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Abstract

We show that the Hamming graph $H(3, q)$ with diameter three is uniquely determined by its spectrum for $q \geq 36$. Moreover, we show that for given integer $D \geq 2$, any graph cospectral with the Hamming graph $H(D, q)$ is locally the disjoint union of D copies of the complete graph of size $q - 1$, for q large enough.

1 Introduction

We consider the question whether the Hamming graph $H(D, q)$ with diameter D is characterized by its spectrum if $3 \leq D < q$. It is known that for $D \geq q \geq 3$, ($D \geq 4$ and $q = 2$) or ($D \geq 2$ and $q = 4$), the Hamming graph $H(D, q)$ is not uniquely determined by its spectrum, whereas for ($2 \leq D \leq 3$ and $q = 2$) or ($q \geq D = 2$ and $q \neq 4$), the Hamming graph $H(D, q)$ is uniquely determined by its spectrum (cf. [3], [4], [5], and [6]).

For a regular graph we obtain bounds on the number of common neighbours of two distinct vertices in terms of the eigenvalues of the graph (Lemma 2.1). Applying these bounds and a result of Metsch (Proposition 3.3) to the Hamming graphs, we show that for fixed D, and if q is large enough, then any graph cospectral with the Hamming graph $H(D, q)$ is locally the disjoint union of D cliques of size $q - 1$ (Theorem 3.4).

Bang and Koolen ([1, Theorem 1.1]) showed that if a graph cospectral with $H(3, q)$ has the same local structure as $H(3, q)$, then it is either the Hamming graph $H(3, q)$ or the dual graph of $H(3, 3)$. As a consequence we obtain that the Hamming graph $H(3, q)$ is uniquely determined by its spectrum for $q \geq 36$ (Theorem 4.5).

In the last section, we discuss spectral characterizations of other distance-regular graphs with classical parameters.

2 Preliminaries

All the graphs considered in this paper are finite, undirected and simple.

Suppose that Γ is a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For a vertex x, let $\Gamma_i(x)$ be the set of vertices at distance i from x.

Let A be the adjacency matrix of Γ. Then the eigenvalues of Γ are the eigenvalues of A. Let $\theta_0, \theta_1, \cdots, \theta_n$ be the distinct eigenvalues of Γ and m_i be the multiplicity of θ_i ($i = 0, 1, \ldots, n$). Then the multiset $\{\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_n^{m_n}\}$ is called the *spectrum* of Γ. Two non-isomorphic graphs are called *cospectral* if they have the same spectrum.

A sequence of vertices $W = w_0, w_1, \cdots, w_\ell$, which are not necessarily mutually distinct, is called an ℓ-*walk* if w_i and w_{i+1} are adjacent for any $i = 0, \cdots, \ell - 1$. The number of ℓ-walks from x to y is given by $A_{(x,y)}^\ell$, where $A_{(x,y)}^\ell$ is the (x, y)-entry of matrix A^ℓ. If $w_0 = w_\ell$ then W is called a *closed* ℓ-*walk*. Let $\text{Tr}(A^\ell)$ denote the trace of A^ℓ (i.e., the sum of the diagonal entries of A^ℓ). Then

$$(1) \qquad \sum_{i=0}^n m_i \theta_i^\ell = \text{Tr}(A^\ell) = |\{W \mid W \text{ is a closed } \ell\text{-walk in } \Gamma\}| \quad (\ell \geq 0).$$

Recall that a *clique* of a graph is a set of mutually adjacent vertices and a *coclique* of a graph is a set of mutually non-adjacent vertices.
For a vertex x, the *local graph* of x (in Γ) is the subgraph induced by $\Gamma_1(x)$. For a positive integer n, Γ is called *locally* the disjoint union of n cliques of size m if for each vertex x the local graph of x in Γ is the disjoint union of n cliques of size m. For a graph on v vertices, let I and J denote the identity and all-one matrix of size $v \times v$, respectively.
For disjoint subsets X and Y of the vertex set, let $e(X, Y)$ be the number of edges between X and Y. If $X = \{a\}$ then put $e(a, Y) := e(\{a\}, Y)$. The number of edges within X is denoted by $e(X)$.

For two distinct vertices x and y, we denote the number of common neighbours by $\lambda(x, y)$ if x and y are adjacent, and by $\mu(x, y)$ if they are not. Following [3], a graph is called *strongly regular* if there exist non-negative integers λ and μ such that $\lambda(x, y) = \lambda$ for any two adjacent vertices x and y, and $\mu(x, y) = \mu$ for any two distinct non-adjacent vertices x and y.
A connected graph Γ with diameter D is called *distance-regular* if there are non-negative integers b_i, c_i, $i = 0, 1, \ldots, D$, such that for any two vertices $x, y \in V(\Gamma)$ at distance i, there are precisely c_i neighbours of y in $\Gamma_{i-1}(x)$ and b_i neighbours of y in $\Gamma_{i+1}(x)$.

Lemma 2.1 *Let Γ be a regular graph on v vertices with valency k. Assume that no eigenvalues are in the open interval (s, r) for $r > s$. Then*

(i) any two adjacent vertices x and y satisfy

$$-k + r + s - rs + \frac{2(k-r)(k-s)}{v} \leq \lambda(x,y) \leq k + r + s + rs \ ,$$

(ii) any two distinct non-adjacent vertices x and y satisfy

$$\mu(x,y) \leq k + rs \ .$$

Proof: Let A be the adjacency matrix of Γ, and define the matrix $M :=$ $(A - rI)(A - sI) - \frac{(k-r)(k-s)}{v}J$. Then

$$M_{(x,y)} = \begin{cases} k + rs - \frac{(k-r)(k-s)}{v} & \text{if } x = y \\ \lambda(x,y) - (r+s) - \frac{(k-r)(k-s)}{v} & \text{if } x \text{ and } y \text{ are adjacent} \\ \mu(x,y) - \frac{(k-r)(k-s)}{v} & \text{if } x \text{ and } y \text{ are not adjacent} \end{cases} .$$

Since each eigenvalue θ of Γ satisfies either $\theta \geq r$ or $\theta \leq s$, the matrix M is positive semidefinite, and thus any principal submatrix of M is also positive semidefinite. The result now follows by considering the principal submatrix indexed by x and y. ∎

Note that if Γ is a strongly regular graph with valency k and eigenvalues $k \geq r > s$, then $\lambda(x,y) = k + r + s + rs$ for any two adjacent vertices x and y, and $\mu(x,y) = k + rs$ if x and y are distinct and non-adjacent.

3   Graphs cospectral with $H(D,q)$

For integers $D > 1$ and $q > 1$, the Hamming graph $H(D,q)$ is the graph whose vertex set is the cartesian product of D copies of a set Q with $|Q| = q$, where two vertices are adjacent if they differ in precisely one coordinate. Note that the Hamming graph $H(D,q)$ is a distance-regular graph with diameter D, and has spectrum $\{\theta_i^{m_i} \mid i = 0, \ldots, D\}$ where $\theta_i = q(D-i) - D$ are the eigenvalues and $m_i = \binom{D}{i}(q-1)^i$ the corresponding multiplicities. (For more information, see [3, Theorem 9.2.1].)

A consequence of the next proposition is that in any graph Γ cospectral with the Hamming graph $H(D,q)$, we can bound the number of common neighbours of a pair of distinct vertices.

Proposition 3.1 *For integers $n > 1$ and $q > 1$, let Γ be a connected regular graph on v vertices with valency $n(q - 1)$. If each eigenvalue θ of Γ satisfies either $\theta \leq -n$ or $\theta \geq q - n$, then*
(i) any two adjacent vertices x and y satisfy

$$-n^2 - n + q + 2n(n - 1)q^2/v \leq \lambda(x, y) \leq n^2 - 3n + q \ ,$$

(ii) any two distinct non-adjacent vertices x and y satisfy

$$\mu(x, y) \leq n^2 - n \ .$$

Proof: This follows from Lemma 2.1.　　　　　　　　　　　　　　　　■

Below we will show that for fixed D, and if q is large enough, then any graph cospectral with the Hamming graph $H(D, q)$ is locally the disjoint union of D cliques of size $q - 1$. For this we need the following two results.

Lemma 3.2 *For integers $D > 1$ and $q > 1$, let Γ be a graph cospectral with the Hamming graph $H(D, q)$, and let $\mathcal{C}$ be a set of cliques of Γ. If every vertex lies in at most D cliques in $\mathcal{C}$ and each edge lies in a unique element of $\mathcal{C}$ then Γ is locally the disjoint union of D cliques of size $q - 1$.*

Proof: Denote by $s_C + 1$ the size of a clique $C \in \mathcal{C}$, and by m_x the number of cliques in $\mathcal{C}$ containing x. Since Γ is cospectral with $H(D, q)$ and $\sum_{C \in \mathcal{C}, x \in C} s_C = D(q - 1)$ for each x, we then have that

$$
\begin{aligned}
& q^D D(q - 1)(q - 2) \\
= \ & \mathrm{Tr}(A^3) \\
\geq \ & \sum_x \sum_{C \in \mathcal{C}, x \in C} s_C(s_C - 1) \geq \sum_x \left(\frac{1}{m_x} \left(\sum_{C \in \mathcal{C}, x \in C} s_C \right)^2 - \sum_{C \in \mathcal{C}, x \in C} s_C \right) \\
= \ & D^2(q - 1)^2 \sum_x \frac{1}{m_x} - q^D D(q - 1) \\
\geq \ & q^D D(q - 1)(q - 2)
\end{aligned}
$$

where A is the adjacency matrix of Γ. Thus equality holds in all inequalities, which implies that all cliques containing x have the same size and that $m_x = D$ for all vertices x, and the statement follows.　　　　　　■

Proposition 3.3 *([9, Result 2.2])*
Let $\mu \geq 1$, λ_1, λ_2 and t be integers. Assume that Γ is a connected graph with the following properties:
(i) Any two adjacent vertices x, y satisfy $\lambda_1 \leq \lambda(x, y) \leq \lambda_2$;
(ii) Any two non-adjacent vertices have at most μ common neighbours;
(iii) $2\lambda_1 - \lambda_2 > (2t - 1)(\mu - 1) - 1$;
(iv) Every vertex has fewer than $(t + 1)(\lambda_1 + 1) - \frac{1}{2}t(t + 1)(\mu - 1)$ neighbours.
Define a line to be a maximal clique C satisfying $|C| \geq \lambda_1 + 2 - (t - 1)(\mu - 1)$. Then every vertex is on at most t lines, and any two adjacent vertices lie in a unique line.

Theorem 3.4 *Let $D \geq 2$ be an integer. If $q > \frac{D^4 + 2D^3 + 2D^2 - 5D - 4}{2}$ then any graph cospectral with the Hamming graph $H(D, q)$ is locally the disjoint union of D cliques of size $q - 1$.*

Proof: Let Γ be a graph cospectral with $H(D, q)$. By Proposition 3.1, we may use Proposition 3.3 for $(\lambda_1, \lambda_2, \mu, t) = (-D^2 - D + q + 1, D^2 - 3D + q, D^2 - D, D)$. Define a line to be a maximal clique of size at least $\lambda_1 + 2 - (t-1)(\mu-1)$. Then it follows that if $q > \frac{D^4 + 2D^3 + 2D^2 - 5D - 4}{2}$ then every vertex is in at most D lines and every edge lies in a unique line. The result then follows immediately from Lemma 3.2. ∎

One can improve Theorem 3.4 by using Proposition 3.3 with $t \approx \frac{4}{3}D$. It then follows that for a graph cospectral with the Hamming graph $H(D, q)$ with $q > \frac{8}{3}D^3 + O(D^2)$, there are lines (cliques) such that every vertex is in at most t of them, and each edge is in a unique one. Now suppose that each vertex is in at most τ lines, and there exists a vertex x which lies in exactly τ lines C_i ($1 \leq i \leq \tau$), where $D < \tau \leq t$. For any $i \in \{1, \ldots, \tau\}$, each vertex $y \in C_i \setminus \{x\}$ has at most $(\tau - 1)^2$ neighbours in $\Gamma_1(x) \setminus C_i$. By $\lambda_1 \leq \lambda(x, y) \leq (|C_i| - 1) + (\tau - 1)^2$, we have that $|C_i| \geq q + 2 - D^2 - D - (\tau - 1)^2$ and therefore $D(q - 1) = |\Gamma_1(x)| \geq \tau(q + 2 - D^2 - D - (\tau - 1)^2)$. Thus it follows that

$$q \leq \frac{\tau((\tau - 1)^2 + D^2 + D - 2) - D}{\tau - D},$$

which is impossible for $q > \frac{8}{3}D^3 + O(D^2)$ and $D < \tau \leq t$. Hence every vertex lies in at most D lines, and so by Lemma 3.2 we may conclude that for $q > \frac{8}{3}D^3 + O(D^2)$, any graph cospectral with the Hamming graph $H(D, q)$ is locally the disjoint union of D cliques of size $q - 1$.

4 Graphs cospectral with $H(3, q)$

In this section, we will improve the result of the previous section for the case $D = 3$. For this case, the following result was obtained by the first and last author:

Theorem 4.1 *([1, Theorem 1.1])*
Let $q > 1$ be an integer. A graph cospectral with the Hamming graph $H(3, q)$ that is locally the disjoint union of three cliques must be either the Hamming graph $H(3, q)$ or the dual graph of $H(3, 3)$ (i.e., the graph whose vertices are the triangles of $H(3, 3)$ and two triangles are adjacent if they intersect.).

Using this, we will show that the Hamming graph $H(3, q)$ is characterized by its spectrum for $q \geq 36$.

Let Γ be a connected regular graph on v vertices with valency k. If Γ has exactly four distinct eigenvalues $k = \theta_0 > \theta_1 > \theta_2 > \theta_3$ then the adjacency matrix A of Γ satisfies the following (cf. [6] and [2, Corollary 3.3]) :

$$(2) \qquad A^3 - (\sum_{i=1}^{3} \theta_i)A^2 + (\sum_{1 \leq i < j \leq 3} \theta_i \theta_j)A - \theta_1 \theta_2 \theta_3 I = \frac{\prod_{i=1}^{3}(k - \theta_i)}{v} J \ .$$

We can use this relation among others by counting 3-walks between adjacent vertices x and y. Indeed, if we put $Z := \Gamma_1(x) \cap \Gamma_1(y)$, $X := \Gamma_1(x) \cap \Gamma_2(y)$ and $Y := \Gamma_2(x) \cap \Gamma_1(y)$, then

$$(3) \qquad\qquad A^3_{(x,y)} = k + \sum_{z \in Z} \lambda(x, z) + \sum_{w \in Y} \mu(x, w).$$

Lemma 4.2 *Let Γ be a connected regular graph on v vertices with valency k. Assume that Γ has exactly four distinct eigenvalues $k = \theta_0 > \theta_1 > \theta_2 > \theta_3$. If λ_1 is the minimum number of common neighbours of any two adjacent vertices, then*

$$\lambda_1^2 - (1 + \sum_{i=1}^{3} \theta_i)\lambda_1 \leq \frac{1}{v}\prod_{i=1}^{3}(k - \theta_i) - 2k + 1 - \sum_{1 \leq i < j \leq 3} \theta_i \theta_j \ .$$

Proof: Let x and y be adjacent vertices satisfying $\lambda(x, y) = \lambda_1$. Then we obtain from (3) that

$$A^3_{(x,y)} \geq k + \lambda_1^2 + (k - 1 - \lambda_1)$$

holds. On the other hand, (2) implies

$$A^3_{(x,y)} = \lambda_1 \sum_{i=1}^{3} \theta_i - \sum_{1 \leq i < j \leq 3} \theta_i \theta_j + \frac{1}{v} \prod_{i=1}^{3} (k - \theta_i) \ ,$$

which gives the result. ∎

Proposition 4.3 *Let Γ be a graph cospectral with the Hamming graph $H(3, q)$. Then*
(i) any two adjacent vertices x and y satisfy $q - 4 \leq \lambda(x, y) \leq q$,
(ii) any two distinct non-adjacent vertices x and y satisfy $\mu(x, y) \leq 6$.

Proof: The statement follows immediately from Proposition 3.1 and Lemma 4.2, as the Hamming graph $H(3, q)$ has four distinct eigenvalues $3q - 3, 2q - 3, q - 3$ and -3. ∎

Proposition 4.4 *Let Γ be a graph cospectral with the Hamming graph $H(3, q)$. If $q \geq 27$ then any two adjacent vertices x and y satisfy*

$$\lambda(x, y) \geq q - 3 \ .$$

Proof: Suppose that there exist two adjacent vertices x and y satisfying $\lambda(x, y) = q - 4$. Put $Z := \Gamma_1(x) \cap \Gamma_1(y)$, $X := \Gamma_1(x) \cap \Gamma_2(y)$ and $Y := \Gamma_2(x) \cap \Gamma_1(y)$. By considering $A^3_{(x,y)}$ in (2) and (3), the following holds:

$$\sum_{z \in Z} \lambda(x, z) + \sum_{w \in Y} \mu(x, w) = q^2 - 6q + 18.$$

This implies that

$$\sum_{z \in Z} (\lambda(x, z) - q + 4) + e(X, Y) + e(Z, Y) = 2,$$

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and a similar equation holds where the roles of x and y have been interchanged. In particular, we have that $e(Z, X) \leq 2$.

Claim 1 *The induced subgraph on $\{y\} \cup Z$ is a clique of size $q - 3$.*
Proof of Claim 1: Suppose that z_1 and z_2 are non-adjacent distinct vertices in Z. By Proposition 4.3 we have that $|\Gamma_1(z_1) \cap \Gamma_1(z_2) \cap Z| \leq \mu(z_1, z_2) - |\{x, y\}| \leq 4$ and $|\Gamma_1(z_i) \cap Z| \geq \lambda(y, z_i) - 1 - e(Z, Y) \geq q - 7$ $(i = 1, 2)$. Hence

$$q - 6 = |Z \setminus \{z_1, z_2\}| \geq \sum_{i=1}^{2} |\Gamma_1(z_i) \cap Z| - |\Gamma_1(z_1) \cap \Gamma_1(z_2) \cap Z| \geq 2(q - 7) - 4,$$

which is impossible as $q \geq 27$. ∎

Claim 2 *There exists a coclique of size three in X.*
Proof of Claim 2: Since $3q^2 - 9q + 6 = 2e(\Gamma_1(x)) \geq 2e(\{y\} \cup Z) + 2e(X) = (q - 4)(q - 3) + 2e(X)$ holds by (2) and Claim 1, we have $q(q - 1) - 3 \geq e(X)$. Applying Mantel's theorem [7] (cf. [10, Theorem 1.3.23]) (which states that a triangle-free graph on n vertices has at most $n^2/4$ edges) to the complement of the graph induced on X, we find that X contains a coclique of size three. ∎

Now let $\{r_i \mid i = 1, 2, 3\}$ be a coclique of size three in X. Note that non-adjacent vertices in X have at most five common neighbours in X, so the number of neighbours of r_3 in X that are not neighbours of r_1 or r_2 is at most

$$2q - 3 - (\lambda(x, r_1) - e(r_1, Z) + \lambda(x, r_2) - e(r_2, Z) - 5) \leq 10 + e(r_1, Z) + e(r_2, Z).$$

Thus, $q - 4 \leq \lambda(x, r_3) \leq 10 + e(r_3, Z) + 10 + e(r_1, Z) + e(r_2, Z) \leq 20 + e(X, Z) \leq 22$, which contradicts the fact that $q \geq 27$. ∎

Theorem 4.5 *For $q \geq 36$, the Hamming graph $H(3, q)$ is uniquely determined by its spectrum.*

Proof: For $q \geq 36$, let Γ be a graph cospectral with $H(3, q)$. By Propositions 4.3 and 4.4, we may consider Proposition 3.3 with $(\lambda_1, \lambda_2, \mu, t) = (q - 3, q, 6, 3)$. Then every vertex is in at most three cliques of $\mathcal{C}$, and each edge lies in a unique clique in $\mathcal{C}$, where $\mathcal{C}$ is the set of all maximal cliques

of size at least $q - 11$. Hence, Γ is locally the disjoint union of three cliques of size $q - 1$ by Lemma 3.2, and this shows that Γ must be isomorphic to $H(3, q)$ by [1, Theorem 1.1]. ∎

There are exactly four graphs with the same spectrum as $H(3, 3)$ (cf. [5]). Two of them are locally the disjoint union of three cliques of size two, namely $H(3, 3)$ and the dual graph of $H(3, 3)$. The two others do not even have regular local graphs.

There are two graphs which have the same spectrum as $H(3, 4)$, namely $H(3, 4)$ itself and a Doob graph. Whether there are more graphs cospectral with $H(3, 4)$ is unknown.

For $5 \leq q \leq 35$, the only graph known to have the same spectrum as $H(3, q)$ is $H(3, q)$ itself.

5 Graphs with classical parameters

Many of the above methods can also be applied to other distance-regular graphs with so-called classical parameters. A distance-regular graph Γ is said to have *classical parameters* (D, b, α, β) whenever the diameter of Γ is D, and the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i - 1 \\ 1 \end{bmatrix} \right) \quad (1 \leq i \leq D),$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \quad (0 \leq i \leq D - 1)$$

where $\begin{bmatrix} j \\ 1 \end{bmatrix} := \sum_{i=0}^{j-1} b^i$ is a b-ary (Gaussian) binomial coefficient (cf. [3, p. 193]). The Hamming graph $H(D, q)$ is a distance-regular graph with classical parameters $(D, 1, 0, q - 1)$. The Johnson graphs $J(n, D)$, Grassmann graphs $J_q(n, D)$, and the bilinear forms graphs $H_q(n, D)$ are also examples of distance-regular graphs with classical parameters (with parameters $(D, 1, 1, n - D)$, $(D, q, q, \begin{bmatrix} n-D+1 \\ 1 \end{bmatrix} - 1)$ and $(D, q, q - 1, q^n - 1)$, respectively). The spectrum of a distance-regular graph with classical parameters (D, b, α, β) is $\{\theta_i^{m_i} \mid i = 0, \ldots, D\}$ where

$$\theta_i := \begin{bmatrix} D - i \\ 1 \end{bmatrix} \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) - \begin{bmatrix} i \\ 1 \end{bmatrix} \quad \text{and its multiplicity}$$

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$$m_i := \frac{\prod_{j=0}^{i-1} \alpha_j \left(1 + \begin{bmatrix} D-2i \\ 1 \end{bmatrix}\alpha + b^{D-2i}\beta\right)}{\prod_{j=1}^{i} \beta_j \left(1 + \begin{bmatrix} D \\ 1 \end{bmatrix}\alpha + b^D\beta\right)}$$

where

$$\alpha_i = b\begin{bmatrix} D-i \\ 1 \end{bmatrix}(\beta - \begin{bmatrix} i \\ 1 \end{bmatrix}\alpha)(1 + \begin{bmatrix} D-i \\ 1 \end{bmatrix}\alpha + b^{D-i}\beta),$$

$$\beta_i = \begin{bmatrix} i \\ 1 \end{bmatrix}(\beta - \begin{bmatrix} i \\ 1 \end{bmatrix}\alpha + b^i)(1 + \begin{bmatrix} D-i \\ 1 \end{bmatrix}\alpha)$$

([3, Corollary 8.4.2, Theorem 8.4.3]). We call any graph with such spectrum a *graph with classical spectrum with parameters* (D, b, α, β) (even if no distance-regular graph with such classical spectrum exists). We remark that it can be shown that the parameter b is a real algebraic integer.

Let Γ be a graph with classical spectrum with parameters (D, b, α, β). Then Γ is called *geometric* if there are lines such that each edge is in a unique line, and each vertex is in at most $t = \begin{bmatrix} D \\ 1 \end{bmatrix}$ lines.

By using Lemma 2.1 with r and s the smallest non-negative, and largest negative eigenvalue, respectively, one can obtain bounds for the number of common neighbours of two vertices.

Application of Metsch's result (Proposition 3.3) using these bounds gives conditions for Γ to be geometric. Unfortunately, we do not know how to generalize Lemma 3.2, i.e., to show that the lines have the same size, and that each vertex is in the same number of lines. For this purpose it would be useful to have a tight bound on the line size in a graph that is cospectral with a distance-regular graph. Maybe, there isn't such a (useful) bound, since for example, the Johnson graph $J(6, 3)$ has line size exactly 4 (from the eigenvalues and distance-regularity), while there is a cospectral mate with a clique (line) of size 5.

Still, the (known) geometric graphs cospectral with the Johnson and Grassmann graphs as described in [4], all have constant line sizes and constant number of lines through a vertex.

As with the Hamming graphs, we can do better for the case $D = 3$, because we can use Lemma 4.2. For example, Lemma 2.1 applied with the two smallest eigenvalues for the Johnson graphs $J(n, 3)$ (which is distance-regular with classical parameters $(3, 1, 1, n-3)$) gives that $n-22+\frac{72}{n} \leq \lambda(x, y) \leq n+2$ for adjacent vertices x and y and $\mu(x, y) \leq 12$ for non-adjacent vertices x and

y. Lemma 4.2 however implies that $\lambda_1 \geq n - 8$ for $n \geq 26$, so that applying Metsch's result with $\lambda_1 = n - 8$, $\lambda_2 = n + 2$, $\mu = 12$ and $t = 3$ gives that a cospectral mate is geometric for $n > 85$.

One further remark we would like to make is that Metsch [8, Corollary 1.3] proved a more general result than the one we used. Although we have not been able to take advantage of this more general result, it is not unconceivable that it can be used to weaken our conditions for cospectral mates to be geometric.

For the other families of distance-regular graphs with classical parameters, not much is known about the existence of cospectral mates. In this light, we challenge the reader to construct cospectral mates for the bilinear forms graphs.

References

[1] S. Bang and J. H. Koolen, Graphs cospectral with $H(3, q)$ which are locally disjoint union of at most three complete graphs, submitted.

[2] N. Biggs, Algebraic Graph Theory, Cambridge Tracts in Mathematics, No. 67, *Cambridge University Press, London,* 1974.

[3] A. E. Brouwer, A. M. Cohen and A. Neumaier, Distance-Regular Graphs, *Springer-Verlag, Berlin,* 1989.

[4] E. R. van Dam, W. H. Haemers, J. H. Koolen, and E. Spence, Characterizing distance-regularity of graphs by the spectrum, *J. Combinatorial Th. A* **113** (2006) 1805–1820.

[5] W. H. Haemers and E. Spence, Graphs cospectral with distance-regular graphs, *Linear Multilin. Alg.* **39** (1995) 91–107.

[6] A. J. Hoffman, On the polynomial of a graph, *Amer. Math. Monthly* **70** (1963) 30–36.

[7] W. Mantel, Vraagstuk XXVIII, *Wiskundige Opgaven met de Oplossingen* **10** (1907) 60–61.

[8] K. Metsch, Improvement of Bruck's completion theorem, *Designs Codes Crypt.* **1** (1991) 99–116.

[9] K. Metsch, A characterization of Grassmann graphs, *European J. Combin.* **16** (1995) 639–644.

[10] Douglas B. West, Introduction to graph theory, *Prentice Hall, Inc., Upper Saddle River, NJ,* 1996.