Applications of bulk queues to group testing models with incomplete identification

Bar-Lev, S.K.; Parlar, M.; Perry, D.; Stadje, W.; van der Duyn Schouten, F.A.

Published in:
European Journal of Operational Research

Publication date:
2007

Available online at www.sciencedirect.com
ScienceDirect
European Journal of Operational Research 183 (2007) 226–237
EUROPEAN JOURNAL OF OPERATIONAL RESEARCH
www.elsevier.com/locate/ejor

Stochastics and Statistics

Applications of bulk queues to group testing models with incomplete identification

Shaul K. Bar-Lev [a], Mahmut Parlar [b,1], David Perry [c], Wolfgang Stadje [d,*], Frank A. Van der Duyn Schouten [e]

[a] *Department of Statistics, University of Haifa, Haifa 31905, Israel*
[b] *DeGroote School of Business, McMaster University, Hamilton, Ont., Canada L8S 4M4*
[c] *Department of Statistics, University of Haifa, Haifa, 31905, Israel*
[d] *Department of Mathematics and Computer Science, University of Osnabrück, 49069 Osnabrück, Germany*
[e] *Center for Economic Research, Tilburg University, 5000 LE Tilburg, The Netherlands*

Received 13 April 2006; accepted 25 September 2006
Available online 21 December 2006

Abstract

A population of items is said to be ''group-testable'', (i) if the items can be classified as ''good'' and ''bad'', and (ii) if it is possible to carry out a simultaneous test on a batch of items with two possible outcomes: ''Success'' (indicating that all items in the batch are good) or ''failure'' (indicating a contaminated batch). In this paper, we assume that the items to be tested arrive at the group-testing centre according to a Poisson process and are served (i.e., group-tested) in batches by one server. The service time distribution is general but it depends on the batch size being tested. These assumptions give rise to the bulk queueing model $M/G^{(m,M)}/1$, where m and $M(>m)$ are the decision variables where each batch size can be between m and M. We develop the generating function for the steady-state probabilities of the embedded Markov chain. We then consider a more realistic finite state version of the problem where the testing centre has a finite capacity and present an expected profit objective function. We compute the optimal values of the decision variables (m, M) that maximize the expected profit. For a special case of the problem, we determine the optimal decision explicitly in terms of the Lambert function.

Keywords: Applied probability; Queueing; Quality control

1. Introduction

Consider a population of items, each of which can be classified into one of two categories: good with probability q or defective with probability $p = 1 - q$. These items are said to be group-testable if for any subset of them it is possible to carry out a simultaneous test (group test) with two possible outcomes: ''success'',

* Corresponding author. Tel.: +49 541 9692530; fax: +49 541 9692770.
E-mail address: wolfgang@mathematik.uni-osnabrueck.de (W. Stadje).
[1] Research supported by the Natural Sciences and Engineering Research Council of Canada.

0377-2217/$ - see front matter
doi:10.1016/j.ejor.2006.09.086

indicating that all items in the subset are good, and "failure", indicating a contaminated subset, i.e., at least one of the items is defective without knowing which or how many are defective. Traditional group-testing policies aim at a "complete identification" of the population, i.e., to separate all defective units from the good units. Obviously, there are many policies which achieve this goal. An optimal policy is usually defined to be one for which the expected number of group tests is minimized.

Dorfman [11] was the first to introduce a group-testing policy for detecting syphilis in blood samples taken from US draftees during the Second World War. His suggestion was to pool a fixed number of blood samples and test the pooled sample, instead of individual samples, for the presence of the syphilis. Since the prevalence rate of syphilis in the draftees' population was small, great savings were obtained when syphilis was only present in very few pooled samples. If the disease was present then all individual samples were tested separately. Since the pioneering work of Dorfman [11], numerous variants of group testing procedures have been introduced and discussed in the literature. For a comprehensive review and key references on the subject the reader is referred to the monograph by Du and Hwang [12].

Group testing procedures are applied in medicine for analyzing blood or urine samples to detect HIV, the variants of hepatitis, as well as other viruses and bacteria. Applications to HIV screening are given, among others, by Hammick and Gastwirth [13], Litvak, Tu and Pagano [15], Tu, Litvak and Pagano [21], Wein and Zenios [23]. Hung and Swallow [14] used binomial grouping in hypotheses testing for the classification of quantitative covariables. Combinatorial questions in the context of DNA library screening were studied by Macula [16,17]. Various group testing models with features like incomplete identification, refined cost functionals, optimization under probabilistic constraints and testing errors have been studied by Bar-Lev, Stadje and Van der Duyn Schouten [4–7]. These papers also provide a detailed discussion of the literature.

Recently, Abolnikov and Dukhovny [2,3] studied deterministic and stochastic models for optimal group testing with complete identification (i.e., with exact determination of all contaminated items). In particular, they considered static optimization for pooled screening of a fixed population followed by individual or by subgroup testing. For randomly arriving items and random testing times they proposed a queueing approach of which the model underlying our analysis is a special case. In contrast to Abolnikov and Dukhovny [2,3] we are interested in optimization under incomplete identification, which is based on the determination of several long-run average performance functionals.

In this study we consider a queueing model, denoted by $M/G^{(m,M)}/1$, and apply it to group testable items. Each item, independently of the others, can be either good with probability q or defective with probability $p = 1 - q$. Items arrive in a storage system according to a Poisson process with rate λ and are served (tested) in batches (groups) by one server (kit). The service time distribution is general and may depend on the batch size being served. Each batch size can range between m and M, where both m and M are considered to be decision variables satisfying $1 < m \leqslant M \leqslant M_0$, for some pre-known kit capacity M_0. If there are less than m items present at a service termination, the system waits until there are m ones and then starts serving them in a batch. If there are between m and M items present, all items are tested together as a group, and if their number exceeds M, a group of M is tested next. The service time distribution of a batch of size j is G_j and has mean $1/\mu_j$, $j = m, \ldots, M$. We assume that the G_j's, as well as q, are known. For every served batch there are two possible outcomes: "clean", implying that all items in the group are good, or "contaminated", implying that at least one of them in the group tested has to be defective. Batches which are found clean are kept and recorded for meeting a possible demand requirement. Contaminated batches are set aside but recorded for, perhaps, other possible uses.

The above bulk queueing model was suggested by Abolnikov and Dukhovny [2,3] with the additional feature of i.i.d. batch arrivals. In accordance with their aim of complete identification they considered service time distributions of a special form. In the case of single item arrivals considered here this system has already been studied in detail by Nair and Neuts [18] and Neuts [19]. Bulking is of course a classical topic in queueing theory; see e.g., Chaudhry and Templeton [9,10] and the more recent Bocharov et al. [8]. From the queueing analysis we need for our purposes only the steady-state distribution of the embedded Markov chain of the queue lengths left behind by departing customers. Neuts [19] gives much more general analytical results, which however seem intractable for optimization purposes and in particular for numerical calculations. Below we briefly indicate how to compute the steady-state distribution by standard queueing methods. This short derivation is

similar to that in Abolnikov and Dukhovny [2,3]; relations (6) and (7) can be obtained as special cases from Theorem 5.1 in Abolnikov and Dukhovny [1] (see also their Theorem 2 in [2]). We then specialize to the case of a finite waiting room, which seems general enough to cover group testing applications. The important new feature of this paper is the detailed presentation of several long-run average profit and cost functionals which are needed to assess the efficiency of the system in the case of incomplete identification: acquisition cost, testing cost, revenue from sales of good items and revenue from sales of contaminated batches. We show that all these functionals can be expressed in terms of the steady-state distribution of the embedded Markov chain. These results pave the way towards numerical optimization. Our decision variables are the threshold m for the minimum size of a batch that is pooled and the maximum group size M.

In this paper, we do not consider complete identification which would require retesting of all items in those groups that have been found contaminated. As is reflected in our choice of profit and cost functionals, our objective is to study the group testing system from a purely economic (profit-raising) point of view. In many medical applications retesting is called for because the aim is to establish a diagnosis for all patients involved. However, in industrial and also in blood bank applications the further processing of contaminated groups may not be required or even possible, for example because items can be unavailable for retesting or they would be destroyed in the process. Moreover, there may be a residual economic value, however reduced, to items belonging to contaminated groups.

The paper is organized as follows. In Section 2 we present a detailed and formal description of the queueing model and determine the steady-state distribution of the embedded Markov chain. The core of this paper is Section 3 in which all pertinent profit and cost functionals are expressed in terms of this distribution. We take into account purchase costs of arriving items, testing costs of batches, idleness costs, and the revenues from sales of clean as well as contaminated items. The resulting optimization problem aims at searching for the optimal lower and upper batch sizes m and M that maximize the total net reward. Since only numerical optimization seems possible, we present in Section 4 some computational examples, including an analysis of the dependence of the objective function and the optimal solutions on the system parameters.

2. Model description and related functionals

Consider the queueing model $M/G^{(m,M)}/1$ as described in the previous section. Let X_n, $n \in \mathbb{N}$, be the number of items left behind in line immediately after the departure of the nth batch. Then, if $X_n < m$ the server will wait until at least m items are accumulated in line. When the number of items becomes m, a batch service of size m is immediately started. If $m < X_n \leqslant M$ all items in line will start immediately their batch service. Finally, if $X_n > M$, then M items will immediately start service. Let $Y_n^{(j)}$, $n \in \mathbb{N}_0$, $j \in A_{m,M} \doteq \{m, m+1, \ldots, M\}$ be the number of items arrived into the system during a service of type j of the nth batch. The dynamics of $\mathbf{X} = \{X_n : n \geqslant 0\}$ is then given by the recursion

$$X_{n+1} = \begin{cases} Y_{n+1}^{(m)}, & \text{if } X_n = 0, 1, 2, \ldots, m, \\ Y_{n+1}^{(m+1)}, & \text{if } X_n = m+1, \\ \vdots & \vdots \\ Y_{n+1}^{(M)}, & \text{if } X_n = M, \\ X_n - M + Y_{n+1}^{(M)}, & \text{if } X_n = M+1, \ldots \end{cases}$$

Due to the memoryless property of the exponential distribution, the random variables $Y_n^{(j)}$, $n \in \mathbb{N}_0$, $j \in A_{m,M}$, are independent, while for any fixed j they are also identically distributed. Moreover, they are independent of $\{X_i : i \leqslant n-1\}$. Consequently, the process $\mathbf{X} = \{X_n : n \geqslant 1\}$, embedded at departure times of batches, is a Markov chain. Moreover, $\mathbf{X}$ is a stable process if and only if

$$\frac{\lambda}{M\mu_M} < 1, \tag{1}$$

see Neuts [19, Theorem 2]. The transition probability matrix of the Markov chain $\mathbf{X}$ is given by

$$\mathbf{P}=\begin{bmatrix} a_0^{(m)} & a_1^{(m)} & a_2^{(m)} & \cdots & a_m^{(m)} & a_{m+1}^{(m)} & \cdots & a_M^{(m)} & a_{M+1}^{(m)} & a_{M+2}^{(m)} & \cdots \\ a_0^{(m)} & a_1^{(m)} & a_2^{(m)} & \cdots & a_m^{(m)} & a_{m+1}^{(m)} & \cdots & a_M^{(m)} & a_{M+1}^{(m)} & a_{M+2}^{(m)} & \cdots \\ a_0^{(m)} & a_1^{(m)} & a_2^{(m)} & \cdots & a_m^{(m)} & a_{m+1}^{(m)} & \cdots & a_M^{(m)} & a_{M+1}^{(m)} & a_{M+2}^{(m)} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ a_0^{(m)} & a_1^{(m)} & a_2^{(m)} & \cdots & a_m^{(m)} & a_{m+1}^{(m)} & \cdots & a_M^{(m)} & a_{M+1}^{(m)} & a_{M+2}^{(m)} & \cdots \\ a_0^{(m+1)} & a_1^{(m+1)} & a_2^{(m+1)} & \cdots & a_m^{(m+1)} & a_{m+1}^{(m+1)} & \cdots & a_M^{(m+1)} & a_{M+1}^{(m+1)} & a_{M+2}^{(m+1)} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ a_0^{(M)} & a_1^{(M)} & a_2^{(M)} & \cdots & a_m^{(M)} & a_{m+1}^{(M)} & \cdots & a_M^{(M)} & a_{M+1}^{(M)} & a_{M+2}^{(M)} & \cdots \\ 0 & a_0^{(M)} & a_1^{(M)} & \cdots & a_{m-1}^{(M)} & a_m^{(M)} & \cdots & a_{M-1}^{(M)} & a_M^{(M)} & a_{M+1}^{(M)} & \cdots \\ 0 & 0 & a_0^{(M)} & \cdots & a_{m-2}^{(M)} & a_{m-1}^{(M)} & \cdots & a_{M-2}^{(M)} & a_{M-1}^{(M)} & a_M^{(M)} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (2)$$

where the rows and columns are indexed as $0, 1, 2, \ldots, m, m+1, \ldots, M, M+1, M+2, \ldots$and

$$a_k^{(j)}=\int_0^\infty \frac{e^{-\lambda t}(\lambda t)^k}{k!}\,dG_j(t), \quad k\in\mathbb{N}_0, \quad j\in A_{m,M}$$

is the probability of k arrivals during a service of type j.

It is useful to note here that the probabilities $a_k^{(j)}$ can be computed using the Laplace transform $\tilde{g}_j(s)=\int_0^\infty e^{-st}\,dG_j(t)$ of the service time distribution $G_j(t)$. Denote by $A^{(j)}(z)=\sum_{k=0}^\infty a_k^{(j)}z^k$ the generating function of $Y_n^{(j)}$. This simplifies to

$$\begin{aligned} A^{(j)}(z)&=\sum_{k=0}^\infty a_k^{(j)}z^k=\sum_{k=0}^\infty z^k\int_0^\infty \frac{e^{-\lambda t}(\lambda t)^k}{k!}\,dG_j(t)=\int_0^\infty e^{-\lambda t}\left(\sum_{k=0}^\infty \frac{(\lambda tz)^k}{k!}\right)dG_j(t)\\ &=\int_0^\infty e^{-\lambda t(1-z)}\,dG_j(t)=\tilde{g}_j(\lambda(1-z)). \end{aligned}$$

Thus, the required probabilities $a_k^{(j)}$ can be generated using the relation

$$a_k^{(j)}=\frac{1}{k!}\frac{d^k\tilde{g}_j(\lambda(1-z))}{dz^k}\bigg|_{z=0}, \quad k=0,1,\ldots$$

For example, if $Y_n^{(j)}$ is Erlang(j, μ)—as we will assume in Section 4—then the LT is given by

$$\tilde{g}_j(\lambda(1-z))=\left(\frac{\mu}{\mu+\lambda(1-z)}\right)^j$$

which can be easily differentiated with respect to z.

Let π_i, $i = 0, 1,\ldots$be the stationary probabilities of $\mathbf{X}$ so that

$$\pi_i=\lim_{n\to\infty}\Pr(X_n=i), \quad i=0,1,\ldots,$$

and denote by $\Pi(z)=\sum_{i=0}^\infty \pi_i z^i$ the corresponding generating function. $\Pi(z)$ can be determined as follows.

Assuming that (1) holds, the stationary probabilities satisfy the set of linear equations

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j = 0, 1, \ldots \tag{3}$$

$$\sum_{j=0}^{\infty} \pi_j = 1. \tag{4}$$

The equation for $j = 0$ contains only the probabilities $\pi_0, \ldots, \pi_M$:

$$\pi_0 = a_0^{(m)}\pi_0 + \cdots + a_0^{(m)}\pi_m + a_0^{(m+1)}\pi_{m+1} + \cdots + a_0^{(M)}\pi_M. \tag{5}$$

After a short calculation (3) yields

$$\Pi(z) = A^{(m)}(z)\sum_{i=0}^{m}\pi_i + \sum_{i=m+1}^{M}\pi_i A^{(i)}(z) + \frac{A^{(M)}(z)}{z^M}\left(\Pi(z) - \sum_{i=0}^{M}\pi_i z^i\right).$$

Isolating $\Pi(z)$ we obtain

$$\Pi(z) = \frac{1}{A^{(M)}(z) - z^M}\left(A^{(M)}(z)\sum_{i=0}^{M}\pi_i z^i - z^M A^{(m)}(z)\sum_{i=0}^{m}\pi_i - \sum_{i=m+1}^{M}\pi_i A^{(i)}(z)\right). \tag{6}$$

Clearly, (6) provides an expression for $\Pi(z)$ in terms of its first $M+1$ coefficients $\pi_0, \ldots, \pi_M$, which we have to determine.

Let us assume that $A^{(M)}(z) = \sum_{n=0}^{\infty} a_n^{(M)} z^n$ has a radius of convergence larger than 1; by our assumptions it also satisfies $a_n^{(M)} > 0$, $\sum_{n=0}^{\infty} a_n^{(M)} = 1$ and $(\mathrm{d}/\mathrm{d}z)A^{(M)}(1) = \sum_{n=0}^{\infty} n a_n^{(M)} < M$. Then, Rouché's Theorem (Saaty [20, p. 87]) implies that $A^{(M)}(z) - z^M$ has exactly M zeros with absolute value $\leqslant 1$. To prove this, set $f(z) = z^M$ and $g(z) = -A^{(M)}(z)$. On the circle $|z| = 1 + \varepsilon$ in the complex plane we have, for small $\varepsilon > 0$,

$$|g(z)| = |A^{(M)}(z)| \leqslant A^{(M)}(|z|) = A^{(M)}(1+\varepsilon) = 1 + \frac{\mathrm{d}}{\mathrm{d}z}A(^{(M)}1)[\varepsilon + \mathrm{o}(\varepsilon)] < 1 + M\varepsilon < (1+\varepsilon)^M = |f(z)|.$$

Thus, by Rouché's Theorem, the function $f(z) + g(z)$ has the same number of roots inside the circle $|z| = 1 + \varepsilon$ as $f(z)$. Letting $\varepsilon \to 0$ we see that $A^{(M)}(z) - z^M$ has exactly M zeros satisfying $|z| \leqslant 1$. One of them is of course $z = 1$.

Denote the roots by $z_0 = 1, z_1, \ldots, z_{M-1}$ and assume that they are distinct. The numerator on the right side of (6) has to vanish for $z \in \{z_1, \ldots, z_{M-1}\}$, which gives us for the unknowns $\pi_0, \ldots, \pi_M$ the $M-1$ linear equations

$$A^{(M)}(z_k)\sum_{i=0}^{M}\pi_i z_k^i - z_k^M A^{(m)}(z_k)\sum_{i=0}^{m}\pi_i - \sum_{i=m+1}^{M}\pi_i A^{(i)}(z_k) = 0, \quad k = 1, \ldots, M-1. \tag{7}$$

Setting $z = 1$ in (6) the numerator vanishes trivially, but factoring out $z - 1$ in the numerator and in the denominator (i.e., applying l'Hôpital's rule) gives us one more equation because $\Pi(1) = 1$. Together with (5) and (7) we arrive at $M+1$ linear equations from which $\pi_0, \ldots, \pi_M$ can be computed. In the case of multiple roots one has to take derivatives in (6) to obtain the necessary number of equations.

This general solution requires the calculation of the roots of $A^{(M)}(z) - z^M$, which in practice can result in numerical inaccuracies especially when the decision variable M assumes a large value. Therefore we examine in Section 4 the finite-state case of the problem in which the (waiting room) capacity of the group testing center is bounded by a finite number B. For example, when $(m, M) = (2, 4)$ and $B = 7$, the transition matrix assumes the form

$$\mathbf{P}_{8\times 8}=\begin{bmatrix} a_0^{(2)} & a_1^{(2)} & a_2^{(2)} & a_3^{(2)} & a_4^{(2)} & a_5^{(2)} & a_6^{(2)} & 1-\sum_{j=0}^{6} a_j^{(2)} \\ a_0^{(2)} & a_1^{(2)} & a_2^{(2)} & a_3^{(2)} & a_4^{(2)} & a_5^{(2)} & a_6^{(2)} & 1-\sum_{j=0}^{6} a_j^{(2)} \\ a_0^{(2)} & a_1^{(2)} & a_2^{(2)} & a_3^{(2)} & a_4^{(2)} & a_5^{(2)} & a_6^{(2)} & 1-\sum_{j=0}^{6} a_j^{(2)} \\ a_0^{(3)} & a_1^{(3)} & a_2^{(3)} & a_3^{(3)} & a_4^{(3)} & a_5^{(3)} & a_6^{(3)} & 1-\sum_{j=0}^{6} a_j^{(3)} \\ a_0^{(4)} & a_1^{(4)} & a_2^{(4)} & a_3^{(4)} & a_4^{(4)} & a_5^{(4)} & a_6^{(4)} & 1-\sum_{j=0}^{6} a_j^{(4)} \\ 0 & a_0^{(4)} & a_1^{(4)} & a_2^{(4)} & a_3^{(4)} & a_4^{(4)} & a_5^{(4)} & 1-\sum_{j=0}^{5} a_j^{(4)} \\ 0 & 0 & a_0^{(4)} & a_1^{(4)} & a_2^{(4)} & a_3^{(4)} & a_4^{(4)} & 1-\sum_{j=0}^{4} a_j^{(4)} \\ 0 & 0 & 0 & a_0^{(4)} & a_1^{(4)} & a_2^{(4)} & a_3^{(4)} & 1-\sum_{j=0}^{3} a_j^{(4)} \end{bmatrix}$$

where the rows and columns are indexed as 0, 1, ..., 7.

When the testing centre capacity B assumes large values, the problem becomes computationally very challenging since the entries $a_k^{(j)}$ of the transition matrix $\mathbf{P}$ of (2) require symbolic differentiation of the LT $\tilde{g}_j(\lambda(1-z))$. In order to construct the matrix $\mathbf{P}$ and to solve for the stationary probabilities we have employed

Table 1
CPU time to compute the stationary probabilities for large values of B

B	m	M	CPU time (seconds)
100	60	80	2.2
300	180	240	46.1
500	300	400	268.2

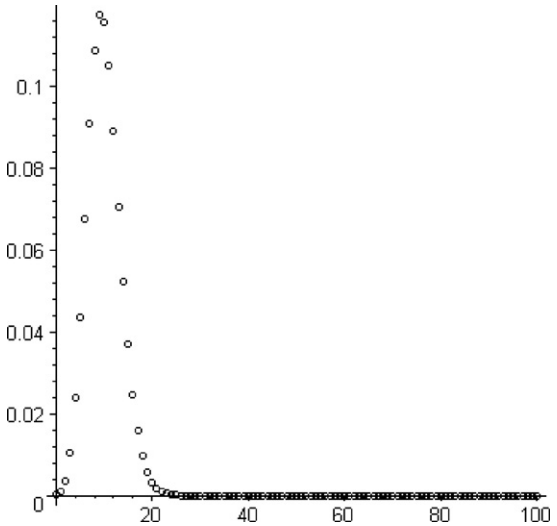

Fig. 1. The stationary distribution π_i, $i=0, 1, \ldots, B$ when $B=100$ and $(m, M)=(60, 80)$.

the computer algebra system Maple 10 http://www.maplesoft.com/products/maple/ which can perform symbolic differentiation of expressions and solve large-scale linear systems. As an example of a non-trivial problem we considered a case where items arrive according to a Poisson process with rate $\lambda = 0.5$ units per unit time and where the time to complete the testing of k items is assumed to be Erlang (k, μ) with $\mu = 3$ completions per unit time. We solved this problem on a Pentium 4 machine running on Windows XP with 2.53 GHz CPU and 512 MB memory for three different values of B and calculated the stationary probabilities. Our numerical calculations were extremely accurate—the solution of the linear system $\pi_j = \sum_{i=0}^{B} \pi_i P_{ij}$, $j = 0,1,\ldots,B$ with $\sum_{j=0}^{B} \pi_j = 1$ produced nonnegative probabilites which added up to 0.9999999 even for the case with $B = 500$. See Table 1 for the input values of B and (m, M) and the corresponding CPU times to compute the probabilities; also see Fig. 1 for the plot of the probabilities when $B = 100$ with $(m, M) = (60, 80)$.

3. The objective function

Let $Z_{m,M}$ be the size of a tested batch in steady-state, taking values in $\{m, m+1, \ldots, M\}$ with probabilities

$$P(Z_{m,M} = j) = \begin{cases} \sum_{i=0}^{m} \pi_i, & \text{if } j = m, \\ \pi_j, & \text{if } m+1 \leqslant j \leqslant M-1, \\ \sum_{i=M}^{\infty} \pi_i, & \text{if } j = M, \end{cases}$$

In the objective function we need its expected value $E(Z_{m,M})$ which is clearly given by

$$E(Z_{m,M}) = m \sum_{i=0}^{m} \pi_i + \sum_{i=m+1}^{M-1} i\pi_i + M \sum_{i=M}^{\infty} \pi_i, \tag{8}$$

where $\sum_{i=M}^{\infty} \pi_i = 1 - \sum_{i=0}^{M-1} \pi_i$.

The objective function of expected net profit [denoted by $\mathscr{P}(m, M)$] is based on the long-run average criterion and is defined as follows:

$$\mathscr{P}(m, M) = G(m, M) + C(m, M) - A(m, M) - T(m, M) - I(m, M), \tag{9}$$

where $G(m, M)$ is the expected revenue per time unit obtained from the sales of "Good" items, $C(m, M)$ is the expected revenue per time unit obtained from the sales of "Contaminated" items, $A(m, M)$ is the "Acquisition" cost of the incoming items per time unit, $T(m, M)$ is the "Testing" cost per time unit and $I(m, M)$ is the "Idleness" cost per time unit.

In the following we assume $B < \infty$, i.e., the waiting room is finite. We now elaborate on each of the terms on the right-hand side of (9).

1. *Acquisition cost.* The acquisition cost of an arriving item is \$$c$/unit. But since only the fraction $1 - \pi_B$ of arriving units enter the testing center (and hence are purchased), the long-run average cost is

   $$A(m, M) = \lambda c(1 - \pi_B),$$

   where, in general, $\pi_B = \pi_B(m, M)$.
2. *Testing cost.* Let the batch testing cost (regardless of the batch size) be \$$b$/batch. Hence, the testing cost is \$$b/E(Z_{m,M})$ per item where $E(Z_{m,M})$ is given by (8). Thus, the long-run average testing costs are

   $$T(m, M) = \frac{\lambda(1 - \pi_B)b}{E(Z_{m,M})}.$$

It is important to note that when optimizing the expected profit there exists a trade-off between the revenue from sales $G(m, M)$ and the testing costs $T(m, M)$. That is, while greater batch sizes result in smaller testing costs per item, they also result in a smaller probability that all items in the batch will be good (and hence, a smaller revenue from sales of clean items).

3. *Idleness cost.* We impose a cost of \$$s$/time unit on idleness of the testing center. Thus, the associated long-run average idleness cost is

$$I(m,M) = s\sum_{i=0}^{m-1}\pi_i$$

since the tester will be idle for the fraction $\sum_{i=0}^{m-1}\pi_i$ of time.

4. *Revenue from sales of good items.* First, note that each unit of a good (clean) item is sold for \$$r$/unit and the effective rate of arrival is $\lambda(1-\pi_B)$ units per time and thus $\lambda(1-\pi_B)r$ would be the expected revenue *if all incoming items were good*. However, since some items are not good, we need to calculate the probability that a batch is clean before we can determine the expected revenue from sales of good items.
Let W be defined as the event that a batch is good. By conditioning on the number X of units left behind by a departing batch, we have

$$\Pr(W) = \sum_{j=0}^{\infty}\Pr(W \mid X=j)\Pr(X=j) = q^m\sum_{j=0}^{m}\pi_j + \sum_{j=m+1}^{M}\pi_j q^j + q^M\sum_{j=M+1}^{B}\pi_j.$$

Note that if j items are left behind after a batch service completion, then the next batch size is m, if $j \leqslant m$, with probability $\pi_0+\pi_1+\cdots+\pi_m$. Hence, $q^m\sum_{i=0}^{m}\pi_i$ is the probability that a batch of size m is clean if $j \leqslant m$. Similarly, $\sum_{j=m+1}^{M}\pi_j q^j$ is the probability that a batch of size of j is clean if $m < j \leqslant M$, and is $q^M\sum_{j=M+1}^{B}\pi_j$ the probability that a batch of size of M is clean if $j > M$. Thus, the expected revenue from sales of good (clean) items is found to be

$$G(m,M) = \lambda(1-\pi_B)r\left(q^m\sum_{j=0}^{m}\pi_j + \sum_{j=m+1}^{M}\pi_j q^j + q^M\sum_{j=M+1}^{B}\pi_j\right) \tag{10}$$

5. *Revenue from sales of contaminated (bad) items.* Let r_0 be the sale price of an item belonging to a contaminated batch, where clearly $r_0 \ll r$. In a manner similar to that used to compute (10), it can be immediately seen that the long-run average sale price of a contaminated batch is

$$C(m,M) = \lambda(1-\pi_B)r_0\left[(1-q^m)\sum_{j=0}^{m}\pi_j + \sum_{j=m+1}^{M}\pi_j(1-q^j) + (1-q^M)\sum_{j=M+1}^{B}\pi_j\right].$$

Hence, by combining all of the above five factors, the long-run average profit is obtained as

$$\begin{aligned}\mathscr{P}(m,M) = {} & \lambda(1-\pi_B)\left[(r-r_0)\left(q^m\sum_{j=0}^{m}\pi_j + \sum_{j=m+1}^{M}\pi_j q^j + q^M\sum_{j=M+1}^{B}\pi_j\right) + r_0\right] \\ & - \lambda(1-\pi_B)\left[c + b\left(m\sum_{i=0}^{m}\pi_i + \sum_{i=m+1}^{M-1} i\pi_i + M\sum_{i=M}^{B}\pi_i\right)^{-1}\right] - s\sum_{i=0}^{m-1}\pi_i.\end{aligned}$$

Now, the optimization problem we consider is

$$\max_{1\leqslant m\leqslant M\leqslant M_0}\mathscr{P}(m,M)$$

where, as indicated before, M_0 is the pre-known kit capacity.

4. A numerical analysis

We now present a specific example in detail. We use the following base values for the revenue and cost parameters:

r	r_0	c	b	s
50	3	10	15	10

. (11)

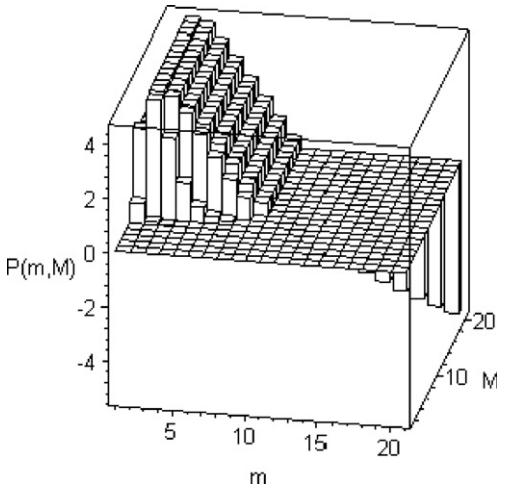

Fig. 2. The plot of the objective function $\mathscr{P}(m,M)$ defined over the feasible region $\mathscr{M} = \{(m,M) : m \leqslant M \leqslant B\}$ when $(r, r_0, c, b, s) = (50, 3, 10, 15, 10)$, $B = M_0 = 20$, $\lambda = 0.5$ and $\mu = 3$. The optimal solution is found as $(m^*, M^*) = (2, 8)$ with a maximum expected profit of $\mathscr{P}^* = 4.52$.

We also assume that the capacity of the testing centre is $B = 20$ (which also equals the kit size, i.e., $M_0 = B$) and that the items arrive according to a Poisson process with rate $\lambda = 0.5$ units per unit time. The time to complete the testing of k items is assumed to be Erlang(k, μ) with $\mu = 3$ completions per unit time. Thus, it takes an average of k/μ time units to complete the testing of k items.

With these parameter values the objective function $\mathscr{P}(m,M)$ is maximized over the region $\mathscr{M} = \{(m,M) : m \leqslant M \leqslant B\}$, which gives $(m^*, M^*) = (2, 8)$ with a maximum expected profit of $\mathscr{P}^* = 4.52$. See Fig. 2 for a graph of the objective function over the feasible region $\mathscr{M}$ and Fig. 3 for a graph of the stationary distribution π_i, $i = 0, 1, \ldots, B$.

To obtain additional insights into the nature of the optimal solution we varied the parameters around their base values given in (11) and solved the resulting problems as shown in Table 2.

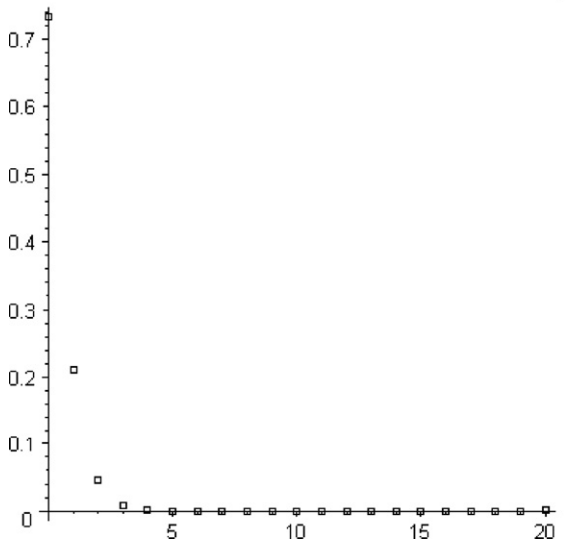

Fig. 3. The stationary distribution π_i, $i = 0, 1, \ldots, B$ when $(m^*, M^*) = (2, 8)$ arising from using $(r, r_0, c, b, s) = (50, 3, 10, 15, 10)$, $B = M_0 = 20$, $\lambda = 0.5$ and $\mu = 3$.

Table 2
Sensitivity of the optimal solution to changing parameter values

r	r_0	c	b	s	m^*	M^*	$\mathscr{P}(m^*, M^*)$
35					5	5	0.00
50					**2**	**8**	**4.52**
65					2	13	11.29
	1				2	8	4.43
	3				**2**	**8**	**4.52**
	5				2	8	4.62
		5			2	10	7.02
		10			**2**	**8**	**4.52**
		20			5	5	0.00
			5		1	8	7.81
			15		**2**	**8**	**4.52**
			25		3	9	2.71
				5	3	13	9.26
				10	**2**	**8**	**4.52**
				15	1	6	0.05

The results corresponding to the base values $(r, r_0, c, b, s) = (50, 3, 10, 15, 10)$ are indicated by *bold* font.

It is worth noting here that both M and the difference $M - m$ increases in the unit revenue r. Such an increase in batch sizes (and the potentially high cost resulting from a higher probability of rejecting a large batch) is compensated by the high unit revenue. The re-sale price of contaminated items r_0 does not affect the optimal solution for the narrow range we have examined. On the other hand, lower values of the unit purchase cost c have the same effect on the optimal batch sizes as the unit revenue: lower the purchase price, higher the allowed batch sizes (despite the increased probability of rejecting a batch). When the testing cost per batch b increases we observe a parallel increase in the batch sizes which results in a reduction of the testing cost per item. Finally, for higher values of the idleness cost s, we observe a narrowing of the difference $M - m$ which would have the effect of reducing the idleness of the tester.

Remark 1. A (nearly) closed-form solution to the above model can be obtained under the simplifying assumptions that, (i) the idleness cost is zero, i.e., $s = 0$, (ii) B is large so that $\pi_B = 0$, (iii) and when it is decided, a priori, that $m = M$. In this case, the individual revenue and cost terms simplify to $E(Z_{m,m}) = m$, $A(m, m) = \lambda c$, $T(m, m) = \lambda b/m$, $I(m, m) = 0$, $G(m, m) = \lambda r q^m$ and $C(m, m) = \lambda r_0(1 - q^m)$, so that the approximate objective function becomes

$$\hat{\mathscr{P}}(m, m) = \lambda(r - r_0)q^m - \frac{\lambda b}{m} - \lambda(c - r_0),$$

where the last term $\lambda(c - r_0) \geqslant 0$ is a constant. With a further simplification, we assume that m is a continuous variable and differentiate $\hat{\mathscr{P}}(m)$ with respect to m, to obtain

$$\frac{\mathrm{d}\hat{\mathscr{P}}(m)}{\mathrm{d}m} = \lambda(r - r_0)q^m \ln(q) + \frac{\lambda b}{m^2}, \tag{12}$$

and

$$\frac{\mathrm{d}^2\hat{\mathscr{P}}(m)}{\mathrm{d}m^2} = \lambda(r - r_0)q^m[\ln(q)]^2 - \frac{2\lambda b}{m^3}.$$

Solving (12) for m, we find the optimal solution for the approximate problem as

$$\hat{m} = \frac{2\mathscr{L}\left(-\frac{1}{2}\sqrt{-\frac{\ln(q)b}{r - r_0}}\right)}{\ln(q)} > 0, \tag{13}$$

where $\mathscr{L}(x)$ is defined as the Lambert function, which is the solution to the nonlinear equation $ye^y = x$ for a given x. (For a detailed description of the Lambert function, see Valluri, Jeffrey and Corless [22].) It can be shown that

$$\left.\frac{\mathrm{d}^2\hat{\mathscr{P}}(m)}{\mathrm{d}m^2}\right|_{m^*} < 0$$

and that $\hat{\mathscr{P}}(m)$ is a unimodal function. Thus, $\hat{m}$ found from (13) will be the globally optimal solution for the approximate problem.

With the parameter values used in the detailed example of this Section, i.e., $(r, r_0, c, b, s) = (50, 3, 10, 15, 10)$, $(\lambda, \mu) = (0.5, 3)$ and $q = 0.95$, the approximate profit function $\hat{\mathscr{P}}(m)$ is maximized at $\hat{m} = 2.67 \simeq 3$.

5. Summary and conclusions

In this paper we use a bulk queueing model to analyze a problem in group testing. We assume that the items to be tested arrive at the group-testing centre according to a Poisson process and are served (i.e., group-tested) in batches by one server. The service time distribution of this queue depends on the batch size being tested. With these assumptions we obtain the bulk queueing model $M/G^{(m,M)}/1$ where m and $M(>m)$ are the decision variables to be determined. We develop first find the generating function for the steady-state probabilities of the embedded Markov chain of the $M/G^{(m,M)}/1$ system and then consider a more realistic finite state version of the problem. We compute the optimal values of the decision variables (m, M) that maximize the expected profit. For a special case of the problem, we determine the optimal decision explicitly in terms of the Lambert function.

The model presented in this paper can be generalized to accommodate other cases. It is useful to note that distinguishing between m and M is also important in the case of deteriorating items. In such a situation—occurring in blood banks, for example—there is a trade-off between losing items due to deterioration and testing (too) small groups. This aspect would lead us to the realm of bulk queues with reneging. Another interesting extension of our model comes into the picture when we make the demand process more explicit (taking lost sales or backorders into account). Also this situation seems to be quite natural in blood banks. Incurring costs for non-delivery could be another reason to work with distinction between m and M.

Acknowledgement

We thank the referee for his/her careful reading of our manuscript and for bringing to our attention some earlier papers on group testing that are related to our model.

References

[1] L. Abolnikov, A. Dukhovny, Markov chains with transition delta-matrix; ergodicity conditions, invariant probability measures and applications, Journal of Applied Mathematics and Stochastic Analysis 5 (1) (1992) 83–98.
[2] L. Abolnikov, A. Dukhovny, Queueing processes and optimization problems in quality control systems with a group-individual testing procedure, Engineering Simulation 16 (1999) 165–178.
[3] L. Abolnikov, A. Dukhovny, Optimization in HIV screening problems, Journal of Applied Mathematics and Stochastic Analysis 16 (4) (2003) 361–374.
[4] S.K. Bar-Lev, W. Stadje, F.A. Van der Duyn Schouten, Group testing procedures with incomplete identification and unreliable testing results, Applied Stochastic Models in Business and Industry 22 (2006) 281–296.
[5] S.K. Bar-Lev, W. Stadje, F. Van der Duyn Schouten, Hypergeometric group testing with incomplete information, Probability in the Engineering and Informational Sciences 17 (2003) 335–350.
[6] S.K. Bar-Lev, W. Stadje, F. Van der Duyn Schouten, Optimal group testing with processing times and incomplete identificationm, Methodology and Computing in Applied Probability 6 (2004) 55–72.
[7] S.K. Bar-Lev, W. Stadje, F. Van der Duyn Schouten, Multinomial group testing models with incomplete identification, Journal of Statistical Planning and Inference 135 (2005) 384–401.
[8] P.P. Bocharov, C. D'Apice, A.V. Pechinkin, S. Salerno, Queueing Theory, Brill Academic Publishers, Utrecht, 2004.

[9] M.L. Chaudhry, J.G.C. Templeton, A First Course in Bulk Queues, Wiley, New York, 1983.
[10] M.L. Chaudhry, J.G.C. Templeton, Bulk Queues, Selecta Statistica Canadiana, No. V., Department of Mathematics, McMaster University, Hamilton, Ontario, 1986.
[11] R. Dorfman, The detection of defective members of large populations, Annals of Mathematical Statistics 14 (1943) 436–440.
[12] D.-Z. Du, F.K. Hwang, Combinatorial Group Testing and Its Applications, World Scientific, Singapore, 2000.
[13] P.A. Hammick, J.L. Gastwirth, Group testing for sensitive characteristics: Extension to higher prevalence levels, International Statistical Review 62 (1994) 319–331.
[14] M.C. Hung, W.H. Swallow, Use of binomial group testing in tests of hypotheses for classification or quantitative covariables, Biometrics 56 (2000) 204–212.
[15] E. Litvak, X.M. Tu, M. Pagano, Screening for the presence of a disease by pooling sera samples, Journal of the American Statistical Association 89 (1994) 424–434.
[16] A.J. Macula, Probabilistic nonadaptive and two-stage group testing with relatively small pools and DNA library screening, Journal of Combinatorial Optimization 2 (1999) 385–397.
[17] A.J. Macula, Probabilistic nonadaptive group testing in the presence of errors and DNA library screening, Annals of Combinatorics 3 (1999) 61–69.
[18] S.S. Nair, M.F. Neuts, Distribution of occupation time and virtual waiting time of a general class of bulk queues, Sankhyā, Series A 34 (1972) 17–22.
[19] M.F. Neuts, A general class of bulk queues with Poisson input, Annals of Mathematical Statistics 38 (1967) 759–770.
[20] T.L. Saaty, Mathematical Methods of Operations Research, McGraw-Hill, New York, 1959.
[21] X.M. Tu, E. Litvak, M. Pagano, On the informativeness and accuracy of pooled testing in estimating prevalence of a rare disease: Application to HIV screening, Biometrika 82 (1995) 287–297.
[22] S.R. Valluri, D.J. Jeffrey, R.M. Corless, Some applications of the Lambert W function to physics, Canadian Journal of Physics 78 (2000) 823–831.
[23] L.M. Wein, S.A. Zenios, Pooled testing for HIV screening: Capturing the dilution effect, Operations Research 44 (1996) 543–569.