected* in an integral triangulation if they are vertices of the same simplex of the triangulation. One of the most well-known integral triangulations of $\mathbb{R}^n$ is the K-triangulation with grid size 1, due to Freudenthal (1942). This triangulation is defined to be the collection of all n-dimensional simplices $\sigma(x,\pi)$ with vertices $x^1, \ldots, x^{n+1}$, where $x \in \mathbb{Z}^n$, $\pi = (\pi(1), \ldots, \pi(n))$ is a permutation of the elements $1, 2, \ldots, n$, and the vertices are given by $x^1 = x$ and $x^{i+1} = x^i + e(\pi(i))$, $i = 1, \ldots, n$.

We are now ready to introduce two new classes of discrete functions.

Definition 2.1. A function $f : \mathbb{Z}^n \to \mathbb{R}^n$ is maximum positive component sign preserving if for any cell-connected points x and y in $\mathbb{Z}^n$, $f_j(x) = \max_{h \in N} f_h(x) > 0$ implies $f_j(y) \geqslant 0$.

The maximum positive component sign preservation condition concerns only those components of the function that have maximum positive value and requires that within a cell a component of the function value vector should not jump from a positive maximum to a negative value. This condition replaces continuity of a function in case the domain is not discrete.

It is useful to compare the new class of functions with the existing classes of functions due to Iimura (2003) and Yang (accepted for publication, 2004). From Iimura (2003) a function $f : \mathbb{Z}^n \to \mathbb{R}^n$ is *direction preserving* if for any two cell connected points x and y in $\mathbb{Z}^n$,

$$f_j(x) f_j(y) \geqslant 0 \quad \text{for all } j \in N;$$

and from Yang (accepted for publication, 2004) that a function $f : \mathbb{Z}^n \to \mathbb{R}^n$ is *locally gross direction preserving* if for any two cell connected points x and y in $\mathbb{Z}^n$,

$$f(x) \cdot f(y) \geqslant 0.$$

Clearly, the class of maximum positive component sign preserving functions is substantially more general than the class of direction preserving functions and so is the class of locally gross direction preserving functions. However, the following examples show that positive maximum component sign preserving functions and locally gross direction preserving functions are incomparable in the sense that they do not imply each other.

Example 1. Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be defined by $f(x) = (3, 2)$ for $x = (1, 0)$, $f(x) = (-1, 2)$ for $x = (0, 1)$, and $f(x) = (0, 0)$ otherwise. Clearly, f is locally gross direction preserving but not positive maximum component sign preserving.

Example 2. Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be defined by $f(x) = (3, 2)$ for $x = (1, 0)$, $f(x) = (1, -2)$ for $x = (0, 1)$, and $f(x) = (0, 0)$ otherwise. Clearly, f is positive maximum component sign preserving but not locally gross direction preserving.

Positive maximum component sign preservingness can be relaxed by imposing the condition only on any two vertices of a same simplex in some integral triangulation of $\mathbb{R}^n$.

Definition 2.2. A function $f : \mathbb{Z}^n \to \mathbb{R}^n$ is simplicially positive maximum component sign preserving if there exists an integral triangulation $\mathcal{T}$ of $\mathbb{R}^n$ such that for any simplicially connected vertices x and y of $\mathcal{T}$, $f_j(x) = \max_{h \in N} f_h(x) > 0$ implies $f_j(y) \geqslant 0$.

It is easy to see that if a function is positive maximum component sign preserving, it must be simplicially positive maximum component sign preserving with respect to any integral triangulation of $\mathbb{R}^n$. Consider Example 1 again. This example shows that a simplicially positive maximum component sign preserving function need not be positive maximum component sign preserving. Clearly, f is simplicially positive maximum component sign preserving with respect to the K-triangulation of $\mathbb{R}^2$ but not positive maximum component sign preserving, because for the cell-connected points $(0, 1)$ and $(1, 0)$ it holds that $f_1(1, 0) = \max_h f_h(1, 0) = 3$ and $f_1(0, 1) = -1 < 0$. However, these points are not vertices of any simplex of the K-triangulation of $\mathbb{R}^2$.

In Yang (2004), the more general but more complex class of interior zero point excludable functions is introduced. A function $f : \mathbb{Z}^n \to \mathbb{R}^n$ is *interior zero point excludable* if for any cell connected points $x^1, \ldots, x^t$ and any $\beta_1 \geqslant 0, \ldots, \beta_t \geqslant 0$, $\sum_{i=1}^t \beta_i = 1$ and $\sum_{i=1}^t \beta_i f(x^i) = 0^n$ imply $f(x^i) = 0^n$ for some i. Danilov and Koshevoy (forthcoming) independently introduce a similar class of such functions using the concept of pointed cone. Notice that both classes of functions are introduced for cell-connected points.

3. A discrete multivariate mean value theorem

In this section we establish the following discrete multivariate mean value theorem and give a constructive and combinatorial proof for the theorem.

Theorem 3.1. *Let* $f : \mathbb{Z}^n \to \mathbb{R}^n$ *be a simplicially positive maximum component sign preserving function. If there exist* $l, u \in \mathbb{Z}^n$ *with* $u_h > l_h + 1$ *for every* h *such that for every* $x \in \mathbb{Z}^n$, $x_j = l_j$ *implies* $f_j(x) \leqslant 0$ *and* $x_j = u_j$ *implies* $f_j(x) \geqslant 0$, *then* f *has a discrete zero point* $x^* \in \mathbb{Z}^n$.

To prove the theorem, we adapt the $2n$-ray algorithm of Laan van der and Talman (1981), which was originally introduced to approximate a fixed point of a continuous function, to the current discrete setting. Let f be a simplicially positive maximum component sign preserving function with respect to the integral triangulation $\mathcal{T}$ of $\mathbb{R}^n$. In case f is positive maximum component sign preserving, we can take any integral triangulation of $\mathbb{R}^n$. Let v be any integral vector in $\mathbb{Z}^n$ lying between the lower bound l and the upper bound u as stated in the theorem, i.e., $l_h < v_h < u_h$ for all $h \in N$. The point v will be the starting point of the algorithm. If $f(v) = 0^n$, then v is a discrete zero point of the function f and the algorithm immediately stops with this solution. In the following let us assume that $f(v) \neq 0^n$. For a nonzero sign vector $s \in \{-1, 0, +1\}^n$, the subset $A(s)$ of $\mathbb{R}^n$ is defined by

$$A(s) = \{x \in \mathbb{R}^n | x = v + \sum_{h \in N} \alpha_h s_h e(h), \quad \alpha_h \geqslant 0, h \in N\}.$$

Clearly, the set $A(s)$ is a t-dimensional subset of $\mathbb{R}^n$, where t is the number of nonzero components of the sign vector s, i.e., $t = |\{i | s_i \neq 0\}|$. Since $\mathscr{T}$ is an integral triangulation of $\mathbb{R}^n$, it triangulates every set $A(s)$ into t-dimensional integral simplices. For some s with t nonzero components, denote $\{h_1, \ldots, h_{n-t}\} = \{h | s_h = 0\}$ and let $\sigma = < x^1, \ldots, x^{t+1}>$ be a t-simplex of the triangulation in $A(s)$. Following Laan van der and Talman (1981) and Todd (1980), we say that σ is *almost s-complete* if there is an $(n+2) \times (n+1)$ matrix W satisfying

$$\begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ f(x^1) & \cdots & f(x^{t+1}) & -e(h_1) & \cdots & -e(h_{n-t}) & s \end{bmatrix} W = I \tag{3.1}$$

and having rows $w^1, \ldots, w^{n+2}$ such that $w^h \succeq 0^{n+1}$ for $1 \leqslant h \leqslant t+1$, and $w^{n+2} \succeq w^i$ and $w^{n+2} \succeq -w^i$ for $t+1 < i \leqslant n+1$, and $w^{n+2} \succeq 0^{n+1}$. Here I denotes the identity matrix of rank $n+1$ and 0^{n+1} is the $(n+1)$-vector of zeroes. If $w_1^{n+2} = 0$, then we say that the simplex σ is *complete*. Further, let τ be a facet of σ, and, without loss of generality, index the vertices of σ such that $\tau = < x^1, \ldots, x^t>$. We say that τ is *s-complete* if there is an $(n+1) \times (n+1)$ matrix W satisfying

$$\begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ f(x^1) & \cdots & f(x^t) & -e(h_1) & \cdots & -e(h_{n-t}) & s \end{bmatrix} W = I \tag{3.2}$$

and having rows $w^1, \ldots, w^{n+1}$ such that $w^h \succeq 0^{n+1}$ for $1 \leqslant h \leqslant t$, and $w^{n+1} \succeq w^i$ and $w^{n+1} \succeq -w^i$ for $t+1 \leqslant i \leqslant n$, and $w^{n+1} \succeq 0^{n+1}$. If $w_1^{n+1} = 0$, then we say that τ is *complete*.

The 0-dimensional simplex $\langle v \rangle$ is an s^0-complete facet of a unique 1-simplex in $A(s^0)$, where s^0 is uniquely determined by the function value of f at v as follows. Let $\alpha = \max_h |f_h(v)|$. Since $f(v) \neq 0^n$, $\alpha > 0$. If $f_h(v) = -\alpha$ for some h, then we let $s_k^0 = 1$ where k is the smallest index h such that $f_h(v) = -\alpha$, and let $s_j^0 = 0$ for $j \neq k$. If $f_h(v) > -\alpha$ for all h, then we take $s_k^0 = -1$ where k is the largest index h such that $f_h(v) = \alpha$, and let $s_j^0 = 0$ for $j \neq k$. Let $\sigma^0 = \langle v, x^+ \rangle$ be the unique simplex in $A(s^0)$ having $\langle v \rangle$ as its facet. Starting with the point v, the algorithm proceeds by pivoting $(1, f(x^+))$ into the system (3.1). Clearly, σ^0 is an almost s^0-complete 1-simplex in $A(s^0)$. The general steps of the algorithm can be described as follows. When for some nonzero sign vector s a t-simplex $\sigma = \langle x^1, \ldots, x^{t+1} \rangle$ in $A(s)$ is almost s-complete, the system (3.1) has two "basic solutions". At each of these solutions exactly one condition on the rows of the solution W is binding. If $w_1^{n+2} = 0$, then σ is complete. If $w^h \succeq 0^{n+1}$ is binding for some h, $1 \leqslant h \leqslant t+1$, then the facet τ of σ opposite the vertex x^h is s-complete, and either (i) τ is the 0-dimensional simplex $< v >$, or (ii) τ is a facet of precisely one other almost s-complete t-simplex σ' of the triangulation in $A(s)$, or (iii) τ lies on the boundary of $A(s)$ and is an almost s'-complete $(t-1)$-simplex in $A(s')$ for some unique nonzero sign vector s' with

$t-1$ nonzero elements differing from s in only one element. If $w^{n+2} \succeq w^i$ ($w^{n+2} \succeq -w^i$) is binding for some $t+1 < i \leqslant n+1$, σ is an s'-complete facet of precisely one almost s'-complete $(t+1)$-simplex in $A(s')$ for some nonzero sign vector s' differing from s in only the ith element, namely $s_i' = +1(-1)$.

Starting with σ^0 the $2n$-ray algorithm generates a sequence of adjacent almost s-complete simplices in $A(s)$ with s-complete common facets for varying sign vectors s. Moving from one s-complete facet of an almost s-complete simplex in $A(s)$ to the next s'-complete facet corresponds to making a lexicographic linear programming pivot step from one of the two basic solutions of system (3.1) to another. Using the well-known Lemke-Howson's argument, one can show that the algorithm will never visit any simplex more than once and thus it cannot cycle; see Laan van der and Talman (1979, 1981) and Reiser (1981) in detail. The algorithm stops as soon as it finds a complete simplex. We will show that in that case one of its vertices is a discrete zero point of the function f.

Lemma 3.2. *Suppose that f is a simplicially positive maximum component sign preserving function. Then any complete simplex contains a discrete zero point of the function f.*

Proof. Let $x^1, \ldots, x^{k+1}$ be the vertices of a complete simplex σ in $A(s)$ and let t be the number of non-zeros in s. Notice that $k = t - 1$ or $k = t$ depending on whether σ is a t-simplex in $A(s)$ or a facet of a t-simplex in $A(s)$. From the system (3.1) or (3.2) it follows that there exists $\lambda_1 \geqslant 0, \ldots, \lambda_{k+1} \geqslant 0$ with sum equal to one such that $\sum_{j=1}^{k+1} \lambda_j f(x^j) = 0^n$. Let $L = \{h \in N | \lambda_h > 0\}$. Clearly, L is not empty. Now we can rewrite $\sum_{j=1}^{k+1} \lambda_j f(x^j) = 0^n$ as

$$\sum_{h \in L} \lambda_h f_j(x^h) = 0^n \quad \text{for all } j \in N.$$

Suppose that $f(x^h) \neq 0^n$ for all $h \in L$. Then there exist $h^* \in L$ and $j_1 \in N$ such that $f_{j_1}(x^{h^*}) = \max_{i \in N} f_i(x^{h^*}) > 0$. Since f is simplicially positive maximum component sign preserving, we obtain $f_{j_1}(x^h) \geqslant 0$ for every $h = 1, \ldots, k+1$. Then it follows from $\sum_{h \in L} \lambda_h f_{j_1}(x^h) = 0$ that $f_{j_1}(x^h) = 0$ for all $h \in L$.

Again if $f(x^h) \neq 0^n$ for all $h \in L$, there must exist $h^* \in L$ and $j_2 \in N \backslash \{j_1\}$ such that $f_{j_2}(x^{h^*}) = \max_{i \in N} f_i(x^{h^*}) > 0$. Since f is simplicially positive maximum component sign preserving, we obtain $f_{j_2}(x^h) \geqslant 0$ for every $h = 1, \ldots, k+1$. Then it follows from $\sum_{h \in L} \lambda_h f_{j_2}(x^h) = 0$ that $f_{j_2}(x^h) = 0$ for all $h \in L$. Repeat this procedure until the elements of N have been exhausted. So in the end we must have $f(x^h) = 0^n$ for all $h \in L$, i.e., for every $h \in L$ the point x^h is a discrete zero point of the function f. $\square$

Because the algorithm cannot cycle, it either terminates with a complete simplex yielding a solution in a finite number of iterations or the sequence of simplices generated by the algorithm goes to infinity. The next lemma shows that under the boundary condition of the theorem, the latter

case can be prevented from happening and thus ensures the existence of a solution. Let $C^n = \{x \in \mathbb{R}^n \mid l \leqslant x \leqslant u\}$.

Lemma 3.3. *Under the condition of Theorem 3.1, the algorithm will find a complete simplex in a finite number of steps.*

Proof. We will show that the algorithm does not traverse the boundary of the set C^n. By definition of integral triangulation, $\mathscr{T}$ triangulates the set C^n and also the set $A(s) \cap C^n$ for any sign vector s into integral simplices.

For some nonzero sign vector s, let τ be an s-complete facet in $A(s)$ with vertices $x^1, \ldots, x^t$, where t is the number of non-zeros in s. We first show that τ is complete if it is on the boundary of C^n. From system (3.2) it follows that there exist $\lambda_1 \geqslant 0, \ldots, \lambda_t \geqslant 0$ with sum equal to one, $\beta \geqslant 0$, and $-\beta \leqslant \mu_i \leqslant \beta$ for $s_i = 0$, such that $\bar{f}_i(z) = -\beta$ if $s_i = 1$, $\bar{f}_i(z) = \beta$ if $s_i = -1$, and $\bar{f}_i(z) = \mu_i$ if $s_i = 0$, where $z = \sum_{i=1}^t \lambda_i x^i$ and $\bar{f}(z) = \sum_{i=1}^t \lambda_i f(x^i)$, i.e., $\bar{f}$ is the piecewise linear extension of f with respect to $\mathscr{T}$. Since τ lies on the boundary of C^n, there exists an index h such that either $x_h^j = l_h$ for all j or $x_h^j = u_h$ for all j. In case $x_h^j = l_h$ for all j, we have $s_h = -1$ and therefore $\bar{f}_h(z) = \beta$. Furthermore, by the Assumption, we have $f_h(x^j) \leqslant 0$ for all j and so $\bar{f}_h(z) \leqslant 0$. On the other hand $\bar{f}_h(z) = \beta \geqslant 0$. Therefore $\bar{f}_h(z) = 0$ and also $\beta = 0$. Since $w_1^{n+1} = \beta$ we obtain that τ is complete. Similarly, we can show that the same results hold for the case of $x_h^j = u_h$ for all j.

Due to the lexicographic pivoting rule and the properties of a triangulation, the algorithm will never visit any simplex more than once. So, because the number of simplices in C^n is finite, the algorithm finds within a finite number of steps a complete simplex. Since f is simplicially positive maximum component sign preserving, Lemma 3.2 shows that at least one of the vertices of the complete simplex is a discrete zero point of the function f. $\square$

As a consequence, we obtain a constructive and combinatorial proof for Theorem 3.1. As an immediate corollary of Theorem 3.1, we have the following discrete fixed point theorem. For any given $l, u \in \mathbb{Z}^n$ with $l_i < u_i - 1$ for all $i \in N$, let $D^n = \{x \in \mathbb{Z}^n \mid l \leqslant x \leqslant u\}$.

Theorem 3.4. *Let $f : D^n \to Co(D^n)$ be a function such that the function $g : \mathbb{Z}^n \to \mathbb{R}^n$ given by $g(x) = x - f(x)$ is simplicially positive maximum component sign preserving. Then f has at least one fixed point.*

4. Applications

Our first application concerns the complementarity problem. Given a function $f : \mathbb{R}_+^n \to \mathbb{R}^n$, the problem is to find a point $x^* \in \mathbb{R}_+^n$ such that

$$f(x^*) \geqslant 0^n, x^* \cdot f(x^*) = 0.$$

This problem has long been one of the most important problems in the field of mathematical programming and intensively studied for the case where f is continuous; see

for example Cottle et al. (1992), Facchinei and Pang (2003) and Kojima et al. (1991). The discrete counterpart of this problem is to replace the domain $\mathbb{R}_+^n$ by the discrete lattice $\mathbb{Z}_+^n$ and is called *the discrete complementarity problem*.

In the following we establish a theorem on the existence of solution to the discrete complementarity problem. For any $x \in \mathbb{Z}_+^n$, define $S^+(x) = \{h \mid x_h > 0\}$.

Definition 4.1. A function $f : \mathbb{Z}_+^n \to \mathbb{R}^n$ is simplicially positive maximum component sign preserving in $\mathbb{Z}_+^n$ if there exists an integral triangulation $\mathscr{T}$ of $\mathbb{R}_+^n$ such that for any simplicially connected vertices x, y of $\mathscr{T}$, $x_k = 0$ implies $f_k(x)f_k(y) \geqslant 0$, and $x_k > 0$ and $f_k(x) = \max_{h \in S^+(x)} f_h(x) > 0$ imply $f_k(y) \geqslant 0$.

Now we present an existence theorem for the discrete nonlinear complementarity problem.

Theorem 4.2. *Let $f : \mathbb{Z}_+^n \to \mathbb{R}^n$ be a simplicially positive maximum component sign preserving function in $\mathbb{Z}_+^n$. If there exists a vector $u \in \mathbb{Z}^n$ with $u_h > 1$ for every h such that for any $x \in \mathbb{Z}_+^n$ with $x \leqslant u$, $x_k = u_k$ implies $f_k(x) \geqslant 0$, then the discrete complementarity problem has a solution.*

We will give a constructive and combinatorial proof for this result by adapting the algorithm described in Section 3 to the current problem. First, the origin 0^n is taken as the starting point v of the algorithm. Since 0^n is on the boundary of $\mathbb{R}_+^n$, the sets $A(s)$ and s-completeness are only defined for nonnegative non-zero sign vectors s. Notice that $A(s) = \{x \in \mathbb{R}_+^n \mid x_i = 0 \text{ whenever } s_i = 0\}$.

Next, we adapt the concepts of an almost s-complete simplex and an s-complete facet. For some sign vector s with $t > 0$ positive components, denote $\{h_1, \ldots, h_{n-t}\} = \{h \mid s_h = 0\}$ and let $\sigma = \langle x^1, \ldots, x^{t+1} \rangle$ be a t-simplex of the triangulation in $A(s)$. Then σ is *almost s-complete* if there is an $(n + 2) \times (n + 1)$ matrix W being a solution to system

$$\begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ f(x^1) & \cdots & f(x^{t+1}) & -e(h_1) & \cdots & -e(h_{n-t}) & s \end{bmatrix} W = I \tag{4.3}$$

and having rows $w^1, \ldots, w^{n+2}$ such that $w^h \succeq 0^{n+1}$ for $1 \leqslant h \leqslant t + 1$, and $w^{n+2} \succeq -w^i$ for $t + 1 < i \leqslant n + 1$, and $w^{n+2} \succeq 0^{n+1}$. If $w_1^{n+2} = 0$, then we say that the simplex σ is *complete*. For τ a facet of σ, without loss of generality, letting $\tau = \langle x^1, \ldots, x^t \rangle$, τ is s-complete if there is an $(n + 1) \times (n + 1)$ matrix W being a solution to system

$$\begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ f(x^1) & \cdots & f(x^t) & -e(h_1) & \cdots & -e(h_{n-t}) & s \end{bmatrix} W = I \tag{4.4}$$

and having rows $w^1, \ldots, w^{n+1}$ such that $w^h \succeq 0^{n+1}$ for $1 \leqslant h \leqslant t$, and $w^{n+1} \succeq -w^i$ for $t + 1 \leqslant i \leqslant n$, and $w^{n+1} \succeq 0^{n+1}$. If $w_1^{n+1} = 0$, then we say that τ is *complete*.

With respect to the starting point 0^n, let $\alpha = \min_h f_h(0^n)$. If $\alpha \geqslant 0$, then 0^n is a solution and the algorithm stops. Otherwise, let s^0 be the sign vector with $s_k^0 = 1$, where k is

the smallest index h such that $f_h(0^n) = \alpha$, and $s_j^0 = 0$ for $j \neq k$. Similarly as in Section 3, it can be shown that the simplex $\langle 0^n \rangle$ is an s^0-complete facet of the unique 1-dimensional simplex σ^0 in $A(s^0)$ having $\langle 0^n \rangle$ as one of its facets. Furthermore σ^0 is almost s^0-complete.

Let $\mathscr{T}$ be the integral triangulation of $\mathbb{R}_+^n$ underlying the function f. Let $C^n = \{x \in \mathbb{R}_+^n \mid x \leqslant u\}$. Clearly, $\mathscr{T}$ subdivides the set $A(s) \cap C^n$ for every nonnegative non-zero sign vector s into t-dimensional integral simplices. Starting with σ^0, the algorithm now generates a unique sequence of adjacent almost s-complete simplices in $A(s)$ with s-complete common facets for varying nonnegative non-zero sign vectors s. The algorithm stops when a complete simplex is found.

Lemma 4.3. *Under the assumption of Theorem 4.2, the algorithm finds in a finite number of steps a solution to the discrete nonlinear complementarity problem.*

Proof. First, we prove that the algorithm cannot cross the boundary of $A(s) \cap C^n$ for any nonnegative non-zero sign vector s. We have to consider two cases. In the first case we show that the algorithm cannot cross the lower boundary of C^n. Suppose the algorithm generates an s-complete facet τ of a t-dimensional simplex in $A(s)$ on the lower boundary of C^n, then it is easy to see that τ is a $(t-1)$-dimensional simplex in $A(s')$ for some unique s', and the algorithm continues with τ in $A(s')$. In the second case we show that the algorithm cannot cross the upper boundary of C^n. To do this, it is sufficient to prove that if τ is an s-complete facet of a simplex in $A(s)$ lying on the upper boundary of C^n, then τ is complete. Let $\tau = \langle x^1, \ldots, x^t \rangle$, where $t > 0$ is the number of positive components of s. It follows from system (4.4) that there exist $\lambda_1 \geqslant 0, \ldots, \lambda_t \geqslant 0$ with sum equal to one, $\beta \geqslant 0$, and $\mu_i \geqslant -\beta$ for $s_i = 0$, such that $\bar{f}_i(z) = -\beta$ when $s_i = 1$ and $\bar{f}_i(z) = \mu_i$ when $s_i = 0$, where $z = \sum_{i=1}^t \lambda_i x^i$ and $\bar{f}$ is the piecewise linear extension of f with respect to $\mathscr{T}$. Since τ lies on the upper boundary of C^n, there exists an index h such that $x_h^j = u_h$ for all j. But then $x_h^j > 0$ and so we must have $s_h = 1$ and therefore $\bar{f}_h(z) = -\beta \leqslant 0$. On the other hand, since $x_h^j = u_h$, we also have $f_h(x^j) \geqslant 0$ for all j. Hence, we obtain $\bar{f}_h(z) \geqslant 0$. Consequently, $\beta = 0$, i.e., τ is complete.

Due to the lexicographic pivoting rule and the properties of a triangulation, the algorithm cannot visit a simplex more than once. So, because the number of simplices in C^n is finite and we showed that the algorithm cannot cross the boundary of C^n, the algorithm finds in a finite number of steps either a complete simplex or a complete facet of a simplex in $A(s) \cap C^n$ for some nonnegative non-zero sign vector s.

Let $\sigma = \langle x^1, \ldots, x^h \rangle$ be a complete simplex or a complete facet of a simplex in some set $A(s) \cap C^n$ with $h = t$ or $t + 1$ for some nonnegative non-zero sign vector s, where $t > 0$ is the number of positive components of s. It follows from system (4.3) or (4.4) that there exist $\lambda_1 \geqslant 0, \ldots, \lambda_h \geqslant 0$ with sum equal to one, and $\mu_i \geqslant 0$ for $s_i = 0$, such that $\bar{f}_i(z) = 0$

when $s_i = 1$ and $\bar{f}_i(z) = \mu_i$ when $s_i = 0$, where $z = \sum_{i=1}^h \lambda_i x^i$. Since $z \in A(s)$, we also have $z_i = 0$ if $s_i = 0$ and $z_i \geqslant 0$ if $s_i = 1$. So, $\bar{f}_i(z) \geqslant 0$ if $z_i = 0$ and $\bar{f}_i(z) = 0$ if $z_i > 0$, i.e., z solves the nonlinear complementarity problem with respect to $\bar{f}$. Without loss of generality, let $\rho = \langle x^1, \ldots, x^k \rangle$ be the unique face of σ containing z in its relative interior. Hence, there exist unique positive numbers $\lambda_1, \ldots, \lambda_k$ summing up to 1 such that $z = \sum_{j=1}^k \lambda_j x^j$ and $\bar{f}(z) = \sum_{j=1}^k \lambda_j f(x^j)$. Take any j^* between 1 and k. We will show that x^{j^*} is a solution of the problem.

Suppose first that $z_i = 0$ and $\bar{f}_i(z) > 0$ for some i. Clearly, $x_i^j = 0$ for all $j = 1, \ldots, k$. Since $\bar{f}_i(z) = \sum_{j=1}^k \lambda_j f_i(x^j)$ there exists h such that $f_i(x^h) > 0$. Since x^h and x^{j^*} are simplicially connected and $x_i^{j^*} = 0$, we have that $f_i(x^h) f_i(x^{j^*}) \geqslant 0$, and therefore $x_i^{j^*} = 0$ and $f_i(x^{j^*}) \geqslant 0$. Suppose next that $z_i = 0$ and $\bar{f}_i(z) = 0$ for some i. Again, $x_i^j = 0$ for all $j = 1, \ldots, k$. Since $\bar{f}_i(z) = \sum_{j=1}^k \lambda_j f_i(x^j)$ and $\bar{f}_i(z) = 0$, we obtain $\sum_{j=1}^k \lambda_j f_i(x^j) = 0$ and therefore $\sum_{j=1}^k \lambda_j f_i(x^j) f_i(x^{j^*}) = 0$. Since for all j it holds that x^j and x^{j^*} are simplicially connected and $x_i^{j^*} = 0$, we have $f_i(x^j) f_i(x^{j^*}) \geqslant 0$, and so each term in the summation must be zero. In particular, it holds that $\lambda_{j^*} f_i^2(x^{j^*}) = 0$. Since $\lambda_{j^*} > 0$, this implies $f_i(x^{j^*}) = 0$.

Thus far we have shown that if $z_i = 0$ then $x_i^{j^*} = 0$ and $f_i(x^{j^*}) \geqslant 0$. We will now show that if $z_i > 0$ for some i then $x_i^{j^*} \geqslant 0$ and $f_i(x^{j^*}) = 0$. Let i be such that $z_i > 0$. Then $x_i^{j^*} \geqslant 0$ and $\sum_{j=1}^k \lambda_j f_i(x^j) = \bar{f}_i(z) = 0$. First, suppose $x_i^h = 0$ for some h. Since x^j and x^h are simplicially connected for all j and $x_i^h = 0$, it holds that $f_i(x^j) f_i(x^h) \geqslant 0$ for all j. From $\sum_{j=1}^k \lambda_j f_i(x^j) = 0$, it follows that $\sum_{j=1}^k \lambda_j f_i(x^j) f_i(x^h) = 0$, and so $\lambda_j f_i(x^j) f_i(x^h) = 0$ for all $j = 1, \ldots, k$. In particular, $\lambda_h f_i^2(x^h) = 0$. Since $\lambda_h > 0$, it follows that $f_i(x^h) = 0$. Consequently, $f_i(x^h) = 0$ whenever $x_i^h = 0$. In particular, if $x_i^{j^*} = 0$, then $f_i(x^{j^*}) = 0$. Next, suppose $x_i^h > 0$ and $f_i(x^h) > 0$ for some h. Then there exists $h^* \in S^+(x^h)$ such that $f_{h^*}(x^h) = \max_{j \in S^+(x^h)} f_j(x^h) > 0$ and therefore $f_{h^*}(x^j) \geqslant 0$ for all $j \neq h$. Hence, $\bar{f}_{h^*}(z) = \sum_{j=1}^k \lambda_j f_{h^*}(x^j) > 0$. On the other hand, $z_{h^*} = \sum_{j=1}^k \lambda_j x_{h^*}^j > 0$, which implies $\bar{f}_{h^*}(z) = 0$, yielding a contradiction. Finally, suppose $x_i^h > 0$ and $f_i(x^h) < 0$ for some h. Since $x_i^h > 0$ implies $z_i = \sum_{j=1}^k \lambda_j x_i^j > 0$, we must have $\sum_{j=1}^k \lambda_j f_i(x^j) = \bar{f}_i(z) = 0$. From above it follows that $f_i(x^j) = 0$ whenever $x_i^j = 0$. Therefore there exists h^* such that $x_i^{h^*} > 0$ and $f_i(x^{h^*}) > 0$, but we just showed that this is not possible. Therefore, $f_i(x^{j^*}) = 0$ whenever $x_i^{j^*} > 0$. This completes the proof that any vertex of ρ solves the discrete complementarity problem. $\square$

As a result, we have given a constructive and combinatorial proof for Theorem 4.2.

Our second application concerns the existence problem of equilibrium in a competitive exchange economy with indivisible goods and money. For related economic models, we refer to Kelso and Crawford (1982), Kaneko and Yamamoto (1986), Bevia et al. (1999), Gul and Stacchetti (1999), Laan van der et al. (2002), Sun and Yang (2006) among others. In the economy, there are a finite number, say, n, of indivisible commodities like houses, cars and

computers, and a finite number of agents, each of whom initially owns a certain amount of indivisible goods and money. The price of money is equal to one. Exchange of indivisible goods is carried out by their prices and via money. All agents exchange their goods to achieve their maximal utility under their budget constraints. This economy can be captured by the excess demand function $z : \mathbb{Z}_+^n \to \mathbb{Z}^n$, where $z_k(p)$ denotes the aggregated excess demand of indivisible commodity k, $k \in N$, at discrete price vector $p \in \mathbb{Z}_+^n$. A vector $p^* \in \mathbb{Z}_+^n$ is called *a discrete Walrasian equilibrium* if $z(p^*) = 0^n$. That is, at equilibrium, the demand is equal to the supply for every indivisible good.

It is natural to assume that *the desirability condition* holds for every indivisible good, i.e., $p_k = 0$ implies $z_k(p) > 0$. That is, if the price of an indivisible good is zero, then the supply of that good cannot meet the demand for that good. It is also natural to assume that *the limited value condition* holds for every indivisible good. That is, there exists some $M > 1$ such that if $p_k \geqslant M$, then $z_k(p) < 0$. In other words, if the price of an indivisible good is too high, no agent will demand that good and thus the supply will exceed the demand. The next assumption replaces continuity.

Assumption 4.4. An excess demand function $z : \mathbb{Z}_+^n \to \mathbb{Z}^n$ is said to be minimum component sign preserving, if there exists an integral triangulation $\mathcal{T}$ of $\mathbb{R}_+^n$ such that for any simplicially connected vertices p, q of $\mathcal{T}$, $z_k(p) = \min_{h \in N} z_h(p) < 0$ implies $z_k(q) \leqslant 0$.

This assumption states that if at price vector p an indivisible good k is highest in excess supply, the demand for that good will not exceed its supply as long as the prices deviate from the price vector p at most one unit for each good.

The following equilibrium existence theorem follows immediately from Theorem 3.1 by letting $f = -z$, $l = 0^n$ and $u_h = M$ for all $h \in N$.

Theorem 4.5. *The exchange economy has a discrete Walrasian equilibrium under the conditions of desirability and limited value and Assumption 4.4.*

5. Concluding remarks

In this paper we have demonstrated a discrete multivariate mean value theorem by using the $2n$-ray simplicial algorithm, which actually finds an exact discrete zero point. The theorem holds for the class of positive maximum component sign preserving functions. In addition, we have established an existence theorem for the discrete nonlinear complementarity problem and an equilibrium existence theorem for a discrete exchange economy. We proved both results in a constructive manner. Other closely related algorithms include those of Eaves (1972), Eaves and Saigal (1972), Laan van der and Talman (1979), Saigal (1983), Freund (1984), and Yamamoto (1984). It will be interesting to know if these algorithms can also find a discrete zero point

of a positive maximum component sign preserving function under conditions similar to those we have studied here.

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