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In this paper, we introduce a general framework for situations with decision making under uncertainty and cooperation possibilities. This framework is based upon a two stage stochastic programming approach. We show that under relatively mild assumptions the cooperative games associated with these situations are totally balanced and, hence, have non-empty cores. Finally, we consider several example situations, which can be studied using this general framework.

Key words: Two-stage stochastic programming; cooperative game theory; core.

JEL classification code: C71.

1. Introduction   In this paper, we consider situations with multiple players, who choose strategies to influence their expected profits. We assume two decision epochs for an individual player. First, he decides on a strategy to play under uncertainty of the future state of the world, which affects the outcome of the played strategy. After the uncertainty is resolved, the player can take a recourse action that compensates for any adverse effects that might have been experienced as a result of the chosen strategy. The optimal strategies and recourse actions for the players are determined by the solution of a two stage stochastic optimization problem. These players can also cooperate in a coalition. In this case, the players in the coalition coordinate their strategies and recourse actions to maximize their total expected profit.

Many real life situations with decision making under uncertainty can be modeled using two stage stochastic programming. Several applications appeared in the supply chain literature. One example is the analysis of multi-product inventory systems with substitution. A series of papers analyzed these systems with random demand (see Bassok et al. [2] and Rao et al. [18]) and random yield (see Hsu and Bassok [11]), where in the first period a production decision is made and after uncertainty is resolved an allocation decision follows. Another application concerns inventory systems with transshipment. Herer and Rashit [10] considered a two-location inventory system with fixed and joint replenishment costs and they developed the properties of optimal decisions. Besides the above applications, Doğru et al. [7] studied a base stock policy for an assemble to order system where the products have common components. They developed an heuristic where the stock levels are set by solving a two-stage stochastic program. Moreover, the solution provides a lower-bound for the system performance. van Mieghem and Rudi [23] introduced a class of models, called newsvendor networks, that provide a framework to study various problems of stochastic capacity investment and inventory management. Their approach is based on a similar two-stage

stochastic programming technique as in this paper. All of the papers above, different from us, assume single ownership of the problem and focus on the determination of optimal decisions or developing effective heuristics. Anupindi et al. [1], Granot and Sošić [8] and Rudi et al. [19] analyzed the performance of decentralized systems, where the centralized (benchmark) performance is given by the solution of a two-stage stochastic program.

In this paper, we provide a general framework for situations in which multiple players collaborate by coordinating their strategies and recourse actions to maximize their total profits. A main question is how the increased profits should be shared among the members of the cooperation. Cooperative game theory mainly studies this issue and proposes the core concept for stability of the cooperation. The core is the set of all stable profit divisions such that no group of players would like to split off from cooperation and form a smaller coalition. We provide sufficient conditions for the associated cooperative games to have non-empty cores. From a similar point of view, several papers studied cooperation in a newsvendor setting to benefit from inventory pooling (see Hartman et al. [9], Müller et al. [13], Özen et al. [15], Özen et al. [16], Özen and Sošić [17], Slikker et al. [21] and Slikker et al. [22]). We remark that these studies fit into our general framework. Moreover, our framework covers several other situations in which the uncertainty deals with other aspects in the system, e.g., random yield. Nonemptiness of the core is also investigated in the literature dealing with investments. Borm et al. [4] studied firms' cooperative investments in capital deposits and de Waegenaere et al. [6] considered a cooperative investment situation where the firms bundle their resources to invest in long term projects. Both studies assume a deterministic setting. In this paper, we consider a two-stage stochastic variant of these problems as well.

The rest of the paper is organized as follows. In section 2, we give preliminaries on positively homogeneous functions and cooperative game theory. In section 3, we introduce a framework for situations with decision making under uncertainty and cooperation possibilities, and we focus on a special class of situations, called stochastic cooperative decision situations. This class captures a broad range of cooperation situations under uncertainty. We show that the cooperative games associated with these situations are totally balanced and, hence, they have non-empty cores. Afterwards, in section 4, we provide some example situations that can be analyzed in this framework. We conclude the paper with further discussions in section 5.

2. Preliminaries   In this section, we give preliminaries on positively homogeneous functions and cooperative game theory.

A function f on $\mathbb{R}^n$ is called *positively homogeneous* (of degree 1) if for every $x \in \mathbb{R}^n$ and $\lambda \in (0, \infty)$

$$f(\lambda x) = \lambda f(x).$$

Note that if a function f is positively homogeneous, then $f(0) = 0$. Moreover, all linear functions are positively homogeneous.

THEOREM 1.   *Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}$. If f is a positively homogeneous concave function, then for every $\lambda_1 \geq 0, ..., \lambda_m \geq 0$ and $x_1, ..., x_m \in \mathbb{R}^n$*

$$f(\lambda_1 x_1 + ... + \lambda_m x_m) \geq \lambda_1 f(x_1) + ... + \lambda_m f(x_m). \tag{1}$$

Proof. Without loss of generality, we may assume that $\lambda_i > 0$ for every $i \in \{1, .., m\}$. We have

$$f(\lambda_1 x_1 + ... + \lambda_m x_m) = f(\sum_{i=1}^{m} \lambda_i x_i)$$

$$= \sum_{j=1}^{m} \lambda_j \left(f \left(\sum_{i=1}^{m} \frac{\lambda_i}{\sum_{j=1}^{m} \lambda_j} x_i \right) \right)$$

$$\geq \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{m} \frac{\lambda_i}{\sum_{j=1}^{m} \lambda_j} f(x_i) \right)$$

$$= \sum_{i=1}^{m} \lambda_i f(x_i)$$

$$= \lambda_1 f(x_1) + ... + \lambda_m f(x_m).$$

The second equality follows since f is positively homogeneous. The inequality follows from concavity of f. This completes the proof. $\qquad\square$

Cooperative game theory deals with situations, where a group of players cooperate by coordinating their actions to obtain a joint profit. It is usually assumed that binding agreements between the players are the mean of the cooperation. A main question of concern is how this profit will be divided among the cooperating players.

Let N be a finite set of players, $N = \{1, ..., n\}$. A subset of N is called a *coalition*. A function v, assigning a value $v(S)$ to every coalition $S \subseteq N$ with $v(\emptyset) = 0$, is called a *characteristic function*. The value $v(S)$ is interpreted as the maximum total profit that coalition S can obtain through cooperation. Assuming that the benefit of a coalition S can be transferred between the players of S, a pair (N, v) is called a *cooperative game with transferable utility* (TU-game) or *a game in coalitional form*. For a game (N, v), $S \subset N$ and $S \neq \emptyset$, the *subgame* $(S, v_{|S})$ is defined by $v_{|S}(T) = v(T)$ for each coalition $T \subseteq S$.

In reality, the players are not primarily interested in benefits of a coalition but in their individual benefits that they make out of that coalition. A division is a *payoff vector* $y = (y_i)_{i \in N} \in \mathbb{R}^N$, specifying for each player $i \in N$ the benefit y_i. A division y is called *efficient* if $\sum_{i \in N} y_i = v(N)$ and *individually rational* if $y_i \geq v(\{i\})$ for all $i \in N$. Individual rationality means that every player gets at least as much as what he could obtain by staying alone. The set of all individually rational and efficient divisions constitutes the *imputation set*:

$$I(v) = \{y \in \mathbb{R}^N | \textstyle\sum_{i \in N} y_i = v(N) \text{ and } y_i \geq v(\{i\}) \text{ for each } i \in N\}.$$

If these rationality requirements are extended to all coalitions, we obtain the *core*:

$$Core(v) = \{y \in \mathbb{R}^N | \textstyle\sum_{i \in N} y_i = v(N) \text{ and } \sum_{i \in S} y_i \geq v(S) \text{ for each } S \subseteq N\}.$$

Thus, the core consists of all imputations in which no group of players has an incentive to split off from the grand coalition N and form a smaller coalition, because they collectively receive at least as much as what they can obtain by cooperating on their own. Note that the core of a game can be empty.

Bondareva [3] and Shapley [20] independently made a general characterization of games with a non-empty core by the notion of balancedness. Let us define the vector e^S for all $S \subseteq N$ by $e_i^S = 1$ for all $i \in S$ and $e_i^S = 0$ for all $i \in N \backslash S$. A map $\kappa : 2^N \backslash \{\emptyset\} \to [0, 1]$ is called a *balanced map* if $\sum_{S \in 2^N \backslash \{\emptyset\}} \kappa(S) e^S = e^N$. Further, a game (N, v) is called *balanced* if for every balanced map $\kappa : 2^N \backslash \{\emptyset\} \to [0, 1]$ it holds that $\sum_{S \in 2^N \backslash \{\emptyset\}} \kappa(S) v(S) \leq v(N)$. The following theorem is due to Bondareva [3] and Shapley [20].

THEOREM 2. *Let (N, v) be a TU-game. Then $Core(v) \neq \emptyset$ if and only if (N, v) is balanced.*

A TU-game (N, v) is called *totally balanced* if it is balanced and each of its subgames is balanced as well.

3. Model Let N be a set of players. Each player $i \in N$ is subject to uncertainty, which is represented by a random variable (or signal) X_i taking values in Υ_i. The players can either work alone or they can cooperate in a coalition. Consider a coalition $S \in 2^N \backslash \{\emptyset\}$. Let $x^S = (x_i)_{i \in S}$ be a realization of random vector $X^S = (X_i)_{i \in S}$ taking values in $\Upsilon^S = \prod_{i \in S} \Upsilon_i$. Before the realization of the random vector X^S, the players in the coalition jointly choose a strategy q^S from the strategy space $Q^S \subseteq \mathbf{R}^B$ with $B \in \mathbf{N}$. After observing realization x^S of X^S, the players decide on a joint recourse action m^S from the action space $M^S(q^S, x^S) \subseteq \mathbf{R}^{B \times N}$, which depends on q^S and x^S. We assume for all $m^S \in M^S(q^S, x^S)$ that $m_i^S = 0$ if $i \notin S$ since the coordinates of the players outside of coalition S are irrelevant for the coalition. We denote the map $(q^S, x^S) \mapsto M^S(q^S, x^S)$ shortly by M^S. There is a cost associated with each strategy of the coalition. Suppose the coalition plays strategy q^S, then it pays a cost $C(q^S)$, where $C : \mathbf{R}^B \to \mathbf{R}$. If the coalition plays recourse action m^S for realization x^S of the random vector, each player in the coalition makes a revenue $H^i(m_i^S, x_i)$, where $m_i^S \in \mathbf{R}^B$ is the i'th coordinate of m^S and $H^i : \mathbf{R}^B \times \Upsilon_i \to \mathbf{R}$. Hence, the coalition's total revenue is

$$Z^S(m^S, x^S) = \sum_{i \in S} H^i(m_i^S, x_i). \tag{2}$$

A tuple $(N, (X_i)_{i \in N}, (Q^S, M^S)_{S \in 2^N \backslash \{\emptyset\}}, C, (H^i)_{i \in N})$ with entities as above is called a *stochastic cooperative decision situation* if the following conditions hold:

(i) For all $T \subseteq N$, all $q^T \in Q^T$ and every realization x^T of X^T, $\max_{m \in M^T(q^T, x^T)} Z^T(m, x^T)$ exists.

(ii) For all $T \subseteq N$ and all $q^T \in Q^T$, $E_{X^T}[\max_{m \in M^T(q^T, \cdot)} Z^T(m, \cdot)]$ exists.

(iii) For all $T \subseteq N$, $\max_{q \in Q^T} \left(-C(q) + E_{X^T} \left[\max_{m \in M^T(q, \cdot)} Z^T(m, \cdot) \right] \right)$ exists.

(iv) For all $T \subseteq N$, every balanced map $\kappa^T : 2^T \backslash \{\emptyset\} \to [0, 1]$ and every $(q^S)_{S \subseteq T} \in \prod_{S \subseteq T} Q^S$, it holds that

$$\sum_{S \in 2^T \backslash \{\emptyset\}} \kappa^T(S) q^S \in Q^T.$$

(v) For all $T \subseteq N$, every balanced map $\kappa^T : 2^T \backslash \{\emptyset\} \to [0, 1]$, every realization x^T of X^T, every $(q^S)_{S \subseteq T} \in \prod_{S \subseteq T} Q^S$ and every $(m^S)_{S \subseteq T} \in \prod_{S \subseteq T} M^S(q^S, x^S)$, it holds that

$$\sum_{S \in 2^T \backslash \{\emptyset\}} \kappa^T(S) m^S \in M^T \left(\sum_{S \in 2^T \backslash \{\emptyset\}} \kappa^T(S) q^S, x^T \right).$$

(vi) For all $i \in N$ and all realizations x_i of X_i, $H^i(\cdot, x_i)$ is a concave function.

(vii) C is a positively homogeneous convex function.

The first three conditions guarantee that optimal strategies and recourse actions that maximize the total expected profit exist for every coalition, and that the expected profit of a coalition under optimal strategies and recourse actions is well defined. Conditions (iv) and (v) simply state that every balanced collection of strategies and recourse actions of coalitions $S \subseteq T$ determines a feasible strategy and recourse action for coalition T. Condition (vi) states that the revenue of a player is a concave function of his recourse action. The last condition states that the cost function is a positively homogenous convex function.

Suppose that coalition S plays strategy $q^S \in Q^S$, and realization x^S of random vector X^S occurs. Let $m^{S*}(q^S, x^S)$ be an optimal recourse action, which exists by condition (i). We refer to $Z^S(m^{S*}(q^S, x^S), x^S)$ as $z^S(q^S, x^S)$, which is the maximum total revenue that coalition S can achieve by playing its optimal recourse action.

In a stochastic cooperative decision situation, individual players or coalitions are assumed to be interested in their expected profits, while choosing their strategies and recourse actions. If coalition S plays strategy q^S and optimal recourse actions, its expected profit is given by

$$\tau^S(q^S) = -C(q^S) + E_{X^S}\left[z^S(q^S, \cdot)\right].$$

From condition (ii), we know that the expectation is well defined, and hence the expected profit function too. Moreover, condition (iii) assures that every coalition has an optimal strategy that maximizes its expected profit.

For a stochastic cooperative decision situation, the associated cooperative game (N, v) is defined by

$$v(S) = \max_{q^S} \tau^S(q^S).$$

In other words, the value of a coalition is the maximum expected profit that the coalition can obtain playing its optimal strategy and recourse actions. Therefore, we call the associated cooperative game an *expected profit game*.

The following theorem states that expected profit games associated with stochastic cooperative decision situations are totally balanced.

THEOREM 3. *Let $(N, (X_i)_{i \in N}, (Q^S, M^S)_{S \in 2^N \setminus \{\emptyset\}}, C, (H^i)_{i \in N})$ be a stochastic cooperative decision situation. Then the associated expected profit game is totally balanced.*

Proof. Consider a coalition $T \subseteq N$. Let $\kappa^T : 2^T \setminus \{\emptyset\} \to [0, 1]$ be a balanced map. Let $(q^{S*})_{S \subseteq T : S \neq \emptyset}$ and $(m^{S*}(q^{S*}, x^S))_{S \subseteq T : S \neq \emptyset, x^S \in \Upsilon^S}$ be the optimal strategies and recourse actions of different coalitions, respectively. Let $t^T = \sum_{S \in 2^T \setminus \{\emptyset\}} \kappa^T(S) q^{S*} \in Q^T$ (condition (iv)). Then, for every realization x^T of X^T,

$$
\begin{aligned}
Z^T\left(m^{T*}\left(t^T, x^T\right), x^T\right) &\geq Z^T\Big(\sum_{S \in 2^T \setminus \{\emptyset\}} \kappa^T(S) m^{S*}(q^{S*}, x^S), x^T\Big) \\
&= \sum_{i \in T} H^i\left(\sum_{S \in 2^T \setminus \{\emptyset\} : i \in S} \kappa^T(S) m_i^{S*}(q^{S*}, x^S), x_i\right) \\
&\geq \sum_{i \in T} \sum_{S \in 2^T \setminus \{\emptyset\} : i \in S} \kappa^T(S) H^i\left(m_i^{S*}(q^{S*}, x^S), x_i\right) \\
&= \sum_{S \in 2^T \setminus \{\emptyset\}} \kappa^T(S) \sum_{i \in S} H^i\left(m_i^{S*}(q^{S*}, x^S), x_i\right) \\
&= \sum_{S \in 2^T \setminus \{\emptyset\}} \kappa^T(S) Z^S\left(m^{S*}(q^{S*}, x^S), x^S\right).
\end{aligned}
$$

The first inequality follows since $m^{T*}(t^T, x^T) \in M(t^T, x^T)$ is an optimal recourse action and $\sum_{S \in 2^T \setminus \{\emptyset\}} \kappa^T(S) m^{S*}(q^{S*}, x^S) \in M(t^T, x^T)$ from condition (v). The first equality holds by definition of Z^T and since $m_j^{S*} = 0$ for all $j \notin S$ for all $S \subseteq T$. The second inequality holds since $H^i(\cdot, x^S)$ is concave for all $i \in N$ and realization x^S of X^S (condition (vi)), $0 \leq \kappa^T(S) \leq 1$ for all $S \in 2^T \setminus \{\emptyset\}$,

and $\sum_{S \in 2^T \setminus \{\emptyset\} : i \in S} \kappa^T(S) = 1$ for all $i \in T$. The second equality follows from interchanging the order of summation. The last equality holds by definition of Z^S.

Taking the expectation of both sides, we find that

$$E_{X^T}\left[Z^T\left(m^{T*}(t^T,\cdot),\cdot\right)\right] \geq \sum_{S \in 2^T \setminus \{\emptyset\}} \kappa^T(S) E_{X^S}\left[Z^S\left(m^{S*}(q^{S*},\cdot),\cdot\right)\right]. \tag{3}$$

Then,

$$\begin{aligned}
v(T) &= -C(q^{T*}) + E_{X^T}\left[Z^T(m^{T*}(q^{T*},\cdot),\cdot)\right] \\
&\geq -C(t^T) + E_{X^T}\left[Z^T(m^{T*}(t^T,\cdot),\cdot)\right] \\
&\geq -C(t^T) + \sum_{S \in 2^T \setminus \{\emptyset\}} \kappa^T(S) E_{X^S}\left[Z^S\left(m^{S*}(q^{S*},\cdot),\cdot\right)\right] \\
&\geq -\sum_{S \in 2^T \setminus \{\emptyset\}} \kappa^T(S) C(q^{S*}) + \sum_{S \in 2^T \setminus \{\emptyset\}} \kappa^T(S) E_{X^S}\left[Z^S\left(m^{S*}(q^{S*},\cdot),\cdot\right)\right] \\
&= \sum_{S \in 2^T \setminus \{\emptyset\}} \kappa^T(S)\left(-C^S(q^{S*}) + E_{X^S}\left[Z^S\left(m^{S*}(q^{S*},\cdot),\cdot\right)\right]\right) \\
&= \sum_{S \in 2^T \setminus \{\emptyset\}} \kappa^T(S) v(S).
\end{aligned}$$

The first equality holds since q^{T*} and $m^{T*}(q^{T*},\cdot)$ are an optimal strategy and optimal recourse actions for coalition T. The first inequality follows from $t^T \in Q^T$ (condition (iv)). The second inequality holds by (3). The third inequality follows from Theorem 1 since $-C$ is a positively homogeneous concave function (from condition (vii)) and $\kappa^T(S) \geq 0$ for all $S \in T$ in balanced map κ^T. The last equality holds since q^{S*} and $m^{S*}(q^{S*},\cdot)$ are an optimal strategy and optimal recourse actions for coalition S, respectively. $\square$

We remark that from the proof of Theorem 3, the associated cooperative game is balanced even if conditions (iv) and (v) only hold for optimal strategies and recourse actions with $T = N$.

From Theorems 2 and 3, the following corollary follows immediately.

COROLLARY 1. *Let* $(N, (X_i)_{i \in N}, (Q^S, M^S)_{S \in 2^N \setminus \{\emptyset\}}, C, (H^i)_{i \in N})$ *be a stochastic cooperative decision situation. Then the associated expected profit game has a non-empty core.*

4. Examples In this section, we present several example situations that fit into the general framework in section 3. The first one concerns newsvendor situations with multiple warehouses (see Özen et al. [16]).

EXAMPLE 1. *Newsvendor situations with multiple warehouses.* Consider a set of retailers N, who sell the same product in separate markets with stochastic demand. Because of long production and transportation lead times, the retailers have to place their orders before the start of the selling period without knowing the realization of stochastic demand, but knowing its distribution. After the lead time elapses, the demand is realized and the orders become available at the warehouses. Finally, the orders are sent to the retailers and the demand is satisfied as much as possible. Let X_i be the stochastic demand of retailer i and let F_i be its cumulative distribution function. Moreover, let x^S denote a realization of demand vector $X^S = (X_i)_{i \in S}$. The retailers can give their orders to several warehouses. Let W be the set of warehouses and let $W_i \subseteq W$ be the set of warehouses from which retailer i can supply the goods. Let k_w, $f_{wi} \geq 0$ and $p_i \geq 0$ be the unit cost of ordering to

warehouse $w \in W$, the unit cost of transporting goods from warehouse $w \in W$ to retailer $i \in N$ and selling price of retailer i, respectively. This newsvendor situation can be represented by the tuple $\Gamma = (N, W, (X_i)_{i \in N}, (k_w)_{w \in W}, (f_{wi})_{w \in W, i \in N}, (W_i)_{i \in N}, (p_i)_{i \in N})$. In this newsvendor situation, the retailers can increase their expected total profit, if they cooperate. By cooperating, they decide on a joint order and can allocate the joint order that becomes available at the warehouse after demand realization. Hence, they benefit from inventory pooling and coordinated ordering. Consider coalition $S \subseteq N$. Forming a coalition, the manufacturers can give a joint order to any manufacturing facility that is available for any of them. Let $W_S = \cup_{i \in S} W_i$ be the set of available facilities for coalition S. Then, the set of possible order vectors is given by

$$Q^S := \{q \in \mathbb{R}_+^W | q_w = 0 \text{ for all } w \notin W_S \}.$$

Suppose that coalition S gives an order $q^S \in Q^S$ at a cost of $C(q^S) = \sum_{w \in W_S} k_w q_w^S$ and the realization of random vector X^S appears to be x^S. An allocation of q^S is represented by a matrix $m^S \in \mathbb{R}_+^{W \times N}$ with

$$m_{wi}^S = 0 \text{ if } i \in N \backslash S \text{ or } w \in W \backslash W_S \text{ ;}$$
$$\sum_{i \in S} m_{wi}^S = q_w^S \text{ for all } w \in W_S.$$

Here, m_{wi}^S denotes the amount of products that is sent from warehouse w to retailer i. We denote the set of all possible allocations for order vector q^S and realization x^S of X^S by $M^S(q^S, x^S)$. For an allocation $m^S \in M^S(q^S, x^S)$, coalition S makes a total revenue given by

$$R^S(m^S, x^S) = \sum_{i \in S} \left(-\sum_{w \in W} f_{wi} m_{wi}^S + p_i \min\{\sum_{w \in W} m_{wi}^S, x_i\} \right).$$

Let $m^{S*}(q^S, x^S) \in M(q^S, x^S)$ be an optimal allocation maximizing $R^S(m^S, x^S)$. In the remainder of this example, we refer to $R^S(m^{S*}(q^S, x^S), x^S)$ as $r^S(q^S, x^S)$. The expected total profit of coalition S is given by

$$\pi(q^S) = -C(q^S) + E_{X^S}[r^S(q^S, \cdot)].$$

In this newsvendor situation, we assume that the retailers in coalition S choose an order vector and an allocation after demand realization to maximize their expected total profit. The associated cooperative game (N, v^Γ) is given by

$$v^\Gamma(S) = \max_{q \in Q^S} \pi(q)$$

In the following part of the example, we will show that (N, v^Γ) is an expected profit game. Consider a situation represented by $\Lambda = (N, (X_i)_{i \in N}, (Q^S, M^S)_{S \in 2^N \backslash \{\emptyset\}}, C, (H^i)_{i \in N})$ with N, X_i, Q^S, M^S and C as in newsvendor situation Γ. Moreover, $H^i : \mathbb{R}^W \times \Upsilon_i \to \mathbb{R}$ is defined by

$$H^i(m_i, x_i) = -\sum_{w \in W} f_{wi} m_{wi} + p_i \min\{\sum_{w \in W} m_{wi}, x_i\}$$

for all $i \in N$.

In the following part of the example, we will check conditions $(i) - (vii)$ for Λ to be a stochastic cooperative decision situation. Conditions (i) and (iii) hold since the respective functions $Z^S(\cdot, x^S)$ and $\tau^S(\cdot)$ are continuous and we can easily derive that each maximum is obtained in a compact set. Condition (ii) holds under the very mild and natural assumption: each random variable X_i has a finite expectation. We refer to Özen [14] for the formal proofs of conditions (i)-(iii). It is

easy to check that conditions (iv), (vi) and (vii) hold as well. The only condition left for Λ to be a stochastic cooperative decision situation is condition (v). Consider coalition T and realization x^T of X^T. Let $\kappa^T : 2^T \backslash \{\emptyset\} \to [0, 1]$ be a balanced map. Moreover, let $(q^S)_{S \subseteq T} \in \prod_{S \subseteq T} Q^S$ be a collection order quantities and let $(m^S)_{S \subseteq T} \in \prod_{S \subseteq T} M^S(q^S, x^S)$ be a collection of allocations of production quantities. We can rewrite condition (v) for this situation as

$$\sum_{S \in 2^T \backslash \{\emptyset\}} \kappa^T(S) m^S_{wi} = 0 \text{ if } i \in N \backslash T \text{ or } w \in W \backslash W_T \ ; \tag{4}$$

$$\sum_{i \in T} \sum_{S \in 2^T \backslash \{\emptyset\} | i \in S} \kappa^T(S) m^S_{wi} = \sum_{S \in 2^T \backslash \{\emptyset\}} \kappa^T(S) q^S_w \text{ for all } w \in W_T. \tag{5}$$

It is easy to verify that (4) holds. Equation (5) holds since $\sum_{i \in S} m^S_{wi} = q^S_w$ for all $w \in W$ and $S \subseteq T$. Therefore, we conclude that the game (N, w^Λ) associated with Λ is an expected profit game. Moreover, (N, w^Λ) and (N, v^Γ) coincide since for any coalition $S \subseteq N$, order vector $q^S \in Q^S$, realization x^S of X^S and allocation $m^S \in M^S(q^S, x^S)$, $Z^S(m^S, x^S) = R^S(m^S, x^S)$ and, moreover, $\tau^S(q^S) = \pi^S(q^S)$. Therefore, cooperative game (N, v^Γ) is an expected profit game, which is totally balanced. $\diamondsuit$

The following example considers a production situation with random yield.

EXAMPLE 2. *Cooperative production with random yield.* Consider a set of manufacturers N, who produce an identical product every period to satisfy their deterministic demand. Production can be performed in one or more production facilities, whose availability differs for each manufacturer. Let B denote the set of production facilities and let $B_i \subseteq B$ be the set of production facilities available for manufacturer $i \in N$. Because of the nature of the production process, the production yield is random in each production facility. Let Y_j be the random variable representing the uncertainty in production facility j for all $j \in B$. We assume that Y_j is taking values in $[0, \infty)$ usually close to 1. The production yield of an order $q \in \mathbb{R}_+$ given to production facility j depends on the realization y_j of Y_j and is given by $L^j_{y_j}(q) = y_j q$. Let c_j, p_i, v_i, b_i and D_i be the unit ordering cost at facility j, selling price, salvage value, penalty cost and deterministic demand of manufacturer i, respectively. We assume that $c_j \geq 0$ for all $j \in B$, $p_i, v_i, b_i, D_i \geq 0$ and $p_i + b_i > v_i$ for all $i \in N$. This production situation can be represented by a tuple $\Gamma = (N, B, (Y_j)_{j \in B}, (c_j)_{j \in B}, (B_i)_{i \in N}, (p_i)_{i \in N}, (v_i)_{i \in N}, (b_i)_{i \in N})$. In this production situation, the manufacturers can either work alone or they can cooperate in a coalition. Being alone, a manufacturer gives an ordering decision and satisfies his demand with the random output as much as possible. On the other hand, if a group of manufacturers cooperates, they decide on a joint order and can allocate the total production quantity to maximize their total expected profit. Hence, they benefit from reduced risk of random production and coordinated ordering. This production situation fits into the general framework in section 3 and it can be represented as the tuple $\Lambda = (N, (X_i)_{i \in N}, (Q^S, M^S)_{S \in 2^N \backslash \{\emptyset\}}, C, (H^i)_{i \in N})$ with entities specified below.

Consider coalition $S \subseteq N$. Forming a coalition, the manufacturers can give a joint order to any manufacturing facility that is available for any of them. Let $B_S = \cup_{i \in S} B_i$ be the set of available facilities for coalition S. Then, the set of possible order vectors is given by

$$Q^S := \{q \in \mathbb{R}_+^B | q_j = 0 \text{ for all } j \notin B_S \ \}.$$

Let $X_i = (Y_j)_{j \in B_i}$ be the vector representing randomness faced by manufacturer i and $X^S = (X_i)_{i \in S}$. Suppose that coalition S gives an order $q^S \in Q^S$ at a cost of $C(q^S) = \sum_{j \in B_S} c_j q^S_j$ and the realization of random vector X^S appears to be x^S. This leads to production output $L^j_{y_j}(q_j)$ where y_j is the coordinate of some x_i with $j \in B_i$ associated with $j \in B_S$. Note that y_j is independent of selection of

i with manufacturing facility $j \in B_i$. After the production quantities are known, the manufacturers make an allocation decision represented by a matrix $m^S \in \mathbb{R}^{B \times N}$ with

$$m^S_{ji} = 0 \text{ if } i \in N \backslash S \text{ or } j \in B \backslash B_S \ ;$$
$$\sum_{i \in S} m^S_{ji} = L^j_{y_j}(q_j) \text{ for all } j \in B_S.$$

Here, m^S_{ji} denotes the amount of products that is assigned in facility j to manufacturer i. We denote the set of all possible allocations for order vector q^S and realization x^S of X^S by $M^S(q^S, x^S)$. For an allocation $m^S \in M^S(q^S, x^S)$, manufacturer i makes a revenue given by

$$H^i(m^S_i, x_i) = p_i \min\{ \sum_{j \in B_S} m_{ji}, D_i \} + v_i (\sum_{j \in B_S} m_{ji} - D_i)^+ - b_i (D_i - \sum_{j \in B_S} m_{ji})^+,$$

and hence the total revenue of coalition S is $Z^S(m^S, x^S) = \sum_{i \in S} H^i(m^S_i, x_i)$.

In the following part of the example, we will show that (N, v^Λ) is an expected profit game by checking the conditions $(i) - (vii)$ for Λ to be a stochastic cooperative decision situation[1]. Using similar arguments and assumptions (i.e., $E[Y_j] < \infty$ for all $j \in B$) as in Example 1, conditions (i), (ii) and (iii) are met. It is easy to check that conditions (iv), (vi) and (vii) hold as well. The only condition left for Λ to be a stochastic cooperative decision situation is condition (v). Consider coalition T and realization x^T of X^T. Let $\kappa^T : 2^T \backslash \{\emptyset\} \to [0, 1]$ be a balanced map. Moreover, let $(q^S)_{S \subseteq T} \in \prod_{S \subseteq T} Q^S$ be a collection of order quantities and let $(m^S)_{S \subseteq T} \in \prod_{S \subseteq T} M^S(q^S, x)$ be a collection of allocations of the production quantities. We can rewrite condition (v) for this situation as

$$\sum_{S \in 2^T \backslash \{\emptyset\}} \kappa^T(S) m^S_{ji} = 0 \text{ if } i \in N \backslash T \text{ or } j \in B \backslash B_T \ ; \tag{6}$$

$$\sum_{i \in T} \sum_{S \in 2^T \backslash \{\emptyset\} | i \in S} \kappa^T(S) m^S_{ji} = L^j_{y_j}(\sum_{S \in 2^T \backslash \{\emptyset\}} \kappa^T(S) q^S_j) \text{ for all } j \in B_S. \tag{7}$$

It is easy to verify that (6) holds. (7) holds since

$$\sum_{i \in T} \sum_{S \in 2^T \backslash \{\emptyset\} | i \in S} \kappa^T(S) m^S_{ji} = \sum_{S \in 2^T \backslash \{\emptyset\}} \sum_{i \in S} \kappa^T(S) m^S_{ji}$$
$$= \sum_{S \in 2^T \backslash \{\emptyset\}} \kappa^T(S) L^j_{y_j}(q^S_j)$$
$$= L^j_{y_j}(\sum_{S \in 2^T \backslash \{\emptyset\}} \kappa^T(S) q^S_j).$$

The last equality follows since $L^j_{y_j}$ is linear. Therefore, we conclude that the game (N, v^Λ) associated with Λ is an expected profit game, which is totally balanced. $\diamondsuit$

The following example considers a cooperative borrowing situation.

EXAMPLE 3. *Cooperative borrowing.* Consider a set of companies N, who need to get loans to support their operations in the following two periods. In the first period, each firm $i \in N$ needs deterministic amount of loan D^1_i, whereas the need for the second period is stochastic and it is denoted by random variable X_i. In the first period, the firms have two loan options: 1-period loan and 2-period loan. Let c_1 and c_2 be the unit costs of borrowing 1-period loan and 2-period loan,

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[1] We remark that the value of coalition S in (N, v^Λ) is the maximum expected profit that coalition S can obtain by joint ordering and allocating the output optimally.

respectively. We assume that the cost of 2-period loan is lower than borrowing 1-period loan in two periods separately, i.e., $c_2 \leq 2c_1$. We also assume that firm i makes a fixed revenue p_i form its operations in the two periods and a unit revenue r by investing unused loans on several options in each period. Moreover, we assume $c_2 - r > c_1$. For ease of presentation, we disregard the time value of money. The sequence of events is as follows. At the start of the first period, the firms decide on how much 1-period and 2-period loan they would like to borrow. Afterwards, at the start of the second period each firm observes his loan need and borrows additional 1-period loan accordingly. For each period, the available amount of loan should be at least what the firms need. This borrowing situation can be represented by a tuple $\Gamma = (N, (D_i, X_i)_{i \in N}, c_1, c_2, (p_i)_{i \in N}, r)$. In this borrowing situation, the firms can cooperate in a coalition to decrease their total expected borrowing cost by making joint borrowing decisions, and hence increase their total expected profit. This borrowing situation fits into the general framework in section 3 and it can be represented as a tuple $\Lambda = (N, (X_i)_{i \in N}, (Q^S, M^S)_{S \in 2^N \setminus \{\emptyset\}}, C, (H^i)_{i \in N})$ with entities specified below.

Consider coalition $S \subseteq N$. Forming a coalition, the companies can decide jointly how much 1-period and 2-period loan to borrow. Observe that for the companies in S, it is optimal to order the exact total amount of loan to cover their needs $\sum_{i \in S} D_i$, since ordering more would create extra cost for them, i.e., ordering extra 1-period loan is redundant since $r < c_2 - c_1 \leq c_2/2 \leq c_1$, and ordering extra 2-period loan is redundant since $c_2 - r > c_1$ and hence it is better to borrow 1-period loan to cover possible needs in the second period. The problem can be formulated as how much 2-period loan to borrow in the first period. Therefore, the set of possible loan vectors is given by

$$Q^S := \{q \in \mathbb{R}_+ | q \leq \sum_{i \in S} D_i\}.$$

Here, q denotes the amount of 2-period (long term) loan borrowed and the rest of the needs will be covered by 1-period loans, which amount is $\sum_{i \in S} D_i - q$. Let $X^S = (X_i)_{i \in S}$ be the random vector representing randomness faced by coalition S. Suppose that coalition S borrows $q^S \in Q^S$ at an additional cost of $C(q^S) = (c_2 - c_1)q^S$. Note that the companies could also cover their needs in the first period by borrowing 1-period loan with a fixed cost $c_1 \sum_{i \in S} D_i$. We will consider this cost in the companies revenue functions and hence function C denotes the additional cost of ordering 2-period loan instead of 1-period loan in the first period. Moreover, suppose that the realization of random vector X^S appears to be x^S. Hence, the firms need in total $\sum_{i \in S} x_i^S$ loan for the second period. Since they already have q^S amount of loan from period 1, they can allocate this amount among themselves and borrow additional 1-period loan if necessary. An allocation is represented by a vector $m^S \in \mathbb{R}_+^N$ with

$$m_i^S = 0 \text{ if } i \in N \setminus S;$$
$$\sum_{i \in S} m_i^S = q^S.$$

We denote the set of all possible allocations for borrowing q^S and realization x^S of X^S by $M^S(q^S, x^S)$. For an allocation $m^S \in M^S(q^S, x^S)$, firm i makes a revenue given by

$$H^i(m_i^S, x_i) = p_i - c_1(x_i - m_i^S)^+ + r(m_i^S - x_i)^+ - c_1 D_i,$$

and hence the total revenue of coalition S is $Z^S(m^S, x^S) = \sum_{i \in S} H^i(m_i^S, x_i)$.

In the following part of the example, we will show that (N, v^Λ) is an expected profit game by checking the conditions $(i) - (vii)$ for Λ to be a stochastic cooperative decision situation[2]. From

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[2] We remark that the value of coalition S in (N, v^Λ) is the maximum expected profit that coalition S can obtain by joint borrowing.

similar arguments and assumptions (i.e., $E[X_i] < \infty$ for all $i \in N$) as in Example 1, conditions (i), (ii) and (iii) are met. It is easy to check that conditions (iv), and (vii) hold as well. Condition (vi) holds since $c_1 > r$. The only condition left for Λ to be a stochastic cooperative decision situation is condition (v). Consider coalition T and realization x^T of X^T. Let $\kappa^T : 2^T \setminus \{\emptyset\} \to [0,1]$ be a balanced map. Moreover, let $(q^S)_{S \subseteq T} \in \prod_{S \subseteq T} Q^S$ be a collection of loans and let $(m^S)_{S \subseteq T} \in \prod_{S \subseteq T} M^S(q^S, x)$ be a collection of allocations. We can rewrite condition (v) for this situation as

$$\sum_{S \in 2^T \setminus \{\emptyset\}} \kappa^T(S) m_i^S = 0 \text{ if } i \in N \setminus T; \tag{8}$$

$$\sum_{i \in T} \sum_{S \in 2^T \setminus \{\emptyset\} : i \in S} \kappa^T(S) m_i^S = \sum_{S \in 2^T \setminus \{\emptyset\}} \kappa^T(S) q^S. \tag{9}$$

It is easy to verify that (8) holds. (9) holds similar as (7) in Example 2. Therefore, we conclude that the game (N, v^Λ) associated with Λ is an expected profit game, which is totally balanced. $\Diamond$

5. Concluding comments   In this paper, we provided a framework for cooperative situations under uncertainty. We focused on two types of structural elements in the construction of the framework: strategy and recourse action spaces, and cost and revenue functions. We identified a set of sufficiency conditions on these elements for the cooperative games associated with the situations in this framework to have non-empty cores. We call the situations that satisfy these conditions and their associated cooperative games stochastic cooperative decision situations and expected profit games, respectively. Some of these conditions have already been used implicitly to show that the core is non-empty for special newsvendor situations in the literature (Müller et al. [13] and Slikker et al. [22]). Our proof technique differs from theirs in the following way. Our technique requires the revenue functions to be concave whereas their technique requires profit functions to be positively homogeneous on order quantity and demand realization (a stronger condition). Moreover, with our technique, it is also possible to handle convex cost structures.

After knowing that the cores of expected profit games are non-empty, the first question one would ask is whether there is a simple algorithm to determine a core division of total profit. Unfortunately, the way that we prove this result does not imply an algorithm for this purpose. There are two recent studies studying this issue in newsvendor situations. In a recent study, Montrucchio and Scarsini [12] showed that the core of a simple newsvendor game is non-empty by identifying a core element. Chen and Zhang [5] considered cost games associated with the newsvendor situation with multiple warehouses as introduced by Özen et al. [16]. They offered a way to find a core element by solving the dual of a stochastic linear program. Moreover, they proved that it is NP-hard to determine whether a given allocation of total profit is in the core for the associated games, even in a very simple setting. This result also holds for the expected profit games considered in this paper. A promising direction of research would be the development of heuristics methods to identify core elements.

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