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SHARE OPPORTUNITY SETS AND COOPERATIVE GAMES

By E.Caprari, F. Patrone, L. Pusillo, S. Tijs, A. Torre

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Share Opportunity Sets and Cooperative Games

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Abstract

In many share problems there is a priori given a natural set of possible divisions to solve the sharing problem. Cooperative games related to such share sets are introduced, which may be helpful in solving share problems. Relations between properties of share sets and properties of games are investigated. The average lexicographic value for share sets and for cooperative games is studied.

KEYWORDS: Cooperative games, bankruptcy games, average lexicographic value, opportunity sets.

JEL code C71

1 Introduction

Inspiration for this work came from the bankruptcy literature (Thomson [T]) on bankruptcy problems and games. In their famous paper [AM] Aumann and Maschler considered the bankruptcy situations described in the Talmud, where the proposals for dividing the estate were for centuries a mystery. For understanding the Talmudic rule, the cooperative games related to share opportunity sets of the bankruptcy problems are helpful. Surprisingly the nucleolus of the cooperative bankruptcy games was the key for understanding the Talmudic examples.

Share opportunity sets can arise from many other practical situations, such as taxation problem or the airport landing fee problem, where cooperative games which may be helpful in giving a reasonable solution for the share

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problems can be also constructed.

In this paper we make a systematic study of the interaction between share opportunity sets and related cooperative games. The average lexicographic vector for share sets and the average lexicographic value for balanced cooperative games [Ti] will also play a role. It turns out that the average lexicographic vector for bankruptcy share opportunity sets, coincides with the run to the bank rule discussed in O'Neill [O'N].

The outline of the paper is as follows. Section 2 is devoted to preliminaries and notations. In section 3, share opportunity sets are introduced and examples of situations from which such share opportunity sets naturally arise are given. Furthermore continuity properties of the average lexicographic vector are studied. In section 4, we introduce operators which associate to share opportunity sets cooperative games. Special attention is paid to perfect opportunity sets which coincide with the core of related minimum right games. In section 5 we tackle the question which properties of share opportunity sets guarantee that the corresponding minimum right game is of a special type e.g. convex, big boss, simplex or dual simplex game. In section 6 we propose a method to extend classical concepts of solutions for TU-games to balanced partially defined games.

2 Preliminaries and notations

An n-person cooperative game ([O]) $\langle N, v \rangle$ with player set $N = \{1, 2, ..., n\}$ is a map $v : 2^N \longrightarrow \mathbb{R}$ with $v(\emptyset) = 0$, where 2^N is the collection of subsets of N. Let us denote with G^N the set of all n-person cooperative games.

Given the game $\langle N, v \rangle$, its dual $\langle N, v^* \rangle$ is the game defined by $v^*(S) = v(N) - v(N \backslash S)$, $S \subseteq N$. Let $\mathcal{F} \subseteq 2^N$ such that $\emptyset \in \mathcal{F}$, $\{i\} \in \mathcal{F}$ for all $i \in N$ and $N \in \mathcal{F}$.

An n-person $\mathcal{F}$-partially defined game (or simply partially defined game) $\langle N, v, \mathcal{F} \rangle$ is a map $v : \mathcal{F} \longrightarrow \mathbb{R}$ with $v(\emptyset) = 0$. If v is a partially defined game, the partial core $C_{\mathcal{F}}(v)$ is the bounded polyhedral set

$$C_{\mathcal{F}}(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N), \ x(S) \geq v(S) \text{ for each } S \in \mathcal{F}\},$$

where $x(S) = \Sigma_{i \in S} x_i$. If v is a game, $C_{2^N}(v) = C(v)$ is the core of v. Games with non empty core are called balanced games, while partially defined games with nonempty partial core are called partially balanced games. The dual core of $\langle N, v \rangle$ is the set $C^*(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N), \ x(S) \leq v(S) \text{ for each } S \in 2^N\}$.

The imputation set of $\langle N, v \rangle$ is the set

$$I(v) = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} x_i = v(N) \,, x_i \geq v(\{i\}) \, \forall i \in N \right\},$$

and the dual imputation set is

$$I^*(v) = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} x_i = v(N) \,, x_i \leq v^*(\{i\}) \, \forall i \in N \right\}.$$

Note that

$$C(v^*) = C^*(v).$$

Given $x \in \mathbb{R}^n$, we denote with x_{-j} the vector belonging to $\mathbb{R}^{n-1}$ obtained from x by deleting its $j - th$ coordinate.

A game $\langle N, v \rangle$ is called:

- a **monotonic game** if $v(S) \leq v(T)$ for all $S \subseteq T$;

- a **convex game** if $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $S \subseteq T \subseteq N \setminus \{i\}$;

- a **simplex game** if $I(v) = C(v)$;

- a **dual simplex game** if $I^*(v) = C(v)$;

- a **big boss game** (BBG for short) with big boss 1 if:
  1) $v(S) = 0$ if $1 \notin S$;
  2) v is monotonic;
  3) $v(N) - v(N \setminus S) \geq \Sigma_{i \in S}(v(N) - v(N \setminus \{i\}))$ if $1 \notin S$.

- an **exact game** if the core $C(v)$ of $\langle N, v \rangle$ is nonempty and for every $S \subseteq N$ there exists $x \in C(v)$ such that $x(S) = v(S)$ (see [S]).

Given a balanced game $\langle N, v \rangle$, its exactification is the game $\langle N, \bar{v} \rangle$ with $\bar{v}(S) = \min_{x \in C(v)} x(S)$ for each $S \in 2^N$.

Given an ordering $\sigma = (\sigma(1), \sigma(2), ..., \sigma(n))$ in N and a compact subset A of $\mathbb{R}^n$, the **Lexicographic maximum** of A with respect to σ is the vector $L^\sigma(A) \in A$ such that:
- $(L^\sigma(A))_{\sigma(1)} = \max\{x_{\sigma(1)} \mid x \in A\}$,
- $(L^\sigma(A))_{\sigma(2)} = \max\{x_{\sigma(2)} \mid x \in A \,, x_{\sigma(1)} = (L^\sigma(A))_{\sigma(1)}\}$,
.

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.
- $(L^\sigma(A))_{\sigma(n)} = \max\{x_{\sigma(n)} \mid x \in A , (x_{\sigma(1)}, ..x_{\sigma(n-1)}) =$
  $= ((L^\sigma(A))_{\sigma(1)}, ..(L^\sigma(A))_{\sigma(n-1)})\}.$

The **Average Lexicographic maximum** $AL(A)$ of A is the average over all $L^\sigma(A)$ i.e. $AL(A) = \frac{1}{n!}\Sigma_{\sigma \in \Pi(N)} L^\sigma(A)$, where $\Pi(N)$ denotes the set of all possible orderings in N. Given a balanced game $\langle N, v \rangle$, we denote by $AL(v)$ the vector $AL(C(v))$ (see Tijs in [Ti]).

3   Share opportunity sets and share problems

Let $\alpha \in \mathbb{R}$ and $H_\alpha = \{x \in \mathbb{R}^n \mid x(N) = \alpha\}$. Let $\mathcal{K}_\alpha^n$ the family of all compact subsets of H_α and

$$\mathcal{K}^n = \cup_{\alpha \in \mathbb{R}} \mathcal{K}_\alpha^n.$$

Definition 1 *We call each $D \in \mathcal{K}^n$ **share opportunity set (SOS)** .*

For each $D \in \mathcal{K}_\alpha^n$, the corresponding **share problem** is the problem of dividing α among $1, 2, ..., n$, where D is the set of all possible allocations.

Definition 2 *Let $\mathcal{F} \subseteq 2^N$ be such that $\emptyset \in \mathcal{F}$, $\{i\} \in \mathcal{F}$ for every $i \in N$ and $N \in \mathcal{F}$. We say that $D \subseteq H_\alpha$ has a **perfect structure** if for all $S \in \mathcal{F}$ there exists $\beta_S \in \mathbb{R}$ such that*

$$D = \bigcap_{S \in \mathcal{F}} \{x \in \mathbb{R}^n \mid x(S) \geq \beta_S\}.$$

Remark 1 *If D has a perfect structure, then it belongs to $\mathcal{K}^n$ and*

$$D = \bigcap_{S \in 2^N} \{x \in \mathbb{R}^n \mid x(S) \geq \beta_S\},$$

where $\beta_S = min\{x(S) \mid x \in D\}$ if $S \notin \mathcal{F}$.

Let

$$\mathcal{P}_\alpha^n = \{D \in \mathcal{K}_\alpha^n \mid D \text{ has a perfect structure}\}$$

and

$$\mathcal{P}^n = \cup_{\alpha \in \mathbb{R}} \mathcal{P}_\alpha^n.$$

Here we describe some examples of share opportunity sets related to well-known problems and games .

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Example 1 (**Bankruptcy Problem**) ([AM]) Consider

$$N = \{1, 2, ..., n\}, \ d \in \mathbb{R}^n_+ \ , E \in \mathbb{R}_+ \ \ s.t. \ \ 0 \le E \le \sum_{i \in N} d_i;$$

the bankruptcy problem associated to (N, d, E) is the problem of finding a point in the share set $D = \{x \in \mathbb{R}^n \ \ | \ \ 0 \le x_i \le d_i \ \forall i \in N, \ \ \Sigma_{i \in N} x_i = E\}$. In this case D represents the set of all possible agreements among $1, 2, ...n$.

Example 2 (**Airport landing strip Problem**) ([LT]) Let $N = \{1, 2, 3\}$ and consider the airport problem such that the costs of parts of the landing strip are as follows:
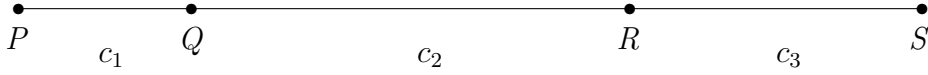
and where the first player needs the part PQ of the landing strip, the second one needs the part PR of the landing strip and the third one needs the whole landing strip PS. Then the share set is

$$D = \{x \in \mathbb{R}^n \ | \ \ 0 \le x_1 \le c_1, \ 0 \le x_2 \le c_1 + c_2,$$

$$0 \le x_1 + x_2 \le c_1 + c_2, \ x_1 + x_2 + x_3 = c_1 + c_2 + c_3\},$$

that represents the set of all possible agreements among $1, 2, 3$ in order to divide the cost of the landing strip.

Example 3 Let $N = \{1, 2, 3\}$. $D = \{(2, 0, 1), (1, 2, 0), (0, 1, 2)\}$ is a share set with $\alpha = 3$. In this case also the smallest convex set containing D is a share set.

Example 4 Given a balanced game $\langle N, v \rangle$, we can consider the share problem related to the core of $\langle N, v \rangle$.
     Here $D = C(v)$ and $\alpha = v(N)$.

Example 5 Given a partially balanced game $\langle N, v, \mathcal{F} \rangle$ we can consider the share problem related to $C_{\mathcal{F}}(v)$.
     Here $D = C_{\mathcal{F}}(v)$ and $\alpha = v(N)$.

A **solution** ϕ **for share problems** with share α is a rule that associates to every $D \in \mathcal{K}^n_\alpha$ a point $\phi(D) \in H_\alpha$. Here $\phi(D)$ represents the chosen share division of the share α. Let us consider the following solutions:

- $L^\sigma : \mathcal{K}^n \to \mathbb{R}^n$ that associates to each $D \in \mathcal{K}^n$ the vector $L^\sigma(D) \in D$. This is a natural solution if there is a fixed ordering σ of the players;

- $AL : \mathcal{K}^n \to \mathbb{R}^n$ that associates to each $D \in \mathcal{K}^n$ the vector $AL(D)$. This is a natural solution if we want to avoid discrimination among $1, 2, ..., n$. $AL(D) \in D$ whenever D is convex.

We endow $\mathcal{K}^n$ with the Hausdorff topology ([KT]) in order to study continuity properties of L^σ and AL on $\mathcal{K}^n$. Let $A \in \mathcal{K}^n$. We denote with $\arg\max(A)$ the $\arg\max\{x_{\sigma(1)} \mid x \in A\}$. The following lemma is an easy consequence of Berge's theorem (see [B] pag. 122).

Lemma 1 *The multifunction* $\arg\max : \mathcal{K}^n \rightrightarrows \mathbb{R}^n$ *that associates to each* $A \in \mathcal{K}^n$ *the set* $\arg\max(A)$ *is upper semicontinuous.*

In spite of its upper semicontinuity, $\arg\max$ is not continuous on $\mathcal{K}^n$ and L^σ and AL are not continuous either, as it is shown in the folowing remark. On the other hand, we have continuity on $\mathcal{P}^n$: see Theorem 1.

Remark 2 L^σ *and* AL *are not continuous on* $\mathcal{K}^n$ *w.r.t. the Hausdorff topology, as we illustrate in the following. Let* $\sigma = (1,2,3)$, $A = co\{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2})\}$ *and* $A_k = co\{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2} + \frac{1}{k}, 0, \frac{1}{2} - \frac{1}{k})\}$, *for* $k \in \mathbb{N}$. *The sequence* $\{A_k\}_{k\in\mathbb{N}}$ *converges to* A *w.r.t. the Hausdorff topology, but* $\lim_{k\to\infty} L^\sigma(A_k) = \lim_{k\to\infty}(\frac{1}{2} + \frac{1}{k}, 0, \frac{1}{2} - \frac{1}{k}) = (\frac{1}{2}, 0, \frac{1}{2}) \neq (\frac{1}{2}, \frac{1}{2}, 0) = L^\sigma(A)$ *and* $\lim_{k\to\infty} AL(A_k) = \lim_{k\to\infty}(\frac{1}{2} + \frac{2}{3k}, \frac{1}{6}, \frac{1}{3} - \frac{2}{3k}) = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3}) \neq (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}) = AL(A)$.

To prove continuity of L^σ and AL on $\mathcal{P}^n$, we use the following lemma:

Lemma 2 *The multifunction* $\arg\max : \mathcal{P}^n \rightrightarrows \mathbb{R}^n$ *is lower semicontinuous on* $\mathcal{P}^n$.

Proof The lower semicontinuity of $\arg\max$ on $\mathcal{P}^n$ follows directly from Theorem 4.3.5 pag 70 of [BGKKT]. It is enough to notice that linear functions belong to the class of "weakly analytic" functions, as defined in [BGKKT]. $\square$

Theorem 1 L^σ *and* AL *are continuous on* $\mathcal{P}^n$.

Proof In lemma 1 and 2 we noticed that $\arg\max$ is upper and lower semicontinuous in $\mathcal{P}^n$, so it is continuous. If $A \in \mathcal{P}^n$, then the set

$$A_{\sigma(1)} = \left\{ x_{-\sigma(1)} \mid x \in \arg\max(A) \right\}$$

belongs to $\mathcal{P}^{n-1}$. By lemma 1 and 2 , the multifunction $\arg\max_{\sigma(2)} : \mathcal{P}^{n-1} \rightrightarrows \mathbb{R}^{n-1}$ that associates to $A_{\sigma(1)}$ the $\arg\max\{x_{\sigma(2)} \mid x \in \arg\max(A_{\sigma(1)})\}$ is continuous on $\mathcal{P}^{n-1}$. Repeating these arguments we prove continuity of L^σ. As $AL(A)$ is the average of all $L^\sigma(A)$, also AL is continuous on $\mathcal{P}^n$. $\square$

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4 Operators from share opportunity sets to cooperative games

Let us consider the operators $m, u : \mathcal{K}^n \to G^N$, defined by

$$m(D)(S) = \min\{x(S) \mid x \in D\},$$

$$u(D)(S) = \max\{x(S) \mid x \in D\},$$

for each $D \in \mathcal{K}^n$, $S \subseteq N$, $N = \{1, 2, ..., n\}$.

Definition 3 *We call the game $\langle N, m(D) \rangle$ the **minimal right game related to** D.*

Definition 4 *We call the game $\langle N, u(D) \rangle$ the **utopia game related to** D.*

Note that $m(D) = (u(D))^*$ and $C(m(D)) = C^*(u(D))$.

Remark 3 *The game $m(D)$ is superadditive and the game $u(D)$ is subadditive since, if $S \cap T = \emptyset$, then by definition $m(D)(S \cup T) \geq m(D)(S) + m(D)(T)$, while $u(D)(S \cup T) \leq u(D)(S) + u(D)(T)$.*

Theorem 2 *Suppose $D \in \mathcal{K}^n$. The following conditions are equivalent:*
a) D is the core of a partially defined game;
b) $D \in \mathcal{P}^n$;
c) $D = C(m(D))$.

Proof It is obvious that $c \implies a \implies b$. Then it is sufficient to prove that $b \implies c$. Suppose $D \in \mathcal{P}^n$. Note that $x \in D$ if and only if for every $S \subseteq N$, we have $x(S) \geq \min\{x(S) \mid x \in D\} = m(D)(S)$ and $x(N) = m(D)(N)$. $\square$

Definition 5 *Let $D \in \mathcal{K}_\alpha^n$. The **perfect closure** $pc(D)$ **of** D is the set*

$$pc(D) = \cap_{T \in \mathcal{P}_\alpha^n}\{T \mid T \supseteq D\}.$$

Note that for each $D \in \mathcal{K}_\alpha^n$ there is at least one $E \in \mathcal{P}_\alpha^n$ with $E \supseteq D$.

Theorem 3 *Let $D \in \mathcal{K}_\alpha^n$. Then*
a) $pc(D) = C(m(D))$;
b) $pc(D)$ is the smallest set in $\mathcal{P}^n$ that contains D.

Proof a) $pc(D) = \cap_{T \in \mathcal{P}_\alpha^n}\{T \mid T \supseteq D\} = \cap\{C(m(T)) \mid T \supseteq D\} = C(m(D))$ where the second equality follows from theorem 2.
b) Note that b) follows from the facts that $pc(D) \supseteq D$ and $pc(D) = C(m(D)) \in \mathcal{P}_\alpha^n$. $\square$

Theorem 4 *The game m(D) is an exact game.*

Remark 4 *If D has a perfect structure then $\mathrm{m}(D)$ is the unique exact game such that $C\left(\mathrm{m}(D)\right) = D$.*

Remark 5 *In general m is not injective even if we consider only convex sets. For example, take D convex such that $C(\mathrm{m}(D)) \supsetneq D$. In this case $m(D) = m(C(\mathrm{m}(D)))$.*

Proposition 1 *The restriction of m to $\mathcal{P}^n$, i.e. $\mathrm{m} : \mathcal{P}^n \to G^N$, is injective.*

Proof Take $D_1, D_2 \in \mathcal{P}^n$ with $D_1 \neq D_2$. Suppose by contradiction that $m\left(D_1\right) = m\left(D_2\right)$. Then $C\left(\mathrm{m}(D_1)\right) = C\left(\mathrm{m}(D_2)\right)$. By theorem 2, $C\left(\mathrm{m}(D_j)\right) = D_j$. This implies $D_1 = D_2$. $\qquad \square$

Proposition 2 *m is continuous on $\mathcal{K}^n$.*

Proof We must prove that $D_k \overset{H}{\to} D$ implies $m\left(D_k\right)\left(S\right) \to m\left(D\right)\left(S\right)$ for all $S \subset N$, where $\overset{H}{\to}$ means convergence with respect to the Hausdorff topology. This is a consequence of Berge's maximum theorem (see [B], pag 122). $\qquad \square$

Remark 6 *Remark 5 and propositions 1 and 2 hold also for the operator u.*

Remark 7 *The function $pc : \mathcal{K}^n \longrightarrow \mathcal{P}^n$ that associates to each $D \in \mathcal{K}^n_\alpha$ the perfect closure $pc(D) = C(\mathrm{m}(D))$ is continuous with respect to the Hausdorff topology. In fact, it is a composition of continuous functions (see proposition 2 and [LPTT]). The functions $\phi, \psi : \mathcal{K}^n \longmapsto \mathbb{R}$ defined as $\phi(D) = AL((\mathrm{m}(D))$ and $\psi(D) = L^\sigma(\mathrm{m}(D))$ are also continuous for each $D \in \mathcal{K}^n_\alpha$.*

Remark 8 *In general $AL(D) \neq AL(\mathrm{m}(D))$. For example, $D = \{(x,y,z) \mid x + y + z = 1, \ x,y,z \geq 0, \ y - z \leq 0\}$ has not a perfect structure and $AL(D) = (\frac{2}{6}, \frac{1}{6}, \frac{3}{6}) \neq (\frac{5}{12}, \frac{1}{6}, \frac{5}{12}) = AL(\mathrm{m}(D))$.*

$\mathcal{K}^n$ is a convex cone with respect to the Minkowski sum and the scalar product. On $\mathcal{K}^n$ are also defined intersection, union and inclusion. We are interested in the behavior of m and u with respect to these operations.

Proposition 3 *Let $D_1 \in \mathcal{K}^n_\alpha$ and $D_2 \in \mathcal{K}^n_\beta$ and $\lambda \in \mathbb{R}^+ \cup \{0\}$. Then:*
a) $D_1 + D_2 \in \mathcal{K}^n_{\alpha+\beta}$ and $m(D_1 + D_2) = m(D_1) + m(D_2)$, $u(D_1 + D_2) = u(D_1) + u(D_2)$;
b) $\lambda D_1 \in \mathcal{K}^n_{\lambda\alpha}$ and $m(\lambda D_1) = \lambda m(D_1)$, $u(\lambda D_1) = \lambda u(D_1)$;
c) $-D_1 \in \mathcal{K}^n_{-\alpha}$ and $m(-D_1) = -u(D_1)$.

Let us set $m(D_1) \wedge m(D_2) = \min \{m(D_1), m(D_2)\}$ and $m(D_1) \vee m(D_2) = \max \{m(D_1), m(D_2)\}$.

Proposition 4 *Let $D_1, D_2 \in \mathcal{K}_\alpha^n$. Then*
a) $m(D_1 \cup D_2) = m(D_1) \wedge m(D_2)$, $u(D_1 \cup D_2) = u(D_1) \vee u(D_2)$;
b) $m(D_1 \cap D_2) \geq m(D_1) \vee m(D_2)$, $u(D_1 \cap D_2) \leq u(D_1) \wedge u(D_2)$.

Remark 9 *Observe that even if $D_1,\ D_2 \in \mathcal{P}^n$ not necessarily $m(D_1 \cap D_2) = m(D_1) \vee m(D_2)$. For example take $D_1 = co\{(1,0,0),\ (0,1,0)\}$ and $D_2 = co\{(0,0,1),\ (0,1,0)\}$. In this case $m(D_1 \cap D_2)(\{2\}) = 1 > m(D_1)(\{2\}) = m(D_2)(\{2\}) = 0$*

5 Properties of share opportunity sets and related games

Now we study relationships between properties of share sets D and those of the games $m(D)$, and $u(D)$. From now on we consider $D \in \mathcal{K}^n$.

5.1 Monotonic games

Proposition 5 $m(D)$ *is monotonic if and only if for all $i \in N$*

$$min_{x \in D} x_i \geq 0. \tag{1}$$

Proof If (1) holds then $m(D)(T) \geq m(D)(S)$ when $S \subset T$, because we add nonnegative elements to $x(S)$ in calculating $m(D)(T)$. Suppose that $m(D)$ is monotonic and suppose that there exists $\bar{i}$ s.t.

$$min_{x \in D} x_{\bar{i}} < 0. \tag{2}$$

Then we have that

$$m(D)(\emptyset) = 0 > m(D)(\bar{i}) > 0$$

which implies that the game is not monotonic. $\square$

5.2 Simplex and dual simplex games

Let us set

$$
\begin{aligned}
a^i &= (a_1^i, a_2^i, ..., a_n^i), \\
b^i &= (b_1^i, b_2^i, ..., b_n^i),
\end{aligned}
$$

where

$$a_j^i = \min\{x_j \mid x \in D\}, \qquad j \in N \setminus \{i\},$$
$$a_i^i = \alpha - \Sigma_{k \in N \setminus \{i\}} a_k^i,$$

$$b_j^i = \alpha - \min\{\Sigma_{k \in N \setminus \{j\}} x_k \mid x \in D\}, \qquad j \in N \setminus \{i\},$$
$$b_i^i = \alpha - \Sigma_{k \in N \setminus \{i\}} b_k^i.$$

- We say that **property $P^i(D)$ is satisfied if $a^i \in D$;**

- We say that **property $Q^i(D)$ is satisfied if $b^i \in D$.**

Theorem 5 *Given $i \in N$, we have that $a^i \in C(m(D))$ if and only if $a^i \in D$ and $b^i \in C(m(D))$ if and only if $b^i \in D$.*

Proof We present the proof only for a^i. ($\Longleftarrow$) holds because $D \subseteq C(m(D))$. Now we prove ($\Longrightarrow$).

Note that a^i is the unique point in $I(m(D))$ where $x(N \setminus \{i\})$ is minimal. As $a^i \in C(m(D))$ it is also unique in $C(m(D))$.

Further, $m(D)(N \setminus \{i\}) = \Sigma_{j \in N \setminus \{i\}} a_j^i$ because $m(D)$ is exact and so there exists $\hat{x} \in D \subseteq C(m(D))$ such that $\hat{x}(N \setminus \{i\}) = \Sigma_{j \in N \setminus \{i\}} a_j^i$. Due to the uniqueness of this minimal point in $C(m(D))$ we have that $\hat{x} = a^i$, so $a^i \in D$. $\square$

Now we give conditions on D in order to have that $m(D)$ is a simplex game.

Theorem 6 *The game $m(D)$ is a simplex game if and only if the property $P^i(D)$ holds for each $i \in N$.*

Proof We must prove that $I(m(D)) \subset C(m(D)) \Longleftrightarrow \forall i \in N, P^i(D)$ holds.

Suppose that for all $i \in N$, $P^i(D)$ holds. Take $\hat{x} \in I(m(D))$. We must prove that $\hat{x} \in C(m(D))$, i.e. for all $S \subset N$, $\sum_{j \in S} \hat{x}_j \geq m(D)(S) = \min\left\{\sum_{j \in S} x_j \mid x \in D\right\}$. If $S = N$ the equality holds because $\sum_{j=1}^n \hat{x}_j = m(D)(S) = \alpha$. Consider then a coalition $S \subsetneq N$ and let us take $a^i \in D$ with $i \notin S$. Then, by definition, $a^i \in C(m(D))$ because $\sum_{j \in S} a_j^i = m(D)(S) = \min\left\{\sum_{j \in S} x_j \mid x \in D\right\}$. As $\hat{x}_j \geq a_j^i$ for all $j \in S$, it must be $\sum_{j \in S} \hat{x}_j \geq m(D)(S)$, that is $\hat{x} \in C(m(D))$. As property $P^i(D)$ holds for all $i \in N$, then $I(m(D)) \subset C(m(D))$.

Suppose now that $I(m(D)) = C(m(D))$. By definition, a^i belongs to $I(m(D))$ and then also to $C(m(D))$. Then $a^i \in D$ by theorem 5, and so property $P^i(D)$ holds for all $i \in N$. $\square$

The following theorem can be proved using arguments similar to the ones of the previous theorem.

Theorem 7 $m(D)$ *is a dual simplex game if and only if for each* $i \in N$ $Q^i(D)$ *holds.*

5.3 Big boss games

Let us consider now a characterization of big boss games. Let us consider the family MG^N of all n-person monotonic games. Given $v \in MG^N$, in [MNPT] Muto et al. defined the set

$$H(v) = \{x \in \mathbb{R}^n \mid \quad x(N) = v(N), \quad 0 \le x_i \le M_i, \quad \text{for all} \quad i = 2, ..., n\},$$

where $M_i = v(N) - v(N \setminus \{i\})$, and then they proved the following theorem.

Theorem 8 *If* v *belongs to* MG^N, *then* v *is a big boss game with big boss* 1 *if and only if*
$$H(v) = C(v).$$

For share opportunity sets the following theorem holds.

Theorem 9 *Let* $D \in \mathcal{K}^n$ *satisfying*

$$min_{x \in D} x_i = 0 \quad \text{for every} \quad i \ne 1. \tag{3}$$

Then $H(m(D)) = C(m(D))$ *if and only if*

$$a^1 = (\alpha, 0, 0, ...0) \in D,$$
$$b^1 = (\alpha - (M_2 + ... + M_n), M_2, ..., M_n) \in D.$$

Proof Suppose that $H(m(D)) = C(m(D))$. By definition of $H(m(D))$, $a^1, b^1 \in H(m(D)) = C(m(D))$ and then , by theorem 5, $a^1, b^1 \in D$.
Now suppose that $a^1, b^1 \in D \subseteq C(m(D))$. We must prove that $H(m(D)) = C(m(D))$. Let $x \in H(m(D))$. If we take a coalition S such that $1 \notin S$, then $a^1 \in D$ implies $m(D)(S) \le 0$, while (3) implies $m(D)(S) \ge 0$ and then $m(D)(S) = 0$. This means $x(S) \ge m(D)(S)$. If we take a coalition S such that $1 \in S$, as $b^1 \in D$, we have that

$$m(D)(S) \le \alpha - \Sigma_{i \in N \setminus S} M_i. \tag{4}$$

As $x \in H(m(D))$, $\Sigma_{i \in N \setminus S} x_i \le \Sigma_{i \in N \setminus S} M_i$, and then $\alpha = \Sigma_{i \in N} x_i \le \Sigma_{i \in S} x_i + \Sigma_{i \in N \setminus S} M_i$. By (4) $\Sigma_{i \in S} x_i \ge \alpha - \Sigma_{i \in N \setminus S} M_i \ge m(D)(S)$. So, $x \in C(m(D))$. Suppose now $x \in C(m(D))$. If $i \ne 1$, then by (3) we have $x_i \ge m(D)(\{i\}) = 0$, and as $m(D)(N \setminus \{i\}) \le \Sigma_{j \ne i} x_j$, we have $x_i = \alpha - \Sigma_{j \ne i} x_j \le \alpha - m(N \setminus \{i\}) = M_i$, that is $x \in H(m(D))$. $\qquad\square$

For share opportunity sets and related big boss games we have the following characterization.

Theorem 10 *Let $D \in \mathcal{K}^n$ satisfy condition (3). Then $m(D)$ is a big boss game with big boss 1 if and only if $P^1(D)$ and $Q^1(D)$ hold.*

Proof Theorem 5 and condition (3) assure that $m(D)$ is monotonic and so it is possible to use theorem 8. If $P^1(D)$ and $Q^1(D)$ hold, then $a^1, b^1 \in D$ and $m(D)$ is a big boss game by theorems 9 and 8. Suppose now that $m(D)$ is a big boss game with 1 as big boss. Then, by theorems 8 and 9, $H(m(D)) = C(m(D))$ and a^1, b^1 must belong to D, that is $P^1(D)$ and $Q^1(D)$ hold. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5.4   Convex games

Let $\sigma = (\sigma(1), ..., \sigma(n))$ be an ordering in N and let $\bar{\sigma} = (\sigma(n), \sigma(n-1), ..., \sigma(1))$ be the reverse ordering of σ. Given $D \in \mathcal{K}^n$, we define $r_\sigma(D) = (r_{\sigma(1)}, ..., r_{\sigma(n)})$ by

$$
r_{\sigma(k)} = \begin{cases}
\min \left\{ x_{\sigma(1)} \,|\, x \in D \right\} & k = 1 \\[2ex]
\min \left\{ \sum_{j=1}^{k} x_{\sigma(j)} \,|\, x \in D \right\} - \min \left\{ \sum_{j=1}^{k-1} x_{\sigma(j)} \,|\, x \in D \right\} & k = 2, ... n
\end{cases}
$$

Remark 10 *Note that*

$$
\begin{aligned}
r_\sigma(D) &\in \arg\min \left\{ x_{\sigma(1)} \,|\, x \in D \right\}, \\
r_\sigma(D) &\in \arg\min \left\{ x_{\sigma(1)} + x_{\sigma(2)} \,|\, x \in D \right\}, \\
&\qquad\qquad . \\
&\qquad\qquad . \\
&\qquad\qquad . \\
r_\sigma(D) &\in \arg\min \left\{ x_{\sigma(1)} + x_{\sigma(2)} ... + x_{\sigma(n)} \,|\, x \in D \right\}.
\end{aligned}
$$

- We say that **property $R_\sigma(D)$ is satisfied** if $r_\sigma(D) \in D$.

Theorem 11 *If for all $\sigma \in \Pi(N)$ property $R_\sigma(D)$ is satisfied, then $m(D)$ is convex.*

Proof Suppose that $r_\sigma(D) \in D$ for all $\sigma \in \Pi(N)$. Then, we must prove that $m(D)$ is convex, i.e for all $S \subset T \subset N \smallsetminus \{i\}$

$$
m(D)(T \cup \{i\}) - m(D)(T) \geq m(D)(S \cup \{i\}) - m(D)(S).
$$

Choose an ordering σ of the form $(S, T \smallsetminus S, i, N \smallsetminus (T \cup \{i\}))$, which means that players in S enter first, then the players in $T \setminus S$ followed by player i. Then $r_\sigma(D) = \widehat{x} \in D$ and

$$\min\left\{\sum_{j \in T \cup \{i\}} x_j \,|\, x \in D\right\} - \min\left\{\sum_{j \in T} x_j \,|\, x \in D\right\} = \sum_{j \in T \cup \{i\}} \widehat{x}_j - \sum_{j \in T} \widehat{x}_j = \widehat{x}_i,$$

and if we choose an ordering σ' of the form $(S, \{i\}, T \smallsetminus S, N \smallsetminus (T \cup \{i\}))$, then $r_{\sigma'}(D) = \widehat{x}' \in D$ and

$$\min\left\{\sum_{j \in S \cup \{i\}} x_j \,|\, x \in D\right\} - \min\left\{\sum_{j \in S} x_j \,|\, x \in D\right\} = \sum_{j \in S \cup \{i\}} \widehat{x}'_j - \sum_{j \in S} \widehat{x}'_j = \widehat{x}'_i$$

with $\widehat{x}'_i \leq \widehat{x}_i$. $\qquad\square$

The converse doesn't hold as we can see in the following example.

Example 6 *Let $D = \{(2,1,0),\ (0,2,1),\ (1,0,2)\}$. In this case, $m(D)(\{1,2,3\}) = 3, m(D)(\{1,2\}) = m(D)(\{2,3\}) = m(D)(\{1,3\}) = 1, m(D)(\{1\}) = m(D)(\{2\}) = m(D)(\{3\}) = 0$. $m(D)$ is convex but $r^{(1,2,3)}(D) = (0,1,2) \notin D$.*

Theorem 12 *Let $D \in \mathcal{P}^n$ be such that $m(D)$ is a convex game. Then $R_\sigma(D)$ holds for all $\sigma \in \Pi(N)$.*

Proof As in this case $C(m(D)) = D$, we have

$$r_\sigma(D) = L^{\bar{\sigma}}(C(m(D))) \in C(m(D)) = D.$$

$\qquad\square$

Corollary 1 *Suppose that $D \in \mathcal{P}^n$. Then $m(D)$ is convex if and only if for all $\sigma \in \Pi(N)$, $R_\sigma(D)$ holds.*

6 Partially defined games

Consider a partially balanced game $\langle N, v, \mathcal{F} \rangle$. The partial core $C_\mathcal{F}(v)$ of this partially balanced game is a set with a perfect structure and it is possible to define the minimum right game associated to $C_\mathcal{F}(v)$ as $m(C_\mathcal{F}(v))$. Hence, if γ is a solution defined for cooperative games, we can extend this solution to partially balanced games as follows. If $\langle N, v, \mathcal{F} \rangle$ is a partially balanced game, then $\gamma(\langle N, v, \mathcal{F} \rangle) = \gamma(m(C_\mathcal{F}(v)))$. In particular, if we consider the

definition of L^σ and AL given by Tijs in [Ti], we can extend such definitions to partially balanced games. In section 3 we gave the definition of AL and L^σ also for subsets of $\mathbb{R}^n$ and we proved their continuity with respect to the Hausdorff topology on $\mathcal{P}^n$.

Like any partially balanced game $\langle N, v, \mathcal{F} \rangle$, $C_\mathcal{F}(v)$ is a set with a perfect structure, then AL and L^σ are continuous. More precisely, the following theorems hold.

Theorem 13 *Let BG^N be the set of balanced n-person games. Let $\langle N, v_k \rangle$, $k = 1, 2, ...$ be a sequence of games converging to $\langle N, v \rangle$ (i.e. $v_k(S)$ converges to $v(S)$ for every $S \in 2^N$). Then $L^\sigma(\langle N, v_k \rangle)$ converges to $L^\sigma(\langle N, v \rangle)$ and $AL(\langle N, v_k \rangle)$ converges to $AL(\langle N, v \rangle)$.*

Theorem 14 *Let $BG_\mathcal{F}^N$ be the set of partially balanced n-person games. Let $\langle N, v_k, \mathcal{F} \rangle$ $k = 1, 2, ...$ be a sequence of partially balanced games converging to $\langle N, v, \mathcal{F} \rangle$. (i.e. $v_k(S)$ converges to $v(S)$ for every $S \in \mathcal{F}$). Then $L^\sigma(\langle N, v_k, \mathcal{F} \rangle)$ converges to $L^\sigma(\langle N, v, \mathcal{F} \rangle)$ and $AL(\langle N, v_k, \mathcal{F} \rangle)$ converges to $AL(\langle N, v, \mathcal{F} \rangle)$.*

7   Concluding remarks

Each share opportunity set gives rise to a minimum right game and a utopia game. For the subclass of perfect share opportunity sets (i.e. opportunity sets with a perfect structure) the relation between share opportunity sets and the corresponding games is a continuous relation and also the lattice structures on share opportunity sets and on games fit nicely.

For bankruptcy games the minimum right games are convex games and the utopia games are concave games. Further the average lexicographic value coincides with with run to the bank rule and the Shapley value.

For airport games, the corresponding games are concave and convex respectively. Further the average lexicographic value coincides with the Shapley value of the airport game.

References

References

[AM]   Aumann R.J., Maschler M. "Game theoretic analysis of a bank-ruptcy problem from the Talmud", *J. Econom. Theory*, 36, 195-213, (1985);

[BGKKT] Bank B., Guddat J., Klatte D., Kummer B., Tammer K. "Non-linear Parametric Optimization", *Birkhäuser Verlag, Basel, Boston, Stuttgart*, (1983);

[B]   Berge C. "Espaces topologiques, fonctions multivoques", *Dunod, Paris*, (1959);

[KT]   Klein E., Thompson A.C. "Theory of correspondences", *Wiley, New York*, (1984);

[LT]   Littlechild S.C., Thompson G.F. "Aircraft landing fees: a game the-ory approach", *Bell Journal of Economics*, 8, 86-204, (1977);

[LPTT] Lucchetti R.E., Patrone F., Tijs S., Torre A. "Continuity Properties of solution concepts for cooperative games", *O.R. Spektrum*, 9, 101-107, (1987);

[MNPT] Muto S., Nakayama M., Potters J., Tijs S. "On Big Boss Games", *Economic Studies Quarterly*, 39, 303-321, (1988);

[O'N]   O'Neill B. "A problem of rights arbitration from the Talmud", *Math. Soc. Sci.*, 2, 345-371, (1982);

[O]   Owen G. "Game theory", *Academic press, New York*, (1982);

[S]   Schmeidler D. "Cores of exact games,I,", *J. of Math Anal. Applica-tion,* 40, 214-225, (1972);

[T]   Thomson W. "Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey", *Math. Soc. Sci.*, 45, 249-297, (2003);

[Ti]   Tijs S. "The first steps with Alexia, the average lexicographic value", *CentER Discussion paper* 2005-123, Tilburg University, The Nether-lands.