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Optimality of irreversible pollution accumulation

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Abstract

This paper considers optimal pollution accumulation when the decay function has an inverted-U shape. Such decay functions have empirical relevance but they lead to nonconvexities in dynamic optimization. The nonconvexity problem is handled here by applying a two-stage optimization procedure. The analysis shows that two qualitatively different optimality candidates may exist simultaneously. We identify cases where the choice can be made on a priori grounds and cases where it requires computation of the present values of both optimality candidates. An optimal emission trajectory leading to irreversible pollution is typically nonmonotonic.

Key words: Nonconvexities; Externalities; Pollution control
JEL classification: Q20; Q25; Q28

1. Introduction

Material flows from the economy back to the environment normally have long-term environmental consequences. This is most obvious in the case of emissions like CO_2, nuclear or organic waste, or certain toxic substances, like PCB, which accumulate as a pollution stock in the environment. The economic implications of pollution accumulation have been studied in the environmental economics literature since the works of Keeler et al. (1972) and Plourde (1972). More recently the analysis has been extended to cover stochastic and catastrophic features (Clarke and Reed, 1994) and the cleanup of hazardous wastes

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(Caputo and Wilen, 1995). A key assumption in the specification defining the pollution accumulation process is the 'constant exponential rate of decay' (cf. Forster, 1975), which defines the rate of pollution decay as a linear (or monotonically increasing) function of the pollution stock. This specification has been criticized by many authors (e.g.., Comolli, 1977; Dasgupta, 1982; Fiedler, 1992; Forster, 1975; Pethig, 1993), who argue that high pollution stock levels may destroy the environment's self-purification processes. A classical reference in the biological literature is Holling (1973), who gives several examples where nutrient enrichment of lakes has caused new equilibria, with the consequence that the lake is incapable of recovery to its original state even if emissions were to be eliminated.

Several authors (e.g., Ström, 1972; Forster, 1975; Comolli, 1977; Dasgupta, 1982; Pethig, 1993; Cesar,1994) have proposed an alternative formulation which includes the feature that a pollution stock level that is sufficiently high will reduce the rate of decay to zero. Such decay functions may be called 'inverted-U-shaped decay functions'. In line with Holling (1973), Forster (1975) argues that this formulation is relevant, e.g., in approximating the accumulation of organic wastes in lakes. The interesting question related to this specification is whether it might be optimal to let pollution accumulate beyond the level where the rate of decay is zero and pollution accumulation is irreversible. Note that this question does not arise when the decay function is nondecreasing. The difficulty in studying this question arises from the fact that an inverted-U-shaped decay function is neither concave nor convex. Furthermore, the decay function may be nondifferentiable at the critical pollution level beyond which decay is zero (see Odum, 1971, p. 434). As is well-known these kinds of properties may cause problems in optimal control applications. In a widely cited paper Forster (1975) suggests that irreversible pollution might be optimal under the assumption that the necessary conditions are also sufficient. However, his optimality candidates are obtained without fully taking into account the nonconvexity problem. Nor is the nonconvexity problem studied by Pethig (1993), Cesar (1994), or Cesar and de Zeeuw (1994). As a consequence, the optimality candidates may give a local maximum only.

We propose a formulation where the state space is divided into a reversible and an irreversible region. In the reversible region the decay function is strictly concave and in the irreversible region decay is zero. We first solve for the optimal pollution control program when the initial pollution level equals the critical level. The maximum benefits of this program are then taken to be a scrap value function of the problem where the initial pollution level is in the reversible region. The overall problem is then to determine the optimal emission time path when the option exists of letting the pollution stock increase to the irreversible region.

Our analysis shows that there may exist simultaneously two types of optimality candidates: an optimal reversible solution and an optimal irreversible solution. Because of the nonconvexity property, the choice cannot be made, in general, by the usual marginal analysis but instead requires computation of the present values of both paths. However, we identify several cases where the choice can be made on a priori grounds. For example, a high initial level of pollution may imply the optimality of irreversible pollution, although there simultaneously exists a (locally) optimal path towards a steady state where the rate of decay will remain positive. We also show that a typical irreversible emission trajectory is nonmonotonic. Our qualitative analysis is supported by a numerical simulation which demonstrates that two completely different emission control programs can yield equal present value benefits. Finally, we emphasize that these findings may have strong implications in any field where dynamic pollution externalities are studied.

The paper is organized as follows. The next section specifies the model and presents the results in three subsections. Section 3 presents some illustrative numerical examples and Section 4 concludes the paper.

2. Optimal reversible and irreversible pollution control

2.1. Problem formulation

Our analysis will concentrate on the implications of the nonconcave decay function, and we will keep the other properties of the model as simple as possible. Thus the model will resemble the specification applied by Keeler et al. (1972), Forster (1975), Clarke and Reed (1994), and several others. Denote the rate of emissions at an instant of time by y, the stock of pollution at an instant of time by z, and the rate of pollution decay by $\alpha(z)$. We denote by $\bar{z}$ the critical pollution level beyond which the rate of decay is zero. The accumulation of pollution is described by the following differential equation:

$$\dot{z} = y - \alpha(z), \quad \text{with} \quad z(0) = z_0 \text{ (given)}, \quad \text{and} \quad z_0 < \bar{z}, \tag{1}$$

where $\alpha(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and satisfies:

(A.1) $\alpha(0) = 0$; there exists $\bar{z} > 0$ such that $\alpha(z) = 0$ for all $z \in [\bar{z}, \infty]$, $\alpha(z) > 0$ for all $z \in (0, \bar{z})$; $\alpha''(z) \leq 0$ for all $z \in [0, \bar{z})$.

This formulation of the decay function coincides with the one by Ström (1972) or Forster (1975). We employ the following welfare functional:

$$W = \int_0^\infty [U(y) - D(z)]\, e^{-\delta t}\, dt, \tag{2}$$

where δ is the rate of time preference, $U(y)$: $\mathbb{R}_+ \to \mathbb{R}_+$ is the utility of the activity that generates emissions, and $D(z)$: $\mathbb{R}_+ \to \mathbb{R}_+$ is the disutility of pollution. We have assumed that emissions are proportional to the production of the commodity that yields utility; so we can proceed as if emissions per se yield utility. With respect to $U(y)$ and $D(z)$ we assume:

(A.2) $U(0) = 0$; $U''(y) < 0$ for all $y \in [0, \infty)$; $U'(0) < \infty$; there exists $\bar{y} > 0$ such that $U'(\bar{y}) = 0$.

(A.3) $D(0) = 0; D'(0) = 0; D''(z) > 0$ for all $z \in [0, \infty)$; $\lim_{z \to \infty} D'(z) = \infty$.

We emphasize that this model describes a local pollution problem (e.g., a problem related to a single lake, pulp factory, and a fishing industry) and not a problem of a whole economy. To give a specific example, let q be the firm's output and U the profit function. Assume that $U(q) = pq - c(q)$, where p is a constant output price and c is a convex cost function. Assume that emissions y are proportional to q, specifically $y = q$. Now $\bar{q}$ is the profit-maximizing output and $\bar{y}$ the corresponding emission level. It is quite innocuous from an empirical point of view to assume that $\bar{y}$ is finite. In the partial equilibrium context it also is possible that pollution does not affect the activity that generates emissions, i.e., the marginal utility from emission is independent of the level of pollution as in the model by Plourde (1972) and many of his followers. In the following we assume that $\max\{\alpha(z)\} > \bar{y}$, which implies irreversible pollution accumulation due to uncontrolled emissions. This is in contrast to the case of constant rate of decay where uncontrolled pollution is always reversible. Our solutions will determine an optimal Pigouvian tax trajectory which can be used as a policy instrument for forcing the firm's emissions below $\bar{y}$.

The objective is to find a trajectory (z, y): $[0, \infty) \to \mathbb{R}^2_+$ such that (2) is maximized subject to (1). Mathematically, this is a standard optimal control problem, except that the decay function is neither concave nor convex nor differentiable at $z = \bar{z}$. These properties may cause problems when there occurs a transition from pollution levels below $\bar{z}$ to the critical level or beyond it. We approach this problem by dividing the state space in two regions. First, we consider the subproblem of maximizing benefits from some instant of time T onwards, where the stock at T equals $\bar{z}$. This problem will be referred to as the *second-period* problem. The overall problem is then to determine the optimal level of emissions given any $z_0 < \bar{z}$, taking into account maximum benefits in the case where pollution reaches the critical level. The second-period problem is defined as follows:

$$\underset{\{y\}}{\text{Maximize}}\; W_T = \int_T^{\infty} [U(y) - D(z)] e^{-\delta t} \, dt$$

$$\text{subject to} \quad \dot{z} = y, \quad z(T) = \bar{z}.$$

Let z^{∞} denote the stock level which satisfies $D'(z) - \delta U'(0) = 0$. Note that our assumptions imply that z^{∞} exists. Let the pair (z_T, y_T) denote the path which solves the problem given above and ψ_T the costate variable for the pollution stock. The optimal solution can be characterized as follows:

Lemma 1. *If* $D'(\bar{z}) - \delta U'(0) \geq 0$, *then* $(z_T, y_T) = (\bar{z}, 0)$ *for all* $t \in [T, \infty)$; *if* $D'(\bar{z}) - \delta U'(0) < 0$, *then* $(z_T, y_T) \to (z^{\infty}, 0)$ *when* $t \to \infty$ *and* $\dot{y}_T < 0$ *for all* $t \in [T, \infty)$.

Proof. The necessary conditions for this problem include: $U'(y_T) + \psi_T \leq 0$, $[U'(y_T) + \psi_T]y_T = 0$, $y_T \geq 0$, $\dot{\psi}_T = D'(z_T) + \delta\psi_T$. Because the Hamiltonian is strictly concave in (z, y), the problem satisfies the concavity requirements and the solution coverging to a steady state is the unique optimal solution. When $D'(\bar{z}) - \delta U'(0) \geq 0$, the choice of $\psi_T = -D'(\bar{z})/\delta$ implies $\dot{\psi}_T = y_T = \dot{z}_T = 0$, for all $t \in [T, \infty)$, and satisfies the infinite-horizon sufficiency requirements. When $D'(\bar{z}) - \delta U'(0) < 0$, the fact that the steady state at $z_T = z^{\infty}$, $y_T = 0$, $\psi_T = -D'(z^{\infty})/\delta$ is a saddle point implies the existence of a unique $\psi_T(T)$ such that $-U'(0) < \psi_T(T) < 0$ with $\dot{\psi}_T < 0$, $y_T > 0$, and $\dot{y}_T < 0$ for all $t \in [T, \infty)$. ■

The case where $D'(\bar{z}) - \delta U'(0) < 0$ is depicted in Fig. 1. The necessary conditions imply $\dot{y} = -[D'(z) - \delta U'(y)]/U''(y)$ and, in addition, that the locus of $\dot{y} = 0$ must be decreasing.

By $\hat{y}$ we denote the second-period optimal initial rate of emissions, i.e., $\hat{y} = y_T(T)$. Note that $\hat{y}$ is independent of T. Along the optimal path the

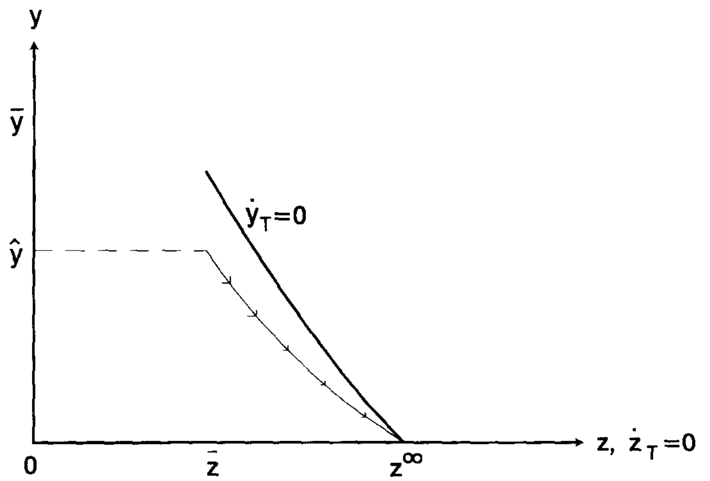

Fig. 1. Optimal trajectory when $z \geq \bar{z}$ and $D'(\bar{z}) - \delta U'(0) < 0$.

marginal benefits from emissions must equal the present value marginal damage. When $y > 0$, the present value marginal pollution damage increases monotonically implying that emissions must decrease monotonically. If $D'(\bar{z})/\delta - U'(0) \geq 0$, it cannot be optimal to increase the pollution stock over $\bar{z}$ because the level of present value pollution damage exceeds the benefits from the first units of emissions.

Next we consider the following problem, again with fixed $T > 0$:

$$\underset{\{y\}}{\text{Maximize}}\, W^T = \int_0^T [U(y) - D(z)]\mathrm{e}^{-\delta t}\,\mathrm{d}t$$

$$\text{subject to} \quad \dot{z} = y - \alpha(z), \quad z(0) = z_0,$$

$$z(T) = \bar{z} \quad \text{if} \quad T < \infty,$$

$$\lim_{t \to \infty} z(t) \leq \bar{z} \quad \text{if} \quad T = \infty.$$

Note that cases $z(T) < \bar{z}$ with $T < \infty$ are excluded without ruling out any optimality candidate. If $z(T) < \bar{z}$ and $T < \infty$, the solution must continue as the first-period solution after T as $t \to \infty$ or as $\bar{z}$ is reached in finite time. Let us denote a solution to this *first-period* problem by (z^T, y^T) and the corresponding costate variable by ψ^T. Among the necessary conditions we have

$$U'(y^T) + \psi^T \leq 0, \qquad y^T[U'(y^T) + \psi^T] = 0, \qquad y^T \geq 0, \tag{3}$$

$$\dot{\psi}^T = D'(z^T) + \psi^T[\delta + \alpha'(z^T)]. \tag{4}$$

In the following, $\hat{W}^T$ and $\hat{W}_T$ denote the maximized W^T and W_T. Total maximized benefits, with finite T, are $W(T) = \hat{W}^T + \hat{W}_T$. If $T = \infty$, we define $W(\infty) = \hat{W}^\infty$ and $\hat{W}_\infty = 0$. Both subproblems satisfy the conditions required to perform a sensitivity analysis with respect to the horizon (see Seierstad and Sydsæter, 1986, Thm. 3.9). Hence

$$\mathrm{e}^{\delta T}\partial \hat{W}^T/\partial T = U[y^T(T)] - D(\bar{z}) + \psi^T(T)y^T(T), \tag{5a}$$

$$-\mathrm{e}^{\delta T}\partial \hat{W}_T/\partial T = U[y_T(T)] - D(\bar{z}) + \psi_T(T)y_T(T). \tag{5b}$$

Now suppose that there exists an optimal finite time T^* to reach $\bar{z}$. Then it must hold that

$$\partial W/\partial T\,|_{T^*} = \partial \hat{W}^T/\partial T\,|_{T^*} + \partial \hat{W}_T/\partial T\,|_{T^*} = 0. \tag{6a}$$

If it is not optimal to reach $\bar{z}$ within finite time, then

$$\limsup_{T\to\infty} \partial W/\partial T = \limsup_{T\to\infty} \partial \hat{W}^T/\partial T + \limsup_{T\to\infty} \partial \hat{W}_T/\partial T \geq 0. \tag{6b}$$

Conditions (5a), (5b), (6a), (3) and the fact that $U(y) - yU'(y)$ is an increasing function imply that when the optimal T is finite it must hold that

$$y^T(T) = y_T(T)(\equiv \hat{y}), \tag{7}$$

i.e., it is necessary that the level of emissions is continuous at T^*. Condition (6b) requires that for $T = \infty$ to be optimal it is necessary that W does not decrease when T increases without limit.

When T is free, conditions (5a)–(7) are only necessary for optimality, i.e., these conditions may merely give a local maximum and there may exist multiple local maxima. It is possible, however, to derive a sufficient condition for optimality (Seierstad, 1988). Assume we have found T^* such that either (6a) or (6b) is satisfied. Assume further that

$$\partial W/\partial T \geq 0 \quad \text{for all} \quad T < T^*, \qquad \partial W/\partial T \leq 0 \quad \text{for all} \quad T^* < T. \tag{8}$$

Then T^* is globally optimal. In view of (3), (5a), (5b) it follows that $e^{\delta T}\partial W/\partial T = e^{\delta T}\partial \hat{W}^T/\partial T + e^{\delta T}\partial \hat{W}_T/\partial T = U[y^T(T)] - U'[y^T(T)]\,y^T(T) - U[y_T(T)] + U'[y_T(T)]y_T(T)$. Because $\partial W/\partial T$ is increasing in $y^T(T)$ it is therefore sufficient that $y^T(T) \geq \hat{y}$ for $T < T^*$ and $y^T(T) \leq \hat{y}$ for $T > T^*$.

We will define solutions which reach $\bar{z}$ in finite time as *irreversible solutions* and solutions satisfying $z < \bar{z}$ for all $t \in [0, \infty)$ as *reversible solutions*. We next identify the solutions maximizing $\hat{W}^T + \hat{W}_T$. We proceed in steps, making a distinction between reversible and irreversible solutions .

2.2. Optimal reversible solutions

To find optimal reversible solutions we begin by studying the occurrence of steady states below the pollution stock level $\bar{z}$. By $\alpha'(\bar{z})$ we denote the left-hand derivative.

Lemma 2. *If* $D'(\bar{z}) - U'(0)[\delta + \alpha'(\bar{z})] \geq 0$ (<0), *the number of optimal reversible steady states is odd* (*even*).

Proof. Given $y > 0$, any optimal path must satisfy $\dot{z} = y - \alpha(z)$, $\dot{y} = -\{D'(z) - U'(y)[\delta + \alpha'(z)]\}/U''(y)$. The locus of points (z, y) with $\dot{z} = 0$ is given by $y = \alpha(z)$. The locus of points (z, y) with $\dot{y} = 0$ and $0 < y < \bar{y}$ is given by $D'(z) - U'(y)[\delta + \alpha'(z)] = 0$. By assumptions (A.2) and (A.3) this locus hits the y-axis at $y = \bar{y}$. For $y = 0$ we must have $D'(z) - U'(0)[\delta + \alpha'(z)] = 0$. If $D'(\bar{z}) - U'(0)[\delta + \alpha'(\bar{z})] \geq 0$, the locus of $\dot{y} = 0$ must reach the z-axis below or at $\bar{z}$ because $D''(z) - U'(0)\alpha''(z) > 0$. Thus, there must exist at least one steady state. But because both $\dot{z} = 0$ and $\dot{y} = 0$ have negative slope when $\alpha'(z) < 0$, multiple, or an odd number of, equilibria cannot be ruled out. If instead $D'(\bar{z}) - U'(0)[\delta + \alpha'(\bar{z})] < 0$, then $D'(z) - U'(0)[\delta + \alpha'(z)] < 0$ for all $z \in [0, \bar{z}]$ and the locus of $\dot{y} = 0$ does not reach the z-axis below $\bar{z}$. In this case either there exist no steady states or the steady states below $\bar{z}$ occur in pairs. ■

Notice that in Lemma 2 we have omitted the degenerate cases where isoclines $\dot{z} = 0$ and $\dot{y} = 0$ are tangent to each other but do not cross. We next give a sufficient condition for the case when the optimal solution is reversible.

Proposition 1. Given $D'(\bar{z}) - \delta U'(0) \geq 0$, the optimal solution is reversible and converges to a saddle point steady state.

Proof. If $D'(\bar{z}) - \delta U'(0) \geq 0$, then $D'(\bar{z}) - U'(0)[\delta + \alpha'(\bar{z})] > 0$, which implies by Lemma 2 that there exists an odd number of reversible steady states with positive y and z. The steady states with the lowest and highest levels of z are saddle points, and between two saddle point steady states there is always an unstable node. This implies that given any $z_0 \in [0, \bar{z})$ there exists a saddle point path converging to a steady state. A typical phase diagram of this case is depicted in Fig. 2. Note that by Lemma 1 $D'(\bar{z}) - \delta U'(0) \geq 0$ implies that $\hat{y} = 0$.

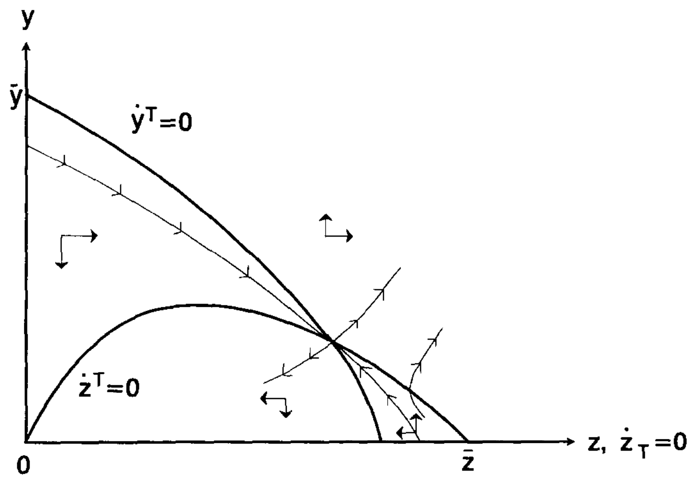

Fig. 2. Global optimality of reversible pollution when $D'(\bar{z}) - \delta U'(0) \geq 0$.

Thus there does not exist an irreversible solution satisfying condition (7). We can apply an artificial restriction, $z \leq \bar{z}$ for all t, which rules out nonconvexity but which does not restrict the optimality candidates. This implies that the concavity of the Hamiltonian in the region $z \leq \bar{z}$ guarantees that a path converging toward a steady state can be taken as a globally optimal solution. ∎

Note that the condition $D'(\bar{z}) - \delta U'(0) \geq 0$ is sufficient only and reversible solutions may well be optimal when the reverse holds. However, in such cases the choice between reversible and irreversible solutions will be less obvious.

2.3. *Optimal irreversible solutions*

We turn to study the less straightforward case where $D'(\bar{z}) - \delta U'(0) < 0$. As shown in Lemma 1 the optimal emission level, given $z \geq \bar{z}$, is now positive, i.e., $\hat{y} > 0$, and irreversible solutions cannot be ruled out by arguments similar to those used in Proposition 1. When $D'(\bar{z}) - \delta U'(0) < 0$, the sign of $D'(\bar{z}) - U'(0)[\delta + \alpha'(\bar{z})]$ is either positive or negative.

Let us first consider the case where $D'(\bar{z}) - U'(0)[\delta + \alpha'(\bar{z})] \geq 0$. By Lemma 2 there exists an odd number of reversible steady states. When $z < \bar{z}$, it holds that $D'(z) - U'(0)[\delta + \alpha'(z)]$ is increasing in z and the locus of points (z, y) for which $\dot{y} = -\{D'(z) - U'(y)[\delta + \alpha'(\bar{z})]\}/U''(y) = 0$ must reach the z-axis with $z < \bar{z}$. This implies that the steady state with the highest z is a saddle point and that we can depict the locus $\dot{y}^T = 0$ as in Fig. 3.

Assume that either path $\hat{y}^1$ or $\hat{y}^2$ is the optimal initial emission level for the second-period problem. Path $\hat{y}^2$ may be associated with a lower rate of discount than path $\hat{y}^1$. The essential difference between these cases is that $\hat{y}^1$ exists above path (a), which radiates from the saddle point steady state, whereas $\hat{y}^2$ exists below this path. Let us first consider the case where $\hat{y} = \hat{y}^1$. From Eq. (7) we know that path (1) satisfies the necessary conditions for optimal irreversible solutions. Let us show that it also satisfies the sufficient condition (8). Consider any path which solves the first-period problem but reaches $\bar{z}$ sooner than path (1). All these paths must reach $\bar{z}$ above path (1), which implies that $\partial W/\partial T > 0$. Similarly, any path that solves the first-period problem but reaches $\bar{z}$ later than path (1) must exist below path (1), which implies that $\partial W/\partial T < 0$. Thus path (1) satisfies the sufficiency condition (8).

Assume next that $\hat{y} = \hat{y}^2$. Let us denote by z_1 the stock level where $\dot{z} = 0$ along path (2) (see Fig. 3). When $z_0 \in [0, z_1]$, all paths which solve the first-period problem with finite T reach $\bar{z}$ above path (2), implying that (8) is satisfied for the saddle point path denoted by (3). Thus a reversible solution like (4) or (3) is the globally optimal solution. Finally, we have the case where $z_0 \in (z_1, \bar{z})$. Again any path which reaches $\bar{z}$ sooner than path (2) reaches $\bar{z}$ above $\hat{y}^2$. However, when T is high enough, paths like (b) imply $y^T(T) > \hat{y}^2$ and thus $\partial W/\partial T > 0$. This means that trajectory (2) does not satisfy the sufficiency condition (8). Thus the

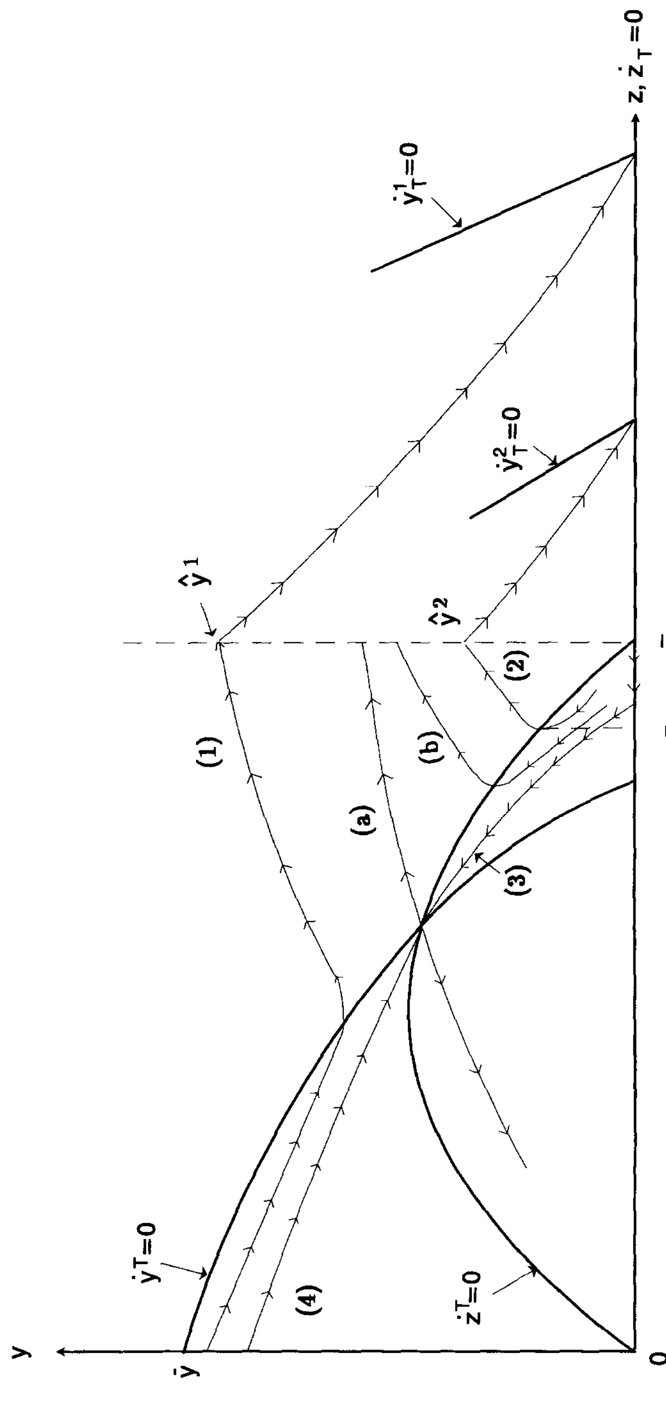

Fig. 3. Choice between reversible and irreversible solutions when $D'(\bar{z}) - \delta U'(0) < 0$ and $D'(\bar{z}) - U'(0)[\delta + \alpha'(\bar{z})] \geq 0$.

globally optimal solution is one of the reversible or irreversible solutions that yields the highest present value benefits (Seierstad and Sydsæter, 1987, Thm. 2.13, Note 27). The property that the optimal choice must be made by comparing the present value contributions is an implication of the nonconvexities in the problem.

Eq. (4) shows that $\dot{\psi}^T(T^*_-) > \dot{\psi}_T(T^*_+)$, which implies by Eq. (3) that $\dot{y}^T(T^*_-) > \dot{y}_T(T^*_+)$, i.e., that the slope of the optimal trajectory is discontinuous at $\bar{z}$, as illustrated by paths (1) and (2). Note also that along the optimal irreversible paths the development of emission level is nonmonotonic. If path (1) is optimal, the emission level reaches a local maximum at moment T^*, and if z_0 is low, there is another local maximum at $t = 0$. Compared to the properties of the pollution accumulation model with constant rate of decay these features are unique. To interpret the properties of the optimal trajectory in the vicinity of the critical pollution level consider path (1) in Fig. 3. When $z > \bar{z}$, it is natural that the level of emissions is lower the higher is the level of pollution, because higher pollution levels imply higher present value of marginal damage costs. When the optimal solution approaches $\bar{z}$ from below, an additional emission unit increases the present value of marginal pollution damage but, in addition, decreases the rate of decay. Compared to the critical stock level $\bar{z}$ the latter component represents extra costs. The value related to the rate of decay is, however, declining when $z \to \bar{z}$. Thus it is natural that the optimal trajectory has positive slope when the critical pollution level is approached from below.

If there is a multiple, odd number of steady states, the analysis is similar. The crucial point for sufficiency is always whether for a given z_0 the path radiating from the relevant saddle point steady state reaches $\bar{z}$ above or below $\hat{y}$. Note also that similar arguments show that it cannot be optimal to approach a steady state at $z = \bar{z}$.

Assume next that $D'(\bar{z}) - U'(0)[\delta + \alpha'(\bar{z})] < 0$. In this case the locus of points (z, y) where $\dot{y} = -\{D'(z) - U'(y)[\delta + \alpha'(z)]\}/U''(y) = 0$ cannot reach the z-axis below $\bar{z}$. One possibility is that no reversible steady state exists, as depicted in Fig. 4. The assumption that $D'(\bar{z}) - U'(0)\delta < 0$ implies that $\hat{y}$ is positive. In addition, given any $z_0 \in [0, \bar{z})$ there exists a unique path like (1) which reaches $(\bar{z}, \hat{y})$ in finite time. Consider any path like (a) which reaches $\bar{z}$ sooner than path (1). These paths imply that $\partial W/\partial T > 0$. Because paths which reach $\bar{z}$ later than path (1) [i.e., paths like (b)] imply that $\partial W/\partial T < 0$, the sufficiency condition (8) is satisfied and the irreversible path (1) is globally optimal.

As shown in Lemma 2, when $D'(\bar{z}) - U'(0)[\delta + \alpha'(\bar{z})] < 0$, there exists, in general, an even number of steady states. A case with two steady states is depicted in Fig. 5. The steady state with the highest stock level, i.e., with z_3^{ζ}, must be an unstable node. Accordingly, z_4^{ζ} is a saddle point. Let us again analyze whether the reversible or irreversible solution is globally optimal.

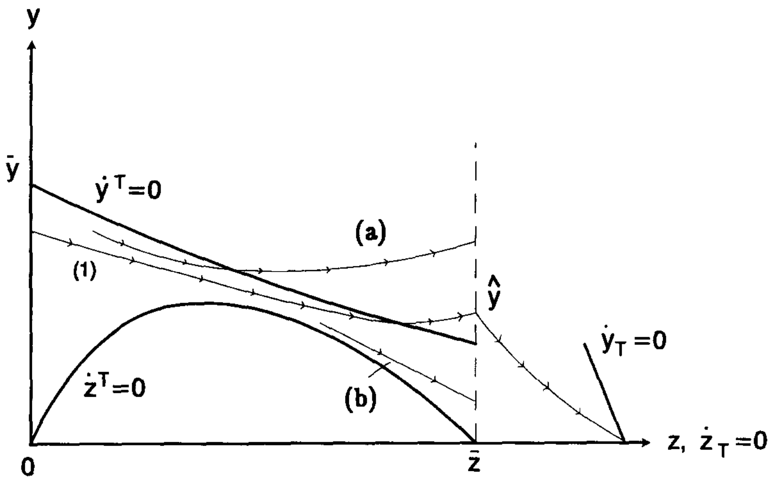

Fig. 4. Optimality of irreversible solutions when no reversible steady states exists.

Assume first that $\hat{y}^1$ is the optimal initial emission level for the second-period problem. In this case path (1) satisfies the necessary conditions for an optimal irreversible solution. By arguments similar to those for Fig. 3, this path satisfies the sufficiency requirement (8). Thus, if $\hat{y}$ exists above the path radiating from the saddle point steady state, the irreversible solution is optimal given any z_0. Assume next that $\hat{y} = \hat{y}^2$. If $z_0 \in [0, z_1]$, it can be shown by an analysis similar to that for Fig. 3 that the reversible solution satisfies the sufficiency condition (8), i.e., it is globally optimal. Consider next cases where $z_0 \in (z_1, z_3^\infty)$. For paths that reach $\bar{z}$ sooner than path (2), $\partial W/\partial T > 0$, as required for sufficiency. However, when T is high enough, paths like (b) imply $y^T(T) > \hat{y}^2$ and violate the sufficiency condition (8). This implies that, when $z_0 \in (z_1, z_3^\infty)$, neither the saddle point path nor the path like (2) can be determined to be a priori globally optimal. Thus the globally optimal solution is the one of these candidates which yields the highest present value benefits.

If $z_0 \in (z_3^\infty, \bar{z})$, an irreversible path satisfying the necessary conditions also satisfies sufficiency and is the globally optimal solution. If $\hat{y}$ is low enough, the emission level decreases monotonically. Finally, we have the case where $z_0 = z_3^\infty$. Along the path that intersects the z-axis at $z = \bar{z}$, $\partial W/\partial T < 0$ holds. This inequality directly implies that the irreversible solution satisfies the sufficiency condition (8) and must be the globally optimal solution. Thus it cannot be optimal to maintain the stock level z_3^∞ forever. Note finally that the existence of the optimality candidates considered above is guaranteed by the existence theorem of ordinary differential equations (Theorems 4.1 and 6.1 in Brock and Malliaris, 1989). We can now summarize the analysis of Section 2.2 as follows.

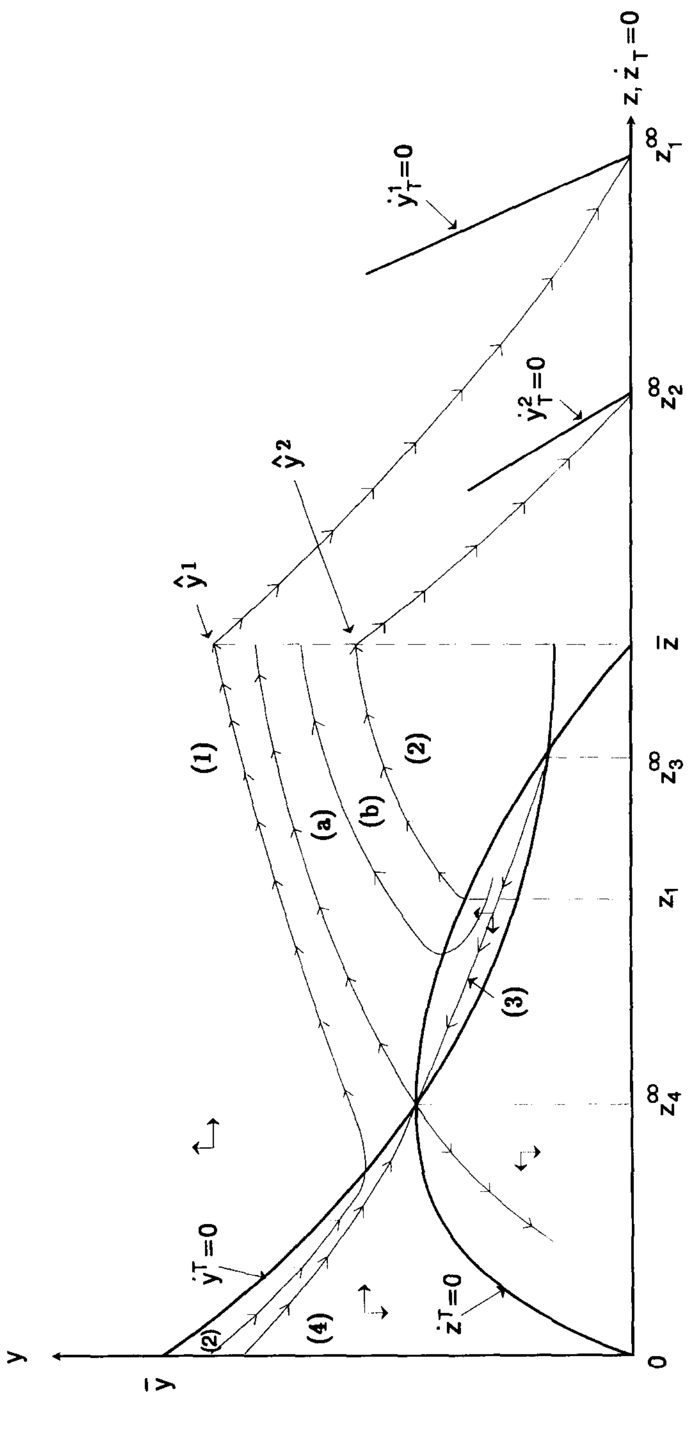

Fig. 5. Choice between reversible and irreversible solutions when $D'(\bar{z}) - U'(0)[\delta + \alpha'(\bar{z})] < 0$.

Proposition 2. *If $D'(\bar{z})/\delta - U'(0) < 0$, the optimal solution is either a reversible solution or an irreversible solution. If neither solution satisfies the sufficiency conditions, the optimal solution is the candidate with highest present value net benefits. Along the optimal irreversible path the pollution stock increases monotonically but emissions either decrease monotonically or there may occur one or two local maxima in the emission rate.*

The result that the optimal emission trajectory may be nonmonotonic should be compared with models like that in Plourde (1972) where the optimal emission level either decreases or increases monotonically. Our analysis shows that the nonconvexity problem of the model is reflected in the choice between irreversible and reversible solutions. Analytically, we consider this choice with the aid of the sufficiency theorem for free horizon problems. The analysis reveals that there are cases where the choice between reversible and irreversible solutions cannot be made without computing the present values of both solution candidates.[1] In other words, there are two locally optimal solutions in terms of the length of T. This result is a natural outcome of the nonconcavity of the decay function. Note that, in contrast to optimal irreversible solutions, the uncontrolled emission path would be a horizontal line at $y = \bar{y}$, which would reach $z = \bar{z}$ and $\alpha(z) = 0$ within a minimum time. Thus, also in the irreversible optimum the emissions must be taxed to keep them below the competitive level.

We can now compare our results with previous work in the field of stock pollution and nonconvexities. Ström (1972) avoids the nonconvexity problem by assuming that $D'(z) \to \infty$ when $z \to \bar{z}$ from below. Forster (1975) does not circumvent the problem and mentions the possibility of multiple optimality candidates. The phase diagram is given in $z - \psi$ state space, and the isocline for $\dot{\psi} = 0$ is drawn to be continuous. This leads to the conclusion that, in a case depicted in Fig. 5, the choice between saddle points z_4^∞, z_3^∞, or z_2^∞ depends on whether $z_0 \gtreqless z_3^\infty$. However, by Eq. (4) isocline $\dot{\psi} = 0$ is discontinuous at $z = \bar{z}$ and must jump upwards. With this feature the extremal dynamics is rather different than in the case were the isocline for $\dot{\psi} = 0$ is continuous. Similar comments apply to Pethig (1993). Cesar (1994) and Cesar and de Zeeuw (1994) assume that $\alpha(z)$ is negative for $z > \bar{z}$. This assumption may guarantee the concavity and differentiability of $\alpha(z)$, but its implication that the pollution stock increases without bound if $\bar{z}$ is exceeded is problematic.

[1] Several properties of this model resemble the nonconvexity implications in renewable resource models (Lewis and Schmalensee, 1977), especially if the model combines pollution control and resource harvesting (Tahvonen, 1989).

2.4. *Global optimality a priori*

In the following, we try to identify properties of the model that oppose or favor irreversible pollution. Proposition 2 already shows that a high initial pollution level may imply the optimality of irreversible pollution. As the rate of discount decreases toward zero, eventually $D'(\bar{z}) - \delta U'(0) > 0$. By Proposition 1 this means that it is optimal to maintain a positive rate of decay for all $z_0 \in [0, \bar{z})$. The same is true if $D'(z)$ is high enough for all $z \in (0, \infty)$. Also, if marginal benefits from the first emission units $U'(0)$ are low enough, it follows that $D'(\bar{z}) - \delta U'(0) > 0$ holds, and it cannot be optimal to let pollution increase above the critical level (Proposition 1). We obtain an opposite situation if we assume that $\delta > -\alpha'(z)$ for all $z \in [0, \bar{z}]$ and instead of the assumption of finite $U'(0)$ [cf. (A2)], assume that $U'(y) \to \infty$ when $y \to 0$. In this case the locus $\dot{y} = 0$ cannot reach the z-axis at a finite z and the highest steady state below $\bar{z}$ is an unstable node, as in Fig. 5. In this case there are always initial pollution levels below $\bar{z}$, from which irreversible pollution is optimal. Irreversible pollution is also optimal on a priori grounds if $D'(z) - \delta U'(0) < 0$ and $\alpha(z)$ is low enough for all $z \in [0, \bar{z}]$. This follows because finally there exist no steady states below $\bar{z}$. This may hold even when steady states exist because a low rate of decay implies that the path radiating from the saddle point steady state reaches $\bar{z}$ above $\hat{y}$ (see Proposition 2). Two somewhat less obvious cases where the choice between irreversible and reversible solutions can be made on a priori ground are described in the following corollaries:

Corollary 1. If the instantaneous net benefits of a reversible steady state associated with the given z_0 are equal to or less than the maximized current value Hamiltonian at $z = \bar{z}$, an irreversible pollution control policy is optimal.

Proof. Let us denote by (z^∞, y^∞) the reversible steady state associated with the given z_0. We show that $U(y^\infty) - D(z^\infty) \le U(\hat{y}) - D(\bar{z}) - U'(\hat{y})\hat{y}$ implies that the path radiating from the saddle point steady state will reach $\bar{z}$ below $\hat{y}$. At the steady state the current value Hamiltonian equals $U(y^\infty) - D(z^\infty)$. Using the necessary conditions (3) and (4), it can be shown that the time derivative of the maximized current value Hamiltonian equals $-\dot{z}U'(y)\delta$. This implies that the time derivative of the Hamiltonian is negative along the path radiating from the saddle point, i.e., along path (a) in Fig. 3. Thus, when path (a) reaches pollution level $\bar{z}$, the level of the Hamiltonian is below the level of $U(\hat{y}) - D(\bar{z}) - U'(\hat{y})\hat{y}$. Thus $\hat{y}$ must exist above a path like (a). In the proof of Proposition 2 we have shown that this implies the optimality of irreversible pollution for all $z_0 \in [0, \bar{z})$. ■

The proof shows that if instantaneous net benefits at the reversible steady state are low compared to the maximized Hamiltonian at $z = \bar{z}$, a path like (1) satisfies all the necessary conditions for optimality (Fig. 3). The solutions below path (1) reach pollution level $\bar{z}$ later than path (1). These paths imply that $\partial W/\partial T < 0$ [see Eq. (10)]. This means that an increase in T towards infinity cannot be optimal. To interpret this result, note that we can decrease the rate of decay below $\bar{z}$ without affecting the optimal solution above $\bar{z}$. When the rate of decay is very low, the reversible solution requires excessively costly investments in the form of low emission levels compared to the irreversible solution.

Corollary 2. If there exists a steady state where $\alpha'(z) > 0$ and at the initial pollution level $\alpha'(z_0) > 0$, reversible pollution control policy is optimal.

Proof. Note from Fig. 2 that we can safely neglect the restriction $y \geq 0$. Let (z^*, y^*) be a solution which together with ψ satisfies (1), (3), and (4) and which converges to the steady state where $\alpha'(z) > 0$. Let (z, y) be any alternative trajectory satisfying (1). Then by (3) and (A.3)

$$\Delta \equiv \int_0^\infty [U(y^*) - U(y) - D(z^*) + D(z)]\mathrm{e}^{-\delta t}\,\mathrm{d}t$$

$$\geq \int_0^\infty [U'(y^*)(y^* - y) - D'(z^*)(z^* - z)]\mathrm{e}^{-\delta t}\,\mathrm{d}t$$

$$= \int_0^\infty \{[\psi[\dot{z} + \alpha(z) - \dot{z}^* - \alpha(z^*)] - D'(z^*)(z^* - z)\}\mathrm{e}^{-\delta t}\,\mathrm{d}t$$

$$= \int_0^\infty \psi \mathrm{e}^{-\delta t}\,\mathrm{d}(z - z^*) + \int_0^\infty \{\psi[\alpha(z) - \alpha(z^*)] - D'(z^*)(z^* - z)\}\mathrm{e}^{-\delta t}\,\mathrm{d}t$$

$$= [z(t) - z^*(t)]\psi \mathrm{e}^{-\delta t}|_0^\infty - \int_0^\infty (z - z^*)(\dot{\psi}\mathrm{e}^{-\delta t} - \delta\psi\mathrm{e}^{-\delta t})\,\mathrm{d}t$$

$$+ \int_0^\infty \{\psi[\alpha(z) - \alpha(z^*)] - D'(z^*)(z^* - z)\}\mathrm{e}^{-\delta t}\,\mathrm{d}t$$

$$= (z - z^*)\psi\mathrm{e}^{-\delta t}|_0^\infty + \int_0^\infty \psi[\alpha(z) - \alpha(z^*) - \alpha'(z^*)(z - z^*)]\mathrm{e}^{-\delta t}\,\mathrm{d}t.$$

The term $(z - z^*)\psi\mathrm{e}^{-\delta t}|_0^\infty$ is nonnegative because we can add an additional restriction of the form $y \leq \bar{y}$ without affecting the optimality candidate. This implies that any admissible z will be finite. Consider next $\psi[\alpha(z) - \alpha(z^*) - \alpha'(z^*)(z - z^*)]$. As long as $z(t) \in [0, \bar{z}]$ the expression is positive by the concavity of $\alpha(z)$. When $z \in (\bar{z}, \infty]$, it holds that $\alpha(z) = 0$ and $z - z^* > 0$. By assumption we have $\alpha'(z^*) > 0$ and therefore $\Delta \geq 0$. ■

To interpret this result, note that $\alpha'(z) > 0$ at the steady state when the rate of discount is near zero or when $D'(z)$ is high enough even with low pollution levels or when marginal steady state pollution control costs, $U'[\alpha(z]$, are low enough. If, in addition, the initial pollution level is not too high, i.e., $\alpha'(z_0) > 0$, the above analysis shows that the reversible policy is optimal. We now characterize the model with the aid of a numerical example.

3. Numerical example

The two crucial consequences of our specification are the occurrence of reversible and irreversible solutions and the nonmonotonic emission time path in the case of irreversible pollution. We can capture these properties with a quadratic linear formulation. For this purpose we assume that $U(y) \equiv ay - by^2$ and $D(z) \equiv cz^2$, where a, b, c are positive constants. We assume that the decay function is linear and declining for $z \in [\hat{z}, z]$, where $\hat{z}$ is some pollution stock level above the level where the rate of decay is at a maximum but below the critical level $\bar{z}$. For $z \in [\bar{z}, \infty)$, $\alpha(z) = 0$. If we assume $z_0 > \hat{z}$ and restrict the parameters of the model so that the optimal pollution level will never reach $\hat{z}$, we can describe the essential properties of the model with a quadratic linear specification which is easily solvable. Thus we assume that $\alpha(z) \equiv \alpha_1 - \alpha_2 z$ when $z \geq \hat{z}$ and that $z_0 \geq \hat{z}$. The optimal solution for this specification is derived in the appendix.

Panels (a)–(c) of Fig. 6 present optimal time paths, which are computed by using the parameter values $\alpha_1 = 20, \alpha_2 = 0.1, a = 18, b = 0.5, c = 0.004, \delta = 0.2$, $z_0 = 101$. These parameters imply that there exists an unstable node at $z = 100$. By Proposition 2, when $z_0 > 100$, no reversible solution exists and the irreversible solution must be globally optimal. Panel (a) shows the optimal emission trajectories with lower ($\delta = 0.2$) and slightly higher ($\delta = 0.205$) rates of discount. Panel (a) shows that the optimal approach path is highly sensitive to the change in the rate of discount although there is only a minor increase in the final irreversible steady state [panel (b)]. By comparing the dotted and solid lines it is possible to show that given any level of the pollution stock emissions are higher with a higher rate of discount. When $\delta = 0.2$, we have $y(0) = 9.911$ and $\hat{y} = 8.761$. When $\delta = 0.205$, the corresponding emission levels are higher, i.e., $y(0) = 9.953$ and $\hat{y} = 8.853$. Consistent with this, the time path for the pollution stock with the higher rate of discount is above the time path with the lower rate of discount. Because $\hat{y}$ is high enough, the path is nonmonotonic before $\bar{z}$ is reached. As we have shown analytically, the time derivatives of these trajectories are discontinuous at T^*. Panel (c) shows the solution (given $\delta = 0.2$) in

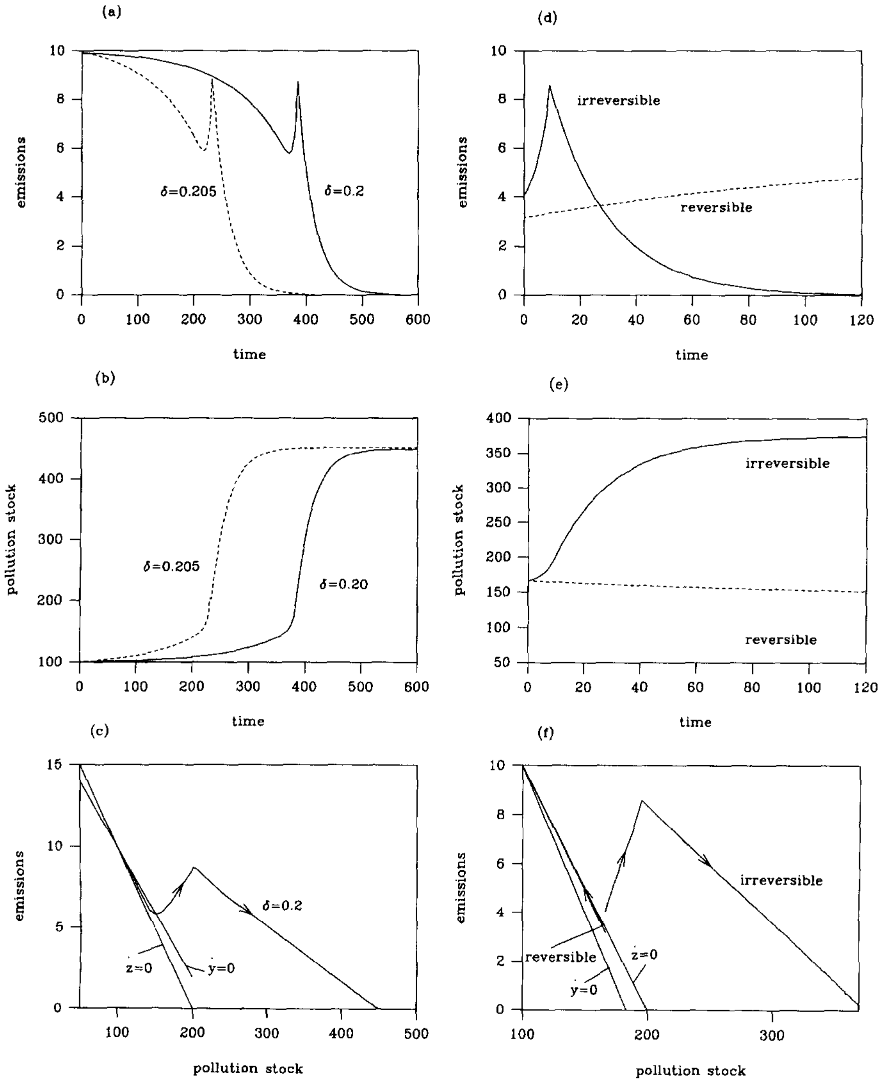

Fig. 6. Optimal trajectories.

pollution–emissions phase space. The case $\delta = 0.205$ is not depicted because in the phase diagram it is very close to the case where $\delta = 0.2$.

Panels (d)–(f) of Fig. 6 demonstrate a case where both irreversible and reversible solution candidates exist and where they give equal benefits. To generate this special case, assume $a = 22$, $b = 0.5$, $c = 0.006$, $\delta = 0.2$, $z_0 = 166.5$.

These parameter values imply that there exists a reversible saddle point steady state at $z = 100$. For $z_0 = 166.5$, there also exists an irreversible solution candidate with $T^* \cong 9.6$ satisfying all the necessary conditions for optimality. Both of these solutions imply the same net benefits ($W \cong -478$). Note from panel (e) that the pollution stock increases along the irreversible path and decreases along the reversible path. Along the irreversible solution path the benefits from higher emission levels accrue at the beginning of the period [panel (d)], whereas the reversible solution gives higher benefits and lower damage in the long run. Finally, panel (f) shows the phase diagram which in this case can be compared to Fig. 3.

4. Conclusions

Assuming a constant rate of decay may be misleading in certain local pollution problems such as nutrition enrichment of lakes, where high pollution concentration may reduce the rate of decay towards zero. In addition, this specification rules out the possibility that pollution may have irreversible consequences for the self-purification process of the given environment. An alternative specification is a decay function with an inverted-U shape. This specification leads to nonconvexity problems, which we dealt with by using the sufficiency theorem by Seierstad (1988). It is shown that there simultaneously may exist two candidates for optimal emission control: an irreversible policy, which finally implies zero emissions and 'high' pollution concentration, and a reversible policy, which maintains a positive rate of decay, strictly positive emissions, and lower pollution concentration. While the latter option may be more appealing, it may be nonoptimal due to costly emission abatement in the beginning of the planning horizon. As emphasized by Forster (1975), this may occur, e.g., with a high rate of discount. We identify several cases where the optimal choice between these policy alternatives can be made a priori. For example, if a steady state exists where the marginal rate of decay is positive and the initial stock level is below the long-run equilibrium level, irreversible pollution is always suboptimal.

As noted by Baumol (1964) and Starrett (1972), nonconvexities complicate the policy for controlling externalities. For example, the optimal cleanup of hazardous wastes, or polluted environment in general (cf. Caputo and Wilen, 1995), may include additional complications if the natural decay function is not concave and the rate of decay is initially close to zero. Studying this issue requires that the assumption $y \geq 0$ must be relaxed, which may imply new complications in the vicinity of the critical stock level, $\bar{z}$.

Although our deterministic model with nonconcave decay does not rule out the optimality of irreversible pollution, the optimal pollution accumulation and the associated Pigouvian tax may be significantly different than in the case where externalities are uncontrolled. Finally, we emphasize that irreversibility implies asymmetry in the possibility of using better information about future preferences (Arrow and Fisher, 1974). Thus, irreversible actions include extra costs which should be incorporated in the framework developed here.

Appendix

Using the specified functional forms, conditions (1), (3), and (4) take the forms: $a - 2by + \psi = 0$ (or $y = 0$), $\dot{\psi} = 2cz + \psi(\delta - \alpha_2)$, $\dot{z} = y - \alpha_1 + \alpha_2 z$, $z(0) = z_0 > \hat{z}$. Let us first solve the second-period problem, i.e., assume $z_0 = \bar{z}$. Now $(\alpha_2, \alpha_1) \equiv 0$. Solving the differential equations and taking into account the transversality condition and the initial level for z yields

$$z_T = (\bar{z} - \delta a/2c)e^{r(t-T)} + \delta a/2c, \tag{9}$$

$$y_T = r(\bar{z} - \delta a/2c)e^{r(t-T)}, \tag{10}$$

where $r = \delta/2 - (\delta^2 + 4c/b)^{1/2}/2(<0)$. Note that Eqs. (9) and (10) give the solution in the case where the optimal emission level is positive, i.e., for $2c\bar{z}/\delta - a > 0$ (cf. Lemma 1).

Consider next the first-period problem and assume $z_0 \in [\hat{z}, \bar{z})$. To assure that the optimal pollution stock remains below $\hat{z}$ we apply the restriction $\delta > \alpha_2$. Solving the differential equation system yields

$$z^T = A_1 e^{v_1 t} + A_2 e^{v_2 t} + z^\infty, \tag{11}$$

$$y^T = (v_1 A_1 - \alpha_2 A_1)e^{v_1 t} + (v_2 A_2 - \alpha_2 A_2)e^{v_1 t} - \alpha_2 z^\infty + \alpha_1, \tag{12}$$

where $v_1, v_2 = \frac{1}{2}\{\delta \pm \{\delta^2 - 4[\alpha_2(\delta - \alpha_2) - c/b]\}^{1/2}\}$ and $z^\infty = (a - 2\alpha_1 b) \times (\delta - \alpha_2)/[2(\alpha_2^2 b - \alpha_2 b\delta + c)]$. A_1 is determined by the initial pollution level z_0 yielding $A_1 = z_0 - z^\infty - A_2$. The determination of A_2 is more complicated because A_2 determines whether the solution will be irreversible or reversible (cf. Proposition 2). If $v_1 < 0$ and $z^\infty \in [\hat{z}, z]$, a reversible saddle point steady state exists, and setting $A_2 = 0$ implies that the solution converges to this steady state. If a solution satisfying (6a) does not exist, the optimal solution has been found. In contrast, an irreversible solution must satisfy $z^T(T) = \bar{z}$ and $y^T(T) = \hat{y}$. The first of these conditions implies that $A_2 = (\bar{z} - z^\infty - A_1 e^{v_1 T})e^{-v_2 T}$. The latter condition implies an analytically unsolvable equation for T^*, i.e., T^* must be solved by iterative methods. If, e.g., $v_1 < 0$ and $v_2 > 0$ and $(a - 2\alpha_1 b)(\delta - \alpha_2)/[2(\alpha_2^2 b - \alpha_2 b\delta + c)] > \bar{z}$, no reversible steady state exists and an irreversible solution can be chosen without present value comparisons.

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