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ANALYSIS OF THE M/GI/1 → ./M/1 QUEUEING MODEL

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Abstract

An M/GI/1 queueing system is in series with a unit with negative exponential service times and infinite waiting room capacity. We determine a closed form expression for the generating function of the joint queue length distribution in steady state. This result is obtained via the solution of a new type of functional equation in two variables.

Keywords

Queues in series, Markov process, generating function, functional equation, Fredholm integral equation

1. Introduction

This study is concerned with the analysis of two queues in series. Customers arrive at the first node according to a homogeneous Poisson process with rate λ and are processed by a single server. Once a customer has been served, he enters a second node which also has a single server. We assume that both waiting rooms have an infinite capacity. The sequence of service times required at the first (respectively second) node constitutes a renewal process with an *arbitrary* (respectively *exponential*) distribution function, with finite mean α^{-1} (respectively μ^{-1}). Furthermore, we assume that the arrival process and the service time processes are all mutually independent. Following Disney and König [4, p. 379], this series of two queues will be denoted by the symbol M/GI/1 → ./M/1.

Our objective is to determine the stationary queue length processes, and more precisely, the generating function (z-transform) for the stationary joint queue length distribution.

The interested reader will find a detailed list of related works in the recent queueing network survey written by Disney and König [4, pp. 379–382]. Of particular interest are the works of Boxma [2] and Neuts [11]. Boxma investigated the tandem queues M/GI/1 → ./GI/1 in the case where the service times of an arbitrary customer at both queues are identical. [2] contains explicit expressions for distributions of sojourn time, actual and virtual waiting time in the second queue (i.e. no transform is needed). Neuts determined the stability condition as well as the generating function of the stationary queue length distribution imbedded at output times for the tandem queueing model M/GI/1 → ./M/1/L where L, $L < \infty$, indicates the capacity of the waiting room at the second node.

Let $F(x, y)$ be the generating function for the joint stationary queue length distribution in the M/GI/1 → ./M/1 queueing system. We show in this paper that $F(x, y)$ can be completely determined via the solution of a functional equation of the following type:

$$K(x, y)\Psi(x, y) = A(y)\Phi(x, y) + B(x, y)\Omega(y), \tag{1.1}$$

which must hold, in particular, for all couples (x, y) with $|x| \leqslant 1$, $|y| = 1$. The functions with roman letters are known. In particular, they are analytic for $|x| < 1$ for every fixed y with $|y| \geqslant 1$, and analytic for $|y| > 1$ for every fixed x with $|x| \leqslant 1$. The function $\Psi(x, y)$ is unknown, but we know that it must be analytic for $|x| < 1$ for every fixed y with $|y| \leqslant 1$, and similarly with x and y interchanged. The function $\Phi(x, y)$ is also unknown, and it must be analytic for $|x| < 1$ for every fixed y with $|y| \geqslant 1$, and analytic for $|y| > 1$ for every fixed x with $|x| \leqslant 1$. Similarly $\Omega(y)$ is unknown, and it must be analytic for $|y| < 1$.

Two-dimensional functional equations frequently arise in the study of random walks with two-dimensional state space. A large class of these equations can be solved by reduction to *boundary value problems* (e.g., Riemann-Hilbert problem). This was first shown by Fayolle and Iasnogorodski [6] and important generalizations have been made by Cohen and Boxma [3].

In this context, equation (1.1) defines a new type of two-dimensional functional equation, mainly because of the presence of *two* bivariate unknown functions. Its solution is the purpose of the present paper.

The paper is organized as follows. In section 2 we establish the equilibrium equations of the system from which equation (1.1) directly follows. Section 3 is devoted to the study of the equation $K(x, y) = 0$, cf. (1.1), which turns out to be a key point of the analysis. This will lead, in section 4, to the factorization and to the solution of (1.1). More precisely, we shall show that (for $\alpha \neq \mu$) the real part of $\Omega(y)$ can be obtained as the unique solution of a *Fredholm integral equation of the second kind.* This result will enable us to derive, in section 5, the generating function $F(x, y)$. Section 6 addresses the computation of the average number of customers at node 2, in steady state.

2. The functional equation

Let us now introduce some notation. For $t > 0$, let

- $X_i(t)$ be the number of customers present at node i $(i = 1, 2)$ at time t, including the one being served, if any;
- $R(t)$ be the residual service time of the customer being served at node 1 at time t if $X_1(t) > 0$; otherwise $R(t) = 0$;
- $B(.)$ be the service time distribution at node 1. We assume that $B(.)$ is not a lattice distribution and that $B(0+) = 0$. It is also assumed that the second order moment of the service times is finite.
- $\beta(s)$ be the Laplace-Stieltjes transform of $B(.)$, $\mathscr{R}(s) \geqslant 0$ ($\mathscr{R}(s)$ denotes the real part of any complex number s);
- $f^{(n)}(.)$ be the nth derivative of a function $f(.)$.

Consider the stochastic process $\boldsymbol{X} := \{(X_1(t), X_2(t), R(t)), t \geqslant 0\}$. This is a *Markov process* with state space $\boldsymbol{N} \times \boldsymbol{N} \times [0, +\infty)$, where $\boldsymbol{N}$ denotes the set of all nonnegative integers. From Borovkov's results [1, p. 7] it is readily seen that under the condition

$$\lambda < \min(\alpha, \mu), \tag{2.1}$$

the Markov process $\boldsymbol{X}$ possesses a unique stationary distribution. From now on, we shall assume that condition (2.1) is fulfilled.

Define for $t > 0$, $\tau > 0$, $(m, n) \in [\boldsymbol{N} - \{0\}] \times \boldsymbol{N}$,

$$p(t; m, n, \tau) := \text{Prob}(X_1(t) = m, X_2(t) = n, R(t) < \tau); \tag{2.2}$$

$$p(t; n) := \text{Prob}(X_1(t) = 0, X_2(t) = n); \tag{2.3}$$

$$q(t; m, n) := \lim_{z \to 0} \frac{\partial}{\partial z} p(t; m, n, z). \tag{2.4}$$

For fixed $m \geqslant 1$, $n \geqslant 0$, $t > 0$, we assume that the partial derivative $\partial p(t; m, n, \tau)/\partial \tau$ exists and is continuous for $\tau > 0$, and that the limit (2.4) is finite. For $t \to +\infty$, the limiting probabilities (2.2)–(2.4) will be denoted by $p(m, n, \tau)$, $p(n)$ and $q(m, n)$, respectively. Note that the existence of these limits is ensured by condition (2.1).

We also introduce the following Laplace-Stieltjes transforms and generating functions:

$$\Xi(x, y, \sigma) := \sum_{m \geq 0} \sum_{n \geq 0} x^m y^n \int_0^{+\infty} e^{-\sigma\tau} \frac{\partial}{\partial\tau} p(m+1, n, \tau) \, d\tau; \tag{2.5}$$

$$\Psi(x, y) := \sum_{m \geq 0} \sum_{n \geq 0} q(m+1, n) x^m y^n; \tag{2.6}$$

$$\Omega(y) := \sum_{n \geq 0} p(n) y^n; \tag{2.7}$$

$$F(x, y) := \lim_{t \to +\infty} \sum_{m \geq 0} \sum_{n \geq 0} \text{Prob}(X_1(t) = m, \ X_2(t) = n) x^m y^n,$$

for $|x| \leq 1$, $|y| \leq 1$, $\mathscr{R}(\sigma) \geq 0$.

From (2.5), (2.7), it is easily seen that the generating function $F(x, y)$ for the joint stationary queue length distribution satisfies the relation

$$F(x, y) = x\Xi(x, y, 0) + \Omega(y), \tag{2.8}$$

for $|x| \leq 1$, $|y| \leq 1$.

The remainder of this section is devoted to the derivation of equation (1.1). To this end, we first derive (proposition (2.1)) the equilibrium equations of the system. Then, we derive a functional equation which relates the three unknown functions defined in (2.5)–(2.7) (proposition 2.2), from which (1.1) directly follows.

PROPOSITION 2.1

For $t \to +\infty$ the state probabilities (2.2)–(2.4) satisfy the following set of differential equations:

for $m \geq 1$, $n \geq 0$, $\tau > 0$,

$$\begin{aligned} \frac{\partial}{\partial\tau} p(m, n, \tau) = & -\lambda p(m-1, n, \tau) \mathbf{1}_{m>1} - \lambda B(\tau) p(n) \mathbf{1}_{m=1} \\ & -\mu p(m, n+1, \tau) + q(m, n) \\ & + (\lambda + \mu \mathbf{1}_{n>0}) p(m, n, \tau) \\ & - B(\tau) q(m+1, n-1) \mathbf{1}_{n>0}; \end{aligned} \tag{2.9}$$

for $n \geq 0$,

$$(\lambda + \mu \mathbf{1}_{n \geq 1}) p(n) = \mu p(n+1) + q(1, n-1) \mathbf{1}_{n>0}, \tag{2.10}$$

where $\mathbf{1}_A$ denotes the indicator function of the event $\{A\}$.

Proof

Let us consider a small time interval $(t, t+h)$. Because the arrival process is Poisson and service times at node 2 are exponentially distributed, we obtain the following equations:

for $m \geqslant 1$, $n \geqslant 0$, $\tau > 0$,

$$\begin{aligned} p(t+h;\, m,\, n,\, \tau) &= \lambda h p(t;\, m-1,\, n,\, \tau+h)\mathbf{1}_{m>1} + \lambda h p(t;\, n)B(\tau)\mathbf{1}_{m=1} \\ &\quad + \mu h p(t;\, m,\, n+1,\, \tau+h) + (1-\lambda h - \mu h \mathbf{1}_{n>0}) \\ &\quad \times\Big(p(t;\, m,\, n,\, \tau+h) - p(t;\, m,\, n,\, h) \\ &\quad + \mathbf{1}_{n>0}\int_0^h B(\tau+h-z)\, \mathrm{d}_z p(t;\, m+1,\, n-1,\, z)\Big) + \mathrm{o}(h); \end{aligned} \tag{2.11}$$

for $n \geqslant 0$,

$$\begin{aligned} p(t+h;\, n) &= \mu h p(t;\, n+1) + (1-\lambda h - \mu h \mathbf{1}_{n \geqslant 1}) \\ &\quad \times\Big(p(t;\, n) + \mathbf{1}_{n>0}\int_0^h \mathrm{d}_z p(t;\, 1,\, n-1,\, z)\Big) + \mathrm{o}(h). \end{aligned} \tag{2.12}$$

Subtracting $p(t;\, m,\, n,\, \tau)$ (respectively $p(t;\, n)$) from both sides of (2.11) (respectively (2.12)), then dividing by h and letting $h \to 0$ and $t \to +\infty$, we obtain (2.9) (respectively (2.10)). ■

PROPOSITION 2.2

The unknown functions $\Xi(x, y, \sigma)$, $\Psi(x, y)$, $\Omega(y)$, cf. (2.5)–(2.7), are related through the following functional equation:

for $|x| \leqslant 1$, $|y| \leqslant 1$, $\mathscr{R}(\sigma) \geqslant 0$,

$$\begin{aligned} &\big(\lambda(1-x) + \mu(1-y^{-1}) - \sigma\big)\Xi(x,\, y,\, \sigma) \\ &\quad = \mu x^{-1}(1-y^{-1})\phi(x,\, \sigma) - \big(1 - yx^{-1}\beta(\sigma)\big)\psi(x,\, y) \\ &\qquad + \big(\lambda(1-x^{-1}) - \mu x^{-1}(1-y^{-1})\big)\beta(\sigma)\Omega(y), \end{aligned} \tag{2.13}$$

where

$$\phi(x,\, \sigma) := x\Xi(x,\, 0,\, \sigma) + \beta(\sigma)\Omega(0). \tag{2.14}$$

Proof

Introduce the transforms (2.5)–(2.7) into equations (2.9), (2.10), by noting that

$$\int_0^{+\infty} \sigma\, \mathrm{e}^{-\sigma\tau} p(m,\, n,\, \tau)\, \mathrm{d}\tau = \int_0^{+\infty} \mathrm{e}^{-\sigma\tau}\, \mathrm{d}_\tau p(m,\, n,\, \tau). \quad ■$$

We are now in position to derive the functional equation announced in section 1 (cf. (1.1)). With condition (2.1) the function $\Xi(x, y, \sigma)$ is bounded for $|x| \leqslant 1$, $|y| \leqslant 1$, $\mathscr{R}(\sigma) \geqslant 0$. Consequently, for

$$\sigma = \lambda(1-x) + \mu(1-y^{-1}),$$

with $|x| \leqslant 1$, $|y| \leqslant 1$ and $\mathscr{R}(\sigma) \geqslant 0$, the right-hand side (r.h.s.) of equation (2.13) must vanish. This yields the following two-dimensional functional equation:

$$K(x, y)\Psi(x, y) = A(y)\Phi(x, y) + B(x, y)\Omega(y), \tag{2.15}$$

for $|x| \leqslant 1$, $|y| \leqslant 1$, $\mathscr{R}(\lambda(1-x) + \mu(1-y^{-1})) \geqslant 0$, with

$$\Phi(x, y) := \phi(x, \lambda(1-x) + \mu(1-y^{-1})); \tag{2.16}$$

$$K(x, y) := x - y\beta(\lambda(1-x) + \mu(1-y^{-1})); \tag{2.17}$$

$$A(y) := \mu(1-y^{-1}); \tag{2.18}$$

$$B(x, y) := -(\lambda(1-x) + \mu(1-y^{-1}))\beta(\lambda(1-x) + \mu(1-y^{-1})). \tag{2.19}$$

The three unknown functions involved in equation (2.15) have the following properties (cf. (2.5)–(2.7), (2.14), (2.16)):

- for every fixed y with $|y| \leqslant 1$, $\Psi(x, y)$ is analytic for $|x| < 1$, continuous for $|x| \leqslant 1$, and similarly for x and y interchanged;
- for every fixed y with $|y| \geqslant 1$, $\Phi(x, y)$ is analytic in x for $|x| < 1$, continuous for $|x| \leqslant 1$;
- for every fixed x with $|x| \leqslant 1$, $\Phi(x, y)$ is analytic in y for $|y| > 1$, continuous for $|y| \geqslant 1$;
- $\Omega(y)$ is analytic for $|y| < 1$, continuous for $|y| \leqslant 1$.

Now consider how the generating function $F(x, y)$ may be determined from $\Psi(x, y)$. Let us assume that $\Psi(x, y)$ is known for $|x| \leqslant 1$ and $|y| \leqslant 1$. Take $\sigma = 0$ in (2.13), and for fixed x with $|x| \leqslant 1$, choose y such that $\lambda(1-x) + \mu(1-y^{-1}) = 0$, i.e.

$$y = \nu(x) := \frac{\mu}{\lambda(1-x) + \mu}.$$

Since $|\nu(x)| \leqslant 1$ for $|x| \leqslant 1$ and since $\Xi(x, y, 0)$ is bounded for $|x| \leqslant 1$ and $|y| \leqslant 1$, cf. (2.5), it follows that the r.h.s. of (2.13) must vanish whenever $(x, y) = (x, \nu(x))$ with $|x| \leqslant 1$. This condition gives the following relation:

$$\phi(x, 0) = \nu(x)\left(\frac{1}{\lambda} - \frac{x}{\mu}\right)\Psi(x, \nu(x)), \quad \text{for } |x| \leqslant 1. \tag{2.20}$$

Replacing $\phi(x, 0)$ by (2.20) into (2.13) and using (2.8), we get for $|x| \leqslant 1$, $|y| \leqslant 1$,

$$F(x, y) = \frac{\mu\left(1 - \frac{1}{y}\right)\left(\frac{1}{\lambda} - \frac{x}{\mu}\right)\nu(x)\Psi(x, \nu(x)) + (y - x)\Psi(x, y)}{\lambda(1 - x) + \mu\left(1 - \frac{1}{y}\right)}. \tag{2.21}$$

The above formula shows that the computation of the generating function $F(x, y)$ is equivalent to the computation of the function $\Psi(x, y)$ for $|x| \leqslant 1$, $|y| \leqslant 1$. The remainder of this study is devoted to the determination of $\Psi(x, y)$ for $|x| \leqslant 1$, $|y| \leqslant 1$. We first consider in the next section the equation $K(x, y) = 0$, cf. (2.17), which will play a fundamental role in the determination of $\Psi(x, y)$. $K(x, y)$ will be called the *kernel* of equation (2.15).

3. Analysis of the kernel

Throughout this paper, $\boldsymbol{U}^+$ (respectively $\boldsymbol{U}^-$) will denote the region lying on the left (respectively right) of any contour $\boldsymbol{U}$, when moving on $\boldsymbol{U}$ in the positive direction (i.e. counter-clockwise).

We recall that a function $f(z)$ is analytic at the point $z = z_0$ if $f(z)$ can be expanded into a power series in the vicinity of z_0.

We define,

$$\delta := \frac{\mu}{\alpha}, \quad 0 < \delta < +\infty;$$

$$\boldsymbol{C}_a := \{ |z| = a \}, \quad \text{for } a > 0;$$

$$\boldsymbol{C} := \boldsymbol{C}_1.$$

Note that $\boldsymbol{C}_a^+ := \{ |z| < a \}$ and $\boldsymbol{C}_a^- := \{ |z| > a \}$.

We shall prove the following results. For fixed δ and under two assumptions bearing on $\beta(.)$ (see (3.3) and (3.4)) there exist a real number y_δ ($y_\delta \geqslant 1$) and a *smooth* contour $\boldsymbol{L}_\delta$ contained in the unit disk, such that:

(R1) for all $x \in \{ |x| \leqslant 1\} \cap \boldsymbol{L}_\delta^-$, the equation $K(x, y) = 0$ has exactly one root $y = Y(x)$ in the region $\{ |y| \geqslant y_\delta \}$;

(R2)′ for all $x \in \boldsymbol{L}_\delta^+$, the equation $K(x, y) = 0$ has no roots in the region $\{ |y| \geqslant y_\delta \}$.

We begin with the following lemma:

LEMMA 3.1

For fixed y with $1 \leqslant |y| \leqslant y_\delta$, the equation $K(x, y) = 0$ has exactly one root $x = X(y)$ in $\{|x| \leqslant 1\}$, where

$$y_\delta := \begin{cases} 1, & \text{if } \delta \leqslant 1; \\ t_\delta, & \text{if } \delta > 1, \end{cases}$$

with t_δ the unique root in $\{|t| > 1\}$ of the equation

$$1 - t\beta\big(\mu(1 - t^{-1})\big) = 0.$$

All the roots have multiplicity one, except whenever $\delta = 1$ and $y = 1$, in which case the equation $K(x, 1) = 0$ has a root of multiplicity two at $x = 1$.

Moreover, $|X(y)| = 1$ if and only if $y = 1$ or $y = y_\delta$, in which case $X(1) = X(y_\delta) = 1$. Finally, $X(y)$ is analytic in $\{1 \leqslant |y| \leqslant y_\delta,\ y \neq 1\}$ and continuous in $\{1 \leqslant |y| \leqslant y_\delta\}$.

Proof

The existence of t_δ for $\delta > 1$ as well as the proof of the lemma for $\delta \leqslant 1$ (respectively $\delta > 1$ and $|y| = 1$) follow from Takacs' lemma [12, pp. 47–48].

Let us consider the case $\delta > 1$ and $1 < |y| \leqslant y_\delta$. Since clearly $\beta(\mu(1 - y^{-1})) \leqslant y^{-1}$ for all real y with $1 \leqslant y \leqslant y_\delta$, we have for $|x| = 1$, $1 < |y| \leqslant y_\delta$, but $y \neq y_\delta$,

$$\left|\beta\left(\lambda(1 - x) + \mu\left(1 - \frac{1}{y}\right)\right)\right| < \beta\left(\mu\left(1 - \frac{1}{|y|}\right)\right) \leqslant \frac{|x|}{|y|}.$$

Consequently, it is seen by *Rouché's theorem* [10, p. 131] that for $1 < |y| \leqslant y_\delta$ with $y \neq y_\delta$, the equation $K(x, y) = 0$ has exactly one root (with multiplicity one) in the region $\{|x| < 1\}$. Let us now examine the equation $K(x, y_\delta) = 0$ for $\delta > 1$. From the relation

$$|\beta^{(1)}(s)| \leqslant \frac{|\beta(s)|}{\alpha},$$

which holds for all $s \geqslant 0$, we derive the following inequalities:

$$\begin{aligned} &\left|\lambda y_\delta \beta^{(1)}\left(\lambda(1 - x) + \mu\left(1 - \frac{1}{y_\delta}\right)\right)\right| \\ &\quad \leqslant \lambda y_\delta \beta^{(1)}\left(\mu\left(1 - \frac{1}{y_\delta}\right)\right) \\ &\quad \leqslant \frac{\lambda}{\alpha} y_\delta \beta\left(\mu\left(1 - \frac{1}{y_\delta}\right)\right) = \frac{\lambda}{\alpha} < 1, \quad \text{for } |x| \leqslant 1. \end{aligned}$$

Consequently, for all $|x| \leqslant 1$ with $x \neq 1$, we have

$$\left|1 - y_\delta \beta\left(\lambda(1-x) + \mu\left(1 - \frac{1}{y_\delta}\right)\right)\right| < |x-1|,$$

which therefore shows that $x = 1$ is the only zero of the function $K(x, y_\delta)$ in $\{|x| \leqslant 1\}$.

Fix now y_0 with $1 \leqslant |y_0| \leqslant y_\delta$, $y_0 \neq 1$, and define $x_0 := X(y_0)$. Since $|X(y_0)| \leqslant 1$, it follows that there exist two real numbers r_1 and r_2 such that $\mathscr{R}(\lambda(1-x_0) + \mu(1-y_0^{-1})) > 0$ for $x \in \{|x - x_0| < r_1\}$ and $y \in \{|y - y_0| < r_2\}$. In other words, the function $K(x, y)$ is analytic in $\{|x - x_0| < r_1\}$ for every fixed y with $|y - y_0| < r_2$, and analytic in $\{|y - y_0| < r_2\}$ for every fixed x with $|x - x_0| < r_1$. Moreover, it is seen from the uniqueness of the solution x_0, that

$$\frac{\partial}{\partial x} K(x, y)\bigg|_{(x_0, y_0)} \neq 0.$$

Consequently, the '*implicit function theorem* for complex variables applies [5, p. 101], which proves the last statement of the lemma. ■

For fixed δ, define

$$\boldsymbol{L}_\delta := X\left(\boldsymbol{C}_{y_\delta}\right). \tag{3.1}$$

Because of the continuity of $X(y)$ on $\boldsymbol{C}_{y_\delta}$ it is seen that $\boldsymbol{L}_\delta$ is a *closed curve*. On the other hand, by differentiating the relation $K(X(y), y) = 0$ (cf. lemma 3.1) we get

$$X^{(1)}(y) = \frac{X(y) + \mu\beta^{(1)}\left(\lambda(1-x) + \mu(1-y^{-1})\right)}{y\left(1 + \lambda y \beta^{(1)}\left(\lambda(1-x) + \mu(1-y^{-1})\right)\right)}, \quad \text{for } y \in \boldsymbol{C}_{y_\delta}. \tag{3.2}$$

Using the uniqueness of the solution $X(y)$ we see that the denominator of $X^{(1)}(y)$ cannot vanish for $y \in \boldsymbol{C}_{y_\delta}$, which shows that the curve $\boldsymbol{L}_\delta$ is everywhere differentiable.

Let us now introduce the technical assumptions on $\beta(.)$ previously announced. For fixed δ, we assume that

$$X^{(1)}(y) \neq 0 \quad \text{for } y \in \boldsymbol{C}_{y_\delta}$$

$$\left(\text{except if } \delta = 1 \text{ and } y = 1 \text{ since in that case } X^{(1)}(1) = 0\right); \tag{3.3}$$

$$X(y_1) \neq X(y_2) \quad \text{for all } (y_1, y_2) \in \boldsymbol{C}_{y_\delta} \times \boldsymbol{C}_{y_\delta} \text{ with } y_1 \neq y_2. \tag{3.4}$$

The assumption (3.3) ensures the *smoothness* of the curve $\boldsymbol{L}_\delta$ [3, p. 7]. The assumption (3.4) ensures that $\boldsymbol{L}_\delta$ is a *simply connected curve* (no double point). Consequently, $\boldsymbol{L}_\delta$ is a smooth contour if (3.3) and (3.4) hold (except at the point $y = 1$ if $\delta = 1$, where $\boldsymbol{L}_1$ has a corner point).

REMARK 3.1

It is easily checked that (3.3) and (3.4) both hold if $\beta(s) := \alpha/(s+\alpha)$ (i.e. the service times in queue 1 are exponentially distributed).

The following lemma contains the key result for the proof of claims (R1) and (R2).

LEMMA 3.2

Let $G(x, y)$ be a complex-valued function defined in $\boldsymbol{D}_1 \times \boldsymbol{D}_2$, where $\boldsymbol{D}_i$ is a connected set of the complex plane, $i = 1, 2$. We assume that $G(x, y)$ is analytic in each of its variables. Let $\boldsymbol{U}$ be a Jordan contour [10, p. 2] contained in $\boldsymbol{D}_2$ such that $G(x, y) \neq 0$ for all $x \in \boldsymbol{D}_1$ and $y \in \boldsymbol{U}$.

Then, for all $(x_1, x_2) \in \boldsymbol{D}_1 \times \boldsymbol{D}_1$ with $x_1 \neq x_2$, the equations $G(x_1, y) = 0$ and $G(x_2, y) = 0$ have exactly the same number of solutions in $\boldsymbol{U}^+$ (respectively in $\boldsymbol{U}^- \cap \boldsymbol{D}_2$).

Proof

Consider the function

$$I(x) := \frac{1}{2\pi \mathrm{i}} \int_U \frac{\frac{\partial}{\partial z} G(x, z)}{G(x, z)} \mathrm{d}z.$$

This is a continuous function in $\boldsymbol{D}_1$ from the above assumptions, which also reads

$$I(x) = \frac{1}{2\pi} \Delta_U \arg\{G(x, y)\}, \quad \text{for any } x \in \boldsymbol{D}_1, \tag{3.5}$$

where $\Delta_U \arg\{G(x, y)\}$ denotes the total variation of the argument of $G(x, y)$ when y describes the contour $\boldsymbol{U}$, for fixed x in $\boldsymbol{D}_1$.

Let us now assume that $G(x_0, y)$ has exactly n zeros in $\boldsymbol{U}^+$ (respectively $\boldsymbol{U}^- \cap \boldsymbol{D}_2$) for fixed $x_0 \in \boldsymbol{D}_1$. Then, the *argument principle* [10, p. 130] together with (3.5) give us that

$$I(x_0) = n, \tag{3.6}$$

since $G(x, y)$ is analytic in $\boldsymbol{D}_2$ for every fixed x in $\boldsymbol{D}_1$.

The proof is now concluded by observing that the (continuous) function $I(x)$ is an integer valued function in $\boldsymbol{D}_1$, cf. (3.5), which implies from (3.6) that

$$I(x) = n, \quad \text{for any } x \in \boldsymbol{D}_1,$$

or, equivalently, that $G(x, y)$ has exactly n zeros in $\boldsymbol{U}^+$ (respectively in $\boldsymbol{U}^- \cap \boldsymbol{D}_2$) $\forall\, x \in \boldsymbol{D}_1$. ■

Lemma 3.2 now enables us to justify claim (R1).

PROPOSITION 3.1

For all $x \in \{|x| \leqslant 1\} \cap \boldsymbol{L}_\delta^-$, the equation $K(x, y)=0$ has exactly one root $y = Y(x)$ in $\{|y| \geqslant y_\delta\}$. Moreover, $Y(x)$ is analytic in $\{|x|<1\} \cap \boldsymbol{L}_\delta^-$ and continuous in $\{|x| \leqslant 1\} \cap \boldsymbol{L}_\delta^-$.

Proof

For $|x|=1$, $x \neq 1$, it is seen by using *Rouché's theorem* that $K(x, y)$ has exactly one zero in $\{|y| \geqslant y_\delta\}$. Moreover, for $|x| \leqslant 1$ with $x \notin \boldsymbol{L}_\delta$ and $|y| = y_\delta$, we know from lemma 3.1 that $K(x, y) \neq 0$. The first part of the proposition then follows from lemma 3.2 with $G(x, y) := K(x, y)$, $\boldsymbol{D}_1 := \{|x| \leqslant 1\} \cap \boldsymbol{L}_\delta^-$, $\boldsymbol{D}_2 := \{|y| \geqslant y_\delta\}$ and $\boldsymbol{U} := \boldsymbol{C}_{y_\delta}$.

The analyticity and the continuity of $Y(x)$ in $\{|x|<1\} \cap \boldsymbol{L}_\delta^-$ and $\{|x| \leqslant 1\} \cap \boldsymbol{L}_\delta^-$ respectively, follow from the *implicit function theorem* for complex variables [5, p. 101]. ■

The following lemma will be needed for proving claim (R2).

LEMMA 3.3

The function $Y(x)$ defined in proposition 3.1 can be analytically continued up to $\boldsymbol{L}_\delta$ (only to $\boldsymbol{L}_1 - \{1\}$ if $\delta = 1$), and moreover,

$$Y(X(y_0)) = y_0, \quad \text{for any } y_0 \in \boldsymbol{C}_{y_\delta}. \tag{3.7}$$

Proof

Fix $y_0 \in \boldsymbol{C}_{y_\delta}$ with $y_0 \neq 1$ if $\delta \leqslant 1$, and define $x_0 := X(y_0)$ (cf. lemma 3.1). It is easily seen from (3.2) and the assumption (3.3) that

$$\frac{\partial}{\partial y} K(x, y)\bigg|_{(x_0, y_0)} = -y_0^{-1}\left(x_0 + \mu\beta^{(1)}\left(\lambda(1-x_0) + \mu\left(1-y_0^{-1}\right)\right)\right) \neq 0.$$

Consequently, the *implicit function theorem* applies to $K(x, y)$ (see an analogous treatment in the proof of lemma 3.1), which shows that there exists a *unique* function $Y_*(x)$ analytic in a neighborhood S_{x_0} of x_0, such that $K(x, Y_*(x))=0$ for any $x \in \boldsymbol{S}_{x_0}$ and $Y_*(x_0) = y_0$. Since the equation $K(x, y)=0$ has exactly one solution in $\{|y| \geqslant y_\delta\}$ for fixed $x \in \{|x| \leqslant 1\} \cap \boldsymbol{L}_\delta^- \cap \boldsymbol{S}_{x_0}$. This shows from the *principle of analytic continuation* that $Y_*(x)$ is the analytic continuation of $Y(x)$ in $\boldsymbol{S}_{x_0}$. In particular, $Y(X(y_0)) = Y(x_0) = y_0$.

If $y_0 = 1$ whenever $\delta \leqslant 1$, then clearly $Y(X(1)) = 1$, which concludes the proof. ■

PROPOSITION 3.2

For every $x \in L_\delta^+$, the equation $K(x, y) = 0$ has no roots in $\{ |y| \geqslant y_\delta \}$.

Proof

The proof is contained in appendix A.

4. The integral equation

In this section we shall prove the following two results:

(R3) the sought function $\Psi(x, y)$ is fully determined in $\{ |x| \leqslant 1, x \notin L_\delta \} \times \{ |y| < y_\delta \}$ by the values $\Omega(y)$ takes on the circle C_{y_δ};

(R4) $\Omega(y)$ satisfies a *homogeneous Fredholm integral equation of the second kind* on C_{y_δ}.

Let us first show that for $\delta > 1$ the functions $\Psi(x, y)$ and $\Omega(y)$ can be both analytically continued up to the contour C_{y_δ}, for every fixed x with $|x| \leqslant 1$.

Recall from section 2 that the unknown functions $\Psi(x, y)$, $\Phi(x, y)$ and $\Omega(y)$ must satisfy the following equation, cf. (2.15),

$$\Psi(x, y) = \frac{A(y)}{K(x, y)} \Phi(x, y) + T(x, y)\Omega(y), \tag{4.1}$$

for $|x| \leqslant 1$, $|y| \leqslant 1$, $\mathscr{R}(\lambda(1-x) + \mu(1-y^{-1})) \geqslant 0$, where $K(x, y)$, $A(y)$ are given in (2.17), (2.18) respectively, and where

$$T(x, y) := \frac{B(x, y)}{K(x, y)}. \tag{4.2}$$

Set $(x, y) = (X(y), y)$ in (2.15) with $|y| = 1$, where $X(y)$ has been defined in lemma 3.1. This immediately yields the following relation

$$\Omega(y) = -\frac{A(y)}{B(X(y), y)} \Phi(X(y), y), \quad |y| = 1. \tag{4.3}$$

For $1 < |y| \leqslant y_\delta$, it is seen that the r.h.s. of (4.3) is a meromorphic function. This follows from the properties of the functions $\Phi(x, y)$ and $X(y)$ (see (2.16) and lemma 3.1). The possible poles of this meromorphic function are the zeros of $B(X(y), y)$ for $1 < |y| \leqslant y_\delta$. Let us show that there do not exist any such zeros. From (2.19) the identity $B(X(y), y) = 0$ implies that

(i) $\lambda(1 - X(y)) + \mu(1 - y^{-1}) = 0$,

or

(ii) $\beta(\lambda(1 - X(y)) + \mu(1 - y^{-1})) = 0$.

Since $X(y)=y\beta(\lambda(1-X(y))+\mu(1-y^{-1}))$ for $1<|y|\leqslant y_\delta$ (cf. the definition of $X(y)$), then (i) necessarily entails that $X(y)=y$, which is impossible if $1<|y|\leqslant y_\delta$, since in that case $|X(y)|\leqslant 1$ by lemma 3.1.

On the other hand, (ii) implies from the definition of $X(y)$ that $X(y)=0$. However, it is seen from (2.14), (2.16), that $\Phi(x, y)$ also vanishes if $x=0$ and $\beta(\lambda(1-x)+\mu(1-y^{-1}))=0$, which therefore shows that the r.h.s. of (4.3) is analytic in $\{1<|y|<y_\delta\}$ and continuous in $\{1\leqslant|y|\leqslant y_\delta\}$.

Consequently, we deduce from the *principle of analytic continuation* that (4.3) gives the analytic continuation of $\Omega(y)$ to $\{|y|\leqslant y_\delta\}$. This, in turn, entails that the r.h.s. of (4.1) defines the analytic continuation of $\Psi(x, y)$ to $\{|y|\leqslant y_\delta\}$, for fixed x with $|x|\leqslant 1$.

We are now ready to prove a part of claim (R3). Set $y=t$ in (4.1) with $t\in C_{y_\delta}$, multiply by $\mathrm{d}t/(2\pi\mathrm{i}(t-y))$ and integrate along the contour C_{y_δ}, where C_{y_δ} is traversed in the *positive direction*. Then,

$$\frac{1}{2\pi\mathrm{i}}\int_{C_{y_\delta}}\frac{\Psi(x, t)}{t-y}\mathrm{d}t=\frac{1}{2\pi\mathrm{i}}\int_{C_{y_\delta}}\frac{A(t)\Phi(x, t)}{K(x, t)(t-y)}\mathrm{d}t+\frac{1}{2\pi\mathrm{i}}\int_{C_{y_\delta}}\frac{T(x, t)\Omega(t)}{t-y}\mathrm{d}t, \tag{4.4}$$

for $|x|\leqslant 1$ with $x\notin L_\delta$ and $|y|<y_\delta$.

Note that $K(x, y)\neq 0$ for $|x|\leqslant 1$ with $x\notin L_\delta$ and $y\in C_{y_\delta}$, cf. lemma 3.1, which ensures that both integrals in the r.h.s. of (4.4) are well defined.

The function $\Psi(x, y)$ being analytic (respectively continuous) for $|y_\delta|<1$ (respectively $|y_\delta|\leqslant 1$) for every fixed x with $|x|\leqslant 1$, the *residue theorem* [10, p. 122] applies to the integral in the left-hand side of (4.4), which gives:

$$\Psi(x, y)=\frac{1}{2\pi\mathrm{i}}\int_{C_{y_\delta}}\frac{A(t)\Phi(x, t)}{K(x, t)(t-y)}\mathrm{d}t+\frac{1}{2\pi\mathrm{i}}\int_{C_{y_\delta}}\frac{T(x, t)\Omega(t)}{t-y}\mathrm{d}t, \tag{4.5}$$

for $|x|\leqslant 1$ with $x\notin L_\delta$ and $|y|<y_\delta$.

By proposition 3.1, we know that $K(x, y)$ has exactly one zero $y=Y(x)$ in $C_{y_\delta}^{-1}$ if $x\in\{|x|\leqslant 1\}\cap L_\delta^-$. Similarly, proposition 3.2 expresses that $K(x, y)$ has no zeros in $C_{y_\delta}^-\cup C_{y_\delta}$ if $x\in L_\delta^+$.

This entails that the function

$$t\to\frac{A(t)\Phi(x, t)}{K(x, t)}$$

is analytic in $C_{y_\delta}^-$ and continuous in $C_{y_\delta}^-\cup C_{y_\delta}$ for any $x\in L_\delta^+$, and that it has exactly one zero $y=Y(x)$ in $C_{y_\delta}^-$ for any $x\in\{|x|\leqslant 1\}\cap L_\delta^-$.

Consequently, the *residue theorem* now applies to the first integral of the r.h.s. of (4.5), which gives, for arbitrary $\delta > 0$,

$$\Psi(x, y) = \begin{cases} \Pi(x, y), & \text{for } x \in \boldsymbol{L}_\delta^+, \quad |y| < y_\delta; \\ \Pi(x, y) + \dfrac{\Gamma(x)}{Y(x) - y}, & \text{for } x \in \{|x| \leqslant 1\} \cap \boldsymbol{L}_\delta^-, \\ & |y| < y_\delta, \end{cases} \tag{4.6}$$

where

$$\Pi(x, y) := \frac{1}{2\pi i} \int_{C_{y_\delta}} \frac{T(x, t)\Omega(t)}{t - y} \mathrm{d}t; \tag{4.7}$$

$$\Gamma(x) := \frac{-\mu(1 - Y(x))\Phi(x, Y(x))}{x + \mu\beta^{(1)}\left(\lambda(1 - x) + \mu\left(1 - \dfrac{1}{Y(x)}\right)\right)}. \tag{4.8}$$

$\Gamma(x)$ is the residue of the function

$$t \to \frac{A(t)\Phi(x, t)}{K(x, t)}$$

at the point $t = Y(x)$, for any $x \in \{|x| \leqslant 1\} \cap \boldsymbol{L}_\delta^-$.

It remains to compute the residue $\Gamma(x)$ in $\{|x| \leqslant 1\} \cap \boldsymbol{L}_\delta^-$ to conclude the proof of claim (R3). To this end, we state the following:

LEMMA 4.1

For any $\delta > 0$, the function $\Gamma(x)$ is continuous in $\{|x| \leqslant 1\} \cap (\boldsymbol{L}_\delta^- \cup \boldsymbol{L}_\delta)$.

Proof

First of all, let us recall that $Y(x)$ is a continuous and non-vanishing function for $x \in \{|x| \leqslant 1\} \cap (\boldsymbol{L}_\delta^- \cup \boldsymbol{L}_\delta)$ (cf. proposition 3.1 and lemma 3.3). Consequently, in view of (4.8) we only have to verify that the denominator of $\Gamma(x)$ does not vanish in $\{|x| \leqslant 1\} \cap (\boldsymbol{L}_\delta^- \cup \boldsymbol{L}_\delta)$, or if it does, that the numerator also vanishes at this point. For $x \in \{|x| \leqslant 1\} \cap \boldsymbol{L}_\delta^-$, it is readily seen that the denominator of $\Gamma(x)$ cannot vanish because of the uniqueness of the solution of the equation $K(x, y) = 0$ (see proposition 3.1). For $x \in \boldsymbol{L}_\delta$ with $x \neq 1$ if $\delta = 1$, the denominator does not vanish because of (3.3) and (3.7). Finally, if $x = 1$ and $\delta = 1$, then $Y(x) = 1$, and the numerator and the denominator of $\Gamma(x)$ both vanish at the point 1, which shows that $\Gamma(1)$ is also well defined in that case. ■

This result can be used as follows. Since the r.h.s. of (4.3) is the analytic continuation of the function $\Omega(y)$ to $\{|y| \leqslant y_\delta\}$, we find from (2.18), (2.19), (3.7), (4.8) and lemma 4.1, that

$$\Gamma(X(y)) = Q(y)\Omega(y), \quad \text{for any } y \in C_{y_\delta}, \tag{4.9}$$

where

$$Q(y) := \frac{X(y)\sigma(y)}{X(y) + \mu\beta^{(1)}(\sigma(y))}; \tag{4.10}$$

$$\sigma(y) := \lambda(1 - X(y)) + \mu\left(1 - \frac{1}{y}\right). \tag{4.11}$$

The idea is now to derive a second relation satisfied by $\Gamma(x)$ for $x \in \boldsymbol{L}_\delta$. Formally, this relation will be obtained by letting successively y tend to 1 and x tend to a point of $\boldsymbol{L}_\delta$ in the second relation of (4.6), and by noting that $\Psi(x, 1)$ is known for any x with $|x| \leqslant 1$. More precisely, it is seen by letting $y = 1$ in (2.15) (cf. (4.2)), that

$$\Psi(x, 1) = \Omega(1)T(x, 1), \quad \text{for } |x| \leqslant 1, \tag{4.12}$$

where

$$\Omega(1) = 1 - \frac{\lambda}{\alpha}, \tag{4.13}$$

from a standard result on the M/GI/1 queueing system (cf. (2.3), (2.7)).

Let us now come back to equations (4.6), (4.7). For $\delta > 1$, the integral $\Pi(x, y)$ is clearly continuous at the point $y = 1$, for fixed x with $|x| \leqslant 1$, $x \notin \boldsymbol{L}_\delta$. However, if $\delta \leqslant 1$, i.e. $C_{y_\delta} = \{|y| = 1\}$, then $\Pi(x, y)$ is a *singular integral* at the point $y = 1$, for fixed x with $|x| \leqslant 1$, $x \notin \boldsymbol{L}_\delta$. Consequently, the behavior of $\Pi(x, y)$ whenever y tends to 1 with $|y| < 1$, must be carefully examined. This is the purpose of the following lemma.

LEMMA 4.2

Let $\delta \leqslant 1$ and $|x| \leqslant 1$ with $x \notin \boldsymbol{L}_\delta$. Then the function

$$y \to \int_{\boldsymbol{C}} \Omega(t) \frac{T(x, t) - T(x, 1)}{t - y} \mathrm{d}t$$

on passing through the point $y = 1$ of the contour $\boldsymbol{C}$ behaves as a continuous function, i.e. this function has a definite limiting value on approaching the point 1 by y from any side of $\boldsymbol{C}$ and along any path.

Proof

Fix δ and x with $\delta \leqslant 1$ and $|x| \leqslant 1$ with $x \notin \boldsymbol{L}_\delta$. It is easily seen that the function $y \to T(x, y)$ possesses a bounded derivative on C_{y_δ}. Therefore, this function satisfies a *Hölder condition* on C_{y_δ}, cf. [7, p. 6]. Moreover, $|\Omega(y)|$ is bounded on C_{y_δ}. Consequently, the proof of the lemma can be constructed in direct analogy with the proof of the *basic lemma* in [7, pp. 20–23], and it is therefore omitted. ■

For $x \in \{|x| \leqslant 1\} \cap L_\delta^-$, $|y| < y_\delta$, let us rewrite $\Psi(x, y)$ as follows (cf. (4.6)):

$$\begin{aligned}\Psi(x, y) &= \frac{1}{2\pi \mathrm{i}} \int_{C_{y_\delta}} \Omega(t) \frac{T(x, t) - T(x, 1)}{t - y} \mathrm{d}t \\ &\quad + \frac{T(x, 1)}{2\pi \mathrm{i}} \int_{C_{y_\delta}} \frac{\Omega(t)}{t - y} \mathrm{d}t + \frac{\Gamma(x)}{Y(x) - y} \\ &= \frac{1}{2\pi \mathrm{i}} \int_{C_{y_\delta}} \Omega(t) \frac{T(x, t) - T(x, 1)}{t - y} \mathrm{d}t + T(x, 1)\Omega(y) \\ &\quad + \frac{\Gamma(x)}{Y(x) - y},\end{aligned} \tag{4.14}$$

by using the *residue theorem* together with the analyticity of $\Omega(y)$ in $C_{y_\delta}^+$.

Now, letting y tend to 1 in (4.14), we get from lemma 4.2 and (4.12), that for arbitrary $\delta > 0$

$$\Gamma(x) = \frac{1 - Y(x)}{2\pi \mathrm{i}} \int_{C_{y_\delta}} \Omega(t) \frac{T(x, t) - T(x, 1)}{t - 1} \mathrm{d}t, \tag{4.15}$$

for any $x \in \{|x| \leqslant 1\} \cap L_\delta^-$.

Substitution of (4.15) into (4.6) yields for $|y| < y_\delta$,

$$\Psi(x, y) = \begin{cases} \Pi(x, y), & \text{for } x \in L_\delta^+; \\ \Pi(x, y) & \\ \quad - \left(\dfrac{1 - Y(x)}{y - Y(x)}\right) \dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_{C_{y_\delta}} \Omega(t) \frac{T(x, t) - T(x, 1)}{t - 1} \mathrm{d}t, & \\ \quad \text{for } x \in \{|x| \leqslant 1\} \cap L_\delta^-. & \end{cases} \tag{4.16}$$

This concludes the proof of claim (R3).

We now focus our attention on the determination of $\Omega(.)$ on the contour C_{y_δ} (see claim (R4)). According to the sketch previously given for deriving a second relation for $\Gamma(x)$ on L_δ, it remains to let x tend to a point of the contour L_δ in (4.15). However, since the function $T(x, t) - T(x, 1)$ has a pole at the point

$x = X(t)$ if $t \in C_{y_\delta}$ (except if $\delta < 1$ and $t = 1$, cf. (4.2) and lemma 3.1), it follows that the integral in (4.15) is a *singular integral* for any $x \in L_\delta$ (except for $x = 1$ if $\delta < 1$). The forthcoming two lemmas will help us to remove this singularity.

LEMMA 4.3

For fixed δ and $y_0 \in C_{y_\delta} - \{y_\delta\}$, there exists a neighborhood V_{y_0} of the point y_0, such that

(1) $X(y)$ is analytic in V_{y_0};

(2) $X(V_{y_0} \cap C_{y_\delta}^-) \subset \{|x| \leqslant 1\} \cap L_\delta^-$;

(3) $X(V_{y_0} \cap C_{y_\delta}^+) \subset L_\delta^+$.

Proof

Let $y_0 \in C_{y_\delta} - \{y_\delta\}$. Since $X(y)$ is analytic at the point $y = y_0$ (cf. lemma 3.1), there exists a number $r > 0$ such that $X(y)$ is analytic in $\{|y - y_0| < r\}$. Moreover, $X^{(1)}(y_0) \neq 0$ from (3.3). Consequently, the *theorem of the inverse function* applies [5, p. 104], which shows that there exists a neighborhood V_{y_0} of the point y_0 ($V_{y_0} \subset \{|y - y_0| < r\}$) such that $X(y)$ has a unique inverse in V_{y_0}. We immediately deduce from lemma 3.3 that this inverse is $Y(x)$. Let us choose V_{y_0} small enough in order that $|X(y)| < 1$ for any $y \in V_{y_0}$, which is always possible since $|X(y_0)| < 1$. Then, statements (2) and (3) directly follow from propositions 3.1 and 3.2. ■

From now on, we shall assume that (see remark 4.2)

$$\delta \neq 1.$$

For convenience, let us introduce the set

$$V := \bigcup_{y_0 \in C_{y_\delta} - \{y_\delta\}} V_{y_0},$$

and let us define for $y \in V$, $t \in C_{y_\delta}$,

$$H(y, t) := \frac{T(X(y), t) - T(X(y), 1)}{t - 1} - \frac{R(y)}{t - y}, \tag{4.17}$$

where

$$R(y) := \begin{cases} \dfrac{Q(y)}{y - 1}, & \text{if } y \neq 1; \\[2ex] \dfrac{\mu - \lambda}{(1 - \lambda\alpha^{-1})(1 - \mu\alpha^{-1})}, & \text{if } y = 1. \end{cases}$$

Note that $R(y)$ is continuous at $y = 1$.

LEMMA 4.4

The function $H(y, t)$ possesses the following properties:

(1) $\forall y \in \boldsymbol{V}$, the mapping $t \to H(y, t)$ is continuous on the circle $\boldsymbol{C}_{y_\delta}$;

(2) $\forall t \in \boldsymbol{C}_{y_\delta}$, the mapping $y \to H(y, t)$ is continuous in $\boldsymbol{V}$.

Proof

The proof is given in appendix B.

We are now ready to prove claim (R4). To this end, relations (4.9) and (4.15) will be used. In (4.15), set $x = X(z)$ for $z \in \boldsymbol{V} \cap \boldsymbol{C}_{y_\delta}^-$. Note that this change of variable is allowed because of property (2) of lemma 4.3. Then, cf. (4.17),

$$\Gamma(X(z)) = \frac{1-z}{2\pi \mathrm{i}} \int_{\boldsymbol{C}_{y_\delta}} \Omega(t) H(z, t)\, \mathrm{d}t + \frac{(1-z)R(z)}{2\pi \mathrm{i}} \int_{\boldsymbol{C}_{y_\delta}} \frac{\Omega(t)}{t-z} \mathrm{d}t,$$

$$= \frac{1-z}{2\pi \mathrm{i}} \int_{\boldsymbol{C}_{y_\delta}} \Omega(t) H(z, t)\, \mathrm{d}t, \tag{4.18}$$

for any $z \in \boldsymbol{V} \cap \boldsymbol{C}_{y_\delta}^-$, from the *residue theorem*.

By evoking an elementary result on the integrals depending on a parameter, we deduce from lemma 4.4 that the integral in the right-hand side of (4.18) is continuous (in particular) on $\boldsymbol{C}_{y_\delta}$, i.e. this integral has a definite limiting value whenever $z \to y$ with $z \in \boldsymbol{V} \cap \boldsymbol{C}_{y_\delta}^-$ and $y \in \boldsymbol{C}_{y_\delta}$. On the other hand, we know that $\Gamma(.)$ is continuous on $\boldsymbol{L}_\delta$ (lemma 4.1) and that $X(.)$ is continuous on $\boldsymbol{C}_{y_\delta}$ (lemma 3.1). Consequently, letting $z \to y$ in both sides of (4.18) with $y \in \boldsymbol{C}_{y_\delta}$, we get

$$\Gamma(X(y)) = \frac{1-y}{2\pi \mathrm{i}} \int_{\boldsymbol{C}_{y_\delta}} \Omega(t) H^*(y, t)\, \mathrm{d}t, \tag{4.19}$$

for any $y \in \boldsymbol{C}_{y_\delta}$, where

$$H^*(y, t) := \begin{cases} H(y, t), & \text{if } t \neq y; \\ H_2(y), & \text{if } t = y, \end{cases}$$

with

$$H_2(y) := \lim_{\substack{z \to y \\ z \in \boldsymbol{V} \cap \boldsymbol{C}_{v_\delta}^-}} H(z, y).$$

Finally, by combining (4.9) and (4.19), we derive the following integral equation:

$$\Omega(y) = \int_{\boldsymbol{C}_{y_\delta}} N(y, t) \Omega(t)\, \mathrm{d}t, \tag{4.20}$$

for any $y \in \boldsymbol{C}_{y_\delta}$, where

$$N(y, t) := \frac{(1-y) H^*(y, t)}{2\pi \mathrm{i} Q(y)}. \tag{4.21}$$

It is shown in remark 4.1 that $N(y, t)$ belongs to $L_2(C_{y_\delta}) \times L_2(C_{y_\delta})$, where $L_2(C_{y_\delta})$ denotes the class of functions which are square integrable on C_{y_δ}. Consequently, (4.20) defines a *homogeneous Fredholm integral equation of the second kind* on the circle C_{y_δ} [8].

REMARK 4.1

For every $t \in C_{y_\delta}$, the function $y \to N(y, t)$ is continuous on C_{y_δ}. This comes from property 2 of lemma 4.4, together with the definition of $H_2(y)$.

Similarly, the function $t \to N(y, t)$ is continuous on $C_{y_\delta} - \{y\}$ (see lemma 4.4, property 1), and discontinuous at point y, for any $y \in C_{y_\delta}$. More precisely, one can observe from (B.1) and (B.2) that

$$H_1(y) \neq H_2(y),$$

for any $y \in C_{y_\delta}$ (to see this, compute for instance the coefficient of $\beta^{(2)}(\sigma(y))$ in both equations (B.1), (B.2)). In particular, these results show that

$$\int_{C_{y_\delta}} \int_{C_{y_\delta}} |N(y, t)|^2 \, \mathrm{d}y \, \mathrm{d}t < +\infty. \tag{4.22}$$

REMARK 4.2

If $\delta = 1$, i.e. $\mu = \alpha$, then the analysis becomes much more cumbersome. Indeed, by expanding $H(y, 1)$ into a power series in the vicinity of 1, it is seen that $H(y, 1)$ has a pole of order two at the point $y = 1$, which entails that the integral in the r.h.s. of (4.20) is a *singular integral* at the point $y = 1$. (Note that this result is basically due to the fact that whenever $\delta = 1$, then the function $K(1, t)$ has a zero of multiplicity two at the point $t = 1$, cf. lemma 3.1). Consequently, this case requires a special analysis which is not reported here for the sake of simplicity.

5. The generating function

Recall the Fredholm integral equation (4.20) to be satisfied by $\Omega(y)$ on C_{y_δ}. We then have the following result:

PROPOSITION 5.1

For fixed $\delta \neq 1$, the real part of $\Omega(y)$ on the circle C_{y_δ} is given as the unique non-zero and continuous solution of the following homogeneous Fredholm integral equation of the second kind:

$$\Omega_R(y) = \int_0^{2\pi} \Omega_R(t) N_R(y, t) \, \mathrm{d}\varphi, \quad \left(t = y_\delta \, \mathrm{e}^{\mathrm{i}\varphi}\right) \tag{5.1}$$

for $y \in \boldsymbol{C}_{y_\delta}$, where

$$\Omega_R(y) := \mathscr{R}(\Omega(y)); \tag{5.2}$$

$$N_R(y, t) := 2\mathscr{R}(\mathrm{i}tN(y, t)). \tag{5.3}$$

Proof

For $|w| = 1$ and $t = y_\delta w$, we clearly have

$$\begin{aligned}\Omega(t) &= 2\mathscr{R}(\Omega(t)) - \overline{\Omega(t)},\\ &= 2\mathscr{R}(\Omega(t)) - \Omega(\bar{t}),\\ &= 2\Omega_R(t) - \Omega\left(\frac{y_\delta}{w}\right),\end{aligned} \tag{5.4}$$

since the coefficients of the generating function $\Omega(y)$ are all real numbers, cf. (2.7) (here $\bar{z}$ denotes the complex conjugate of z).

By substituting (5.4) into (4.20), we get

$$\Omega(y) = 2y_\delta \int_C \Omega_R(t) N(y, t)\, \mathrm{d}w - y_\delta \int_C \Omega\left(\frac{y_\delta}{w}\right) N(y, y_\delta w)\, \mathrm{d}w, \tag{5.5}$$

for $y \in \boldsymbol{C}_{y_\delta}$ and $t = y_\delta w$ with $|w| = 1$.

Since the function $N(y, y_\delta w)\Omega(y_\delta/w)$ is analytic for $|w| > 1$ and continuous for $|w| \geqslant 1$ for fixed $y \in \boldsymbol{C}_{y_\delta}$ (see section 4), it follows from *Cauchy's theorem* [10, p. 84] that the second integral in the r.h.s. of (5.5) vanishes, i.e.

$$\Omega(y) = 2y_\delta \int_C \Omega_R(t) N(y, t)\, \mathrm{d}w. \tag{5.6}$$

Taking now the real part in both sides of (5.6), we obtain (5.1).

Similarly to (4.20), equation (5.1) defines a *homogeneous Fredholm integral equation of the second kind*, since clearly (see (4.22) and (5.3)) $N_R(y, t)$ belongs to $L_2(\boldsymbol{C}_{y_\delta}) \times L_2(\boldsymbol{C}_{y_\delta})$.

From the theory of Fredholm integral equation we know that the integral equation (5.1) has *either* the trivial solution $\Omega(.) \equiv 0$ *or* an infinite number of solutions [8, p. 35].

The *existence* of a non-zero solution to equation (5.1) is a consequence of the ergodicity of the Markov process $\boldsymbol{X}$ under condition (2.1). Indeed, if $\delta \leqslant 1$ then clearly $\Omega(1) > 0$ (see (4.13)). On the other hand, if $\delta > 1$ then, cf. (4.3), (2.16), (2.14),

$$\Omega(y_\delta) = y_\delta \Xi(1, 0, \mu(1 - y_\delta^{-1})) + \Omega(0)$$

is strictly positive, since $\Omega(0) = \operatorname{Prob}(X_1 = 0, X_2 = 0)$ is strictly positive if the Markov process $\boldsymbol{X}$ is ergodic (here X_i stands for the number of customers in queue i at steady state, $i = 1, 2$). Consequently, $\Omega(y_\delta) > 0$ for any $\delta > 0$, which shows that the function $\Omega(y)$ is not identically zero on $\boldsymbol{C}_{y_\delta}$.

The existence of at least one *continuous* solution also follows from the ergodicity of the Markov process $\boldsymbol{X}$, since we have shown in that case that $\Omega(y)$ is necessarily continuous in $\{\,|y| \leqslant y_\delta\}$ (see section 4).

The uniqueness of a (non-zero) *continuous* solution follows from the well known result that a function which is analytic in $\boldsymbol{C}^+_{y_\delta}$ and continuous in $\boldsymbol{C}^+_{y_\delta} \cup \boldsymbol{C}_{y_\delta}$ is completely determined in $\boldsymbol{C}^+_{y_\delta}$ (up to an additive constant which can be determined by using the relation (4.13)) by the values its real part takes on $\boldsymbol{C}_{y_\delta}$. This *boundary value problem* is called the *Dirichlet problem* for the circle [9, pp. 107–108]. Consequently, two distinct non-zero continuous solutions of equation (5.1) will yield two distinct functions $\Omega(y)$, for $y \in \boldsymbol{C}^+_{y_\delta} \cup \boldsymbol{C}_{y_\delta}$, which is clearly impossible if the process $\boldsymbol{X}$ is ergodic. ■

Up to a standard numerical procedure (see for instance [3, pp. 349–350]), proposition 5.1 provides us with the *real part* of the function $\Omega(y)$ on the circle $\boldsymbol{C}_{y_\delta}$. Let us now show that the knowledge of $\Omega_R(y)$ on $\boldsymbol{C}_{y_\delta}$ is actually sufficient for determining the generating function $F(x, y)$.

Using (5.4), the function $\Pi(x, y)$ defined in (4.7) can be rewritten as follows, for $|x| \leqslant 1$ with $x \notin \boldsymbol{L}_\delta$ and $|y| < y_\delta$,

$$\Pi(x, y) = \frac{1}{\pi \mathrm{i}} \int_{\boldsymbol{C}_{y_\delta}} \frac{T(x, t)\Omega_R(t)}{t-y} \mathrm{d}t - \frac{1}{2\pi \mathrm{i}} \int_{\boldsymbol{C}_{y_\delta}} \frac{T(x, t)\Omega(\bar{t})}{t-y} \mathrm{d}t$$

$$= \frac{1}{\pi \mathrm{i}} \int_{\boldsymbol{C}_{y_\delta}} \frac{T(x, t)\Omega_R(t)}{t-y} \mathrm{d}t, \tag{5.7}$$

from *Cauchy's theorem*, since the function

$$w \to \frac{T(x, t)\Omega(y_\delta/w)}{t-y}, \quad t = y_\delta w,$$

is analytic for $|w| > 1$ (respectively continuous for $|w| \geqslant 1$) for fixed x, y with $|x| \leqslant 1$, $x \notin \boldsymbol{L}_\delta$, and $|y| < y_\delta$.

Consequently, (cf. (4.16), (5.7)):

$$\Psi(x, y) = \begin{cases} \dfrac{1}{\pi \mathrm{i}} \displaystyle\int_{\boldsymbol{C}_{y_\delta}} \frac{T(x, t)\Omega_R(t)}{t-y} \mathrm{d}t, & \text{for } x \in \boldsymbol{L}^+_\delta; \\ \dfrac{1}{\pi \mathrm{i}} \displaystyle\int_{\boldsymbol{C}_{y_\delta}} \frac{T(x, t)\Omega_R(t)}{t-y} \mathrm{d}t & \\ \quad - \left(\dfrac{1-Y(x)}{y-Y(x)}\right) \dfrac{1}{\pi \mathrm{i}} \displaystyle\int_{\boldsymbol{C}_{y_\delta}} \Omega_R(t) \frac{T(x, t) - T(x, 1)}{t-1} \mathrm{d}t, & \\ \quad \text{for } x \in \{\,|x| \leqslant 1\} \cap \boldsymbol{L}^-_\delta, & \end{cases} \tag{5.8}$$

for any $|y| < y_\delta$, where $\Omega_R(.)$ is provided by proposition 5.1.

These results are collected in the following theorem:

THEOREM 5.1

For $\delta \neq 1$ and if conditions (2.1), (3.3) and (3.4) hold, then the generating function $F(x, y)$ for the stationary joint distribution of the number of customers in the system is given by

$$F(x, y) = \frac{\mu\left(1 - \frac{1}{y}\right)\left(\frac{1}{\lambda} - \frac{x}{\mu}\right)\nu(x)\Psi(x, \nu(x)) + (y - x)\Psi(x, y)}{\lambda(1 - x) + \mu\left(1 - \frac{1}{y}\right)},$$

for $|x| \leqslant 1$ and $|y| \leqslant 1$, where

- $\Psi(x, y)$ is given in (5.8) for $|x| \leqslant 1$, $x \notin \boldsymbol{L}_\delta$ and $|y| < y_\delta$;
- $\nu(x) := \mu/(\lambda(1 - x) + \mu)$.

6. Average number of customers at node 2

This section is devoted to the computation of $E(X_2)$, the average number of waiting customers at node 2, in steady state. Note that the corresponding quantity at node 1 is known – and given by the so-called *Pollaczek-Khinchin's formula* – since this node is a standard M/GI/1 queue.

Starting from the relation

$$E(X_2) = \frac{\partial}{\partial y} F(x, y)\bigg|_{(x,y)=(1,1)},$$

we easily get by using (2.21) and (4.12), that

$$E(X_2) = \frac{\lambda + \theta}{\mu}, \tag{6.1}$$

where

$$\theta := \frac{\partial}{\partial y} \Psi(x, y)\bigg|_{(x,y)=(1,1)}. \tag{6.2}$$

COMPUTATION OF θ

Since $\Psi(x, y)$ is a continuous function for $|x| \leqslant 1$ and $|y| \leqslant y_\delta$, it turns out that θ can be computed by using either (5.8a) or (5.8b). Let us consider relation (5.8a). First, let us investigate the case $\delta < 1$ (i.e. $y_\delta = 1$ and $\boldsymbol{C}_{y_\delta} = \boldsymbol{C}$).

The function $T(1, t)$ being well defined for any $|t| = 1$ if $\delta < 1$ (see sections 3 and 4), it is readily seen that

$$\Psi(1, y) = \lim_{\substack{x \to 1 \\ x \in L_\delta^+}} \Psi(x, y) = \frac{1}{\pi \mathrm{i}} \int_C \frac{T(1, t) \Omega_R(t)}{t - y} \mathrm{d}t, \tag{6.3}$$

for any y with $|y| < 1$.

It now remains to differentiate equation (6.3) and then to let y tend to 1. For $|y| < 1$, we have

$$\frac{\partial}{\partial y} \Psi(1, y) = \frac{1}{\pi \mathrm{i}} \int_C \frac{T(1, t) \Omega_R(t)}{(t - y)^2} \mathrm{d}t \tag{6.4}$$

$$= \frac{1}{\pi \mathrm{i}} \int_C \Omega_R(t) \left(\frac{T(1, t) - a_0 - (t - 1) a_1}{(t - y)^2} \right) \mathrm{d}t$$

$$+ \frac{a_0}{\pi \mathrm{i}} \int_C \frac{\Omega_R(t)}{(t - y)^2} \mathrm{d}t + \frac{a_1}{\pi \mathrm{i}} \int_C \frac{(t - 1) \Omega_R(t)}{(t - y)^2} \mathrm{d}t, \tag{6.5}$$

where

$$a_0 := \lim_{t \to 1} T(1, t); \left(= \frac{\alpha \mu}{\alpha - \mu} \right) \tag{6.6}$$

$$a_1 := \frac{\partial}{\partial t} T(1, t) |_{t=1}. \left(= -\frac{\alpha^2 \mu}{(\alpha - \mu)^2} (1 - \mu^2/\alpha + \mu^2 \beta^{(2)}(0)/2) \right). \tag{6.7}$$

It is readily checked that (i) the first integral in the r.h.s. of (6.5) has a definite limiting value at the point $y = 1$, (ii) the second integral reduces to $a_0 \Omega^{(1)}(y)$, (iii) the third integral is equal to $a_1[\Omega(y) + (y - 1)\Omega^{(1)}(y)]$ ((ii) and (iii) are obtained by using (5.4), and by applying both *Cauchy's theorem and the residue theorem*). Consequently, we get by letting y tend to 1 in (6.5) (cf. also (4.13)),

$$\theta = \frac{1}{\pi \mathrm{i}} \int_C \Omega_R(t) \left(\frac{T(1, t) - a_0 - (t - 1) a_1}{(t - 1)^2} \right) \mathrm{d}t + a_0 \Omega^{(1)}(1) + a_1 \left(1 - \frac{\lambda}{\alpha} \right), \tag{6.8}$$

where $\Omega^{(1)}(1)$ is obtained by differentiating (5.6).

Let us now examine the case $\delta > 1$ (i.e. $y_\delta > 1$). In (5.8a), set $x = X(z)$ with $z \in V \cap C_{y_\delta}^+$. Note that this change of variable is allowed because of property (3) of lemma 4.3. We then have,

$$\begin{aligned}\left.\frac{\partial}{\partial y}\Psi(x, y)\right|_{y=1} &= \frac{1}{\pi \mathrm{i}}\int_{C_{y_\delta}} \Omega_R(t)\frac{T(X(z), t)}{(t-1)^2}\mathrm{d}t \\ &= \frac{1}{\pi \mathrm{i}}\int_{C_{y_\delta}} \Omega_R(t)\left(\frac{T(X(z), t)}{(t-1)^2} - \frac{Q(z)}{(z-1)^2(t-z)}\right)\mathrm{d}t \\ &\quad + \frac{Q(z)}{\pi \mathrm{i}(z-1)^2}\int_{C_{y_\delta}} \frac{\Omega_R(t)}{t-z}\mathrm{d}t, \end{aligned} \tag{6.9}$$

for any $z \in \boldsymbol{V} \cap \boldsymbol{C}_{y_\delta}^+$ and for $|y| < y_\delta$.

It is readily checked that the first integral in the r.h.s. of (6.9) is continuous at the point $z = y_\delta$ (since $Q(z)$ is the residue of the function $t \to T(X(z), t)$ at the point $t = z$, for any $z \in \boldsymbol{C}_{y_\delta}$) and that the second integral in the r.h.s. of (6.9) is equal to $Q(z)\Omega(z)/(z-1)^2$. Consequently, we get by letting z tend to y_δ in (6.9),

$$\theta = \frac{1}{\pi \mathrm{i}}\int_{C_{y_\delta}} \Omega_R(t)\left(\frac{T(1, t)}{(t-1)^2} - \frac{Q(y_\delta)}{(y_\delta-1)^2(t-y_\delta)}\right)\mathrm{d}t + \frac{Q(y_\delta)\Omega(y_\delta)}{(y_\delta-1)^2}, \tag{6.10}$$

where $\Omega(y_\delta)$ is determined from (5.6).

Appendix A

Proof of proposition 3.2

Define for $|x| \leqslant 1$, $x \notin \boldsymbol{L}_\delta$:

- $$J(x) := \frac{1}{2\pi \mathrm{i}}\int_{C_{y_\delta}} \frac{\frac{\partial}{\partial z}K(x, z)}{K(x, z)}\mathrm{d}z; \tag{A.1}$$
- $n_Z(x)$ the number of zeros of $K(x, z)$ in $\{|z| > y_\delta\}$;
- $n_P(x)$ the number of poles of $K(x, z)$ in $\{|z| > y_\delta\}$.

From the argument principle [10, p. 130], we have

$$J(x) = n_P(x) - n_Z(x),$$

for any $|x| \leqslant 1$, $x \notin \boldsymbol{L}_\delta$.

Note that $n_P(x) = 1$ for any $|x| \leqslant 1$, $x \notin \boldsymbol{L}_\delta$ (cf. (2.17)), so that

$$J(x) = 1 - n_Z(x), \tag{A.2}$$

for any $|x| \leqslant 1$, $x \notin \boldsymbol{L}_\delta$.

Let us show that $J(x)=1$ for any $x \in \boldsymbol{L}_\delta^+$, which will prove proposition 3.2.

Set $z = Y(u)$ in (A.1) for $u \in \boldsymbol{L}_\delta$. Note that this change of variable is allowed from lemma 3.3. It comes

$$J(x) = \frac{1}{2\pi \mathrm{i}} \int_{\boldsymbol{L}_\delta} \frac{\left.\frac{\partial}{\partial z} K(x, z)\right|_{z=Y(u)}}{K(x, Y(u))} Y^{(1)}(u)\, \mathrm{d}u,$$

for any $|x| \leqslant 1$, $x \notin \boldsymbol{L}_\delta$.

We easily see that the function

$$u \to \frac{\left.\frac{\partial}{\partial z} K(x, z)\right|_{z=Y(u)}}{K(x, Y(u))} Y^{(1)}(u)$$

has a pole of multiplicity one at the point $u = x$, for any $x \in \boldsymbol{L}_\delta$; its residue is 1. Let us now rewrite $J(x)$ as follows:

$$J(x) = \frac{1}{2\pi \mathrm{i}} \int_{\boldsymbol{L}_\delta} \left(\frac{\left.\frac{\partial}{\partial z} K(x, z)\right|_{z=Y(u)}}{K(x, Y(u))} Y^{(1)}(u) - \frac{1}{u-x} \right) \mathrm{d}u + \frac{1}{2\pi \mathrm{i}} \int_{\boldsymbol{L}_\delta} \frac{1}{u-x} \mathrm{d}u, \tag{A.3}$$

for any $|x| \leqslant 1$, $x \notin \boldsymbol{L}_\delta$.

The first integral in the right-hand side of (A.3) is clearly continuous in $\{|x| \leqslant 1\}$ from the preceding result. On the other hand, *Plemelj-Sokhotski's* formulas for *singular integrals* [3, p. 32] apply to the second integral. We therefore deduce that

$$J(x) \text{ is continuous in } \boldsymbol{L}_\delta^+ \cup \boldsymbol{L}_\delta \ \left(\text{respectively } \{|x| \leqslant 1\} \cap (\boldsymbol{L}_\delta^- \cup \boldsymbol{L}_\delta)\right), \tag{A.4}$$

and that

$$J^+(x_0) - J^-(x_0) = 1, \tag{A.5}$$

for any $x_0 \in \boldsymbol{L}_\delta$, where

$$J^+(x_0) := \lim_{\substack{x \to x_0 \\ x \in \boldsymbol{L}_\delta^+}} J(x);$$

$$J^-(x_0) := \lim_{\substack{x \to x_0 \\ x \in \{|x| \leqslant 1\} \cap \boldsymbol{L}_\delta^-}} J(x).$$

We know by proposition 3.1 that $n_Z(x) = 1$ for any $x \in \{|x| \leqslant 1\} \cap \boldsymbol{L}_\delta^-$, i.e. (cf. (A.2))

$$J(x) = 0, \quad \text{for any } x \in \{|x| \leqslant 1\} \cap \boldsymbol{L}_\delta^-.$$

By continuity, cf. (A.4), this result necessarily entails that

$$J^-(x_0) = 0, \tag{A.6}$$

for any $x_0 \in \boldsymbol{L}_\delta$, since $J(x)$ is an integer-valued function from (A.2).

Combining now (A.5) and (A.6), we get

$$J^+(x_0) = 1,$$

for any $x_0 \in \boldsymbol{L}_\delta$, and by continuity, cf. (A.4),

$$J(x) = 1, \quad \text{for any } x \in \boldsymbol{L}_\delta^+,$$

which concludes the proof. ∎

Appendix B

Proof of lemma 4.4

If $y \in \boldsymbol{V} - \boldsymbol{C}_{y_\delta}$, then the continuity of $t \to H(y, t)$ on $\boldsymbol{C}_{y_\delta}$ readily follows from definition (4.17). Similarly, if $y \in \boldsymbol{C}_{y_\delta}$ then $H(y, t)$ is clearly continuous on $\boldsymbol{C}_{y_\delta} - \{y\}$. It remains to prove that $H(y, t)$ has a finite limit whenever $t \to y$ if $y \in \boldsymbol{C}_{y_\delta}$.

The existence of this limit follows from the fact that $R(y)$ is the residue of the function

$$t \to \frac{T(X(y), t) - T(X(y), 1)}{t - 1}$$

at the point y if $y \in \boldsymbol{C}_{y_\delta}$. In that case, a tedious but routine calculation shows that

$$H_1(y) := \lim_{\substack{t \to y \\ t \in \boldsymbol{C}_{y_\delta} - \{y\}}} H(y, t)$$

is given by

$$H_1(y) = \begin{cases} -\dfrac{T(X(y), 1)}{y - 1} - \dfrac{Q(y)}{(y-1)^2} \\ \quad + \left(\dfrac{\mu}{y^2(y-1)\left(X(y) + \mu\beta^{(1)}(\sigma(y))\right)}\right) \\ \quad \times \left(\dfrac{-\mu Q(y)\beta^{(2)}(\sigma(y))}{2} + X(y) + y(\sigma(y))\beta^{(1)}(\sigma(y))\right); \\ \quad \text{if } y \neq 1; \\ \left(\dfrac{\alpha\mu}{\mu - \alpha}\right)\left(1 + \dfrac{\mu}{\alpha} + \dfrac{\alpha\mu^2\beta^{(2)}(0)}{2(\alpha - \mu)}\right); \qquad \text{if } y = 1, \end{cases} \tag{B.1}$$

for any $y \in \boldsymbol{C}_{y_\delta}$, which concludes the proof of property 1.

Fix now $t \in C_{y_\delta}$. Then, clearly $y \to H(y, t)$ is continuous in $V - \{t\}$ from definition (4.17). It remains to prove that $H(y, t)$ has a finite limit whenever $y \to t$. After lengthy computations, we find that

$$H_2(t) = \lim_{\substack{y \to t \\ y \in V - \{t\}}} H(y, t) = \begin{cases} \dfrac{W(t) - T(X(t), 1)}{t-1}, & \text{if } t \neq 1; \\ \dfrac{c_0 c_2 - c_1 c_3}{c_2^2} + \frac{1}{2} Q^{(2)}(1), & \text{if } t = 1, \end{cases} \tag{B.2}$$

for any $t \in C_{y_\delta}$, where

$$W(t) := \left(\frac{\lambda X^{(1)}(t)}{K_1(t)} \right) \left(\frac{X(t)}{t} + \beta^{(1)}(\sigma(t))\sigma(t) \right) + \frac{\sigma(t) X(t) K_2(t)}{t K_1(t)^2};$$

$$K_1(t) := \frac{\partial}{\partial y} K(X(y), t) \bigg|_{y=t};$$

$$K_2(t) := \frac{1}{2} \frac{\partial^2}{\partial y^2} K(X(y), t) \bigg|_{y=t};$$

$$c_0 := X^{(1)}(1)^2 \left(-\lambda^2 \mu \left(\frac{1}{\alpha^2} - \frac{\beta^{(2)}(0)}{2} \right) + \frac{2\lambda}{\alpha}(\mu - \lambda) \right) + \tfrac{1}{2} X^{(2)}(1)(\mu - \lambda);$$

$$c_1 := \left(\frac{\alpha - \mu}{\alpha - \lambda} \right)(\mu - \lambda);$$

$$c_2 := \left(1 - \frac{\mu}{\alpha} \right)^2;$$

$$c_3 := \left(1 - \frac{\mu}{\alpha} \right) \left(X^{(2)}(1) \left(1 - \frac{\lambda}{\alpha} \right) - \lambda^2 X^{(1)}(1)^2 \beta^{(2)}(0) \right),$$

which therefore ensures the validity of property 2. ■

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