A vector labeling method for solving discrete zero point and complementarity problems

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A Vector Labeling Method for Solving Discrete Zero Point and Complementarity Problems[1]

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Abstract: In this paper we establish the existence of a discrete zero point of a function from the n-dimensional integer lattice $\mathbb{Z}^n$ to the n-dimensional Euclidean space $\mathbb{R}^n$ under very general conditions with respect to the behaviour of the function. The proof is constructive and uses a combinatorial argument based on a simplicial algorithm with vector labeling and lexicographic linear programming pivot steps. The algorithm provides an efficient method to find an exact solution. We also discuss how to adapt the algorithm for two related problems, namely, to find a discrete zero point of a function under a general antipodal condition, and to find a solution to a discrete nonlinear complementarity problem. In both cases the modified algorithm provides a constructive existence proof, too. We further show that the algorithm for the discrete nonlinear complementarity problem generalizes the well-known Lemke's method to nonlinear environments. An economic application is also presented.

Keywords: Integer lattice, zero point, vector labeling rule, simplicial algorithm, discrete complementarity

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1 Introduction

We consider the problem of finding a point $x^* \in \mathbb{Z}^n$ such that

$$f(x^*) = 0^n$$

where 0^n is the n-vector of zeroes, $\mathbb{Z}^n$ is the integer lattice of the n-dimensional Euclidean space $\mathbb{R}^n$, and f is a function from $\mathbb{Z}^n$ to $\mathbb{R}^n$. Such an integral point x^* is called a *discrete zero point* of f. Recently, the existence problem of an integral solution has been investigated in several papers. These papers were all inspired by the discrete fixed point statement given in Iimura [11]. In Iimura, Murota and Tamura [12] and Danilov and Koshevoy [4], their existence theorems concern functions that exhibit the so-called *direction-preserving property* proposed by Iimura [11], which can be seen as the counterpart of the continuity property for functions defined on the Euclidean space $\mathbb{R}^n$. The existence results in Yang [37] and [38] hold for the class of so-called *locally gross direction-preserving* mappings, which is substantially more general and richer than the class of Iimura's direction-preserving mappings and which contains the results in [4] and [12] as special cases. Besides establishing these more general existence results, Yang also initiated in [37] the study of discrete nonlinear complementarity problems and provided several general theorems for the existence of solutions for this class of problems. All this literature, however, is not concerned with the problem of finding an integral solution. In fact, all these existence proofs are nonconstructive.

To provide constructive proofs based on a combinatorial argument we apply the technique of the so-called simplicial algorithms originally designed to find approximate zero or fixed points of continuous functions or upper semi-continuous mappings. The first of such algorithm was developed by Scarf [28] and subsequent algorithms proposed by Eaves [5], Eaves and Saigal [6], Merrill [23], van der Laan and Talman [17] among others, substantially improved Scarf's original algorithm in terms of efficiency and applicability. For comprehensive treatments on such algorithms we refer to Allgower and Georg [1], Todd [30] and Yang [36]. By van der Laan, Talman and Yang, the $2n$-ray integer labeling algorithm in [18] and [26], has been modified in [20] to find an integral zero point of a function satisfying the direction-preserving property, and in [21] to find a solution of a discrete nonlinear complementarity problem.

The aim of this paper is to provide a combinatorial algorithm for finding an integral zero point of a function satisfying the more general simplicially local gross direction-preserving property. This algorithm is also a modification of the $2n$-ray simplicial algorithm, introduced in [18] and [26]. However, in this case we cannot rely on integer labeling anymore, instead we have to apply the more subtle concept of vector labeling. The modified algorithm makes use of a triangulation of $\mathbb{R}^n$, being a family of integral simplices, constructed in such a way that the set of vertices of the simplices of the triangulation is equal to $\mathbb{Z}^n$

and the mesh size of each simplex in the triangulation is equal to one according to the maximum norm. Starting with some integral point in $\mathbb{Z}^n$, the algorithm leaves the starting point along one out of $2n$ directions and then generates a sequence of adjacent simplices of varying dimension by making lexicographic linear programming pivot steps in a system of linear equations. We show that under a mild convergence condition the algorithm ends in a finite number of steps with an exact integral zero point. It is worth mentioning that in case of a continuous function on $\mathbb{R}^n$, algorithms for finding for a zero (or fixed) point only find an approximate solution, whereas the current algorithm for the discrete case finds an exact solution.

We also discuss how to adapt the algorithm for two related problems, namely, to find a discrete zero point of a function under a general antipodal condition, and to find a solution to a discrete nonlinear complementarity problem. In the first case the antipodal condition guarantees convergency, in the second case we also propose a convergence condition. We show that the modified algorithm for the discrete nonlinear complementarity problem generalizes the well-known Lemke's method. In particular, when the function $f(x)$ is affine, i.e., $f(x) = Mx + q$, where M is an $n \times n$ matrix and q is an integral n-vector, it is shown that the algorithm finds an integral solution provided that M is totally unimodular and copositive-plus, and the system of $Mx + q \geq 0^n$, $x \geq 0^n$ is feasible.

The paper is organized as follows. In Section 2 we introduce the concepts of triangulation and simplicially local gross direction preservingness and describe the algorithm. In Section 3 we state a convergence condition guaranteeing the existence of an integral solution to the discrete zero point problem and provide a constructive proof. In Section 4 we modify the algorithm for the case that the function satisfies a general antipodal condition. In Section 5 we modify the algorithm for the discrete complementarity problem and show that this modified algorithm generalizes Lemke's method. An economic application is discussed in Section 6.

2   A method for solving discrete nonlinear equations

For a given positive integer n, let N denote the set $\{1, 2, \ldots, n\}$. For $i \in N$, $e(i)$ denotes the ith unit vector of $\mathbb{R}^n$. Given a set $D \subset \mathbb{R}^n$, $\mathrm{Co}(D)$ and $\mathrm{Bd}(D)$ denote the convex hull of D and the relative boundary of D, respectively. For any x and y in $\mathbb{R}^n$, we say y is lexicographically greater than x, and denote it by $y \succeq x$, if the first nonzero component of $y - x$ is positive.

Two integral points x and y in $\mathbb{Z}^n$ are said to be *cell-connected* if $\max_{h \in N} |x_h - y_h| \leq 1$, i.e., their distance is less than or equal to one according to the maximum norm. In other words, two integral points x and y are cell-connected if and only if there exists $q \in \mathbb{Z}^n$ such

that both x and y belong to the hyper cube $[0,1]^n + \{q\}$.

For an integer t, $0 \leq t \leq n$, the t-dimensional convex hull of $t+1$ affinely independent points $x^1, \ldots, x^{t+1}$ in $\mathbb{R}^n$ is called a *t-simplex* or simply a *simplex* and will be denoted by $< x^1, \ldots, x^{t+1} >$. The extreme points $x^1, \ldots, x^{t+1}$ of a t-simplex $\sigma = < x^1, \ldots, x^{t+1} >$ are called the *vertices* of σ. The convex hull of any subset of $k+1$ vertices of a t-simplex σ, $0 \leq k \leq t$, is called a *face* or *k-face* of σ. A k-face of a t-simplex σ is called a *facet* of σ if $k = t-1$, i.e., if the number of vertices is just one less than the number of vertices of the simplex. A simplex is said to be *integral* if all of its vertices are integral vectors and are cell-connected. Any two vertices x and y of an integral simplex are said to be *simplicially connected*.

Given an m-dimensional convex set D, a collection $\mathcal{T}$ of m-dimensional simplices is a *triangulation* or *simplicial subdivision* of the set D, if (i) D is the union of all simplices in $\mathcal{T}$, (ii) the intersection of any two simplices of $\mathcal{T}$ is either empty or a common face of both, and (iii) any neighborhood of any point in D only meets a finite number of simplices of $\mathcal{T}$. A facet of a simplex of $\mathcal{T}$ either lies on the boundary of D and is facet of no other simplex of $\mathcal{T}$ or it is a facet of precisely one other simplex of $\mathcal{T}$. A triangulation is called *integral* if all its simplices are integral simplices. One of the most well-known integral triangulations of $\mathbb{R}^n$ is the K-triangulation owing to Freudenthal [8]. This triangulation is the collection of all integral simplices $\sigma(y, \pi)$ with vertices $y^1, \ldots, y^{n+1}$, where for $y \in \mathbb{Z}^n$ and $\pi = (\pi(1), \ldots, \pi(n))$ a permutation of the elements $1, 2, \ldots, n$, the vertices are given by $y^1 = y$ and $y^{i+1} = y^i + e(\pi(i))$, $i = 1, \ldots, n$. Furthermore, a triangulation $\mathcal{T}$ is *symmetric* if $\sigma \in \mathcal{T}$ implies $-\sigma \in \mathcal{T}$. An example of symmetric integral triangulations of $\mathbb{R}^n$ is the K'-triangulation of Todd [31].

Now we introduce the class of simplicially local gross direction preserving functions on $\mathbb{Z}^n$ on which the existence theorems of this paper are based. Locally gross direction preservingness replaces the continuity condition for the existence of a zero point of a function defined on $\mathbb{R}^n$. Let $a \cdot b$ denote the inner product of two n-vectors a and b.

Definition 2.1
(i) *A function $f : \mathbb{Z}^n \to \mathbb{R}^n$ is* locally gross direction preserving *if, for any cell-connected points x and y in $\mathbb{Z}^n$,*

$$f(x) \cdot f(y) \geq 0.$$

(ii) *A function $f : \mathbb{Z}^n \to \mathbb{R}^n$ is* simplicially local gross direction preserving *with respect to some given integral triangulation $\mathcal{T}$ of $\mathbb{R}^n$, if, for any vertices x and y of a simplex of $\mathcal{T}$,*

$$f(x) \cdot f(y) \geq 0.$$

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The locally gross preserving property was originally introduced in Yang [38] and prevents the function from changing too drastically in direction within one cell. The simplicially local gross preserving condition is weaker and only requires that the function does not change too drastically in direction within any integral simplex of the given integral triangulation. Since any two vertices of a simplex of an integral triangulation are cell-connected we have the property that every locally gross direction preserving function is also simplicially local gross direction preserving with respect to any integral triangulation.

To compute a discrete zero point of a simplicially local gross direction preserving function, we adapt the $2n$-ray vector labeling algorithm of van der Laan and Talman [18] (see also Reiser [26] for integer labeling), to the current discrete setting. Let f be a simplicially local gross direction preserving function with respect to some given integral triangulation $\mathcal{T}$ of $\mathbb{R}^n$. Let v be an arbitrarily chosen integral vector in $\mathbb{Z}^n$. The point v will be the starting point of the algorithm. For a nonzero sign vector $s \in \{-1, 0, +1\}^n$, the subset $A(s)$ of $\mathbb{R}^n$ is defined by

$$A(s) = \{x \in \mathbb{R}^n \mid x = v + \sum_{h \in N} \alpha_h s_h e(h), \ \alpha_h \geq 0, \ h \in N\}.$$

Clearly, the set $A(s)$ is a t-dimensional subset of $\mathbb{R}^n$, where t is the number of nonzero components of the sign vector s, i.e., $t = |\{i \mid s_i \neq 0\}|$. Since $\mathcal{T}$ is an integral triangulation of $\mathbb{R}^n$, it triangulates every set $A(s)$ into t-dimensional integral simplices. For some s with t nonzero components, denote $\{h_1, \ldots, h_{n-t}\} = \{h \mid s_h = 0\}$ and let $\sigma = <x^1, \ldots, x^{t+1}>$ be a t-simplex of the triangulation in $A(s)$. Following Todd [32], who improved the original system of equations used by van der Laan and Talman [18], we say that σ is *almost s-complete* if there is an $(n+2) \times (n+1)$ matrix W satisfying

$$\begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ f(x^1) & \cdots & f(x^{t+1}) & e(h_1) & \cdots & e(h_{n-t}) & -s \end{bmatrix} W = I \tag{2.1}$$

and having rows $w^1, \ldots, w^{n+2}$ such that $w^h \succeq 0$ for $1 \leq h \leq t+1$, $w^{n+2} \succeq w^i$ and $w^{n+2} \succeq -w^i$ for $t+1 < i \leq n+1$, and $w^{n+2} \succeq 0$. Here I denotes the identity matrix of rank $n+1$. If $w_1^{n+2} = 0$, then we say that the simplex σ is complete. Further, let τ be a facet of σ, and, without loss of generality, index the vertices of σ such that $\tau = <x^1, \ldots, x^t>$. We say that τ is *s-complete* if there is an $(n+1) \times (n+1)$ matrix W satisfying

$$\begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ f(x^1) & \cdots & f(x^t) & e(h_1) & \cdots & e(h_{n-t}) & -s \end{bmatrix} W = I \tag{2.2}$$

and having rows $w^1, \ldots, w^{n+1}$ such that $w^h \succeq 0$ for $1 \leq h \leq t$, $w^{n+1} \succeq w^i$ and $w^{n+1} \succeq -w^i$ for $t+1 \leq i \leq n$, and $w^{n+1} \succeq 0$. If $w_1^{n+1} = 0$, then we say that τ is complete.

The lemma below says that the 0-dimensional simplex $<v>$ is an s^0-complete facet for a uniquely determined sign vector s^0. Let $\alpha = \max_h |f_h(v)|$. If $f_h(v) = -\alpha$ for some h,

then we take $s_k^0 = -1$ where k is the smallest index h such that $f_h(v) = -\alpha$, and $s_j^0 = 0$ for $j \neq k$. If $f_h(v) > -\alpha$ for all h, then we take $s_k^0 = 1$ where k is the largest index h such that $f_h(v) = \alpha$, and $s_j^0 = 0$ for $j \neq k$. Let σ^0 be the unique 1-dimensional simplex in $A(s^0)$ containing $< v >$ as a facet. Clearly, s^0 contains only one nonzero element.

Lemma 2.2 *The simplex $< v >$ is an s^0-complete facet of σ^0. Moreover, s^0 is uniquely determined.*

Proof: Consider the system

$$
\begin{bmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
f(v) & e(1) & \cdots & e(k-1) & e(k+1) & \cdots & e(n) & -s_k^0 e(k)
\end{bmatrix} V = I.
$$

Clearly, the first matrix on the left-hand side is regular and therefore its inverse exists and equals the matrix V. The rows of V are given by

$$v^1 = (1, 0, \ldots, 0),$$

$$v^h = (-f_{h-1}(v), 0, \ldots, 0, 1, 0, \ldots, 0), \quad h = 2, \ldots, k, \quad \text{if } k > 1,$$

with 1 being the hth component,

$$v^h = (-f_{h-1}(v), 0, \ldots, 0, 1, 0, \ldots, 0), \quad h = k+1, \ldots, n, \quad \text{if } k < n,$$

with 1 being the $(h+1)$th component and

$$v^{n+1} = (s_k^0 f_k(v), 0, \ldots, 0, -s_k^0, 0, \ldots, 0)$$

with $-s_k^0$ being the $(k+1)$th component. Clearly, v^1 is lexicographically positive. Moreover, v^{n+1} is lexicographically positive, because we have either $s_k^0 f_k(v) > 0$ or $s_k^0 f_k(v) = 0$ and $-s_k^0 > 0$. For $j = 2, \ldots, k$, we have $v^{n+1} \succeq v^j$, because $s_k^0 f_k(v) > 0$ and $s_k^0 f_k(v) > -f_{j-1}(v)$, and we also have $v^{n+1} \succeq -v^j$, because $s_k^0 f_k(v) > 0$ and $s_k^0 f_k(v) > f_{j-1}(v)$. For $j = k+1, \ldots, n$, and $s_k^0 = -1$, we have $v^{n+1} \succeq v^j$, because either $s_k^0 f_k(v) > 0$ and $s_k^0 f_k(v) > -f_j(v)$ or $s_k^0 f_k(v) = -f_j(v)$ and the $(j+1)$th component of v^j is 0 but the same component of v^{n+1} is 1, and we also have $v^{n+1} \succeq -v^j$, because either $s_k^0 f_k(v) > 0$ and $s_k^0 f_k(v) > f_j(v)$ or $s_k^0 f_k(v) = f_j(v)$ and the $(j+1)$th component of v^j is 0 but the same component of v^{n+1} is 1. For $j = k+1, \ldots, n$, and $s_k^0 = 1$, we have $v^{n+1} \succeq v^j$, because $s_k^0 f_k(v) > 0$ and $s_k^0 f_k(v) > -f_j(v)$, and we also have $v^{n+1} \succeq -v^j$, because either $s_k^0 f_k(v) > 0$ and $s_k^0 f_k(v) > f_j(v)$ or $s_k^0 f_k(v) = f_j(v)$ and the $(j+1)$th component of $-v^j$ is -1 but the same component of v^{n+1} is 0. Hence, V satisfies all the requirements of the matrix W in system (2.2) and thus $< v >$ is an s^0-complete facet of σ^0. Clearly, there is no other sign-vector s for which $< v >$ is s-complete. $\qquad\square$

We are now able to describe the algorithm for finding an integral solution to the system of equations $f(x) = 0^n$. When for some nonzero sign vector s a t-simplex $\sigma =< x^1, \ldots, x^{t+1} >$ in $A(s)$ is almost s-complete, the system (2.1) has two "basic solutions". At each of these solutions exactly one row of the solution matrix W is binding. If $w_1^{n+2} = 0$, then σ is complete. If $w^h \succeq 0$ is binding for some h, $1 \leq h \leq t+1$, then the facet τ of σ opposite the vertex x^h is s-complete, and so τ is either (i) the 0-dimensional simplex $< v >$ or (ii) a facet of precisely one other almost s-complete t-simplex σ' of the triangulation in $A(s)$ or (iii) τ lies on the boundary of $A(s)$ and is an almost s'-complete $(t-1)$-simplex in $A(s')$ for some unique nonzero sign vector s' with $t-1$ nonzero elements differing from s in only one element. If $w^{n+2} \succeq w^i$ ($w^{n+2} \succeq -w^i$) is binding for some $t+1 < i \leq n+1$, σ is an s'-complete facet of precisely one almost s'-complete $(t+1)$-simplex in $A(s')$ for some nonzero sign vector s' differing from s in only the ith element, namely $s_i' = +1$ (-1).

Since $< v >$ is s^0-complete, σ^0 is an almost s^0-complete 1-dimensional simplex in $A(s^0)$. Starting with σ^0, the $2n$-ray algorithm generates a sequence of adjacent almost s-complete simplices in $A(s)$ with s-complete common facets for varying sign vectors s. Moving from one s-complete facet to the next s'-complete facet corresponds to making a lexicographic linear programming pivot step from one of the two basic solutions of system (2.1) to the other. The algorithm stops as soon as it finds a complete simplex. We will show that in that case one of its vertices is a discrete zero point of the function f.

Lemma 2.3 *Let f be simplicially local gross direction preserving with respect to $\mathcal{T}$. Then any complete simplex contains a discrete zero point of the function f.*

Proof: Let x^1, ..., x^{k+1} be the vertices of a complete simplex σ in $A(s)$ and let t be the number of nonzeros in s. Notice that $k = t-1$ or $k = t$ depending on whether σ is a t-simplex in $A(s)$ or a facet of a t-simplex in $A(s)$. From the system (2.1) or (2.2) it follows that there exists $\lambda_1 \geq 0$, ..., $\lambda_{k+1} \geq 0$ with sum equal to one such that $\sum_{j=1}^{k+1} \lambda_j f(x^j) = 0^n$. Let j^* be such that $\lambda_{j^*} > 0$. Then by premultiplying $f(x^{j^*})$ on both sides of $\sum_{j=1}^{k+1} \lambda_j f(x^j) = 0^n$, we obtain

$$\lambda_1 f(x^1) \cdot f(x^{j^*}) + \ldots + \lambda_{j^*} f(x^{j^*}) \cdot f(x^{j^*}) + \ldots + \lambda_{k+1} f(x^{k+1}) \cdot f(x^{j^*}) = 0.$$

Since f is simplicially local gross direction preserving, it is easy to see that every term in the above expression is nonnegative. Therefore every term is equal to zero. In particular, $f(x^{j^*}) = 0^n$, and so x^{j^*} is a discrete zero point of the function f. $\qquad\square$

Formally, the steps of the above algorithm are given below in detail.

Initial Step: Compute $f(v)$. If $f(v) = 0^n$, then the algorithm terminates with v as a solution. Otherwise $< v >$ is an s^0-complete facet of a unique 1-simplex

$\sigma^0 = \langle v, v^+ \rangle$ in $A(s^0)$. Let $s = s^0$, $t = |\{i \mid s_i \neq 0\}|$ and $\sigma = \sigma^0$. Go to Main Step 1 with the system (2.1) corresponding to σ^0..

Main Step 1: Perform a lexicographic linear programming (l.l.p.) pivot step in the system (2.1) with the column $(1, f(v^+))$. If $w_1^{n+2} = 0$, the algorithm terminates with a complete simplex which yields a solution. Otherwise, in case $w^h \succeq 0$ is binding for some h, $1 \leq h \leq t+1$, then the facet τ of σ opposite the vertex x^h is s-complete and go to Main Step 2. In case $w^{n+2} \succeq w^i$ ($w^{n+2} \succeq -w^i$) is binding for some $t+1 < i \leq n+1$, go to Main Step 3.

Main Step 2: If τ is a facet of precisely one other almost s-complete t-simplex σ' of the triangulation in $A(s)$, let v^+ be the vertex of σ' differing from those of τ and let $\sigma = \sigma'$ and go to Main Step 1. Otherwise, τ lies on the boundary of $A(s)$ and is an almost s'-complete $(t-1)$-simplex in $A(s')$ for some unique nonzero sign vector s' with $t-1$ nonzero elements differing from s in only one element. Let h be the unique element with $s_h \neq 0$ and $s'_h = 0$, $\sigma = \tau$ and $s = s'$, and go to Main Step 4.

Main Step 3: σ is an s'-complete facet of precisely one almost s'-complete $(t+1)$-simplex σ' in $A(s')$ for some nonzero sign vector s' differing from s in only the ith element, namely $s'_i = +1$ (-1). Let v^+ be the vertex of σ' differing from those of σ, $s = s'$ and $\sigma = \sigma'$, and go to Main Step 1.

Main Step 4: Perform an l.l.p. pivot step in the system (2.1) with the column $(0, e(h))$. If $w_1^{n+2} = 0$, the algorithm terminates with a complete simplex which yields a solution. Otherwise, in case $w^h \succeq 0$ is binding for some h, $1 \leq h \leq t+1$, then the facet τ of σ opposite the vertex x^h is s-complete and go to Main Step 2. In case $w^{n+2} \succeq w^i$ ($w^{n+2} \succeq -w^i$) is binding for some $t+1 < i \leq n+1$, go to Main Step 3.

Because all steps are uniquely determined due to the lexicographically pivoting and the properties of a triangulation, the algorithm cannot visit any simplex more than once and therefore the algorithm either terminates in a finite number of iterations with a complete simplex yielding a solution, or the sequence of simplices generated by the algorithm goes to infinity. In the next section we present a convergence condition which prevents the latter case from happening and thus ensures the existence of a solution.

3 Convergence conditions

To present a convergence condition for the algorithm, for $x \in \mathbf{Z}^n$, let $N(x)$ denote the set of integer points being simplicially connected to x.

Assumption 3.1 Convergence Condition

Given a function $f : \mathbb{Z}^n \to \mathbb{R}^n$, there exist vectors m, $M \in \mathbb{Z}^n$, with $m_h < M_h - 1$ for every $h \in N$, such that for every integral vector x on the boundary of the set $C^n = \{z \in \mathbb{R}^n \mid m \le z \le M\}$ the following conditions hold:

(i) If $x_i = m_i$ then $f_i(y) \ge 0$ for all $y \in N(x) \cap C^n$ satisfying $y_i = m_i$ or there exists $j \in N$ such that $f_j(y) < f_i(y)$ for all $y \in N(x) \cap C^n$ satisfying $y_i = m_i$.

(ii) If $x_i = M_i$ then $f_i(y) \le 0$ for all $y \in N(x) \cap C^n$ satisfying $y_i = M_i$ or there exists $j \in N$ such that $f_j(y) > f_i(y)$ for all $y \in N(x) \cap C^n$ satisfying $y_i = M_i$.

The condition means that there exist lower and upper bounds, such that when x is an integral vector on the ith lower (upper) bound, then either $f_i(y)$ is nonnegative (nonpositive) for any integral vector y on the same lower (upper) bound being simplicially connected to x or for some $j \ne i$ $f_i(y)$ is bigger (smaller) than $f_j(y)$ for any integral vector y on the same lower (upper) bound being simplicially connected to x. We show that under this condition any simplicially local gross direction preserving function has a discrete zero point within the bounded set C^n induced by the lower and upper bounds. To do so, the starting point v of the $2n$-ray algorithm is taken to be an arbitrarily chosen integral vector in the interior of the set C^n. Then the constructive proof of Theorem 3.2 is based on the combinatorial argument that under the convergence condition the algorithm can not cross the boundary of the set C^n and therefore it must terminate in a finite number of steps with a simplex having one of its vertices as integral solution to f. It is worth pointing out that while both the lower and upper bounds are part of the condition in the theorem, in our constructive proof we only need the starting point to lie between these bounds without need to know exactly what they are. Typically, in applications these bounds are naturally determined and indicate the domain of interest underlying the problem, see, for instance, the application in Section 6.

Theorem 3.2 *Let $f : \mathbb{Z}^n \to \mathbb{R}^n$ be a simplicially local gross direction preserving function with respect to some integral triangulation $\mathcal{T}$. If f satisfies Assumption 3.1, then f has a discrete zero point.*

Proof: Take any integral vector in the interior of the set C^n as the starting point v of the algorithm. By definition of integral triangulation, $\mathcal{T}$ triangulates the set C^n and also the set $A(s) \cap C^n$ for any sign vector s into integral simplices.

For some nonzero sign vector s, let τ be an s-complete facet in $A(s)$ with vertices $x^1, \cdots, x^t$, where t is the number of nonzeros in s. We first show that τ is complete if it is on the boundary of C^n. From system (2.2) it follows that there exist $\lambda_1 \ge 0, \cdots, \lambda_t \ge 0$ with sum equal to one, $\beta \ge 0$, and $-\beta \le \mu_i \le \beta$ for $s_i = 0$, such that $\bar{f}_i(z) = \beta$ if $s_i = 1$, $\bar{f}_i(z) = -\beta$ if $s_i = -1$, and $\bar{f}_i(z) = \mu_i$ if $s_i = 0$, where $z = \sum_{i=1}^t \lambda_i x^i$ and

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$\bar{f}(z) = \sum_{i=1}^{t} \lambda_i f(x^i)$, i.e., $\bar{f}$ is the piecewise linear extension of f with respect to $\mathcal{T}$. Since τ lies on the boundary of C^n, there exists an index h such that either $x_h^j = m_h$ for all j or $x_h^j = M_h$ for all j. In case $x_h^j = m_h$ for all j, we have $s_h = -1$ and therefore $\bar{f}_h(z) = -\beta$. Furthermore, by Assumption 3.1, we have (i) $f_h(x^j) \geq 0$ for all j or (ii) there exists k such that $f_k(x^j) < f_h(x^j)$ for all j. In case (ii) we obtain $\bar{f}_k(z) < \bar{f}_h(z)$. On the other hand, $\bar{f}_k(z) \geq -\beta = \bar{f}_h(z)$, yielding a contradiction, i.e. this case cannot occur. In case (i) we obtain $\bar{f}_h(z) \geq 0$. On the other hand $\bar{f}_h(z) = -\beta \leq 0$. Therefore $\bar{f}_h(z) = 0$ and also $\beta = 0$. Since $w_1^{n+1} = \beta$ we obtain that τ is complete. Similarly, we can show that the same results hold for the case of $x_h^j = M_h$ for all j.

Now, consider the algorithm as described at the end of the previous section. Due to the lexicographic pivoting rule, the algorithm will never visit any simplex more than once. So, because the number of simplices in C^n is finite, the algorithm finds in a finite number of steps a complete simplex. Since f is simplicially local gross direction preserving, Lemma 2.3 guarantees that at least one of the vertices of this simplex is a discrete zero point of the function f. $\qquad\square$

We conclude this section with an example to illustrate the conditions of the theorem and how the algorithm operates. Consider the function $f : \mathbb{Z}^2 \to \mathbb{R}^2$ defined by $f(x) = (2 - 2x_1, x_1 - x_2^2)$. This function is simplicially locally gross direction preserving with respect to the K-triangulation described in the previous section. It is interesting to note that f is not locally gross direction preserving, since, for example, for the cell-connected points $x = (1, 2)$ and $y = (2, 1)$, we have $f(x) = (0, -3)$ and $f(y) = (-2, 1)$ and so $f(x) \cdot f(y) = -3 < 0$. Further, the example satisfies Assumption 3.1 for any vector $m = (a, a)$ and $M = (b, b)$ with $a < 0$ and $b > 1$, implying that the convergence condition of Theorem 3.2 is satisfied. Hence there exists a solution and in fact $x^* = (1, 1)$ is a discrete zero point. Let the starting point v be $(5, 4)$. Then the sequence of points traced by the algorithm is shown in Figure 1 and given by $x^1 = (5, 3)$, $x^2 = (4, 3)$, $x^3 = (4, 4)$, $x^4 = (3, 3)$, $x^5 = (3, 2)$, $x^6 = (2, 2)$, $x^7 = (2, 3)$, $x^8 = (1, 2)$ and leads to the solution $x^* = (1, 1)$ in 10 function evaluations. Observe that to apply the algorithm, we do not need to fix the bounds a priori.

The following corollary strengthens a result of Yang (2004b) for locally gross direction preserving functions and follows immediately from Theorem 3.2.

Corollary 3.3   *Let $f : \mathbb{Z}^n \to \mathbb{R}^n$ be a simplicially local gross direction preserving function. Suppose that there exist vectors $m, M \in \mathbb{Z}^n$, with $m_h < M_h - 1$ for every $h \in N$, such that for every integral vector x on the boundary of the set $C^n = \{z \in \mathbb{R}^n \mid m \leq z \leq M\}$, $x_i = m_i$ implies $f_i(x) \geq 0$ and $x_i = M_i$ implies $f_i(x) \leq 0$. Then f has a discrete zero point.*

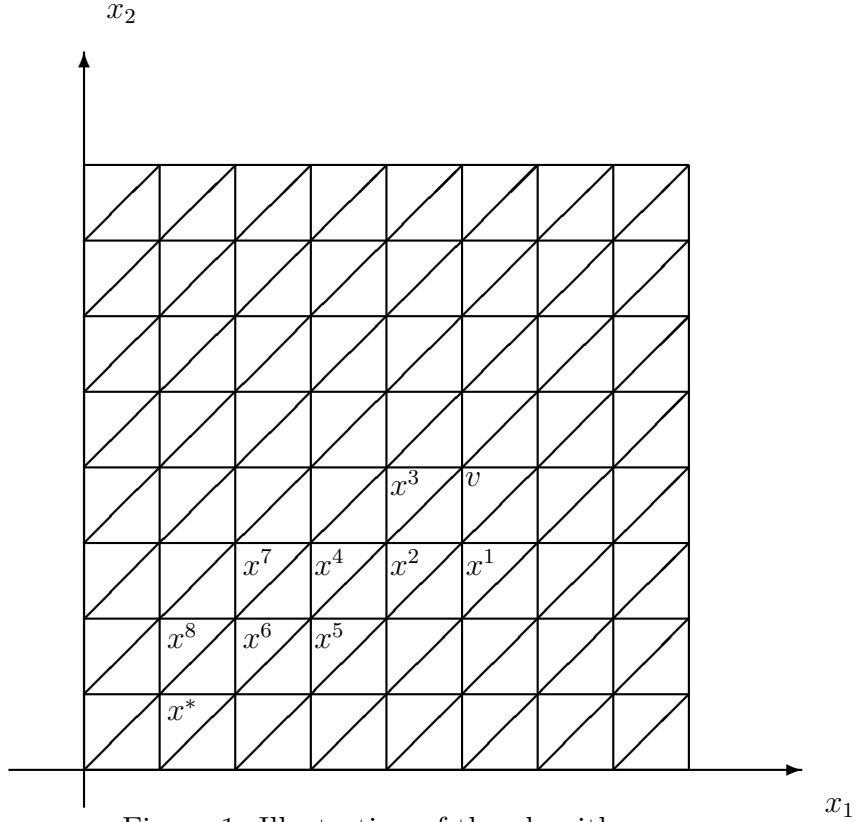Furthermore, we have the following discrete fixed point theorem.

Figure 1: Illustration of the algorithm.

Corollary 3.4   *Let $D^n = \{z \in \mathbf{Z}^n \mid m \leq z \leq M\}$, where m and M are vectors in $\mathbf{Z}^n$ with $m_h < M_h - 1$ for every $h \in N$. Assume that $f : D^n \to \mathrm{Co}(D^n)$ is a function such that $x - f(x)$ is a simplicially local gross direction preserving function in x. Then f has a discrete fixed point.*

Proof: Define the function $g : D^n \to \mathbb{R}^n$ by $g(x) = x - f(x)$. Clearly, g satisfies the condition of Corollary 3.3. So there exists $x^* \in D^n$ such that $g(x^*) = 0$, i.e., $f(x^*) = x^*$.

$\square$

4   Convergency under an antipodal condition

In this section we modify the algorithm in Section 2 to find a discrete zero point under a general antipodal condition to be stated next.

Assumption 4.1 Antipodal Condition

Given a function $f : \mathbf{Z}^n \to \mathbb{R}^n$, there exists a vector $u \in \mathbf{Z}^n$ with $u_h \geq 1$ for all $h \in N$, such that $f(x) \cdot f(-y) \leq 0$ for any cell-connected integral points x and y lying on a same proper face of the set $U^n = \{z \in \mathbb{R}^n \mid -u \leq z \leq u\}$.

This condition is very natural and might be viewed as a discrete analogue of a weak version of the Borsuk-Ulam antipodal condition for a continuous function saying that

$f(x) \cdot f(-x) \leq 0$ when x is on the boundary of U^n. It is known that under the latter condition a continuous function has a zero point; see for instance van der Laan [16] and Yang [36]. Todd and Wright [33] used a modification of the $2n$-ray algorithm to give a constructive proof of the Borsuk-Ulam theorem and Freund and Todd [9] used the modified algorithm to give a constructive proof for a combinatorial lemma of Tucker [34]. Yang [38] proposed the antipodal condition and showed that under the condition a locally gross direction preserving function has a discrete zero point. The next theorem strengthens this result by allowing for simplicially local gross direction preservingness on a symmetric triangulation in the interior of U^n.

Theorem 4.2 *Let $f : \mathbf{Z}^n \to \mathbf{R}^n$ be a simplicially local gross direction preserving function with respect to a symmetric integral triangulation $\mathcal{T}$ of $\mathbf{R}^n$, satisfying Assumption 4.1 and $f(x) \cdot f(y) \geq 0$ for any cell-connected integral points x and y lying on a same proper face of the set U^n. Then f has a discrete zero point.*

The next example illustrates the theorem. Let $f : \mathbf{Z}^2 \to \mathbf{R}^2$ be given by $f(x) = (x_1 - x_2 - 1, \ x_2 - 1)$. Then it is easy to see that f satisfies the antipodal property for $u_1 = u_2 = 4$. Further, it is easy to check that f is simplicially local gross direction preserving with respect to the symmetric integral K'-triangulation of $\mathbf{R}^2$ (see Section 2) on the interior of U^n and f is locally gross direction preserving on the boundary of U^n, as required in the last condition of the theorem. So, f has a discrete zero point and in fact the point $(2, 1)$ is the unique discrete zero point. Observe that f is not locally gross direction preserving in the interior of U^n and therefore the existence does not follow from the result of Yang [38]. For instance, for the cell-connected points $x = (2, 0)$ and $y = (1, 1)$, we have $f(x) \cdot f(y) = -1 < 0$.

Besides the relaxation to simplicially local gross direction preserving, the main contribution of this section is that, in contrast to the nonconstructive proof in [38], below we give a constructive proof for the theorem. We now modify the $2n$-ray algorithm of Section 2 to accomodate the antipodal condition. The modification is based on a lemma on the extension V^n of the set U^n given by

$$V^n = \{x \in \mathbf{R}^n \mid -(u_i + 1) \leq x_i \leq u_i + 1, \ \ \forall i \in N\}.$$

Let the projection function $p : V^n \to U^n$ be defined by

$$p_h(x) = \max\{-u_h, \min\{u_h, x_h\}\}, \ \text{for all } h \in N.$$

Clearly, $p(x) = x$ if $x \in U^n$. Moreover, $p(x) \in U^n \cap \mathbf{Z}^n$ if $x \in V^n \cap \mathbf{Z}^n$. We now extend f to the function $g : V^n \cap \mathbf{Z}^n \to \mathbf{R}^n$ by setting $g(x) = f(x)$ for $x \in U^n$ and $g(x) = f(p(x)) - f(-p(x))$ for $x \in V^n \backslash U^n$. It follows straightforwardly that $g(x) = -g(-x)$ for any $x \in \mathbf{Z}^n \cap \mathrm{Bd}(V^n)$. We now have the following lemma.

Lemma 4.3   *For $f : \mathbf{Z}^n \to \mathbb{R}^n$ as given in Theorem 4.2, the extension g of f to V^n is simplicially local gross direction preserving on $\mathbf{Z}^n \cap V^n$ with respect to the given symmetric triangulation $\mathcal{T}$.*

Proof: Clearly, g is simplicially local gross direction preserving on U^n. It remains to consider the following two cases.

First, let $x, y \in \mathbf{Z}^n$ be two vertices of a simplex of $\mathcal{T}$ on the boundary of V^n. Then $p(x)$ and $p(y)$ are two cell-connected points on a same proper face of U^n and thus satisfy $f(p(x)) \cdot f(p(y)) \geq 0$. The same holds for $-p(x)$ and $-p(y)$. Together with the antipodal Assumption 4.1 this yields

$$
\begin{aligned}
g(x) \cdot g(y) &= (f(p(x)) - f(-p(x))) \cdot (f(p(y)) - f(-p(y))) \\
&= f(p(x)) \cdot f(p(y)) - f(p(x)) \cdot f(-p(y)) - f(-p(x)) \cdot f(p(y)) \\
&+ f(-p(x)) \cdot f(-p(y)) \geq 0.
\end{aligned}
$$

Second, let $x, y \in \mathbf{Z}^n$ be two vertices of a simplex of $\mathcal{T}$ with x on the boundary of U^n and y on the boundary of V^n. Again x and $p(y)$ are two cell-connected points on a same proper face of U^n and thus $f(x) \cdot f(p(y)) \geq 0$. Together with the antipodal condition this again yields

$$
\begin{aligned}
g(x) \cdot g(y) &= f(x) \cdot (f(p(y)) - f(-p(y))) \\
&= f(x) \cdot f(p(y)) - f(x) \cdot f(-p(y)) \geq 0.
\end{aligned}
$$

$\square$

Proof of Theorem 4.2. To prove the theorem, let the set V^n and the function g be defined as above. Take the origin 0^n of $\mathbb{R}^n$ as the starting point v of the algorithm as described in Section 2. The underlying symmetric integral triangulation $\mathcal{T}$ for the function f subdivides each set $A(s)$ into t-simplices such that if σ is a simplex in $A(s)$, then $-\sigma$ is a simplex in $A(-s)$.

Starting with the origin, the algorithm generates a sequence of adjacent almost s-complete simplices with s-complete common facets in $A(s) \cap V^n$ for varying sign vectors s with the following modification. If in the Main Step 2 of the algorithm τ is an s-complete facet lying in $A(s)$ on the boundary of V^n, then the antipodal facet $-\tau$ is a $(-s)$-complete facet in $A(-s)$ on the boundary of V^n, since $g(x) = -g(-x)$ for any $x \in \mathbf{Z}^n \cap \mathrm{Bd}(V^n)$. The algorithm continues with Main Step 1 by letting $s = -s$, σ the unique almost $-s$-complete simplex in $A(-s) \cap V^n$ containing $-\tau$ as facet and v^+ the vertex of σ opposite to facet $-\tau$. The algorithm therefore always stays in V^n and due to the lexicographic pivoting rule will never visit any simplex in V^n more than once. Since the number of simplices in V^n is finite, within a finite number of steps the algorithm terminates with a complete simplex σ^* in V^n. Since g is simplicially local gross direction preserving, by the Lemmas 2.3 and 4.3

it follows that σ^* has a vertex z being a discrete zero point of g. It remains to prove that $p(z) \in U^n$ is a discrete zero point of f. If z is not on the boundary of V^n, then $p(z) = z$ is an integral vector in U^n and $g(z) = f(z)$, and therefore z is a discrete zero point of f. Suppose z is on the boundary of V^n. Since $g(z) = 0^n$, this implies

$$
\begin{aligned}
0 &= f(p(z)) \cdot g(z) = f(p(z)) \cdot (f(p(z)) - f(-p(z))) \\
&= f(p(z)) \cdot f(p(z)) - f(p(z)) \cdot f(-p(z)),
\end{aligned}
$$

and therefore

$$
0 \leq f(p(z)) \cdot f(p(z)) = f(p(z)) \cdot f(-p(z)) \leq 0,
$$

where the last inequality follows from the antipodal condition on f. Hence, $f(p(z)) \cdot f(p(z)) = 0$ and therefore $p(z)$ is a discrete zero point of f in U^n. $\qquad\square$

5   A method for discrete complementarity problems

The complementarity problem is to find a point $x^* \in \mathbb{R}^n$ such that

$$
x^* \geq 0^n, \ f(x^*) \geq 0^n, \ \text{ and } \ x^* \cdot f(x^*) = 0,
$$

where f is a given function from $\mathbb{R}^n$ into itself. For an arbitrary function f, the problem is called the *nonlinear complementarity problem*. In case f is affine, i.e., $f(x) = Mx + q$ with M an $n \times n$ matrix and q an n-vector, the problem is called the *linear complementarity problem*, denoted by $\mathrm{LCP}(M, q)$. There is by now a voluminous literature on the complementarity problem, see Lemke [22], Cottle [2], Karamardian [13], Moré [24], [25], Kojima [14], van der Laan and Talman [19] among others. For comprehensive surveys on the subject, see for example Kojima *et al.* [15], Cottle *et al.* [3], Facchinei and Pang [7].

In the following we consider the problem that the solution of the complementarity problem is required to be integral or that the function f is defined only on the integer lattice $\mathbb{Z}^n$ of $\mathbb{R}^n$. In this case we call the problem the *discrete complementarity problem*, denoted by $\mathrm{DCP}(f)$. We first give sufficient conditions under which the general case $\mathrm{DCP}(f)$ has a solution and we will give a constructive proof of this existence result by modifying the system of equations of the algorithm in Section 2 to the current situation. Next we will show that when applied to the linear complementarity problem $\mathrm{LCP}(M, q)$, the algorithm reduces to the well-known Lemke's method [22] and finds an integral solution provided that M is totally unimodular and copositive-plus, and the system of $Mx + q \geq 0^n$, $x \geq 0^n$ is feasible.

In the following, for any $x, y \in \mathbb{R}^n$, let $I(x) = \{i \mid x_i > 0\}$ and let $I(x,y) = I(x) \cup I(y)$. We first modify the definition of simplicially local gross direction preservingness for points on the boundary of the nonnegative orthant $\mathbb{R}^n_+$.

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Definition 5.1 *A function $f : \mathbb{Z}^n \to \mathbb{R}^n$ is* simplicially local gross direction preserving *with respect to some given integral triangulation $\mathcal{T}$ of $\mathbb{R}^n$, if for any two vertices x and y of a simplex of $\mathcal{T}$ in $\mathbb{R}^n_+$ it holds that*

$$f_i(x)f_i(y) \geq 0 \ \ whenever \ x_i = y_i = 0, \ \ and \ \sum_{h \in I(x,y)} f_h(x)f_h(y) \geq 0.$$

The next theorem establishes the existence of a solution to DCP(f) under a natural condition.

Theorem 5.2 *Let $f : \mathbb{Z}^n \to \mathbb{R}^n$ be a simplicially local gross direction preserving function on $\mathbb{Z}^n_+$. If there exists a vector $M \in \mathbb{Z}^n_{++}$ such that for any $x \in \mathbb{Z}^n_+$ with $x \leq M$, $x_i = M_i$ implies $f_i(x) \geq 0$, then DCP(f) has a solution.*

We will provide a constructive proof by applying the algorithm in Section 2 to the current situation. To do so, the origin 0^n is taken as the starting point v. Since 0^n lies on the boundary of $\mathbb{R}^n_+$, the sets $A(s)$ and s-completeness are only defined for nonnegative nonzero sign vectors s. Notice that $A(s) = \{x \in \mathbb{R}^n_+ \mid x_i = 0 \text{ whenever } s_i = 0\}$. Further, to apply the algorithm in this case, we have to adapt the concepts of an almost s-complete simplex and an s-complete facet. For some sign vector s with t positive components, denote $\{h_1, \ldots, h_{n-t}\} = \{h \mid s_h = 0\}$ and let $\sigma = < x^1, \ldots, x^{t+1} >$ be a t-simplex of the triangulation in $A(s)$. Then σ is *almost s-complete* if there is an $(n+2) \times (n+1)$ matrix W being a solution to system

$$\begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ f(x^1) & \cdots & f(x^{t+1}) & -e(h_1) & \cdots & -e(h_{n-t}) & s \end{bmatrix} W = I \tag{5.3}$$

and having rows $w^1, \ldots, w^{n+2}$ such that $w^h \succeq 0$ for $1 \leq h \leq t+1$, and $w^{n+2} \succeq -w^i$ for $t+1 < i \leq n+1$, and $w^{n+2} \succeq 0$. If $w^{n+2}_1 = 0$, then we say that the simplex σ is complete. For τ a facet of σ, without loss of generality, letting $\tau = < x^1, \ldots, x^t >$, τ is *s-complete* if there is an $(n+1) \times (n+1)$ matrix W being a solution to system

$$\begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ f(x^1) & \cdots & f(x^t) & -e(h_1) & \cdots & -e(h_{n-t}) & s \end{bmatrix} W = I \tag{5.4}$$

and having rows $w^1, \ldots, w^{n+1}$ such that $w^h \succeq 0$ for $1 \leq h \leq t$, and $w^{n+1} \succeq -w^i$ for $t+1 \leq i \leq n$, and $w^{n+1} \succeq 0$. If $w^{n+1}_1 = 0$, then we say that τ is complete.

With respect to the starting point 0^n, let $\alpha = \min_h f_h(0^n)$ and let s^0 be the sign vector with $s^0_k = 1$, where k is the smallest index h such that $f_h(0^n) = \alpha$, and $s^0_j = 0$ for $j \neq k$. To avoid triviality, we may assume that $f(0^n) \not\succeq 0^n$. Similarly as in Section 2, it can be shown that the simplex $< 0^n >$ is an s^0-complete facet of the unique 1-dimensional simplex σ^0 in $A(s^0)$ having $< 0^n >$ as one of its facets. Furthermore σ^0 is almost s^0-complete.

We now apply the algorithm as described in Section 2. Starting with σ^0, by applying the steps as given in Section 2 with the system (5.3) the algorithm generates a unique sequence of adjacent almost s-complete simplices in $A(s)$ with s-complete common facets for varying positive sign vectors s. The algorithm stops with a complete simplex in a finite number of steps under the assumption stated in the theorem. As shown below, a complete simplex gives a solution to the problem. Recall that $\bar{f}$ stands for the piecewise linear extension of the function f with respect to $\mathcal{T}$. Let $C^n = \{x \in \mathbb{R}^n \mid 0^n \leq x \leq M\}$.

Lemma 5.3 *For some nonnegative sign vector s, let σ be a simplex in $A(s)$ with an s-complete facet τ on the upper boundary of C^n. Then τ is a complete simplex.*

Proof: From system (5.4) it follows that there exist $\lambda_1 \geq 0, \ldots, \lambda_t \geq 0$ with sum equal to one, $\beta \geq 0$, and $\mu_i \geq -\beta$ for $s_i = 0$, such that $\bar{f}_i(z) = -\beta$ when $s_i = 1$ and $\bar{f}_i(z) = \mu_i$ when $s_i = 0$, where $z = \sum_{i=1}^{t} \lambda_i x^i$. Since τ lies on the upper boundary of C^n, there exists an index h such that $x_h^j = M_h$ for all j. But then we must have $s_h = 1$ and therefore $\bar{f}_h(z) = -\beta \leq 0$. On the other hand, by assumption, we have $f_h(x^j) \geq 0$ for all j. Hence, we obtain $\bar{f}_h(z) \geq 0$. As a result, $\beta = 0$, which implies that τ is a complete simplex by definition. $\quad\square$

Lemma 5.4 *For some nonnegative sign vector s, let σ be a complete simplex in $A(s)$. Then σ contains a solution to the nonlinear complementarity problem for $\bar{f}$.*

Proof: Let $x^1, \cdots, x^{k+1}$ be the vertices of the complete simplex σ in $A(s)$ and let t be the number of nonzeros in s. Note that $k = t - 1$ or $k = t$ depending on whether σ is a t-simplex in $A(s)$ or a facet of a t-simplex in $A(s)$. It follows from the system (5.3) or (5.4) that there exist $\lambda_1 \geq 0, \ldots, \lambda_{k+1} \geq 0$ with sum equal to one and $\mu_i \geq 0$ for $s_i = 0$ such that $\bar{f}_i(z) = 0$ if $s_i = 1$, and $\bar{f}_i(z) = \mu_i$ if $s_i = 0$, where $z = \sum_{i=1}^{k+1} \lambda_i x^i$. Since $z \in A(s)$, we also have $z_i = 0$ if $s_i = 0$ and $z_i \geq 0$ if $s_i = 1$. So, $\bar{f}_i(z) \geq 0$ if $z_i = 0$ and $\bar{f}_i(z) = 0$ if $z_i > 0$, i.e., z solves the nonlinear complementarity problem with respect to $\bar{f}$. $\quad\square$

The next lemma says that, for any complete simplex σ, at least one of its vertices is a solution to DCP(f).

Lemma 5.5 *Let σ be a complete simplex of $\mathcal{T}$ in $A(s)$ for some sign vector s. Then σ contains a vertex being a solution to DCP(f).*

Proof: Because σ is a complete simplex in $A(s)$, as shown in Lemma 5.4, there is a point z in σ being a solution to the nonlinear complementarity problem with respect to $\bar{f}$. Now let $\rho = \langle x^1, \ldots, x^k \rangle$ be the unique face of σ containing z in its relative interior. Namely, there exist unique positive numbers $\lambda_1, \ldots, \lambda_k$ summing up to 1 such that $z = \sum_{j=1}^{k} \lambda_j x^j$

and $\bar{f}(z) = \sum_{j=1}^{k} \lambda_j f(x^j)$. Take any j^* between 1 and k. Suppose first that $z_i = 0$ and $\bar{f}_i(z) > 0$ for some i. Clearly, $x_i^j = 0$ for all $j = 1, \ldots, k$. Since $\bar{f}_i(z) = \sum_{j=1}^{k} \lambda_j f_i(x^j)$ there exists h such that $f_i(x^h) > 0$. Since x^h and x^{j^*} are simplicially connected and $x_i^h = x_i^{j^*} = 0$, we have that $f_i(x^h) f_i(x^{j^*}) \geq 0$, and therefore $x_i^{j^*} = 0$ and $f_i(x^{j^*}) \geq 0$. Suppose next that $z_i = 0$ and $\bar{f}_i(z) = 0$ for some i. Again, $x_i^j = 0$ for all $j = 1, \ldots, k$. Since $\bar{f}_i(z) = \sum_{j=1}^{k} \lambda_j f_i(x^j)$ and $\bar{f}_i(z) = 0$, we obtain $\sum_{j=1}^{k} \lambda_j f_i(x^j) = 0$ and therefore $\sum_{j=1}^{k} \lambda_j f_i(x^j) f_i(x^{j^*}) = 0$. Since for all j it holds that x^j and x^{j^*} are simplicially connected and $x_i^j = x_i^{j^*} = 0$, we have $f_i(x^j) f_i(x^{j^*}) \geq 0$, and so each term in the summation must be zero. In particular, it holds that $\lambda_{j^*} f_i^2(x^{j^*}) = 0$. Since $\lambda_{j^*} > 0$, this implies $f_i(x^{j^*}) = 0$.

Thus far we have shown that whenever $z_i = 0$ both $f_i(x^{j^*}) \geq 0$ and $x_i^{j^*} = 0$ must hold. It remains to show that whenever $z_i > 0$ it holds that $f_i(x^{j^*}) = 0$, and hence that x^{j^*} is a solution to DCP(f). By construction, $\bar{f}_i(z) = \sum_{j=1}^{k} \lambda_j f_i(x^j) = 0$ whenever $z_i > 0$. Note that $I(x^j) \subseteq I(z)$ for any $j = 1, \ldots, k$. Therefore,

$$\sum_{h \in I(z)} \sum_{j=1}^{k} \lambda_j f_h(x^j) f_h(x^{j^*}) = 0$$

and so

$$\sum_{j=1}^{k} (\lambda_j \sum_{h \in I(z)} f_h(x^j) f_h(x^{j^*})) = 0.$$

Since $I(z)$ contains the set $I(x^j, x^{j^*})$ and x^j and x^{j^*} are simplicially connected for all j, by hypothesis we have that each of the k terms between brackets is nonnegative and therefore must be zero. Hence,

$$\lambda_{j^*} \sum_{h \in I(z)} f_h^2(x^{j^*}) = 0.$$

Since $\lambda_{j^*} > 0$, we obtain $f_h(x^{j^*}) = 0$ for all $h \in I(z)$. Therefore $f_i(x^{j^*}) = 0$ if $z_i > 0$. $\quad\square$

Theorem 5.2 now follows from the lemmas stated above by a combinatorial argument.

Proof of Theorem 5.2. Due to the lexicographic pivoting rule, the algorithm will never visit any simplex more than once. Since the number of simplices in C^n is finite, the algorithm terminates in a finite number of steps with a complete simplex in $A(s)$. According to Lemma 5.5, the complete simplex gives a solution to DCP(f). $\quad\square$

In the sequel, we turn our attention to the linear complementarity problem LCP(M, q). Recall that Lemke's method [22] introduces an artificial variable z_0 and operates by moving from one basic solution of the following system of linear equations to another:

$$\begin{aligned} Iz - Mx - z_0 e &= q \\ x_j \geq 0, \ z_j \geq 0, \ z_0 &\geq 0, \ \text{for } j = 1, 2, \ldots, n \\ x_j z_j &= 0, \ \text{for } j = 1, 2, \ldots, n \end{aligned} \tag{5.5}$$

where e is the n-vector of all ones and I is the identity matrix of order n. The algorithm starts with a ray at $x = 0^n$ and terminates in a finite number of pivot steps when a solution is found or when another ray is encountered.

Lemke [22] shows that his method is guaranteed to find a solution of $LCP(M, q)$ if M is copositive-plus and the system of $Mx + q \geq 0^n$ and $x \geq 0^n$ is feasible. Recall that a square matrix B is said to be *copositive* if $x \cdot Bx \geq 0$ for any $x \in \mathbb{R}_+^n$. Furthermore, B is said to be *copositive-plus* if B is copositive and in addition $x \geq 0^n$ and $x \cdot Bx = 0$ imply $(B + B^t)x = 0$, where B^t is the transpose of B. Of course, even if an $LCP(M, q)$ has a solution, it may have no integral solution at all. To guarantee that an $LCP(M, q)$ has an integral solution, we need to impose total unimodularity on the matrix M. Recall that a matrix B is said to be *totally unimodular* if the determinant of every subsquare matrix of B is -1, 0, or 1. Now we establish the following theorem on the existence of an integral solution to $LCP(M, q)$.

Theorem 5.6 *Suppose that M is totally unimodular, copositive-plus, and q is an integral vector, and that the system of $Mx + q \geq 0^n$ and $x \geq 0^n$ is feasible. Then the algorithm defined by system (5.3) reduces to Lemke's method and terminates at an integral solution in a finite number of steps.*

Proof: For the $LCP(M, q)$, we first show that the algorithm defined by system (5.3) reduces to Lemke's method. We may assume that $q \not\geq 0^n$. In the initial step of Lemke's method the system defined by (5.5) at $x = 0^n$ is put in a tableau format and a pivot step is made with the z_0 column on row k, where k is such that $q_k = \min\{q_h \mid h \in N\}$. This corresponds exactly to the initial step of the algorithm defined by system (5.3) at which 0^n is the starting point and the algorithm moves in the set $A(s^0)$, where $s^0 \in \mathbb{R}_+^n$ is the sign vector defined by $s_k^0 = 1$ and $s_j^0 = 0$ for $j \neq k$ with k being the smallest index h such that $q_h = \min q_j$. Here the choice of the smallest index is to avoid degeneracy.

In a general step, let $\sigma = <x^1, \ldots, x^{t+1}>$ be an almost s-complete simplex in $A(s)$. Let $I^0(s) = \{h \mid s_h = 0\}$ and $I^+(s) = \{h \mid s_h = 1\}$. Now it follows from the system (5.3) that there exist $\lambda_h \geq 0$, $h = 1, \ldots, t + 1$, $\mu_0 \geq 0$, and $\mu_0 \geq -\mu_h$ for every $h \in I^0(s)$ such that

$$\sum_j \lambda_j f(x^j) - \sum_{h \in I^0(s)} \mu_h e(h) + \mu_0 s = 0^n \tag{5.6}$$

and $\sum_j \lambda_j = 1$. Let $\beta_0 = \mu_0$ and $\beta_h = -\mu_h$. Let $x = \sum_j \lambda_j x^j$ for $h \in I^0(s)$. Since x is a convex combination of points $x^1, \ldots, x^{t+1}$ in $A(s)$, x also lies in $A(s)$. Further, $f(x) = Mx + q = \sum_j \lambda_j (Mx^j + q) = \sum_j \lambda_j f(x^j)$. Thus equation (5.6) reduces to

$$Mx + q + \sum_{h \in I^0(s)} \beta_h e(h) + \beta_0 s = 0^n$$

where $\beta_0 \geq 0$, and $\beta_0 \geq \beta_h$ for every $h \in I^0(s)$, $x_h = 0$ for $s_h = 0$ and $x_h \geq 0$ for $s_h = 1$. We can rewrite the equation as follows:

$$-Mx + \sum_{h \in I^0(s)} (\beta_0 - \beta_h)e(h) - \beta_0 e = q$$

where $\beta_0 \geq 0$, and $\beta_0 \geq \beta_h$ for every $h \in I^0(s)$, $x_h = 0$ for $s_h = 0$ and $x_h \geq 0$ for $s_h = 1$. Now let $z_h = \beta_0 - \beta_h$ for $h \in I^0(s)$, $z_h = 0$ for $h \in I^+(s)$ and $z_0 = \beta_0$. Then we have

$$
\begin{aligned}
Iz - Mx - z_0 e &= q, \\
z_0 \geq 0, \; z_h &\geq \; 0 \text{ for } h \in I^0(s), \\
z_h &= \; 0 \text{ for } h \in I^+(s), \\
x_h &= \; 0 \text{ for } h \in I^0(s), \\
x_h &\geq \; 0 \text{ for } h \in I^+(s).
\end{aligned}
\tag{5.7}
$$

For this system we have $x_h z_h = 0$ for every $h = 1, \ldots, n$, and the algorithm finds a solution as soon as z_0 becomes zero. This shows that the system above coincides with the system (5.5). As a result, we have proved that the algorithm defined by system (5.3) indeed reduces to Lemke's method. It is worth pointing out that for the $LCP(M, q)$, actually no triangulation is needed for the algorithm. In fact, for given sign vector s, all pivot steps of the $2n$-ray algorithm within the region $A(s)$ reduce to one pivot step in the Lemke algorithm because of the linearity of the function $f(x) = Mx + q$.

Concerning the second statement of the Theorem 5.6 (the termination of the algorithm), it follows from Lemke [22] that because M is copositive plus and the system of $Mx + q \geq 0^n$ and $x \geq 0^n$ is feasible, the algorithm must end up with a solution x^* in a finite number of steps. More precisely, the algorithm stops with a solution $x^* \in A(s)$ for some sign vector $s \in \mathbb{R}_+^n$ corresponding to the $n \times n$ regular matrix $B = [(-M_h, h \in I^+(s)), (e(h), h \in I^0(s))]$, where M_h denotes the hth column of matrix M. Note that $x^* = B^{-1}q$. It remains to show that x^* is integral. Because M is totally unimodular, $[-M, I]$ is totally unimodular and so is B (see Schrijver [29]). Because B^{-1} exists and is also totally unimodular and q is integral, $x^* = B^{-1}q$ is integral. This shows that the algorithm indeed finds an integral solution of the $LCP(M, q)$. $\qquad \square$

The next corollary follows immediately from Theorem 5.6 and answers a question raised by one of the referees.

Corollary 5.7 *Suppose that M is totally unimodular, positive definite and q is an integral vector. Then the algorithm terminates at an integral solution in a finite number of steps.*

Proof: Because M is positive definite, M is copositive-plus. Moreover, because M is positive definite, there exists $x \geq 0^n$ such that $Mx + q \geq 0^n$ (see Cottle *et al.* [3], p.140, Lemma 3.1.3). The conclusion follows immediately. $\qquad \square$

6 Applications

Discrete zero point (or fixed point) problems often occur in economics. For instance, in an exchange economy with n commodities, one of the most studied problems is the existence of a Walrasian equilibrium price system, being a price vector $p \in \mathbb{R}_+^n$ solving the complementarity problem for the excess demand function z defined from the price space $\mathbb{R}_+^n$ into the commodity space $\mathbb{R}^n$, where $z_j(p)$ is the excess demand for commodity j at the (nonnegative) price system p, $j = 1, \ldots, n$. In the literature the existence of an equilibrium price system $p^* \in \mathbb{R}_+^n$ has been studied extensively, nevertheless in almost all real life situations prices are in some monetary unit, implying that actually a price system belongs to $\mathbb{Z}_+^n$ for appropriately chosen units of the components of p. Hence, in fact an equilibrium price system should be a solution to the DCP(z).

In this section we apply Theorem 3.2 to the supermodular games, see for instance Fudenberg and Tirole [10]. A well-known example of such games is the Bertrand price competition model. Here we consider the Cournot oligopoly model with complementary commodities, see Vives [35]. There are n firms, each firm producing its own commodity. The goal of each firm is to choose an amount of product that maximizes its own profit given the production levels chosen by other firms. Let $q_i \geq 0$ denote the quantity of commodity i produced by firm i and let $q_{-i} = (q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_n)$ denote the vector of amounts of commodities produced by all firms but firm i. The price at which firm i can sell its product is decreasing in its own quantity q_i and, due to the complementarities, it is increasing in the quantities q_j, $j \neq i$. It is standard to assume that the price function of each firm $i = 1, \cdots, n$, is linear, i.e.,

$$P_i(q_i, q_{-i}) = a_i - b_i q_i + \sum_{j \neq i} d_{ij} q_j,$$

where all parameters a_i, b_i, d_{ij} are positive. Each firm i has a linear cost function $C_i(q_i) = c_i q_i$ with $a_i > c_i > 0$. For quantities $(q_1, \ldots, q_n)$, the profit π_i of firm i is given by its quantity times price minus its cost of production, i.e.,

$$\pi_i(q_i, q_{-i}) = q_i P_i(q_i, q_{-i}) - c_i q_i.$$

A tuple $(q_1^*, q_2^*, \ldots, q_n^*) \in \mathbb{R}_+^n$ of nonnegative real numbers is a *Cournot-Nash equilibrium* if for every firm i,

$$\pi_i(q_i^*, q_{-i}^*) \geq \pi_i(q_i, q_{-i}^*), \text{ for all } q_i \in \mathbb{R}_+.$$

It is well-known that there exists a Cournot-Nash equilibrium if $2b_i > \sum_{j \neq i} d_{ij}$ for every firm $i = 1, \ldots, n$. However, in reality, it is often the case that the commodities are produced only in integer quantities. Here we will show that under the same condition a discrete

Cournot-Nash equilibrium exists in this model. A tuple $(q_1^*, q_2^*, \ldots, q_n^*)$ of nonnegative integers is a *discrete Cournot-Nash equilibrium* if

$$\pi_i(q_i^*, q_{-i}^*) \geq \pi_i(q_i, q_{-i}^*), \text{ for all } q_i \in \mathbf{Z}_+, \ i = 1, \cdots, n.$$

That is, given the quantities chosen by other firms, each firm chooses an integer quantity that yields a profit which is at least as high as any other integer quantity could give.

For a real number x, the symbol $[x]$ denotes the greatest nearest integer to x. Given non-negative integer quantities q_{-i} of all other firms, firm i maximizes its own profit $\pi_i(q_i, q_{-i})$ over all nonnegative integers q_i and its optimal or reaction integer quantity is given by

$$r_i(q_{-i}) = [\frac{a_i - c_i}{2b_i} + \sum_{j \neq i} \frac{d_{ij}}{2b_i} q_j].$$

Observe that $r_i(q_{-i}) \geq 0$ for all $q \in \mathbf{Z}_+^n$, because $a_i > c_i > 0$. Define the function $f : \mathbf{Z}_+^n \to \mathbf{Z}^n$ by

$$f_i(q_i, q_{-i}) = r_i(q_{-i}) - q_i, \quad i = 1, \ldots, n.$$

Clearly, a discrete zero point of f is a discrete Cournot-Nash equilibrium.

Theorem 6.1 *Suppose that $2b_i > \sum_{j \neq i} d_{ij}$, $i = 1, \ldots, n$. Then there exists a discrete Cournot-Nash equilibrium in the above Cournot oligopoly competition model.*

Proof: We show that the function f satisfies the conditions of Corollary 3.3. First, we show that f satisfies the boundary condition. As a natural lower bound, take $m = 0^n$, and as upper bound take for all i, $M_i = M$, $i = 1, \ldots, n$ with $M > 1$ an integer satisfying $M > \max_i \{\frac{a_i - c_i}{2b_i - \sum_{j \neq i} d_{ij}}\}$. Then for any i and any $q \in \mathbf{Z}_+^n$, $q_i = 0$ implies $f_i(q) = r_i(q_{-i}) \geq 0$. Further, $q_i = M$ and $q_j \leq M$ imply $f_i(q) = [\frac{a_i - c_i}{2b_i} + \sum_{j \neq i} \frac{d_{ij}}{2b_i} q_j - q_i] \leq [\frac{a_i - c_i}{2b_i} + \sum_{j \neq i} \frac{d_{ij}}{2b_i} M - M]$ $\leq [\frac{a_i - c_i - (2b_i - \sum_{j \neq i} d_{ij})M}{2b_i}] \leq 0$, where the last inequality follows from the fact that $M > \frac{a_i - c_i}{2b_i - \sum_{j \neq i} d_{ij}}$.

Second, we show that f is simplicially local gross direction preserving with respect to the K-triangulation as described in Section 2. Since the K-triangulation is given by integral simplices $\sigma(y, \pi)$ with vertices $y^1, \ldots, y^{n+1}$, with $y^1 = y$ and $y^{i+1} = y^i + e(\pi(i))$, $i = 1, \ldots, n$ for given $y \in \mathbf{Z}^n$ and $\pi = (\pi(1), \ldots, \pi(n))$ a permutation of the elements $1, 2, \ldots, n$, we have to check that $f(x) \cdot f(y) \geq 0$ for any pair $x \in \mathbf{Z}_+^n$ and $y = x + \sum_{h=1}^k e(\pi(h))$ for $k = 1, \ldots, n$ and any permuation π. Observe that for any such pair it holds that $y_i \in \{x_i, x_i + 1\}$ for all $i = 1, \ldots, n$. For some pair x, y and $i \in \{1, \ldots, n\}$, define $S_i(x, y) = \{j \neq i | y_j = x_j + 1\}$. Then $r_i(y) = [\frac{a_i - c_i}{2b_i} + \sum_{j \neq i} \frac{d_{ij}}{2b_i} y_j] = [\frac{a_i - c_i}{2b_i} + \sum_{j \neq i} \frac{d_{ij}}{2b_i} (x_j) + \sum_{j \in S_i(x,y)} \frac{d_{ij}}{2b_i}]$. Since $\sum_{j \neq i} \frac{d_{ij}}{2b_i} < 1$, it follows that $r_i(y) \in \{r_i(x), r_i(x) + 1\}$. Hence, since $y_i \in \{x_i, x_i + 1\}$, it follows that $f_i(y) \in \{f_i(x) - 1, f_i(x), f_i(x) + 1\}$. So, when $f_i(x) \geq 1$, then $f_i(y) \geq f_i(x) - 1 \geq 0$

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and when $f_i(x) \leq -1$, then $f_i(y) \leq f_i(x) + 1 \leq 0$. So, $f_i(x)f_i(y) \geq 0$ for all i and thus $f(x) \cdot f(y) \geq 0$.

We have shown that f satisfies all the conditions of Corollary 3.3 and thus has a discrete zero point. As a result, there is a discrete Cournot-Nash equilibrium. $\qquad\square$

It is worth mentioning that f may not be simplicially local gross direction preserving with respect to other triangulations, as shown by the following example with $n = 2$. Let the parameters be given by $a_1 = 4$, $c_1 = 2.5$, $b_1 = 0.5$, $d_1 = d_{12} = 3/4$, $a_2 = 5$, $c_2 = 4$, $b_2 = 1/3$, and $d_2 = d_{21} = 1/12$. These parameters satisfy the stated condition for the model and therefore there is a discrete Cournot-Nash equilibrium. In fact, $(3, 2)$ is the unique discrete Cournot-Nash equilibrium for this example. As shown above, the function f is simplicially local gross direction preserving with respect to the K-triangulation. However, this function is not simplicially local gross direction preserving with respect to the H-triangulation of Saigal [27]. For $\mathbb{R}^2$, this triangulation is given by the simplices $< y^1, y^2, y^3 >$, with $y^1 \in \mathbb{Z}^2$, $y^2 = y^1 + p(\pi(1))$, $y^3 = y^2 + p(\pi(2))$, where $p(1) = (1, 0)$, and $p(2) = (-1, 1)$. Now take $\pi = (2, 1)$, $x = y^1 = (3, 1)$ and $y = y^2 = y^1 + p(2) = (2, 2)$. Since $f(x) = (-1, 1)$ and $f(y) = (1, 0)$, we have that $f(x) \cdot f(y) = -1 < 0$, and so the function is not simplicially local direction preserving with respect to the H-triangulation. Note that x and y do not belong to a same simplex of the K-triangulation.

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