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Exploiting group symmetry in truss topology optimization

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Abstract

We consider semidefinite programming (SDP) formulations of certain truss topology optimization problems, where a lower bound is imposed on the fundamental frequency of vibration of the truss structure. These SDP formulations were introduced in: [M. Ohsaki, K. Fujisawa, N. Katoh and Y. Kanno, Semi-definite programming for topology optimization of trusses under multiple eigenvalue constraints, *Comp. Meth. Appl. Mech. Engng.*, 180: 203–217, 1999]. We show how one may automatically obtain symmetric designs, by eliminating the 'redundant' symmetry in the SDP problem formulation. This has the advantage that the original SDP problem is substantially reduced in size for trusses with large symmetry groups.

Keywords: truss topology optimization, semidefinite programming, group symmetry

AMS classification: 90C22, 20Cxx, 70-08

JEL code: C60

1 Introduction

In this paper we consider semidefinite programming (SDP) formulations of certain truss topology optimization problems. In particular, we consider so called group-symmetric truss designs of the type studied by Kanno et al. [6]; see also [9, 8].

Kanno et al. pointed out that, although a symmetric truss design is desirable in practice, there may exist optimal solutions of the SDP formulation that do not exhibit this symmetry. They therefore proceed to show that certain search directions used in interior point algorithms for SDP preserve symmetry. This means that the interior point algorithms generate a sequence of iterates that are group symmetric, given that the starting point is group symmetric.

In this paper we show how one may automatically obtain symmetric designs, by eliminating the 'redundant' symmetry in the SDP problem formulation. In particular, we perform pre-processing to restrict the feasible set of the SDP problem to symmetric designs. This is in the spirit of work by Schrijver [10], Gatermann and Parrilo [4], De Klerk *et al.* [2], De Klerk, Pasechnik and Schrijver [3], and others, who have shown how 'group symmetric' SDP problems may be reduced in size using representation theory.

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We illustrate our approach by considering a family of dome truss structures, and show that the resulting SDP problems may be greatly reduced in size via symmetry reduction.

Outline of the paper

We begin with a discussion of finite groups and their linear orthogonal representations in Section 2. The next two sections deal with matrix algebras and their representations. In particular, the case where the matrix algebra in question is the commutant of a linear representation of a finite group is of interest to us in this paper. In Section 5 we recall the notion of SDP problems with 'group symmetric data', and how these problems may be reduced in size using the algebraic techniques described in the previous sections. We then describe the SDP formulation of a truss topology optimization problem due to Ohsaki et. al. [9] in Section 6. We explain in which sense these SDP problems have group symmetric data, and show how they may be reduced using the techniques described in the previous section. Finally, we illustrate our results on a family of dome truss structures in Section 7.

Notation

The space of $p \times q$ real matrices is denoted by $\mathbb{R}^{p\times q}$, and the space of $k \times k$ symmetric matrices is denoted by $\mathcal{S}_k$, and the space of $k \times k$ positive semidefinite matrices by $\mathcal{S}_k^+$. We will sometimes also use the notation $X \succeq 0$ instead of $X \in \mathcal{S}_k^+$, if the order of the matrix is clear from the context.

We use I_n to denote the identity matrix of order n, and omit the subscript if the order is clear from the context.

The *Kronecker product* $A \otimes B$ of matrices $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{r\times s}$ is defined as the $pr\times qs$ matrix composed of pq blocks of size $r\times s$, with block ij given by $A_{ij}B$ $(i = 1, \ldots, p)$, $(j = 1, \ldots, q)$.

The following properties of the Kronecker product will be used in the paper, see e.g. [5],

$$(A \otimes B)^T = A^T \otimes B^T \tag{1}$$

$$(A \otimes B)(C \otimes D) = AC \otimes BD, \tag{2}$$

for all $A \in \mathbb{R}^{p\times q}, B \in \mathbb{R}^{r\times s}, C \in \mathbb{R}^{q\times k}, D \in \mathbb{R}^{s\times l}$.

Finally, let $E_{ij} \in \mathbb{R}^{n\times n}$ denote the matrix with 1 in position ij and zero elsewhere.

2 On finite groups and their representations

The next definition recalls the fact that finite groups may be represented by multiplicative groups of orthogonal matrices.

Definition 1. **[7]** *Let V be a real, m–dimensional vector space and identify $\mathbb{R}^{m\times m}$ (respectively, $\mathcal{O}_m$) as the space of all (respectively, orthogonal) $m \times m$ matrices. An (*orthogonal*)* linear representation *of a group $\mathcal{G}$ on V is a* group homomorphism $\mathcal{T} : \mathcal{G} \to \mathbb{R}^{m\times m}$ *(respectively, $\mathcal{T} : \mathcal{G} \to \mathcal{O}_m$). In other words for each element $g \in \mathcal{G}$ there exists an invertible $\mathcal{T}_g \equiv \mathcal{T}(g) \in \mathbb{R}^{m\times m}$ (respectively, in $\mathcal{O}_m$) such that $\mathcal{T}(g_1)\mathcal{T}(g_2) = \mathcal{T}(g_1g_2)$.*

In what follows we consider images of SDP data matrices $A_i = A_i^T \in \mathbb{R}^{m\times m}$ under $\mathcal{T}_g$'s. Thus, we have to restrict our attention to orthogonal representations, as in the usual SDP setting one needs (as it will become clear in what follows) that $B_i = \mathcal{T}_g A_i \mathcal{T}_g^{-1}$ are symmetric, i.e. $B_i = B_i^T$. From the representation-theoretic point of view, there is little loss of generality in considering such representations only. Indeed, any real linear representation of a finite group is equivalent, by conjugation with an upper-triangular matrix, to an orthogonal representation[1].

The following theorem shows that, if one has two orthogonal representations of a finite group, one may obtain a third representation using Kronecker products. In representation theory this construction is known as *tensor product of representations.*

Theorem 2. *Let $\mathcal{G}$ be a group and denote two orthogonal linear representations of $\mathcal{G}$ by p_i $(i = 1, \ldots, |\mathcal{G}|)$ and s_i $(i = 1, \ldots, |\mathcal{G}|)$, such that p_i corresponds to s_i $(i = 1, \ldots, |\mathcal{G}|)$.*

Then a third orthogonal linear representation of $\mathcal{G}$ is given by

$$P_i := p_i \otimes s_i \quad (i = 1, \ldots, |\mathcal{G}|).$$

Proof: Let indices $i, j, k \in \{1, \ldots, |\mathcal{G}|\}$ be given such that $p_i p_j = p_k$ (and therefore also $s_i s_j = s_k$). Note that

$$\begin{aligned} P_i P_j &= (p_i \otimes s_i)(p_j \otimes s_j) \\ &= (p_i p_j) \otimes (s_i s_j) \\ &= p_k \otimes s_k \equiv P_k. \end{aligned}$$

Moreover, note that the matrices P_i are orthogonal, since the p_i and s_i's are. □

The *commutant* (or centralizer ring) of a linear representation of $\mathcal{G}$ (still denoted by $\mathcal{G}$ for convenience) is defined by

$$\mathcal{A}_\mathcal{G} := \{X \in \mathbb{R}^{n\times n} \ : \ XP = PX \quad \forall\, P \in \mathcal{G}\}.$$

An alternative, equivalent, defition of the commutant is

$$\mathcal{A}_\mathcal{G} = \{X \in \mathbb{R}^{n\times n} \ : \ R(X) = X\},$$

where

$$R(X) := \frac{1}{|\mathcal{G}|} \sum_{P\in\mathcal{G}} PXP^T, \ X \in \mathbb{R}^{n\times n}$$

is called the *Reynolds operator* (or group average) of $\mathcal{G}$. Thus R is the orthogonal projection onto the commutant. Orthonormal eigenvectors of R corresponding to the eigenvalue 1 form a orthonormal basis of $\mathcal{A}_\mathcal{G}$ (seen as a vector space).

The commutant is a C^*-algebra, i.e. a subspace of $\mathbb{R}^{n\times n}$ that is closed under matrix multiplication and conjugation.

We will study optimization problems where we may assume that the feasible set is contained in some commutant, and we therefore devote one more section to recall some results on representations of matrix $*$-algebras. The basic idea is that we want to obtain the most 'economical' representation of the feasible set of our optimization problem.

[1]This can be seen by modifying, in the obvious way, the standard representation theory argument, that works for any complex representation of $\mathcal{G}$, creating a $\mathcal{T}(\mathcal{G})$-invariant positive definite matrix $A = \sum_{g\in\mathcal{G}} \overline{\mathcal{T}(g)}^T \mathcal{T}(g)$, and conjugating $\mathcal{T}(\mathcal{G})$ by the Cholesky factors of A.

3 Matrix algebras and their representations

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ denote two matrix $*$-algebras. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are *equivalent* if there exists a *unitary* matrix Q (i.e. $Q^*Q = I$) such that

$$\mathcal{A}_2 = \{Q^*XQ \mid X \in \mathcal{A}_1\}.$$

We define the *direct sum* of matrices X_1 and X_2 as

$$X_1 \oplus X_2 := \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}.$$

An algebra $\mathcal{A}$ is called *basic* if

$$\mathcal{A} = \left\{ \bigoplus_{i=1}^{t} M \mid M \in \mathbb{C}^{m\times m} \right\}$$

for some t and m. Finally, the direct sum of two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ is defined as

$$\mathcal{A}_1 \oplus \mathcal{A}_2 := \{X_1 \oplus X_2 \mid X_1 \in \mathcal{A}_1,\ X_2 \in \mathcal{A}_2\}.$$

The following existence theorem gives the so-called completely reduced representation of a matrix $*$-algebra $\mathcal{A}$.

Theorem 3 (Wedderburn [13][2])**.** *Each matrix $*$-algebra is equivalent to a direct sum of basic algebras and a zero algebra.* □

In general this completely reduced representation is not known, but it is known in our case, that is, when our $*$-algebra is the commutant[3] of a finite group representation.

In the next section we give details on how to compute the completely reduced representation of the commutant. The reader may wish to skip this section during a first reading of the paper.

4 Commutant of a group representation

Here we summarise (and use) the relevant material from [12], in particular from Sect. 13.2, where $\mathbb{R}$-representations are treated[4]

Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$, and $\mathcal{T}$ be a $\mathbb{F}$-linear representation of a finite group $\mathcal{G}$ into $\mathbb{F}^{m\times m}$, or more precisely, into the group $GL_m(\mathbb{F})$ of the invertible matrices in $\mathbb{F}^{m\times m}$. The *character* of $\mathcal{T}$ is a function $\chi_{\mathcal{T}} : \mathcal{G} \to \mathbb{F}$ given by $\chi_{\mathcal{T}}(g) := \mathrm{tr}(\mathcal{T}(g))$, that encodes a lot of information about $\mathcal{T}$ and $\mathcal{G}$. For instance, two representations fo $\mathcal{G}$ are equivalent if and only if they have the same character. Note that in order to know $\chi_{\mathcal{T}}$, it suffices to know its values on representatives of *conjugacy classes*[5] of $\mathcal{G}$.

Important (and easy to check) formulae for the characters of the direct sum $\mathcal{T} \oplus \mathcal{Q}$ and of the tensor product $\mathcal{T} \otimes \mathcal{Q}$ of two representations $\mathcal{T}$ and $\mathcal{Q}$ of $\mathcal{G}$ are as follows:

$$\chi_{\mathcal{T}\oplus\mathcal{Q}}(g) = \chi_{\mathcal{T}}(g) + \chi_{\mathcal{Q}}(g), \qquad \chi_{\mathcal{T}\otimes\mathcal{Q}}(g) = \chi_{\mathcal{T}}(g)\chi_{\mathcal{Q}}(g), \qquad g \in \mathcal{G}. \tag{3}$$

[2] A detailed treatment of this material is given in [14]; our statement of the theorem is based on lecture notes by A. Schrijver, available at `http://homepages.cwi.nl/~lex/files/block.pdf`.

[3] Called *commuting algebra* in [12].

[4] Representation theory of finite groups is easiest over the field of complex numbers $\mathbb{C}$. It is in fact beneficial to bear in mind that every $\mathbb{R}$-representation is also a $\mathbb{C}$-representation.

[5] The conjugacy class of $g \in \mathcal{G}$ in $\mathcal{G}$ is the set $\{hgh^{-1} \mid h \in \mathcal{G}\} \subset \mathcal{G}$.

A representation is called *irreducible* if the space $\mathbb{F}^m$ does not contain a proper subspace that is left invariant by all the $\mathcal{T}(g)$, where $g \in \mathcal{G}$. The following is well-known, cf. e.g. [12, p.108].

Theorem 4. *The commutant $C(\mathcal{I})$ of a linear $\mathbb{R}$-irreducible representation $\mathcal{I}$ of $\mathcal{G}$ is isomorphic to a division ring*[6] *over $\mathbb{R}$. Thus $C(\mathcal{I})$ depends upon the decomposition of the representation $\mathcal{I}$ over $\mathbb{C}$. Namely:*

1. *$\mathcal{I}$ is irreducible over $\mathbb{C}$: $C(\mathcal{I}) \cong \mathbb{R}$, $\dim_{\mathbb{R}}(C(\mathcal{I})) = 1$.*
2. *$\chi_{\mathcal{G}}(\mathcal{I}) = \zeta_{\mathcal{G}}(\mathcal{J}) + \overline{\zeta_{\mathcal{G}}(\mathcal{J})}$, with $\mathbb{C}$-valued $\zeta_{\mathcal{G}}(\mathcal{J})$ and a $\mathbb{C}$-representation $\mathcal{J}$ of $\mathcal{G}$: $C(\mathcal{I}) \cong \mathbb{C}$, $\dim_{\mathbb{R}}(C(\mathcal{I})) = 2$.*
3. *$\chi_{\mathcal{G}}(\mathcal{I}) = 2\zeta_{\mathcal{G}}(\mathcal{J})$, with a $\mathbb{R}$-valued $\zeta_{\mathcal{G}}(\mathcal{J})$ and a $\mathbb{C}$-representation $\mathcal{J}$ of $\mathcal{G}$: $C(\mathcal{I}) \cong \mathbb{H}$, $\dim_{\mathbb{R}}(C(\mathcal{I})) = 4$.* □

In our case the group $\mathcal{G}$ will only have irreducible over $\mathbb{R}$ representations that remain irreducible over $\mathbb{C}$, so only the case 1 will occur. One can describe $C(\mathcal{I})$ explicitly. In the case 1 it just consists of the scalar matrices λI, $\lambda \in \mathbb{R}$.

For the curious reader, let us give an example of the case 2. Let $\mathcal{G} = \mathbb{Z}_n$, the cyclic group of order n. All its irreducible representations over $\mathbb{C}$ are 1-dimensional. However $\mathbb{Z}_n = \{a^k \mid k = 1, \dots, n\}$ has 2-dimensional representations over $\mathbb{R}$ that are irreducible over $\mathbb{R}$, e.g.

$$a^k \mapsto \begin{pmatrix} \cos 2\pi k/n & \sin 2\pi k/n \\ -\sin 2\pi k/n & \cos 2\pi k/n \end{pmatrix}.$$

This representation $\mathcal{I}$ has the character $\chi(a^k) = 2\cos 2\pi k/n = e^{2\pi \mathbf{i} k/n} + e^{-2\pi \mathbf{i} k/n} = \zeta(a^k) + \overline{\zeta(a^k)}$, where ζ is the character of the 1-dimensional $\mathbb{C}$-representation $a^k \mapsto e^{2\pi \mathbf{i} k/n}$. We can directly check that

$$C(\mathcal{I}) = \left\langle \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}, \quad y, z \in \mathbb{R} \right\rangle.$$

Going back to the general situation, $\mathcal{T}$ is equivalent to a direct sum of irreducible $\mathbb{F}$-representations $\mathcal{T}_k$, i.e.

$$\mathcal{T}(g) \cong \mathcal{T}_1(g) \oplus \mathcal{T}_2(g) \oplus \cdots \oplus \mathcal{T}_\ell(g), \qquad g \in \mathcal{G}.$$

On the other hand, $\mathcal{G}$ has exactly as many irreducible $\mathbb{C}$-representations as it has conjugacy classes, say $c := c_{\mathbb{C}}$. Theorem 4 implies that when $\mathbb{F} = \mathbb{R}$ then the number $c_{\mathbb{F}}$ of $\mathbb{R}$-irreducible representations is at most $c_{\mathbb{C}}$. By rearranging, if necessary, direct summands, and abbreviating

$$k\mathcal{J}(g) = \underbrace{\mathcal{J}(g) \oplus \cdots \oplus \mathcal{J}(g)}_{k \text{ times}},$$

we obtain a decomposition that is called *explicit* in [12, Sect. 2.7].

$$\mathcal{T}(g) \cong m_1\mathcal{T}_1(g) \oplus m_2\mathcal{T}_2(g) \oplus \cdots \oplus m_c\mathcal{T}_c(g), \quad m_i \geq 0,\ 1 \leq i \leq c, \quad g \in \mathcal{G}, \tag{4}$$

[6] A division ring is an algebraic object that is "just like" a field, except that its multiplication need not be commutative. The finite-dimensional division rings that have $\mathbb{R}$ in the center are classified: such a ring is either $\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{H}$. The latter is the famous Hamilton's *algebra of quaternions*.

where $\mathcal{T}_i$ is not equivalent to $\mathcal{T}_j$ when $i \neq j$. The latter implies in particular that any $x \in C(\mathcal{T})$ must respect the coarse block structure provided by the $\mathcal{T}_k$'s; on the other hand x can have a nontrivial action within any $m_k\mathcal{T}_k$-block. The following completely describes the commutant of such a block.

Theorem 5. *Let $\mathcal{I}$ be an irreducible $\mathbb{R}$-representation of $\mathcal{G}$. Then for any $k \geq 1$ one has $C(k\mathcal{I}) = M_k(\mathbb{R}) \otimes C(\mathcal{I})$, where $C(\mathcal{I})$ is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, depending upon $\mathcal{I}$ in accordance with Theorem 4.* □

The formula $C(k\mathcal{I}) = M_k(\mathbb{R}) \otimes C(\mathcal{I})$ just says that each element $x \in C(k\mathcal{I})$ equals the Kronecker product $x = X \otimes Y$, with $X \in M_k(\mathbb{R})$ and $Y \in C(\mathcal{I})$. In the case 1 of Theorem 4, we have $Y = \lambda I$, $\lambda \in \mathbb{R}$.

Further, we will use the following extremely useful *First Orthogonality Relation for characters*, see [12, Thm. I.3]. Let χ, ζ be characters of two representations of $\mathcal{G}$, and define the scalar product of them to be

$$\langle \chi \mid \zeta \rangle = \frac{1}{|G|} \sum_{g \in \mathcal{G}} \chi(g)\overline{\zeta(g)}.$$

Theorem 6. *Let χ, ζ be characters of two nonequivalent irreducible representations of $\mathcal{G}$. Then $\langle \chi \mid \zeta \rangle = 0$, and $\langle \chi \mid \chi \rangle = 1$.* □

Computing the decomposition

Computing the decomposition (4), more precisely, the isomorphism, that is, a matrix M such that $M^{-1}\mathcal{T}(g)M$ has the form as in (4), between the original representation and the representation (4) is greatly helped by the explicit knowledge of each irreducible occurring there. Let $W := \mathcal{T}_k$ for some k be an irreducible representation of $\mathcal{G}$ of dimension ℓ, given by the matrices $(w_{ij}(g))$ for each $g \in \mathcal{G}$. For each $1 \leq \alpha, \beta \leq \ell$ consider the linear map

$$p_{\alpha\beta} = \frac{\ell}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} w_{\beta\alpha}(g^{-1})\mathcal{T}(g),$$

described in [12, Sect. 2.7], see in particular Prop. 8 there. In particular $p_{\alpha\alpha}$ is a projection. Denote its image by V_α. Then $V_\alpha \cap V_\beta = \{0\}$ for $\alpha \neq \beta$. Moreover $\dim V_\alpha = m_k$ and $\dim V_1 \oplus \cdots \oplus V_\ell = m_k\ell$. The matrices $\mathcal{T}(g)$, $g \in \mathcal{G}$, preserve $V = V_1 \oplus \cdots \oplus V_\ell$. Such subspaces are called *$\mathcal{G}$-stable*, because $\mathcal{G}$, or, more precisely $\mathcal{T}(g)$ for any $g \in \mathcal{G}$, maps each vector in V to a vector in V. Each $\mathcal{T}(g)$ on V is equivalent to $m_kW(g)$.

It remains to specify the m_k subspaces V^s of V that are $\mathcal{G}$-stable, so that on each of them $\mathcal{T}(g)$ is equivalent to $W(g)$. Let V_1 be spanned by $x_1, \ldots, x_{m_k}$. Then for $1 \leq s \leq m_k$ the subspace is spanned by $p_{11}(x_s), p_{21}(x_s), \ldots, p_{\ell,1}(x_s)$.

With this information at hand, it is a routine linear algebra to write down an isomorphism M. As we know the natural basis of the commutant for the representation in the form as at the right-hand side of (4), we can apply M^{-1} to it to obtain a basis for the commutant of $\mathcal{T}$.

5 Group symmetric SDP problems

Assume that the following semidefinite programming problem is given

$$p^* := \min_{X \succeq 0} \{ \ \mathrm{tr}(A_0X) \ : \ \mathrm{tr}(A_kX) = b_k, \quad k = 1, \ldots, m \}, \tag{5}$$

where $A_i \in \mathcal{S}_n$ $(i = 0, \ldots, m)$ are given. The associated dual problem is

$$p^* = \max_{y \in \mathbb{R}^m} \{ b^T y \ : \ A_0 - \sum_{i=1}^m y_i A_i \succeq 0 \}. \tag{6}$$

We assume that both problems satisfy the Slater condition so that both problems have optimal solutions with identical optimal values.

Assumption 1 (Group symmetry)**.** *We assume that there is a nontrivial multiplicative group of orthogonal matrices $\mathcal{G}$ such that the associated Reynolds operator*

$$R(X) := \frac{1}{|\mathcal{G}|} \sum_{P \in \mathcal{G}} P X P^T, \ X \in \mathbb{R}^{n \times n}$$

maps the feasible set of (5) into itself and leaves the objective value invariant, i.e.

$$\mathrm{tr}(A_0 R(X)) = \mathrm{tr}(A_0 X) \text{ if } X \text{ is a feasible point of (5).}$$

Since the Reynolds operator maps the convex feasible set into itself and preserves the objective values of feasible solutions, we may restrict the optimization to feasible points in the commutant $\mathcal{A}'$ of $\mathcal{G}$.

Moreover, the next result shows that one may replace the data matrices A_i $(i = 0, \ldots, m)$ in the SDP formulation (5) by their projections $R(A_i)$ $(i = 0, \ldots, m)$ onto the commutant.

Theorem 7. *One has*

$$p^* = \min_{X \succeq 0} \{ \ \mathrm{tr}(R(A_0)X) \ : \ \mathrm{tr}(R(A_k)X) = b_k \quad k = 1, \ldots, m \} .$$

Proof. The proof is an immediate consequence of Assumption 1 and the observation that $\mathrm{tr}(A_i R(X)) = \mathrm{tr}(R(A_i)X)$ for any i. □

It follows that one may also replace the data matrices in the dual problem (6) by their projections onto the commutant .

Corollary 8. *Under Assumption 1, one has*

$$p^* = \max_{y \in \mathbb{R}^m} \{ b^T y \ : \ R(A_0) - \sum_{i=1}^m y_i R(A_i) \succeq 0 \}.$$

If the completely reduced representation of the commutant is known, this may be used to reduce the size of the SDP problem, by block-diagonalizing the matrix variable $S := R(A_0) - \sum_{i=1}^m y_i R(A_i)$ using the procedure described in the previous section.

This idea has been applied most notably by Schrijver [10, 11], for SDP's arising from coding theory, where the centralizer ring is either the Bose-Mesner algebra [10], or the Terwilliger algebra of the Hamming scheme [11].

We will use the same approach below for an example in truss topology optimization.

6 A truss topology optimization problem

We consider a truss defined by a ground structure of nodes and bars. Let m be the number of bars, and assume that free nodes have 3 degrees of freedom.

Let $b \in \mathbb{R}^m$ be the vector of bar lengths, and $z \in \mathbb{R}^m$ the vector of cross-sectional areas. The topology optimization problem (TOP) is to find a truss of minimum volume such that the fundamental frequency of vibration is higher that some prescribed critical value [6]:

$$\begin{aligned} \text{(TOP)} \qquad \min \quad & \sum_{i=1}^{m} b_i z_i \\ \text{s.t.} \quad & S = \sum_{i=1}^{m} (K_i - \bar{\Omega} M_i) z_i - \bar{\Omega} M_0 \\ & z_i \geq 0 \quad i = 1, \ldots, m \\ & S \succeq 0, \end{aligned}$$

where $\bar{\Omega}$ is a lower bound on the (squared) fundamental frequency of vibration of the truss, and M_0 the so-called non-structural mass matrix. If the same nonstructural mass m is added to each free node, the nonstructural mass matrix M_0 is given by

$$M_0 := \frac{m}{3} \sum_{i \text{ is a free node}} E_{ii} \otimes I_3.$$

(We will only consider this case.)

The matrices $z_i K_i$ and $z_i M_i$ are known as *element stiffness* and *element mass matrices*, respectively. If a bar k has endpoints i and j, these matrices are defined as follows:

If i and j are free nodes, then

$$M_k = \frac{\rho b_k}{6} \left(2(E_{ii} + E_{jj}) + E_{ij} + E_{ji}\right) \otimes I_3,$$

where ρ is the mass density of the bars, and b_k the length of bar k; moreover

$$K_k = \frac{\kappa}{b_k} (E_{ii} + E_{jj}) \otimes d_k d_k^T,$$

where d_k is a unit direction vector of the bar k, and κ the elastic modulus.

If i is a free and j a fixed node, then

$$M_k = \frac{\rho b_k}{6} \left(2E_{ii}\right) \otimes I_3.$$

Moreover,

$$K_k = \frac{\kappa}{b_k} E_{ii} \otimes d_k d_k^T.$$

Note that we have not specified the order of the matrices E_{ij}. It will be convenient to assume that the order of the E_{ij} equals the total number of nodes (as opposed to the usual definition that corresponds to the the number of *free* nodes).

This results in zero rows and columns (corresponding to fixed nodes) in the data matrices. For computational purposes, the zero rows and columns may simply be deleted.

6.1 Truss symmetry

We may formally define the symmetry group $\mathcal{G}$ of a given truss, by viewing the ground structure of the truss as the embedding of a graph in $\mathbb{R}^3$ (for space trusses) or in $\mathbb{R}^2$ (for plane trusses). We will in fact only consider space trusses in this paper.

Now $\mathcal{G}$ is defined as the subgroup of graph automorphisms that:

1. are also isometries (i.e. that also preserve edge (i.e. bar) lengths);
2. map free nodes to free nodes and fixed nodes to fixed nodes.

It will be useful to represent $\mathcal{G}$ in two different ways:

1. As a multiplicative group of 3×3 orthogonal matrices r_i $(i = 1, \ldots, |\mathcal{G}|)$ that are bijections of the set of coordinates of the nodes to itself; in other words, the r_i matrices are rotation or reflection matrices.
2. As a group of permutation matrices p_i $(i = 1, \ldots, |\mathcal{G}|)$ corresponding to the permutations of the nodes in the automorphisms.

Lemma 9. *The matrices $P_i := p_i \otimes r_i$ $(i = 1, \ldots, |\mathcal{G}|)$ form an orthogonal, linear representation of $\mathcal{G}$.*

Proof: Follows immediately from Theorem 2. □

Theorem 10. *If K_j (resp. M_j) corresponds to an element stiffness (resp. mass) matrix of the truss structure, and an index $i \in \{1, \ldots, |\mathcal{G}|\}$ is given, then there is an index k such that*

$$P_i K_j P_i^T = K_k,$$

resp.

$$P_i M_j P_i^T = M_k.$$

Moreover, one has $b_j = b_k$.

Proof. If i and j are free nodes and the endpoints of bar k, then

$$M_k = \frac{\rho b_k}{6}\,(2(E_{ii} + E_{jj}) + E_{ij} + E_{ji}) \otimes I_3 =: \frac{\rho b_k}{6}(D_{ij} \otimes I_3),$$

where $D_{ij} := (2(E_{ii} + E_{jj}) + E_{ij} + E_{ji})$.

Now let p be a permutation matrix corresponding to a permutation of the nodes, and let r be the corresponding 3×3 rotation/reflection matrix. Set $P = p \otimes r$.

One has

$$\begin{aligned} PM_kP^T &= \frac{\rho b_k}{6}(p \otimes r)(D_{ij} \otimes I_3)(p^T \otimes r^T) \\ &= \frac{\rho b_k}{6}(p \otimes r)(D_{ij}p^T \otimes r^T) \\ &= \frac{\rho b_k}{6}(pD_{ij}p^T) \otimes (rr^T) \\ &= \frac{\rho b_k}{6}(pD_{ij}p^T) \otimes I_3. \end{aligned}$$

Assume now that p maps bar $k = (i, j)$ to the bar $k' = (i', j')$. Since $b_k = b_{k'}$ we have

$$PM_kP^T = M_{k'}.$$

If i is a free and j a fixed node, then

$$M_k = \frac{\rho b_k}{6}(2E_{ii}) \otimes I_3,$$

and the proof is similar to the previous case.

The proof for the element stiffness matrices is also similar: Let z_kK_k be an element stiffness matrix corresponding to a bar (i, j). If i and j are free nodes, then

$$K_k = \frac{\kappa}{b_k}(E_{ii} + E_{jj}) \otimes d_k d_k^T,$$

where d_k is a unit direction vector of the bar k, b_k the length of the bar k, and κ the elastic modulus. One has

$$\begin{aligned} PK_kP^T &= \frac{\kappa}{b_k}(p \otimes r)((E_{ii} + E_{jj}) \otimes d_k d_k^T)(p^T \otimes r^T) \\ &= \frac{\kappa}{b_k}(p \otimes r)((E_{ii} + E_{jj})p^T \otimes d_k d_k^T r^T) \\ &= \frac{\kappa}{b_k}(p(E_{ii} + E_{jj})p^T) \otimes ((rd_k)(rd_k)^T). \end{aligned}$$

Now use the fact that rd_k is the direction vector of bar $k' := (i', j')$, say $d_{k'}$, i.e.

$$PK_kP^T = \frac{\kappa}{b_{k'}}(E_{i'i'} + E_{j'j'}) \otimes d_{k'}(d_{k'})^T =: K_{k'}.$$

If i is a free node and j a fixed node, then

$$K_k = \frac{\kappa}{b_k}E_{ii} \otimes d_k d_k^T,$$

and the proof is similar as before. □

Corollary 11. *The dual SDP problem of (TOP) satisfies Assumption 1 for the representation $P_1, \ldots, P_{|\mathcal{G}|}$ of the symmetry group $\mathcal{G}$.*

Remark

We may replace the permutation matrices p_i by their principal submatrices indexed by the free nodes. Indeed, this is equivalent to removing the zero rows and columns that correspond to fixed nodes from the data matrices of problem (TOP).

6.2 Reformulating problem (TOP)

By Corollaries 11 and 8, we may replace the data matrices in the formulation of (TOP) by their projections onto the commutant of our linear representation of $\mathcal{G}$ (see Lemma 9) to obtain

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} b_i z_i \\ \text{s.t.} \quad & S = \sum_{i=1}^{m} R(K_i - \bar{\Omega}M_i)z_i - \bar{\Omega}R(M_0) \\ & z_i \geq 0 \quad i = 1, \ldots, m \\ & S \succeq 0, \end{aligned}$$

where the Reynolds operator R is now given by

$$R(X) = \frac{1}{|\mathcal{G}|} \sum_{i=1}^{|\mathcal{G}|} P_i X P_i^T$$

and the P_i's are as described in Lemma 9.

Note that each data matrix corresponds to a bar, except for the nonstructural mass matrix M_0 which is a multiple of the identity. In particular, $R(M_0) = M_0$.

Now consider a bar k. The projection of the data matrix $(K_k - \bar{\Omega} M_k)$ onto the commutant depends only on the orbit of bar k under the action of $\mathcal{G}$. In particular, if bar k belongs to an orbit o (say), then

$$R(K_k - \bar{\Omega} M_k) = \frac{1}{\ell_o} \sum_{i \in o} (K_i - \bar{\Omega} M_i),$$

where ℓ_o is the length of orbit o, i.e. the number of bars in orbit o.

We may therefore replace the variables z_i that belong to the same orbit o by a single variable ζ_o. Moreover, the b_i values for bars belonging to an orbit o are equal and will be denoted by b_o. Finally, denoting the set of orbits by O, we obtain:

$$\begin{array}{ll} \min & \sum_{o \in O} \zeta_o \ell_o b_o \\ \text{s.t.} & S = \sum_{o \in O} \zeta_o \left(\sum_{i \in o} (K_i - \bar{\Omega} M_i) \right) - \bar{\Omega} M_0 \\ & \zeta_o \geq 0 \quad (o \in O) \\ & S \succeq 0. \end{array}$$

If we know the irreducible representation of the commutant, we may obtain an orthogonal matrix Q that block-diagonalizes it (see Theorem 3).

Thus we obtain the final formulation

$$\left.\begin{array}{ll} \min & \sum_{o \in O} \zeta_o \ell_o b_o \\ \text{s.t.} & \sum_{o \in O} \zeta_o Q^T \left(\sum_{i \in o} (K_i - \bar{\Omega} M_i) \right) Q - \bar{\Omega} M_0 \succeq 0 \\ & \zeta_o \geq 0 \quad (o \in O). \end{array}\right\} \tag{7}$$

Note that the number of scalar variables has changed from the number of bars to the number of orbits of bars. The linear matrix inequality has also been block diagonalized, and this may be exploited by interior point solvers for SDP. The actual sizes of the blocks depends on the structure of the group $\mathcal{G}$.

7 A D_n–symmetric dome

Here we consider lattice dome truss structures with the dihedral symmetry group $\mathcal{G} = D_n$. (Recall that the *dihedral group* D_n is the symmetry group of an n-sided regular polygon for $n > 2$.)

The truss in Figure 1 corresponds to the case $n = 6$ (and was studied in [8]), but it is clear that the example may be generalized to all integer values of $n \geq 6$. The free nodes of the truss structure are denoted by filled circles in the figure, and the remaining nodes are fixed. In the general case there will be $4n$ bars, n fixed nodes and $n + 1$ free nodes. Each

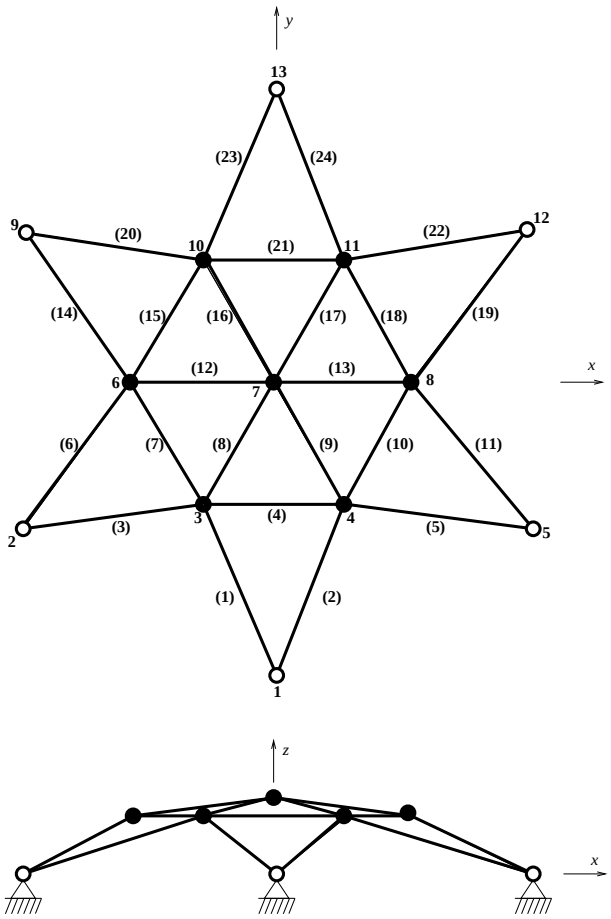

Figure 1: Top and side views of a spherical lattice dome with D_6 symmetry. The black nodes are free and the white nodes fixed.

of the free nodes possesses 3 translational degrees of freedom, giving in total of $3(n+1)$ degrees of freedom for the system.

The symmetry group D_n of the truss consists of n elements corresponding to rotations of the polygon, and n more corresponding to reflections. Therefore we may represent the dihedral group in the following way

$$D_n := \left\{ r\left(\frac{2\pi}{n}k\right), sr\left(\frac{2\pi}{n}k\right) : k = 0, 1, \ldots, n-1 \right\},$$

where $r(\alpha)$ stands for counter–clockwise rotation around the z-axis at an angle α, i.e.

$$r(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and s for the reflections with respect to the xz-plane, i.e.

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We proceed to derive the linear orthogonal representation of $\mathcal{G} = D_n$ as described in Lemma 9. For continuity of presentation, we only state results here; the details we require on different representations of D_n may be found in an appendix to this paper.

Our first representation of D_n is via the rotation/reflection matrices: for $\alpha = 2\pi/n$, set

$$r_i := \begin{cases} r((i-1)\alpha) & : \ i = 1, \ldots, n \\ sr((i-(n+1))\alpha) & : \ i = n+1, \ldots, 2n. \end{cases} \tag{8}$$

This is the representation ρ^1 as defined in (9) in the appendix.

Let p_i $(i = 1, \ldots, 2n)$ be the permutation matrices that correspond to the permutations $\pi_i \in D_n$. Thus the representation of D_n described in Lemma 9 is given by $P_i = p_i \otimes r_i$ $(i = 1, \ldots, 2n)$.

The size the SDP problem (7), is determined by the number of orbits of bars under the action of D_n, and the block sizes of the completely reduced representation of the commutant. The number of orbits equals 3 for the example (independent of n). In particular, the set of bars is tri-partitioned into 3 orbits as follows: those bars connected to a fixed node, those bars connected to the central (hub) node, and the remaining bars (as is clear from Figure 1).

Moreover, we show in Lemma 12 in the appendix that the block sizes of the commutant are as follows:

1. If n is odd: 1 (one block), 4 (one block), and 3 for the remaining $(n-1)/2$ blocks;
2. If n is even: 1 (two blocks), 2 (one block), 4 (one block), and 3 for the remaining $n/2 - 1$ blocks.

Finally, we compare the sizes of the original problem (TOP) and the reduced problem (7) in Table 1 for even values of n.

	# Scalar variables	p.s.d. matrix variable sizes
(TOP)	$4n$	$3(n+1) \times 3(n+1)$
(7)	3	$1 \times 1, 2 \times 2, 4 \times 4, 3 \times 3$ $(n/2 - 1$ times$)$

Table 1: Comparison of the sizes of the SDP problem before (i.e. (TOP)) and after (i.e. (7)) symmetry reduction, for the dome example (for even n).

Note that the biggest gain is in replacing a p.s.d. matrix variable of order $3(n+1)$ by $n/2 + 2$ matrix variables of order at most four. This type of block structure can be exploited very well by interior point methods.

Moreover, the number of nonnegative variables was reduced from $4n$ to only 3. From a computational viewpoint, problem (7) can easily be solved for $n = 1000$, whereas problem (TOP) would be of a size that is very challenging for the present state-of-the-art in interior point method software for semidefinite programming.

Numerical illustration for the case $n = 6$

In order to make the above a bit more concrete, we work out the numerical example for $n = 6$, i.e. for the truss shown in Figure 1, using numerical data from from [8].

The lattice dome in Figure 1 one has 24 bars and 21 degrees of freedom.

Now the rotation r_2 (see (8)) (respectively, the reflection s) corresponds to the permutation $\pi_2 = (3\,4\,8\,11\,10\,6)(7)$ (respectively, to $\pi_7 = (3\,10)(4\,11)(6)(7)(8)$) of the free nodes (our second representation of D_6). The remaining permutations π_i are obtained from π_2 and π_7 by setting $\pi_i = \pi_2^{i-1}$ for $1 \le i \le 6$ and $\pi_i = \pi_7\pi_2^{i-7}$ for $7 \le i \le 12$.

We use nodal coordinates given in Table 2. The material of the members is a particular steel with elastic modulus $\kappa = 205.8$ GPa, and the mass density is $\rho = 7.86 \cdot 10^{-3}$ kg/cm^3. To avoid numerical problems, we scaled κ and ρ so that $\kappa = 1000$ is satisfied. The lower bound on the squared fundamental frequency of vibration the truss $\bar{\Omega}$ is 1000 rad$^2/s^2$.

A nonstructural mass of $2.1 \cdot 10^4$ kg is placed at each free node, and the truss is supported at all the remaining nodes.

The bars are divided into three sets via the orbits of bars under the action of D_6, and these sets are (see Figure 1):

$$\{8, 9, 12, 13, 16, 17\},$$
$$\{4, 7, 10, 15, 18, 21\},$$
$$\{1, 2, 3, 5, 6, 11, 14, 19, 20, 22, 23, 24\}.$$

The irreducible representation of the centralizer ring of our linear orthogonal representation of D_6 was computed using the GAP software [**?**]. This led to a block diagonalization with of the basis of the centralizer ring with blocks of order $3, 1, 2, 1, 8/2$ and $6/2$ respectively. The dimension of the centralizer ring is 40 in this case.

The semidefinite program (7) was solved using the solver SeDuMi [**?**] with the YALMIP interface [**?**]. The optimal cross–sectional areas obtained by solving (7) are given in Table 3.

Node ♯	x	y	z
1	0.00	−500.00	0.00
2	−433.01	−250.00	0.00
3	−125.00	−216.51	62.16
4	125.00	−216.51	62.16
5	433.01	−250.00	0.00
6	−250.00	0.00	62.16
7	0.00	0.00	82.16
8	250.00	0.00	62.16
9	−433.00	250.00	0.00
10	−125.00	216.51	62.16
11	125.00	216.51	62.16
12	433.01	250.00	0.00
13	0.00	500.00	0.00

Table 2: Nodal coordinates (cm) of the 24–bar truss.

ζ_1	ζ_2	ζ_3
49.9444	32.8444	33.5703

Table 3: Optimal cross-sectional areas for D_6–symmetric dome.

8 Conclusion and discussion

We have shown how the semidefinite programming (SDP) formulation of a specific truss topology design problem may be reduced in size by exploiting symmetry conditions.

The approach we used to reformulate the (SDP) is due to Schrijver [10, 11] and described in [4].

An alternative approach for exploiting algebraic symmetry in SDP's was introduced in [3], which uses the so called regular $*$-representation of the commutant as opposed to its completely reduced representation. The advantage is that the regular $*$-representation can be readily obtained (this is not the case for the irreducible representation.) The disadvantage is that it gives less reduction in general than the irreducible representation. Indeed, for the dome example presented in this paper the regular $*$-representation does not give any reduction in problem size.

Appendix: Representations and characters of the dihedral group.

A complete description of irreducible $\mathbb{C}$-representations of the dihedral group D_n can be found in [12, Sect. 5.3]. Here we need to adapt it slightly to our case. We are helped by the fact that all the $\mathbb{C}$-representations of D_n are in fact equivalent to $\mathbb{R}$-representations. The order of D_n is $2n$, and it is generated by the rotation r through the angle $2\pi/n$ and any reflection s (i.e. a linear transformation of order 2 fixing a hyperplane (in this case, a line) through the origin) that preserves the regular n-gon rotated by r. So we know that $r^n = s^2 = (sr)^2 = 1$. Note that it follows that $(sr^k)^2 = 1$ for all k. Each element of D_n can be uniquely written either in the form r^k, or in the form sr^k, for $0 \le k \le n-1$.

The irreducible representations of D_n. The 2-dimensional irreducible $\mathbb{R}$-representation ρ^h of D_n is given by

$$\rho^h(r^k) = \begin{pmatrix} \cos\frac{2\pi hk}{n} & \sin\frac{2\pi hk}{n} \\ -\sin\frac{2\pi hk}{n} & \cos\frac{2\pi hk}{n} \end{pmatrix}, \quad \rho^h(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho^h(sr^k) = \rho^h(s)\rho^h(r^k). \tag{9}$$

It is straightforward to compute its character

$$\chi_h(r^k) = 2\cos 2\pi hk/n, \qquad \chi_h(sr^k) = 0.$$

It obviously depends upon $h \mod n$, not $h \in \mathbb{Z}$. Moreover, as $\chi_h = \chi_{n-h}$, we have that ρ^h is equivalent to ρ^{n-h}. Moreover, when $h = 0$ or $n = 2h$ the representation ρ^h is reducible. So we can take $1 \le h \le \lfloor n/2 \rfloor$ to parameterize the representations ρ^h uniquely.

The number of 1-dimensional irreducible $\mathbb{R}$-representation ρ^h of D_n is four, when n is even, and two, when n is odd. In both cases D_n has two representations (and the characters, as in the 1-dimensional case it is the same thing) ψ_1 (the *trivial representation*), and ψ_2 given by

$$\psi_1(r^k) = \psi_1(sr^k) = 1, \qquad \psi_2(r^k) = -\psi_2(sr^k) = 1, \qquad 0 \le k \le n-1.$$

In the case n even D_n has two more 1-dimensional representations, ψ_3 and ψ_4 given by

$$\psi_3(r^k) = \psi_3(sr^k) = (-1)^k, \qquad \psi_4(r^k) = -\psi_4(sr^k) = (-1)^k, \qquad 0 \le k \le n-1.$$

The natural permutation representation θ of D_n. We need to determine the decomposition of the character χ_θ of the permutation representation θ of D_n acting on the n-gon into the irreducibles ψ_i and ρ^h just described. It turns out that they can occur at most once – one says that θ is *multiplicity-free*. Indeed, θ satisfies the well-known sufficient condition for multiplicity-freeness, cf. e.g. [1], that its 2-orbits, i.e. the orbits on pairs of elements of $\{1,\dots,n\}$, are *symmetric*. The latter means that for any $i,j \in \{1,\dots,n\}$ there exists $g \in D_n$ such that $g(i)=j$, and $g(j)=i$. However, not all of ψ_i and ρ^h will occur - there are simply too many of them. In order to determine the decomposition, we compute χ_θ. After this, we can use Theorem 6 to see which irreducibles occur in χ_θ. (Even without computing, we know that ψ_1, the trivial character, will occur in χ_θ, as θ is a permutation representation, so all the group elements fix the sum of coordinates, an invariant 1-dimensional subspace.)

As χ_θ is the character of a permutation representation, $\chi_\theta(g)$ is simply the number of fixed points of g, when it is considered as a permutation. So in particular $\chi_\theta(r^k)=0$ for all $1 \le k < n$, and $\chi_\theta(r^n)=n$. The case $\chi_\theta(sr^k)$ needs to be treated separately for n odd, resp. n even. In the former case $\chi_\theta(sr^k)=1$.

In the latter case $\chi_\theta(sr^k)$ depends upon the parity of k. The elements of the form sr^k are split into two different conjugacy classes, each of size $n/2$; the elements of first (resp. second) class fix two opposite vertices (resp. edges) of the n-gon, so they have 2 (resp. 0) fixed vertices. So we have $\chi_\theta(sr^{2k})=2$ and $\chi_\theta(sr^{2k+1})=0$ for $0 \le k \le n/2-1$. To summarise:

$$\begin{aligned}
&\chi_\theta(1)=n,\ \chi_\theta(r^m)=0, && 1 \le m \le n-1,\ n \in \mathbb{Z}_+\\
&\chi_\theta(sr^k)=1, && 0 \le k \le n-1,\ n \in \mathbb{Z}_+ - 2\mathbb{Z},\\
&\chi_\theta(sr^k)=1+(-1)^k, && 0 \le k \le n-1,\ n \in 2\mathbb{Z}_+.
\end{aligned}$$

We have $\langle \chi_\theta \mid \psi_2 \rangle = 0$, and Theorem 6 implies that ψ_2 does not occur in θ. Thus for n odd we obtain

$$\chi_\theta = \psi_1 + \sum_{h=1}^{(n-1)/2} \chi_h, \qquad n \in \mathbb{Z}_+ - 2\mathbb{Z}. \tag{10}$$

For n even we compute $\langle \chi_\theta \mid \psi_3 \rangle = 1$, and obtain

$$\chi_\theta = \psi_1 + \psi_3 + \sum_{h=1}^{n/2-1} \chi_h, \qquad n \in 2\mathbb{Z}_+. \tag{11}$$

The representation of D_n used in the dome truss example

Here, we construct the representation $\mathcal{P}$ of $\mathcal{G}=D_n$ as

$$\mathcal{P} = (\psi_1 \oplus \theta) \otimes (\psi_1 \oplus \rho^1), \tag{12}$$

where, as before, ψ_1 denotes the trivial 1-dimensional representation, θ the natural permutation representation, and ρ^1 is as in (9). Note that the representation $\psi_1 \oplus \rho^1$ is equivalent to the one described in (8) for $\alpha = 2\pi/n$.

In order to analyze the block structure of the commutant, as described in Theorem 8, we should find a decomposition of $\mathcal{P}$ into irreducibles. It suffices to compute the decomposition

of the character $\chi_{\mathcal{P}}$ into irreducible characters using (3). Thus, expanding the tensor product, we obtain

$$\mathcal{P} = \psi_1 \oplus \theta \oplus \rho^1 \oplus \bigoplus_h (\rho^h \otimes \rho^1) \oplus \rho^1 \oplus (\psi_3 \otimes \rho^1).$$

The $\oplus$-summation index h ranges as in (10) for n odd, resp. as in (11) for n even, using $t := n$. The last term in this decomposition is 0 when t is odd. Otherwise, computing the character,

$$(\chi_1 \otimes \psi_3)(r^k) = 2(-1)^k \cos\frac{2\pi k}{n} = 2\cos\left(\pi k - \frac{2\pi k}{n}\right) = \chi_{\frac{n}{2}-1}(r^k),$$

we obtain $\psi_3 \otimes \rho^1 = \rho^{\frac{n}{2}-1}$. Similarly one computes

$$(\chi_1 \otimes \chi_h)(r^k) = 4\cos\frac{2\pi k}{n}\cos\frac{2\pi hk}{n} = 2\cos\frac{2\pi(h+1)k}{n} + 2\cos\frac{2\pi(h-1)k}{n} = (\chi_{h+1} + \chi_{h-1})(r^k),$$

deriving $\rho^h \otimes \rho^1 = \rho^{h+1} \oplus \rho^{h-1}$ as long as χ_{h+1} are χ_{h-1} are defined and irreducible. When $h = 1$ we further decompose $\rho^{h-1} = \psi_1 \oplus \psi_2$. When n is even and $h = n/2 - 1$ we further decompose $\rho^{n/2} = \psi_3 \oplus \psi_4$, and when n is odd and $h = (n-1)/2$ we get $\rho^{h+1} = \rho^{n-h-1} = \rho^{(n-1)/2}$.

To summarize, we have the following proposition.

Lemma 12. *Consider the representation $\mathcal{P}$ of D_n, given by (12).*

For n odd, one has:

$$\mathcal{P} = 3\psi_1 \oplus \psi_2 \oplus \rho^1 \oplus 3\bigoplus_{h=1}^{(n-1)/2} \rho^h.$$

The block sizes of the commutant are thus 1 (for ψ_2), 4 (for ρ^1) and 3 for the remaining $(n-1)/2$ irreducibles. Respectively, the dimension of the commutant is $1+4^2+3^2(n-1)/2$.

For n even, one has:

$$\mathcal{P} = 3\psi_1 \oplus \psi_2 \oplus 2\psi_3 + \psi_4 \oplus \rho^1 \oplus 3\bigoplus_{h=1}^{n/2-1} \rho^h.$$

The block sizes of the commutant are thus 1 (for ψ_2 and ψ_4), 2 (for ψ_3), 4 (for ρ^1), and 3 for the remaining $n/2-1$ irreducibles. Respectively, the dimension of the commutant is $2 + 2^2 + 4^2 + 3^2(n/2-1)$.

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