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ABSTRACT. Microscopic simulation models are often evaluated based on visual inspection of the results. This paper presents formal econometric techniques to compare microscopic simulation (MS) models with real-life data. A related result is a methodology to compare different MS models with each other. For this purpose, possible parameters of interest, such as mean returns, or autocorrelation patterns, are classified and characterized. For each class of characteristics, the appropriate techniques are presented. We illustrate the methodology by comparing the MS model developed by Levy, Levy, and Solomon (2000) and the market fraction model developed by He and Li (2005a, b) with actual data.

JEL classification: C10, G12

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1. INTRODUCTION

In financial markets, the observable quantities are usually the consequences of aggregated individual movements at the macro level, but the determinants lie at the micro level. In general, it might be very difficult to describe the individual behavior (decision making under risk and uncertainty), and the implied aggregated phenomena explicitly:

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economics, including financial markets, is a complex system. Very often, it is very difficult, if not impossible, to find analytical solutions for such systems. In order to get some insight into it, a possible approach is to apply Microscopic Simulation (MS). The idea is to study complex systems by representing each of the microscopic elements individually (on a computer) and by simulating the behavior of the entire system, keeping track of all of the individual elements and their interactions over time. Throughout the simulation, the macroscopic variables that are of interest can be recorded, and their dynamics can be investigated.

The growing literature of MS in finance has resulted in various competing microscopic simulation models to explain observed phenomena in real-life financial markets. The works of Arthur *et al.* (1997), Brock and Hommes (1997, 1998), Chiarella and He (2003), Chiarella *et al.* (2005), Levy *et al.* (2000), Lux (1998), Lux and Marchesi (1999, 2001), Manzan and Westerhoff (2005), Westerhoff (2004), Westerhoff and Dieci (2005), among others, provide good examples of various MS models. So far, research has mainly focused on investigating whether a model shares some important characteristics of the actual financial markets, the stylized facts, such as short-term momentum, excess volatility, heavy trading volume, a positive correlation between volume and contemporaneous absolute returns, endogenous market crashes, etc. The typical way to do this is by running a single or at most a few ('representative') realizations of the MS model under consideration, and to analyze whether the outcomes of the variables of interest, like stock returns, share more or less the same patterns as can be found in the actual data. Although much work has already been done along these lines (see, for instance, Chen *et al.*, 2001, Hommes, 2005, LeBaron, 2000, 2005, Lux, 2004, and references therein), to our knowledge, *systematic* procedures to investigate the difference between two MS models, and to judge whether a MS model is realistic or not have not yet been developed.

The relevance of such systematic procedures is clear: First, when comparing different microscopic simulation models, important factors that drive MS economies can be detected and investigated. Second, when the comparison of a MS model with real life data leads to rejection of the MS model, factors might be identified that need to be adapted or integrated to create models that better fit the empirical findings on financial markets. Third, the procedures to compare different MS models can be used to check the sensitivity of the outcomes of a MS model with respect to its initial conditions and parameter settings. In this way we might gain a better understanding of the underlying mechanism of MS models and, in particular, we can find out which parameter values or initial settings are most useful when calibrating a MS model. Note also that confronting a MS model with real life data is not only a way to check the "realism" of the model, which might enhance our knowledge of financial markets, but it is also an essential step from a practical point of view. For example, when MS models are used to evaluate the impact of government policies, or to forecast, we need to explicitly link the MS models with real life data.

The aim of this paper is to develop and apply *econometric* techniques to compare different microscopic simulation models, and, more importantly, to compare data generated by a MS model with real life data. We shall study such comparisons in terms of distribution functions of the variables of interest, focussing on distributional characteristics that are considered to be relevant. The distributional characteristics of an MS economy can be retrieved with an arbitrary level of precision, since we can run the MS model independently as many times as we want. In particular, this possibility of independent runs of a MS model allows us to quantify the simulation inaccuracy in a straightforward way, making comparisons between different MS models a standard exercise in econometrics.

When comparing a MS model with actual data the sampling inaccuracy of the actual data has to be taken into account. The quantification of this sampling inaccuracy usually takes place via time series asymptotics. Exploiting the extra simulation dimension available in MS models, the combined sampling and simulation inaccuracy in the MS outcomes can be made negligibly small, when compared with the actual outcomes, by having the number of simulations sufficiently large. This means that we only have to quantify the estimation inaccuracy of the actual data, making also the comparison of a given MS model with actual data a straightforward exercise in econometrics.

However, usually we do not have available a single MS model, but a whole class, where each MS model in the class corresponds to different parameter values, initial conditions, and so on. Preferably, we would like to be able to estimate the appropriate MS model in this model class. Generally, however, this seems to be infeasible, due to the complexity of the microscopic simulation models, which makes verification of identification rather difficult, and thus proving consistency of estimation troublesome. Moreover, in case consistent estimation is possible, the likely heavily nonlinear relationship between observables and unknown parameters to be estimated might seriously complicate estimation. Therefore, in this paper, we only consider *calibration* of a model in a model class, by choosing some model in the model class that minimizes a distance between particular actual data based parameters and MS model based parameters, restricting attention to a subset of MS models.

To illustrate the methodology, we consider two applications. The first one is the Levy *et al.* (2000) microscopic simulation model, and the second one is the He and Li (2005a, b) Market Fraction model. In the first application, we focus on a single model, the LLS-model, and test its sensitivity to changes in the input variables as well as designed mechanism, and we confront this single model with actual data. In the

second application, we consider a whole model class, and select one model by means of calibration, before confronting it to actual data.

The remainder of this paper is organized as follows. In the next section, we present the econometric methodology, focussing particularly on the case of stationary data. Then we discuss the two applications, the Levy *et al.* (2000) microscopic simulation model in Section 3, and the He and Li (2005a, b) Market Fraction model in Section 4. The final section concludes.

2. ECONOMETRIC BACKGROUND

In this section we present some econometric background. First, we discuss the general set up. Then we specialize to the stationary case, distinguishing between short range and long range dependent distributional characteristics.

2.1. **Set up.** A microscopic model will be denoted by m. A class of microscopic models will be denoted by $\mathcal{M}$. Generally speaking, a microscopic simulation model consists of a designed mechanism of the system, inputs, and outputs. The designed mechanism describes the functioning of the system, and how the dynamics evolve over time; inputs include parameters, initial conditions, and also noise; the outputs are the observations of variables like $-$in our case$-$ stock prices and returns. Usually, models in the same model class will differ in terms of inputs, but not in terms of the designed mechanism. For instance, the Levy, Levy, Solomon (LLS) model class, $\mathcal{M}_{LLS}$, (see Levy *et al.*, 2000) consists of models aimed to study the behavior of individual investors. According to the designed mechanism, these investors make their decision by maximizing a standard expected utility function and they interact via a market mechanism consisting of buying and selling stocks and bonds within a temporary Walrasian equilibrium mechanism. Different input values, like the number of investors, the number of shares, the (parameters of the) utility functions, the initial

dividend value, and the standard deviation of the random noise affecting the investors' decisions, etc., result in different models m in the model class $\mathcal{M}_{LLS}$.

These MS models generate as output observations on variables, including, in our case, stock prices and stock returns, but possibly also other interesting variables, like the subdivision of total wealth among groups of investors, etc. One of the main aims of microscopic simulation models is to be able to give a description of actual data. However, before making this comparison with actual data, it might also be relevant to compare different microscopic models m in the same model class $\mathcal{M}$, for instance, to find out the sensitivity of the output variables with respect to the input values. Both comparisons can be quantified using econometric techniques.

Let the actual data consist of T time period observations on a k-dimensional vector of variables $X_t^a \in \mathbb{R}^k$, $t = 1, ..., T$, where the superscript a refers to *actual*. The actual data is then described by the distribution function $d_{a,T}$, i.e., $(X_1^a, ..., X_T^a) \sim d_{a,T}$. We have $d_{a,T} \in \mathcal{D}_T$, where $\mathcal{D}_T$ is a set of distribution functions. Given a microscopic model class $\mathcal{M}$, we assume a transformation d_T, that assigns to each $m \in \mathcal{M}$ the corresponding distribution function $d_T(m)$, where we assume that $d_T(m) \in \mathcal{D}_T$, thus, $d_T : \mathcal{M} \mapsto \mathcal{D}_T$. The set of all distribution functions that can be generated by the microscopic model class $\mathcal{M}$ is denoted by $d_T(\mathcal{M})$. Notice that the set $\mathcal{D}_T$ is assumed to be so large that it includes both $d_{a,T}$ and $d_T(\mathcal{M})$. Given m, we can generate N simulations, which we assume to be independent from one another. Each simulation run results in T observations on the k variables, one observation for each time period. We denote the observations for microscopic model m in simulation run j for time period t by the vector $X_t^{m,j} \in \mathbb{R}^k$. We then have $\left(X_1^{m,j}, ..., X_T^{m,j}\right) \overset{i.i.d.}{\sim} d_T(m)$, for $j = 1, ..., N$.

In order to compare distributions, we shall make use of *distribution characteristics*, like the mean, other moments, density functions, etc. The set in which these characteristics take their value is denoted by $\mathcal{E}$. To work with distributional characteristics, we assume, given $\mathcal{D}_T$, the presence of a transformation ψ_T, that assigns to each $\delta_T \in \mathcal{D}_T$ the corresponding characteristics $\varphi_T = \psi_T(\delta_T) \in \mathcal{E}$.[1] Define $\varphi_{m,T} = \psi_T(d_T(m))$, the distribution characteristics of microscopic model m. The actual distribution characteristics are given by $\varphi_{a,T} = \psi_T(d_{a,T})$.

Given the above formalization, our aim is to compare $d_T(m)$ for different microscopic models m in $\mathcal{M}$, and to confront $d_T(m)$ with $d_{a,T}$, for some or all $m \in \mathcal{M}$. This will be achieved by using the distribution characteristics of the microscopic model(s) $m \in \mathcal{M}$ under consideration, $\varphi_{m,T}$, and the corresponding actual distribution characteristics $\varphi_{a,T}$. To proceed, notice that we can estimate $d_T(m)$ by the empirical distribution function based on N i.i.d.-simulations. Denote this empirical distribution function by $\widehat{d}_{m,T,N}$. Using this empirical distribution function we can estimate $\varphi_{m,T}$ by $\widehat{\varphi}_{m,T,N} = \psi_T\left(\widehat{d}_{m,T,N}\right)$. Under appropriate regularity conditions, this estimator will be consistent for $N \to \infty$, and, moreover, we will be able to quantify its estimation and simulation inaccuracy. So, using standard econometric estimation and testing

[1]The transformation ψ_T might be defined explicitly or implicitly. In case we consider means, other moments, or density functions of some or all marginal distributions, the transformation ψ_T can be defined explicitly. But in many cases one works with an econometric model class to describe the data, which generally will result in an implicitly defined ψ_T. An econometric model class consists of a set $\mathcal{M}^e$ of econometric models and a transformation $d_T^e : \mathcal{M}^e \mapsto \mathcal{D}_T$, that assigns to each econometric model m^e the corresponding induced distribution $d_T^e(m^e) \in \mathcal{D}_T$. In addition, there will be a transformation $\chi_T : d_T^e(\mathcal{M}^e) \mapsto \mathcal{E}$ (the estimation procedure) that assigns to each distribution in the set $d_T^e(\mathcal{M}^e)$, the set of distribution functions described by the econometric model class $\mathcal{M}^e$, the corresponding distributional characteristic $\varphi_T \in \mathcal{E}$. Assuming that the econometric model class is identified, we have that d_T^e is injective. The model class $\mathcal{M}^e$ is exactly identified in case d_T^e is also one-to-one, in which case ψ_T is immediately determined by χ_T: $\psi_T = \chi_T$. But, generally, we are dealing with the case of overidentification, when d_T^e is not one-to-one, meaning that $d_T^e(\mathcal{M}^e)$ is a strict subset of $\mathcal{D}_T$, and we have to take into account the possibility of misspecification, i.e., $d_{a,T} \notin d_T^e(\mathcal{M}^e)$. In this case the set $\mathcal{D}_T$ has to be partitioned into equivalence classes such that each $\delta_T \in \mathcal{D}_T$ belongs to an equivalence class fully described by a unique $d_T^e(m^e) \in d_T^e(\mathcal{M}^e)$, and $\psi_T : \mathcal{D}_T \mapsto \mathcal{E}$ is then defined by $\psi_T(\delta_T) = \chi(d_T^e(m))$ in case δ_T belongs to the equivalence class of $d_T^e(m^e)$. Usually, the partitioning of $\mathcal{D}_T$ into the equivalence classes follows implicitly from the estimation method employed.

techniques, we will then be able in a straightforward way to make a comparison between $d_T(m_1)$ and $d_T(m_2)$ by means of comparing $\varphi_{m_1,T}$ and $\varphi_{m_2,T}$, on the basis of the corresponding estimators $\widehat{\varphi}_{m_1,T,N}$ and $\widehat{\varphi}_{m_2,T,N}$.

However, for the actual data we only have one realization $(X_1^a, ..., X_T^a)$ drawn from $d_{a,T}$, so that, without further assumptions, we will be limited in our possibilities to confront $\varphi_{m,T}$ with $\varphi_{a,T}$. In order to be able to make a comparison between $d_T(m)$ and $d_{a,T}$, we shall consider the limiting case $T \to \infty$, together with appropriate regularity conditions to ensure that $\varphi_{a,\infty} \equiv \lim_{T\to\infty} \varphi_{a,T}$ is well defined, and $\widehat{\varphi}_{a,T} = \psi_T\left(\widehat{d}_{a,T}\right)$ is a consistent estimator for $\varphi_{a,\infty}$, whose estimation inaccuracy can be quantified asymptotically (as $T \to \infty$). Then it becomes possible to confront $\varphi_{m,\infty}$ with $\varphi_{a,\infty}$ (as approximations of $\varphi_{m,T}$ with $\varphi_{a,T}$, respectively), using their respective estimates $\widehat{\varphi}_{a,T}$ and $\widehat{\varphi}_{m,T,N}$. Notice that now we might have two limits to take into account: the time dimension limit $T \to \infty$ and the simulation limit $N \to \infty$.

We shall make the various comparisons under the assumption of (strict) stationarity of $(X_1^a, ..., X_T^a)$ and $\left(X_1^{m,j}, ..., X_T^{m,j}\right)$ (for each j). Such a stationarity assumption might become more or less realistic after appropriate data transformations, like, for instance, transforming stock prices into returns, or bond prices into yields. We shall distinguish between short range distributional characteristics, like the mean at some time t, and long range distributional characteristics, like a long memory parameter. In the former case a comparison between two microscopic models using econometric techniques is possible for finite T, i.e., only $N \to \infty$-asymptotics suffices. In the latter case we shall need $T \to \infty$, also when comparing two microscopic models.

2.2. **Stationarity with short range distributional characteristics.** Due to the (strict) stationarity assumption, the distribution of $X_{t+k_1}^a, ..., X_{t+k_\ell}^a$ does not depend on t, i.e., we can write

$$\left(X_{t+k_1}^a, ..., X_{t+k_\ell}^a\right) \sim d_{a,(k_1,...,k_\ell)},$$

with $d_{a,(k_1,...,k_\ell)}$ the distribution function of $\left(X^a_{t+k_1},...,X^a_{t+k_\ell}\right)$, the same for each t, and, similarly, we have

$$\left(X^{m,j}_{t+k_1},...,X^{m,j}_{t+k_\ell}\right) \sim d_{(k_1,...,k_\ell)}(m),$$

with $d_{(k_1,...,k_\ell)}(m)$ the distribution function of $\left(X^{m,j}_{t+k_1},...,X^{m,j}_{t+k_\ell}\right)$, also the same for each t. We denote the set of induced $(k \times \ell)$-dimensional distribution functions to which $d_{a,(k_1,...,k_\ell)}$ and $d_{(k_1,...,k_\ell)}(m)$ belong by $\mathcal{D}_{(k_1,...,k_\ell)}$, and the subset generated by the model class $\mathcal{M}$ by $d_{(k_1,...,k_\ell)}(\mathcal{M})$.

Short range dependent distribution characteristics are transformations of distribution functions in $\mathcal{D}_{(k_1,...,k_\ell)}$ for sufficiently large but finite k_ℓ. Thus, let $\psi : \mathcal{D}_{(k_1,...,k_\ell)} \mapsto \mathcal{E}$ be such a transformation, that assigns to each distribution $\delta_{(k_1,...,k_\ell)} \in \mathcal{D}_{(k_1,...,k_\ell)}$ the corresponding distribution characteristics of interest in $\mathcal{E}$. Then we have as actual distribution characteristics

$$\varphi_{a,(k_1,...,k_\ell)} = \psi\left(d_{a,(k_1,...,k_\ell)}\right) \in \mathcal{E}$$

and as distribution characteristics of the corresponding microscopic model m

$$\varphi_{m,(k_1,...,k_\ell)} = \psi\left(d_{(k_1,...,k_\ell)}(m)\right) \in \mathcal{E}.$$

Estimators and their limit distributions—Assuming asymptotic independence,[2] in addition to stationarity, we can estimate the characteristics of the actual data DGP $\varphi_{a,(k_1,...,k_\ell)} = \psi\left(d_{a,(k_1,...,k_\ell)}\right)$ consistently by

$$\widehat{\varphi}_{a,(k_1,...,k_\ell),T} = \psi\left(\widehat{d}_{a,(k_1,...,k_\ell),T}\right),$$

[2]Asymptotic independence corresponds to ergodicity and/or mixing conditions. See, for instance, Billingsley (1968), or, more recently, Bierens (2004).

for $T \to \infty$, assuming additional smoothness conditions on ψ. Here $\widehat{d}_{a,(k_1,...,k_\ell),T}$ denotes the empirical distribution function of $d_{a,(k_1,...,k_\ell)}$, using the time series $(X_1^a, ..., X_T^a)$. Moreover, under appropriate additional regularity conditions we will be able to quantify the (asymptotic) estimation inaccuracy of $\widehat{\varphi}_{a,(k_1,...,k_\ell),T}$. Indeed, the limit distribution of $\widehat{\varphi}_{a,(k_1,...,k_\ell),T}$ for $T \to \infty$ can easily be found by linearization of the transformation $\psi : \mathcal{D}_{(k_1,...,k_\ell)} \mapsto \mathcal{E}$ (assuming that it is smooth enough so that it can be linearized), combined with an appropriate limit distribution[3].

Similarly, and under the same smoothness assumptions regarding ψ, we can estimate $\varphi_{m,(k_1,...,k_\ell)} = \psi\left(d_{(k_1,...,k_\ell)}(m)\right)$ consistently, but here we have various alternative ways to estimate $\varphi_{m,(k_1,...,k_\ell)}$, due to the extra simulation dimension N.

a) We can estimate $\varphi_{m,(k_1,...,k_\ell)}$ consistently by $\widehat{\varphi}^j_{m,(k_1,...,k_\ell),T} = \psi\left(\widehat{d}^j_{m,(k_1,...,k_\ell),T}\right)$, for $T \to \infty$, where $\widehat{d}^j_{m,(k_1,...,k_\ell),T}$ is the empirical distribution function of $d_{(k_1,...,k_\ell)}(m)$, using the time series of the j-th simulation $\left(X_1^{m,j}, ..., X_T^{m,j}\right)$. This is comparable to the way we estimate $\varphi_{a,(k_1,...,k_\ell)}$, and is based on exploiting the asymptotic independence assumption in addition to the stationarity assumption.

b) We can estimate $\varphi_{m,(k_1,...,k_\ell)}$ consistently by $\widehat{\varphi}^t_{m,(k_1,...,k_\ell),N} = \psi\left(\widehat{d}^t_{m,(k_1,...,k_\ell),N}\right)$, for $N \to \infty$, where $\widehat{d}^t_{m,(k_1,...,k_\ell),N}$ is the empirical distribution function of $d_{(k_1,...,k_\ell)}(m)$, using the N simulations $\left(X_{t+k_1}^{m,1}, ..., X_{t+k_\ell}^{m,N}\right)$ for some given t (with $1 \leq t + k_1, ..., t + k_\ell \leq T$). Obviously, this estimator does not require stationarity and an asymptotic independence assumption to yield a consistent estimator for the distribution of $\left(X_{t+k_{m,1}}^1, ..., X_{t+k_\ell}^{m,N}\right)$, for the given t, but only the i.i.d.-assumption over the N simulations.

[3]Essentially, in the general case this means combining Donsker's theorem for dependent sequences (see, for instance, Billingsley (1968), or, more recently, Andrews and Pollard (1994), also for the required regularity conditions), to find the limiting distribution of $\widehat{d}_{a,(k_1,...,k_\ell),T}$ for $T \to \infty$, and the functional delta method (see, for instance, van der Vaart, 1998), applied to the transformation $\varphi : \mathcal{D}_{(k_1,...,k_\ell)} \mapsto \mathcal{E}$, assuming that it is smooth enough to be Hadamard differentiable.

c) We can estimate $\varphi_{m,(k_1,...,k_\ell)}$ consistently by averaging $\widehat{\varphi}^j_{m,(k_1,...,k_\ell),T}$ over the N simulations: $\overline{\varphi}^N_{m,(k_1,...,k_\ell),T} = \frac{1}{N}\sum_{j=1}^N \widehat{\varphi}^j_{m,(k_1,...,k_\ell),T}$ is a consistent estimator, at least, if we take the sequential limit $T \to \infty$, possibly followed by $N \to \infty$.

d) We can estimate $\varphi_{m,(k_1,...,k_\ell)}$ consistently by averaging $\widehat{\varphi}^t_{m,(k_1,...,k_\ell),N}$ over time: $\overline{\varphi}^T_{m,(k_1,...,k_\ell),N} = \frac{1}{T}\sum_{t=1}^T \widehat{\varphi}^t_{m,(k_1,...,k_\ell),N}$ is a consistent estimator, at least, if we take the sequential limit $N \to \infty$, possibly followed by $T \to \infty$.

e) We can estimate $\varphi_{m,(k_1,...,k_\ell)}$ consistently by $\widehat{\varphi}_{m,(k_1,...,k_\ell),N,T} = \psi\left(\widehat{d}_{m,(k_1,...,k_\ell),N,T}\right)$, for $N, T \to \infty$, where $\widehat{d}_{m,(k_1,...,k_\ell),N,T}$ is the empirical distribution function of $d_{(k_1,...,k_\ell)}(m)$ using all observations $X_t^{m,j}$, $t = 1, ..., T$, $j = 1, ..., N$, taking the limit $NT \to \infty$.

Under appropriate additional regularity conditions we are able to quantify the (asymptotic) estimation/simulation inaccuracy and to characterize the asymptotic distribution of the estimator in each of these cases. In particular, the limit distributions of cases c)-e), when both limits $N \to \infty$ and $T \to \infty$ are taken (in appropriate orders) satisfy (with any of the three estimators simply represented by $\widehat{\varphi}_{m,(k_1,...,k_\ell)}$)

$$\sqrt{NT}\left(\widehat{\varphi}_{m,(k_1,...,k_\ell)} - \varphi_{m,(k_1,...,k_\ell)}\right) \xrightarrow{d} \mathcal{N}\left(0, \Sigma_{\varphi_{m,(k_1,...,k_\ell)}}\right),$$

with $\Sigma_{\varphi_{m,(k_1,...,k_\ell)}}$ the same in all three cases (as can easily be checked under appropriate smoothness conditions).

In general terms, the limit distribution of the characteristics based on actual data, $\widehat{\varphi}_{a,(k_1,...,k_\ell),T}$ takes a similar form:

$$\sqrt{T}\left(\widehat{\varphi}_{a,(k_1,...,k_\ell),T} - \varphi_{a,(k_1,...,k_\ell)}\right) \xrightarrow{d} \mathcal{N}\left(0, \Sigma_{\varphi_{a,(k_1,...,k_\ell)}}\right),$$

with $\Sigma_{\varphi_{a,(k_1,...,k_\ell)}}$ the asymptotic covariance matrix. Moreover, in case $d_{(k_1,...,k_\ell)}(m) = d_{a,(k_1,...,k_\ell)}$ it holds that $\Sigma_{\varphi_{m,(k_1,...,k_\ell)}} = \Sigma_{\varphi_{a,(k_1,...,k_\ell)}}$. The difference between the limit

distributions of $\widehat{\varphi}_{a,(k_1,\ldots,k_\ell),T}$ and the three estimators $\widehat{\varphi}_{m,(k_1,\ldots,k_\ell)}$ lies, of course, in the rate of convergence: the microscopic simulation based estimators have the extra dimension N, making them converge much faster than the actual data based estimator, in case $N \to \infty$.

Example 1: The mean $E(X_t^{m,j})$

The mean $\mu_m = E(X_t^m)$ can be rewritten as $\mu_m = \int x dF_m(x) = \psi(F_m) = \varphi_{m,(0)}$, where $F_m \equiv d_{m,(0)}$ stands for the marginal distribution function of X_t^m, the same for each t, and where we have $\ell = 1$ and $(k_1) = (0)$. We assume also that $k = 1$, i.e., $X_t^m \in \mathbb{R}$. The function ψ is defined as $\psi : F \in D_{(0)} \subseteq D(\mathbb{R}) \mapsto \int x dF(x) \in \mathbb{R}$, with $D_{(0)} = \{F \in D(\mathbb{R}),\ \int |z|\, dF(z) < \infty\}$, and with $D(\mathbb{R})$ the set of all non-decreasing right continuous functions $z : \mathbb{R} \mapsto \mathbb{R}$ such that $z(-\infty) = 0$ and $z(+\infty) = 1$.

With T possibly fixed and $N \to \infty$, we can take as estimators b) and d) (with in the latter case possibly no second limit $T \to \infty$). In case of b), we have that the empirical distribution function $\widehat{F}_m^t \equiv \widehat{d}_{m,(0),N}^t \in D(\mathbb{R})$ is given by

$$\widehat{F}_m^t(x) = \frac{1}{N} \sum_{j=1}^{N} 1_{[-\infty,x]}(X_t^{m,j}).$$

The corresponding estimate is given by

$$\widehat{\mu}_m^t \equiv \widehat{\varphi}_{m,(0),N}^t = \psi(\widehat{F}_m^t) = \int x d\widehat{F}_m^t(x) = \frac{1}{N} \sum_j X_t^{m,j}.$$

Averaging over T time periods yields case d)

$$\widehat{\mu}_m^T \equiv \widehat{\varphi}_{m,(0),N}^T = \frac{1}{T} \sum_t \widehat{\mu}_m^t.$$

In the alternative case with N possibly fixed and $T \to \infty$, we can take as estimators a) and c) (with in the latter case possibly no second limit $N \to \infty$). In case of estimator

a), we have that the empirical distribution function $\widehat{F}_m^j \equiv \widehat{d}^j_{m,(0),T} \in D(\mathbb{R})$ is given by

$$\widehat{F}_m^j(x) = \frac{1}{T}\sum_{t=1}^{T} 1_{[-\infty,x]}(X_t^{m,j}).$$

The corresponding estimate is given by

$$\widehat{\mu}_m^j \equiv \widehat{\varphi}^j_{m,(0),T} = \psi(\widehat{F}_m^j) = \int x d\widehat{F}_m^j(x) = \frac{1}{T}\sum_t X_t^{m,j}.$$

Averaging over the N independent simulations yields method c)

$$\widehat{\mu}_m^N \equiv \widehat{\varphi}^N_{m,(0),T} = \frac{1}{N}\sum_j \widehat{\mu}_m^j.$$

In case of estimation method e), we have $\widehat{F}_m \equiv \widehat{d}_{m,N,T}$ given by

$$\widehat{F}_m = \frac{1}{NT}\sum_{j=1}^{N}\sum_{t=1}^{T} 1_{(-\infty,x]}(X_t^{m,j})$$

and

$$\widehat{\mu}_{m,N,T} \equiv \widehat{\varphi}_{m,(0),N,T} = \int x d\widehat{F}_m(x) = \frac{1}{NT}\sum_{j=1}^{N}\sum_{t=1}^{T} X_t^{m,j}.$$

Straightforward calculations show for estimation procedure d)

$$\sqrt{N}(\widehat{\mu}_m^T - \mu_m) \xrightarrow{d} \mathcal{N}\left(0, \left(\frac{1}{T}\gamma_{m,0} + 2\sum_{i=1}^{T-1}\frac{T-i}{T^2}\gamma_{m,i}\right)\right),$$

when $N \to \infty$, with $\gamma_{m,i} = Cov(X_t^{m,j}, X_{t+i}^{m,j})$. For estimation method c), we find

$$\sqrt{T}(\widehat{\mu}_m^N - \mu_m) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{N}\left(\gamma_{m,0} + 2\sum_{i=1}^{\infty}\gamma_{m,i}\right)\right),$$

for $T \to \infty$. And for estimation method e), we have

$$\sqrt{NT}(\widehat{\mu}_{m,N,T} - \mu_m) \xrightarrow{d} \mathcal{N}\left(0, \left(\gamma_{m,0} + 2\sum_{i=1}^{\infty}\gamma_{m,i}\right)\right),$$

when $N \to \infty$ and $T \to \infty$. This latter limit distribution also applies to estimation method d) with additional, sequential limit $T \to \infty$ (and scaling by $\sqrt{NT}$ instead of $\sqrt{T}$) and to method c) with additional, sequential limit $N \to \infty$ (and similar adaptation of scaling).

Comparing microscopic simulation models—A comparison between two microscopic models m_1 and m_2 in $\mathcal{M}$ corresponds to the null hypothesis

$$H_0 : \varphi_{m_1,(k_1,\ldots,k_\ell)} = \varphi_{m_2,(k_1,\ldots,k_\ell)}$$

versus the alternative hypothesis

$$H_1 : \varphi_{m_1,(k_1,\ldots,k_\ell)} \neq \varphi_{m_2,(k_1,\ldots,k_\ell)}.$$

To test these hypotheses, we can use each of the estimators a)-e), so it is possible to use those not requiring $T \to \infty$. As test procedure we can use the standard Wald-test. For instance, when both N and T are assumed to go to ∞, we can take as test statistic

$$NT \left(\widehat{\varphi}_{m_1,(k_1,\ldots,k_\ell)} - \widehat{\varphi}_{m_2,(k_1,\ldots,k_\ell)}\right)' \left(\widehat{\Sigma}_{\varphi_{m_1,(k_1,\ldots,k_\ell)}} + \widehat{\Sigma}_{\varphi_{m_2,(k_1,\ldots,k_\ell)}}\right)^{-1} \times \left(\widehat{\varphi}_{m_1,(k_1,\ldots,k_\ell)} - \widehat{\varphi}_{m_2,(k_1,\ldots,k_\ell)}\right) \xrightarrow{d} \chi^2_\nu$$

where $\widehat{\varphi}_{m_1,(k_1,\ldots,k_\ell)}$ is any of the estimators c)-e), and $\widehat{\Sigma}_{\varphi_{m_a,(k_1,\ldots,k_\ell)}}$ is a consistent estimator for $\Sigma_{\varphi_{m_a,(k_1,\ldots,k_\ell)}}$, $a \in \{1,2\}$, for instance, the Newey-West (1987) estimator. The degree of freedom ν is typically equal to the number of elements in $\varphi_{m_a,(k_1,\ldots,k_\ell)}$.

Comparing a microscopic simulation model with actual data—The question whether some particular microscopic model $m \in \mathcal{M}$ is able to describe the actual data leads to the following null hypothesis

$$H_0 : \varphi_{m,(k_1,\ldots,k_\ell)} = \varphi_{a,(k_1,\ldots,k_\ell)}$$

versus the alternative hypothesis

$$H_1 : \varphi_{m,(k_1,\ldots,k_\ell)} \neq \varphi_{a,(k_1,\ldots,k_\ell)}.$$

To be able to test the implied hypotheses we shall use as estimators

$$\widehat{\varphi}_{a,(k_1,\ldots,k_\ell),T} = \psi\left(\widehat{d}_{a,(k_1,\ldots,k_\ell),T}\right),$$

for $\varphi_{a,(k_1,\ldots,k_\ell)} = \psi\left(d_{a,(k_1,\ldots,k_\ell)}\right)$ and

$$\overline{\varphi}^N_{m,(k_1,\ldots,k_\ell),T} = \frac{1}{N}\sum_{j=1}^{N}\widehat{\varphi}^j_{m,(k_1,\ldots,k_\ell),T} = \frac{1}{N}\sum_{j=1}^{N}\psi\left(\widehat{d}^j_{m,(k_1,\ldots,k_\ell),T}\right)$$

for $\varphi_{m,(k_1,\ldots,k_\ell)} = \psi\left(d_{(k_1,\ldots,k_\ell)}(m)\right)$. When testing the equality of $\varphi_{m,(k_1,\ldots,k_\ell)}$ and $\varphi_{a,(k_1,\ldots,k_\ell)}$ we exploit the simulation dimension in the asymptotics, assuming $N \to \infty$, and ignore the estimation inaccuracy in the microscopic simulation based estimator. We can therefore apply the standard Wald test statistic

$$W = T\left(\widehat{\varphi}_{a,(k_1,\ldots,k_\ell),T} - \overline{\varphi}^N_{m,(k_1,\ldots,k_\ell),T}\right)' \widehat{\Sigma}^{-1}_{\varphi_{a,(k_1,\ldots,k_\ell)}}\left(\widehat{\varphi}_{a,(k_1,\ldots,k_\ell),T} - \overline{\varphi}^N_{m,(k_1,\ldots,k_\ell),T}\right),$$

with $\widehat{\Sigma}_{\varphi_{a,(k_1,\ldots,k_\ell)}}$ a consistent estimator for $\Sigma_{\varphi_{a,(k_1,\ldots,k_\ell)}}$, like the Newey-West estimator.

Comparing a microscopic simulation model class with actual data—Finally, and perhaps most interestingly, we might be interested in finding out whether a particular microscopic model *class* contains some model m that is able to describe the actual data. This leads to the following null hypothesis

$$H_0 : \varphi_{a,(k_1,\ldots,k_\ell)} \in \left\{\varphi_{m,(k_1,\ldots,k_\ell)} : m \in \mathcal{M}\right\}$$

versus the alternative hypothesis

$$H_1 : \varphi_{a,(k_1,\ldots,k_\ell)} \notin \left\{\varphi_{m,(k_1,\ldots,k_\ell)} : m \in \mathcal{M}\right\}.$$

When testing these implied hypotheses, we have to deal with a problem. Estimation of microscopic simulation models is quite often infeasible. First of all, the class $\mathcal{M}$ might be rather complicated, so that we are unable to determine whether the transformation $d_T : \mathcal{M} \mapsto \mathcal{D}_T$ is injective. If d_T is not injective, then the microscopic model class $\mathcal{M}$ is underidentified, i.e., the same distribution

$$\delta_{(k_1,\ldots,k_\ell)} \in d_{(k_1,\ldots,k_\ell)}(\mathcal{M}) \subset \mathcal{D}_{(k_1,\ldots,k_\ell)}$$

might correspond to different microscopic models in $\mathcal{M}$, making (consistent) estimation impossible. But even if the transformation $d_T : \mathcal{M} \mapsto \mathcal{D}_T$ would turn out to be injective (which we might not be able to find out), then still the distribution characteristic

$$\varphi_{m,(k_1,\ldots,k_\ell)} = \psi\left(d_{(k_1,\ldots,k_\ell)}(m)\right)$$

usually will depend in a heavily nonlinear way on m, so that estimation of m by minimizing some distance between, say, $\varphi_{a,(k_1,\ldots,k_\ell)} = \psi\left(d_{a,(k_1,\ldots,k_\ell)}\right)$ and $\varphi_{m,(k_1,\ldots,k_\ell)}$ over all $m \in \mathcal{M}$ is likely to be too complicated.

Instead, we shall *calibrate* by selecting some microscopic model $m \in \mathcal{M}^f \subset \mathcal{M}$, over some subset $\mathcal{M}^f$ of MS models in $\mathcal{M}$. In this subset, we select the model m that minimizes some distance between the sample and simulation analogues of $\varphi_{a,(k_1,\ldots,k_\ell)}$ and $\varphi_{m,(k_1,\ldots,k_\ell)}$, respectively.[4] The null hypothesis

$$H_0 : \varphi_{a,(k_1,\ldots,k_\ell)} \in \left\{\varphi_{m,(k_1,\ldots,k_\ell)} : m \in \mathcal{M}^f\right\}$$

[4]In the application, we solve

$$\widehat{m} \equiv \widehat{m}_{(k_1,\ldots,k_\ell),N,T} \in \arg\min_{m\in\mathcal{M}^f} \frac{1}{N}\sum_{j=1}^{N} \|\widehat{\varphi}^j_{m,(k_1,\ldots,k_\ell),T} - \widehat{\varphi}_{a,(k_1,\ldots,k_\ell),T}\|^2,$$

for the standard Euclidian norm $\|\cdot\|$, using the generalized simplex algorithm.

versus the alternative hypothesis

$$H_1 : \varphi_{a,(k_1,...,k_\ell)} \notin \left\{ \varphi_{m,(k_1,...,k_\ell)} : m \in \mathcal{M}^f \right\}$$

is then tested by comparing $\widehat{\varphi}_{a,(k_1,...,k_\ell),T}$ and $\overline{\varphi}^N_{m,(k_1,...,k_\ell),T}$, for $m = \widehat{m}$, as in case of comparing a given model m with the real data case, but with using the calibrated model $\widehat{m}$ instead of some given m.

2.3. **Stationarity and long range distributional characteristics.** Next to short range distributional characteristics, we might have to deal with long range distributional characteristics. These are distributional characteristics that require the knowledge of the distribution of $(X_1^a, X_2^a, ...)$, or of $\left(X_1^{m,j}, X_2^{m,j}, ...\right)$. Denote the distribution of $(X_1^a, X_2^a, ...)$ by $d_{a,\infty}$, and the distribution of $\left(X_1^{m,j}, X_2^{m,j}, ...\right)$ by $d_{m,\infty}$. Long range distributional characteristics are of the form $\varphi_a = \varphi_{a,\infty} = \psi(d_{a,\infty})$ and $\varphi_m = \varphi_{m,\infty} = \psi(d_{a,\infty})$, and there is no $(k_1, ..., k_\ell)$, with $\ell < \infty$, such that $\varphi_{a,\infty} = \psi(d_{a,(k_1,...,k_\ell)})$ or $\varphi_{m,\infty} = \psi(d_{m,(k_1,...,k_\ell)})$. In this case we need estimators $\widehat{\varphi}_{a,T} = \psi_T(\widehat{d}_{a,T})$ and $\widehat{\varphi}_{m,T} = \psi_T(\widehat{d}_{m,T})$ that are consistent for $\varphi_{a,\infty}$ and $\varphi_{m,\infty}$ as $T \to \infty$, respectively, and whose estimation/simulation inaccuracy can be quantified. Notice that now, even in the case of comparing two microscopic simulation models, we cannot obtain results without the limit $T \to \infty$. We illustrate this case by two examples we will use in the sequel.

Example 2: ARFIMA

Granger (1980) and Hosking (1981) introduced the ARFIMA(p, d, q) process $X_t \in \mathbb{R}$, $t \in \mathbb{Z}$

$$\Phi(L)(1 - L)^d X_t = \Theta(L)\varepsilon_t,$$

where $d \in (-\frac{1}{2}, \frac{1}{2}]$, where L is the lag operator, with the fractional difference operator $(1-L)^d$ defined as

$$(1-L)^d := \sum_{j=0}^{\infty} \begin{pmatrix} d \\ j \end{pmatrix} (-1)^j L^j,$$

with L^j the composition of j lag operators, and where $\Phi(L)$ and $\Theta(L)$ are lag polynomials of orders p and q respectively:

$$\Phi(L) = 1 + A_1 L + A_2 L^2 + ... + A_p L^p$$

and

$$\Theta(L) = 1 + M_1 L + M_2 L^2 + ... + M_q L^q.$$

Thus, a process X_t is said to be fractionally integrated (FI), if, after applying the operator $(1-L)^d$, it follows an ARMA(p, q) process. Generally, it is assumed that the roots of $\Phi(x)$ are simple, and the roots of $\Phi(x)$ and $\Theta(x)$ are outside the unit circle, and $\varepsilon_t \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma^2)$. It is proved in Granger (1980) and Hosking (1981) that when $d \in (-\frac{1}{2}, \frac{1}{2}]$, X_t is (strictly) stationary and ergodic. For $0 < d < \frac{1}{2}$, the process has *long memory* in the sense that its autocovariances are eventually positive and decay slowly (at a hyperbolic rate). In the frequency domain, for small frequencies, ω, an approximation for the spectral density function is given by ω^{-2d}. For $-\frac{1}{2} < d < 0$, the autocovariances are eventually negative and decline slowly. In the frequency domain, the spectral density declines to zero as frequency approaches zero.

When modeling the process of X_t^a or $X_t^{m,j}$ by means of an ARFIMA(p, d, q) process, the unknown parameters are given by $\varphi_b = (d_b, A_{b1}, ..., A_{bp}, M_{b1}, ..., M_{bq}, \sigma_b^2)'$, satisfying $\varphi_b = \varphi_{b,\infty} = \psi_\infty(d_{b,\infty})$ for $b \in \{a, m\}$.

Consistent estimation with quantification of estimation/simulation inaccuracy is possible by means of Maximum Likelihood, cf. Sowell (1992). This yields estimators $\widehat{\varphi}_{a,T}$

for the actual data and $\widehat{\varphi}^j_{m,T}$ for the simulated data of simulation j that both converge at rate $\sqrt{T}$, assuming the number of time periods per simulation equals that of the actual data. For the simulation based estimates, we can construct averages of the form $\overline{\varphi}_{m,T} = \frac{1}{N}\sum_{j=1}^{N}\widehat{\varphi}^j_{m,T}$, which will converge at rate $\sqrt{NT}$, in case we also exploit the limit $N \to \infty$, sequentially after $T \to \infty$. This allows us to construct Wald tests, in which we can ignore the simulation inaccuracy in the simulation based estimator $\overline{\varphi}_{m,T}$, as in the case of short range dependence.

Example 3: FIGARCH

The family of ARCH (Autoregressive Conditionally Heteroskedastic) models was introduced by Engle (1982) and its generalization, the GARCH model, was introduced by Bollerslev (1986). Following their specification, a GARCH(p,q) model is defined by:

$$\begin{cases} X_t = a + bX_{t-1} + \varepsilon_t, & \varepsilon_t = \sigma_t z_t, \\ \sigma_t^2 = \alpha_0 + \alpha(L)\varepsilon_t^2 + \beta(L)\sigma_t^2, & z_t \sim \mathcal{N}(0,1), \end{cases} \tag{1}$$

where L is the lag operator, $\alpha(L) = \sum_{i=1}^{q}\alpha_i L^i$ and $\beta(L) = \sum_{j=1}^{p}\beta_i L^j$. Defining $v_t = \varepsilon_t^2 - \sigma_t^2$, the process can be rewritten as an ARMA(m,p) process

$$[1 - \alpha(L) - \beta(L)]\varepsilon_t^2 = \alpha_0 + [1 - \beta(L)]v_t \tag{2}$$

with $m = \max\{p,q\}$.

Baillie *et al.* (1996) consider the Fractionally Integrated GARCH (FIGARCH) process, where a shock to the conditional variance dies out at a slow hyperbolic rate of decay. Chung (1999) suggested a slightly different parameterization of the model:

$$\phi(L)(1-L)^d(\varepsilon_t^2 - \sigma^2) = \alpha_0 + [1 - \beta(L)]v_t, \tag{3}$$

where $\phi(L) = 1 - \sum_{i=1}^{q} \phi_i L^i$, $\alpha_0 = \phi(L)(1-L)^d \sigma^2$, and σ^2 is the unconditional variance of the corresponding GARCH model.

When modeling the process of X_t^a or $X_t^{m,j}$ by means of an FIGARCH process, the unknown parameters are given by

$$\varphi_b = \varphi_{b,\infty} = (a_b, b_b, \phi_{b0}, ...\phi_{bq}, d_b, \beta_{b1}, ..., \beta_{bp}, \sigma_b^2)' = \psi_\infty(d_{b,\infty})$$

for $b \in \{a, m\}$. Consistent estimation with quantification of estimation/simulation inaccuracy is possible by means of Maximum Likelihood, allowing us to construct as estimators $\widehat{\varphi}_{a,T}$ for the actual data, and $\overline{\varphi}_{m,T} = \frac{1}{N} \sum_{j=1}^{N} \widehat{\varphi}_{m,T}^j$ for the simulated data, so that we can proceed as in the ARFIMA-case.

3. An application to the LLS model

In this section, we illustrate how the proposed econometric tools can be used to analyze the model by Levy *et al.* (2000) (LLS model from now on). First, we briefly describe the LLS model class $\mathcal{M}_{LLS}$, and the LLS-model, m_{LLS}, obtained by choosing the parameter settings and initial conditions according to Levy *et al.* (2000). Then we investigate the sensitivity of m_{LLS} to some initial conditions and parameter values, and we compare it with an extended model, allowing for extra investors, and finally, we confront this model m_{LLS} with real life data.

3.1. **The Levy-Levy-Solomon Model Class.**

In the model class by Levy *et al.* (2000), LLS economy from now on, there are two assets: a stock and a bond. The bond is assumed to be a risk free asset, while the stock is a risky asset. The bond is exogenous with infinite supply, so the investors can buy from it as much as they wish at a given rate of return, r. The stock is in bounded supply. There are N outstanding shares of the

stock.[5] The return on the stock is composed of capital gains and dividend payments. The dividend per share at time t, D_t, is a random variable that follows a multiplicative random walk: $D_t = D_{t-1}(1+z)$, where z is distributed uniformly in the range $[z_1, z_2]$. The overall rate of gross return on the stock in period t, R_t, is then given by

$$R_t = \frac{P_t + D_t}{P_{t-1}} \tag{4}$$

where P_t is the stock price at time t.

The investors are expected utility maximizers, characterized by the utility index $U(W) = W^{1-\alpha}/(1-\alpha)$, which reflects their personal preference. The investors are divided into two groups, the first group will be referred to as the rational informed investors (RII), and the second group will be referred to as the efficient market believers (EMB).

The RII investors—At time t the RII investors believe that the convergence of the price to the fundamental value will occur in period $t+1$. Furthermore, RII investors estimate the next period fundamental value of stock price P^f_{t+1} by

$$P^f_{t+1} = \frac{E_{t+1}[D_{t+2}]}{k-g}, \tag{5}$$

according to Gordon's dividend stream model. Here k is the discount factor, and g is the expected growth rate of the dividend, i.e., $g = E(z)$, which is known to the investors. Using $E_{t+1}[D_{t+2}] = D_{t+1}(1+g)$ and $D_{t+1} = D_t(1+z)$, RII investors thus believe that $P_{t+1} = P^f_{t+1}$ is given by

$$P^f_{t+1} = \frac{D_t(1+z)(1+g)}{k-g}. \tag{6}$$

[5]To keep close to the original notation by LLS, we might have to give the same symbol, like N, different meanings, but we expect that the context makes the correct interpretation straightforward.

Investing a proportion x of wealth in the stock at time t, the expected utility becomes

$$E\left\{U(W_{t+1})\right\} = E\left\{U\left(W_t[(1-x)r + xR_{t+1}]\right)\right\},$$

with W_t the wealth at time t, and where the expectation is over the rate of return $R_{t+1} = (P_{t+1} + D_{t+1})/P_t$, with $P_{t+1} = P^f_{t+1}$. A solution for this optimization problem can be found by solving the first order conditions.

The EMB investor—EMB investor i has only a limited memory, and uses the most recent m^i returns on the stock to estimate the ex ante distribution. At time t, each of these past returns on the stock $R_{j,}\ j = t, t-1, ..., t-m^i+1$ is given an equal probability $1/m^i$ to reoccur in the next period ($t+1$). Therefore, the expected utility of EMB investor i is given by

$$E\left\{U(W_{t+1})\right\} = \frac{1}{m^i}\sum_{j=1}^{m^i}\frac{1}{1-\alpha}[W_t[(1-x)r + xR_{t-j}]]^{1-\alpha}. \tag{7}$$

Maximization of this expected utility yields the optimal proportion of wealth, x^{*i}, that will be invested in the stock by EMB-investor i. To allow for noise around the optimal portfolio choice, LLS assume

$$x^i = x^{*i} + \varepsilon^i$$

where ε^i is a random variable drawn from a normal distribution with mean zero and standard deviation σ. For simplicity, noise is only added to the portfolio share of stocks for the EMB investors.

Given the stock demand of the RII- and EMB-investors, together with the total supply of shares N fixed, the (temporary) Walrasian equilibrium stock price at time t, P_t, can be determined. This price leads to updated expectations and a new equilibrium arises in the next period, and so on.

The parameter values and initial conditions chosen by Levy *et al.* (2000) are as follows.

- Time periods: quarters of a year.
- Number of investors $= 1000,$ with 96% RII investors and 4% EMB investors. There are two types of EMB investors, with memory span 5 and $15,$ respectively. Both groups are equally large.
- At time $t = 1$ each investor is endowed with a total wealth of $\$1000$, which is composed of 10 shares worth an initial price of $\$20.94$ per share, and the remainder in cash.
- The initial dividend is set at $\$0.5$.
- $N = 1000$, $r = 1\%$, $k = 4\%$, $z_1 = -7\%$, $z_2 = 10\%$, $\sigma = 0.2$, $\alpha = 1.5$.

The model with these initial conditions and parameter settings will be referred to as m_{LLS}.

3.2. **Sensitivity analysis.** In this subsection we investigate the sensitivity of the benchmark model m_{LLS} to changes in initial conditions and parameter values. We illustrate this by estimating ARFIMA(p, d, q)-processes, as discussed in *Example 2*, restricting attention to the ARFIMA($0, 1, 0$)- and ARFIMA($1, d, 1$)-cases.

Table 1 summarizes the results of the benchmark model m_{LLS}, by presenting the average results over the simulations, as well as the numbers of significant parameter estimations. We run 5,000 independent simulations over 1,000 time periods, and for each run we use the last 152 observations to estimate the two ARFIMA-models by Maximum Likelihood. The reason that we use the final 152 observations is, first, to wash away the initial noise effects, and, secondly, to match the sample size of the actual data (the S&P 500) that we use later on.

In case of ARFIMA$(0, d, 0)$, we find on average a negative value of d, but which is in only 7.6% out of the 5,000 simulations significant. So, based on a single simulation, we would accept most of the times the hypothesis $d = 0$. However, combining the 5,000 simulations, we find as t-value $t = -23.80$,[6] so that we clearly have to reject the hypothesis $d = 0$!

In case of ARFIMA$(1, d, 1)$ we find in most cases a significantly negative value of d. Moreover, in most cases also the AR-coefficient is significant, while the MA-coefficient in most cases turns out to be insignificant. However, combining again the 5,000 simulations, we find as t-value for the MA-coefficient, $t = 9.66$,[7] so that the MA-coefficient is clearly significantly different from zero.

TABLE 1. Maximum likelihood estimation of ARFIMA(p, d, q) model for the LLS model

	Coefficient	Std.	p-value	95% CI	Sig%
$(0, d, 0)$	-0.0272	0.0808	0.5210	[-0.0294, -0.0250]	7.6
	-0.7444	0.2155	0.0339	[-0.7506, -0.7382]	88.2
$(1, d, 1)$	0.6983	0.1356	0.0151	[0.6911, 0.7055]	93.4
	0.0208	0.1521	0.5342	[0.0147, 0.0269]	5

Note: The reported numbers under Coefficients, Std., and p-value are averages over 5,000 simulations. The 95% confidence interval presents the simulation accuracy. 'Sig%' reports the percentage of the estimates that are significant at 5% level among 5,000 independent simulations. The estimated coefficients of ARFIMA$(1, d, 1)$ model are listed in the order: d, AR coefficient, MA coefficient. This is also true for other tables in this section.

To investigate the sensitivity, we next run the LLS model with different input values, including different initial price ($P_0 \in \{16, 26\}$), initial dividend ($D_0 \in \{0.4, 0.6\}$), different risk aversion parameter ($\alpha \in \{1.45, 1.55\}$), different maximum single-period dividend decrease ($z_1 \in \{-0.08, -0.06\}$), and different initial wealth (50% $W_0 = 500$, 50% $W_0 = 1500$, and W_0 drawn from uniform distribution over $[500, 1500]$) We report

[6] $t = -0.0272/(0.0808/\sqrt{5000}) = -23.80$.
[7] $t = 0.0208/(0.1521/\sqrt{5000}) = 9.66$.

the estimation results of the ARFIMA $(0, d, 0)$ model in Table 2 and in Table 3 we also report the t-test for the difference of estimated d between the benchmark model and the models with different initial parameters.

TABLE 2. Maximum likelihood estimation of ARFIMA$(0, d, 0)$ model for the LLS models

		d	Std.	p-value	95% CI	Sig%
P_0	16	-0.0286	0.0807	0.5201	[-0.0308, -0.0264]	7.7
	26	-0.0282	0.0807	0.5247	[-0.0304, -0.0260]	7.6
D_0	0.4	-0.0272	0.0808	0.5228	[-0.0294, -0.0250]	7.4
	0.6	-0.0247	0.0807	0.5193	[-0.0269, -0.0225]	6.7
α	1.45	-0.0281	0.0808	0.5172	[-0.0303, -0.0259]	7.9
	1.55	-0.0273	0.0808	0.5233	[-0.0295, -0.0251]	7.4
z_1	-0.08	0.0077	0.0837	0.6621	[0.0063, 0.0091]	0.7
	-0.06	-0.1806	0.0707	0.0848	[-0.1827, -0.1785]	70
W_0	unif.	-0.0239	0.0809	0.5241	[-0.0261, -0.0217]	6.9
	50%	-0.0279	0.0808	0.5230	[-0.0301, -0.0257]	7.7

TABLE 3. The t-test for the sensitivity analysis of the LLS models in terms of ARFIMA$(0, d, 0)$ estimates

	P_0		D_0		α		z_1		W_0	
	16	26	0.4	0.6	1.45	1.55	-0.08	-0.06	unif.	50%
t	1.167	0.840	0.047	1.935	0.760	0.125	26.696	105.53	2.546	0.554

We see from Table 3 that the LLS model is rather insensitive with respect to the initial prices, initial dividend, risk aversion parameter in terms of d. However, the changes of maximal one-period dividend decrease z_1 has a big impact. The reason seems to be that the dividend process is the driving force in the LLS-model, and a change in z_1 changes the whole distribution of the dividend process.

In the benchmark model each investor is endowed equally with a total wealth of $\$1000$. If half of the investors is endowed $\$500$ and the other half endowed with $\$1500$, then, compared to the benchmark model, the difference in d is not significant.

However, the difference in d is significant when all of the investors initial wealth is drawn from a uniform distribution on $[500, 1500]$.

We also calculate the estimation results of the ARFIMA $(1, d, 1)$ model, and the Wald test for the difference in the estimated parameters between the benchmark model and the models with different initial parameters (not reported). The results are more or less in line with the ARFIMA$(0, d, 0)$-case, although the insensitivity of the LLS model with respect to the initial conditions and parameters becomes somewhat more ambiguous, due to the fact that now three parameters are being estimated.

A sensitivity analysis, like the one above, may be of help in determining which parameters or initial conditions, in particular, to use when one would like to calibrate (or even estimate) a MS model using actual data. In case of the LLS model, the dividend process seems to be an appropriate choice in a calibration exercise. By modeling it flexibly, one might become able to describe a wide range of potential distribution characteristics, so that a calibration exercise might become successful.

3.3. **Comparing two LLS models.** In the benchmark model m_{LLS}, there are only two types of investors, the RII and the EMB investors. It might be interesting to investigate what will happen when we introduce a new type of investors. Similar to Zschischang and Lux (2001), we consider as deviation from the benchmark model an economy with an extra type of investors, namely the constant portfolio investors, who always invest a constant proportion of their wealth in the stocks. Zschischang and Lux (2001) investigate the LLS model where initially all the investors are EMB investors (consisting of three or more subgroups). The authors found, when the market is invaded by only a small amount of constant portfolio investors (1%), that, even when these new investors are endowed with a small initial wealth and hold 1.5% of their portfolio in the stock, they eventually achieve dominance and asymptotically gain 100% of the available wealth. As an alternative economy, we consider an economy where 0.5% of

the investors are constant portfolio investors instead of RII-investors (having the same initial wealth as the other investors). These constant portfolio investors invest 1.5% of their wealth in the stock. We keep the other characteristics of the economy the same as the benchmark model.

We performed a Wald test to investigate whether the introduction of the constant portfolio investors has a significant impact on the economy. The comparison with the benchmark model are made in terms of log return, log price, and proportion of total wealth held by the two groups EMB investors with different memory span. We considered two cases to see the variations of these quantities over time and their long run behavior. In $CaseI$, the periods under consideration are the last 100 periods, $t = 901, 902, ..., 1000$; in $CaseII$, only six time points are considered, $t = 500, 600, ..., 900, 1000$. The resulting test statistics are summarized in Table 4. Within the column named 'Log Return', the first subrow reports the Wald statistics of the benchmark economy, for instance, 101.88 is the Wald statistic corresponding to the null hypothesis of equality of the average log return in periods $t = 901, 902, ..., 1000$ (with degrees of freedom between brackets), and so on, the second subrow reports the Wald statistics of the new economy, and the third subrow reports the results of comparing the new economy with the benchmark model, and ms stands for memory span. It is clear that none of the comparison statistics is significant, thus the constant portfolio investors do not cause a significant impact on the economy.

Figure 1 explains why. It presents the average proportion of total wealth of the extra constant portfolio investors across 5,000 simulations. As the figure shows, the wealth of the constant portfolio investors decreases gradually. In the Zschischang and Lux-analysis the constant portfolio investors are the only investors who are at the opposite side of the market in case of the cycles, so that eventually they are able to gain all wealth. But in the economy considered here, the RII investors for a large part take

TABLE 4. The comparison results with the benchmark model in terms of the mean

	Log Return	Log Price	Wealth $(ms = 5)$	Wealth $(ms = 15)$
	$101.88(99)$	$42085.5(99)$	$122.20(99)$	$125.10(99)$
$CaseI$	$101.68(99)$	$43413.2(99)$	$124.84(99)$	$113.73(99)$
	$77.95(100)$	$76.95(100)$	$97.29(100)$	$79.13(100)$
	$9.53(5)$	$192436.7(5)$	$383.93(5)$	$772.40(5)$
$CaseII$	$2.65(5)$	$194884.1(5)$	$388.87(5)$	$757.85(5)$
	$8.33(6)$	$7.44(6)$	$3.37(6)$	$2.45(6)$

over this role by buying or selling, depending on the price being lower or higher than its fundamental value, resulting in a gradually decreasing wealth held by the constant portfolio investors.

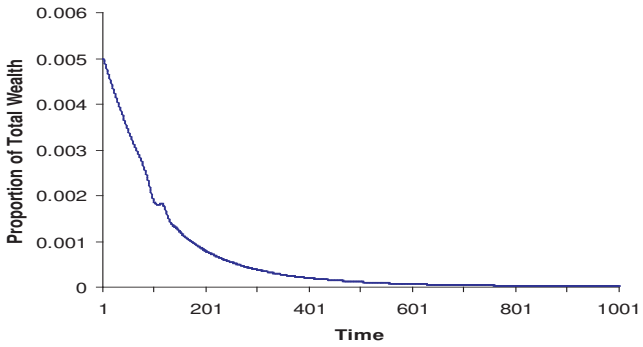

FIGURE 1. Proportion of total wealth held by constant portfolio investors, averaged over 5,000 simulations.

3.4. **Comparing the LLS model with real life data.** We use quarterly data of the S&P 500 from Datastream as representation of the real life situation, which runs from the first quarter of 1965 to the first quarter of 2003. See Table 5 for some descriptive statistics.

TABLE 5. Sample statistics of returns of the S&P 500

Mean	Median	Std.Dev.	Max	Min	Skew.	Kurt.
0.0194	0.0162	0.0848	0.2923	-0.2548	-0.0575	3.800

We consider a comparison in terms of AR coefficients and the coefficients of the ARFIMA(p, d, q) process. First, for the actual data, we calculate the autocorrelations and construct a confidence interval for each autocorrelation, using the Newey-West corrected standard errors. Then we estimate the averaged autocorrelations that comes from LLS model and verify whether the average autocorrelations lie in the confidence intervals for the actual data. As one cannot compute all possible autocorrelations, we focus on the autocorrelation for lags 1 to 60. The results are summarized in Figure 2. The average autocorrelations of the LLS economy lie entirely in the 95% confidence intervals around the real-life data autocorrelations. Hence, we conclude that the LLS model fits the real world very well in terms of the first sixty AR coefficients.

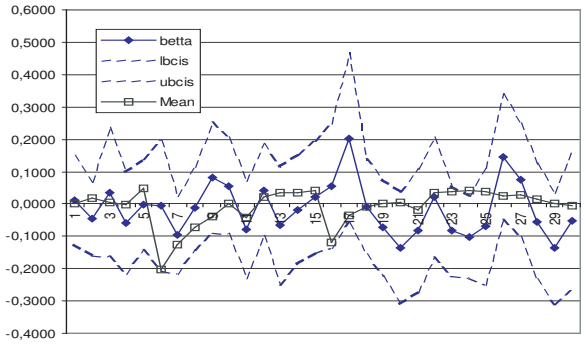

FIGURE 2. The confidence intervals for the autocorrelations of the S&P 500 and the averaged autocorrelation of the LLS model. Note: In this Figure, "betta" means the estimated autocorrelation of the S&P 500, "lbcis" and "ubcis" indicate the lower and upper bounds of the confidence intervals, and "Mean" is the averaged autocorrelation of the LLS model.

Next, we illustrate our comparison method in terms of the coefficients of the ARFIMA(p, d, q) process. We estimate the ARFIMA $(0, d, 0)$ model and the ARFIMA $(1, d, 1)$ model for stock returns and summarize the results for the S&P 500 in Table 6.

We see from the table that in both of the ARFIMA $(0, d, 0)$ and ARFIMA $(1, d, 1)$ model the parameter d is not significant; there is no evidence of long memory in the quarterly stock return process. For the ARFIMA $(0, d, 0)$ model, the estimated d from

TABLE 6. Maximum likelihood estimation of ARFIMA(p, d, q) model for the S&P 500

	Coefficient	Std. Error	t-value	p-value
$(0, d, 0)$	-0.0183	0.0675	-0.272	0.786
	-0.0527	0.0813	-0.648	0.518
$(1, d, 1)$	-0.5668	0.4687	-1.21	0.229
	0.6365	0.4240	1.5	0.135

the LLS model (the average reported in Table 1) lies within the 95% confidence interval of estimates of d from actual data, which is $(-0.1506, 0.114)$. However, for the ARFIMA $(1, d, 1)$ model, there is a significant difference between the actual data and the data of LLS model: The average value of d according to the LLS-model does not lie in the 95% confidence interval around the estimate of d according to the actual data.

So, the 'standard' version of the LLS-model, m_{LLS}, considered here, seems to be able to give, at least, a description of some aspects of the actual data, but not all aspects. However, the sensitivity analysis that we performed, suggests that by choosing a model in the LLS model class $\mathcal{M}_{LLS}$ by calibration (or estimation), particularly using a flexible description of the dividend process, might result in a LLS-model that is able to describe many aspects of the actual data quite well.

4. AN APPLICATION TO THE MARKET FRACTION MODEL CLASS

4.1. **The Market Fraction Model Class.** The market fraction (MF) model is a standard discounted value asset pricing model with heterogeneous agents. It is closely related to the framework of Chiarella and He (2003). We outline the model and refer the readers to He and Li (2005a) for details.

There is one risky and one risk free asset. The risk free asset is perfectly elastically supplied at gross return $R = 1 + r/K$, where r stands for a constant risk-free rate per annum and $K = 250$ stands for the frequency of trading periods (days) per year.

Let P_t be the price (ex dividend) per share of the risky asset at time t and $\{D_t\}$ be the stochastic dividend process of the risky asset. Then the wealth of a typical investor-h at $t+1$ is given by

$$W_{h,t+1} = RW_{h,t} + [P_{t+1} + D_{t+1} - RP_t]z_{h,t}, \tag{8}$$

where $W_{h,t}$ and $z_{h,t}$ are the wealth and the number of shares of the risky asset purchased of investor-h at t, respectively. Let $E_{h,t}$ and $V_{h,t}$ be the "beliefs" of type h traders about the conditional expectation and variance of quantities at $t+1$ based on their information set. Denote by R_{t+1} the excess capital gain on the risky asset at $t+1$, that is

$$R_{t+1} = P_{t+1} + D_{t+1} - R\,P_t. \tag{9}$$

Traders have a constant absolute risk aversion (CARA) utility function with the risk aversion coefficient a_h for type h traders (that is $U_h(W) = -\exp(-a_h W)$) and their optimal demand on the risky asset $z_{h,t}$ are determined by maximizing their expected utility of the wealth, resulting in

$$z_{h,t} = \frac{E_{h,t}(R_{t+1})}{a_h V_{h,t}(R_{t+1})}. \tag{10}$$

Given the heterogeneity and the nature of asymmetric information among traders, we consider two popular trading strategies corresponding to two types of boundedly rational traders—fundamentalists and trend followers. The market fraction of the fundamentalists and trend followers is n_1 and n_2, respectively. Let $m = n_1 - n_2 \in [-1, 1]$. Then, using (10), the aggregate excess demand per investor $z_{e,t}$ is given by

$$z_{e,t} \equiv n_1 z_{1,t} + n_2 z_{2,t} = \frac{1+m}{2}\frac{E_{1,t}[R_{t+1}]}{a_1 V_{1,t}[R_{t+1}]} + \frac{1-m}{2}\frac{E_{2,t}[R_{t+1}]}{a_2 V_{2,t}[R_{t+1}]}. \tag{11}$$

The market is cleared by a market maker, who takes a long (when $z_{e,t} < 0$) or short (when $z_{e,t} > 0$) position so as to clear the market. Let μ be the speed of price adjustment of the market maker. To capture unexpected market news or speculators' excess demand, there is a noisy demand term δ_t which is an i.i.d. normally distributed random variable with mean 0 and variance σ_δ^2. Based on these assumptions and (11), the market price is determined by

$$P_{t+1} = P_t + \frac{\mu}{2}\left[(1+m)\frac{E_{1,t}[R_{t+1}]}{a_1 V_{1,t}[R_{t+1}]} + (1-m)\frac{E_{2,t}[R_{t+1}]}{a_1 V_{2,t}[R_{t+1}]}\right] + \delta_t. \tag{12}$$

Now we turn to discuss the beliefs of fundamentalists and trend followers.

Fundamentalists—Apart from the common information on P_t, $P_{t-1}, \cdots$, D_t , $D_{t-1}, \cdots$, the fundamentalists have *superior* information on the fundamental value. The relative return ($P^*_{t+1}/P^*_t - 1$) of the fundamental value follows a Wiener process,

$$P^*_{t+1} = P^*_t[1 + \sigma_\epsilon \epsilon_t], \qquad \epsilon_t \sim \mathcal{N}(0,1), \qquad \sigma_\epsilon \geq 0, \qquad P^*_0 = \bar{P} > 0, \tag{13}$$

where ϵ_t is independent of the noisy demand process δ_t. The conditional mean and variance of the fundamental traders are assumed to follow

$$E_{1,t}(P_{t+1}) = P_t + \alpha(P^*_{t+1} - P_t), \qquad V_{1,t}(P_{t+1}) = \sigma_1^2, \tag{14}$$

where σ_1^2 stands for a constant variance on the price, and with $\alpha \in [0,1]$ the speed of price adjustment of the fundamentalist towards the fundamental value.

Trend followers—The trend followers extrapolate the latest observed price change over a long-run sample mean price and adjust their variance estimate accordingly. More precisely, their conditional mean and variance are assumed to follow

$$E_{2,t}(P_{t+1}) = P_t + \gamma(P_t - u_t), \qquad V_{2,t}(P_{t+1}) = \sigma_1^2 + b_2 v_t, \tag{15}$$

where $\gamma, b_2 \geq 0$ are constants, and u_t and v_t are sample mean and variance, respectively, which follow the learning processes

$$u_t = \delta u_{t-1} + (1-\delta)P_t, \tag{16}$$

$$v_t = \delta v_{t-1} + \delta(1-\delta)(P_t - u_{t-1})^2, \tag{17}$$

where $\delta \in [0,1]$ is a memory parameter.

The dividend process D_t follows $D_t \sim \mathcal{N}(\bar{D}, \sigma_D^2)$, the expected long-run fundamental value $\bar{P} = \bar{D}/(R-1)$, and the unconditional variances of price and dividend over the trading period are related by $\sigma_D^2 = q\sigma_1^2$, with $\sigma_1^2 = \sigma_{\bar{P}}^2/K$ and $q = r^2$.

The parameters in the model used by He and Li (2005a, b), to be denoted by m_{MF}, are $\overline{P} = 100, q = r^2 = 0.05^2, \sigma = 0.20, \alpha = 0.1$, $\gamma = 0.3$, $a_1 = 0.8$, $a_2 = 0.8$, $\mu = 2$, $m = 0$, $\delta = 0.85$, $b = 1$, $\sigma_\varepsilon = 0.013$, and $\sigma_\delta = 1$.

4.2. **Comparing the MF model class with real life data.** Instead of comparing the original MF-model m_{MF} with real data, we consider a subclass of MF-models, $\mathcal{M}_{MF}^f \subset \mathcal{M}_{MF}$, with the aim to test whether some microscopic simulation model in this subclass is able to give an accurate description of real life data.

The real life data—As real life data we use the S&P 500 price index[8]. There are altogether 5,306 daily observations from Oct 20, 1982 to Oct 27, 2003. Table 7 gives summary statistics for the log returns. We can see from Table 7 that the kurtosis (44.76) is much higher than that of a normal distribution. The kurtosis and studentized range statistics (which is the range divided by the standard deviation) show the characteristic fat-tailed behavior compared with a normal distribution. The Jarque-Bera normality test statistic is far beyond the critical value which suggests that the log return-distribution is far from a normal distribution.

[8]Data are obtained from http://finance.yahoo.com.

TABLE 7. Summary statistics of r_t.

data	sample size	mean	std	skewness	kurtosis	min	max	studentized range	Jarque-Bera
r_t	5305	0.00037	0.0108	-1.933	44.76	-0.229	0.087	29.16	388510

A well known stylized fact of high frequency stock returns is that the log returns themselves contain little serial correlation, but the squared returns r_t^2 and absolute return $|r_t|$ do have positive serial correlation over long lags. For example, Ding *et al.* (1993) investigate autocorrelations of returns (and their transformations) of the daily S&P 500 index over the period 1928 to 1991 and find that the absolute returns and squared returns tend to have very slow decaying autocorrelations, and, further, the sample autocorrelations for the absolute returns are greater than the sample autocorrelations for squared returns at every lag up to at least 100 lags. Figure 3 shows the autocorrelation coefficients for the returns, squared returns, and absolute returns, where the lines from the bottom to the top are the autocorrelation coefficients for the returns, squared returns, and absolute returns respectively. These results coincide with the findings in Ding *et al.* (1993).

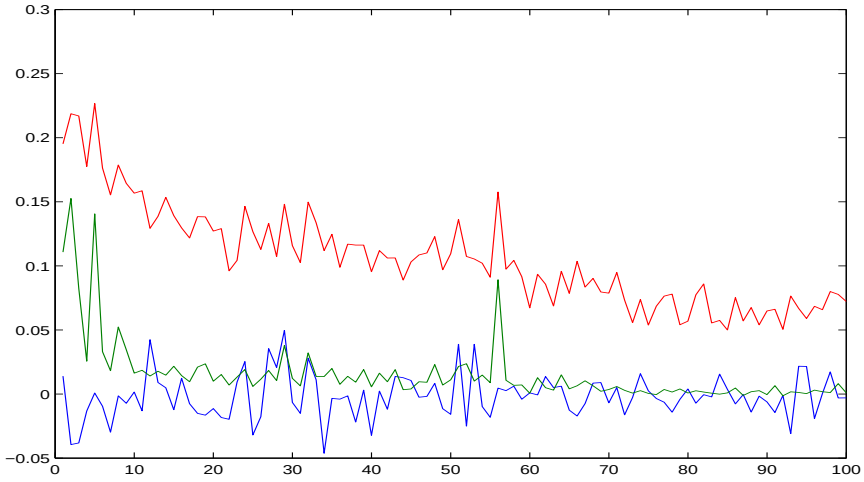

FIGURE 3. Autocorrelations of r_t, r_t^2 and $|r_t|$ for the S&P 500.

Calibration of the MF model—We calibrate the MF-model by minimizing the average distance between the actual ($a = S\&P500$) autocorrelations of the log returns (the squared log returns, the absolute log returns), and the corresponding autocorrelations according to the MF-models in $\mathcal{M}_{MF}^{f}$,[9] based on 1000 independent simulations.[10] Again, it is not possible to use autocorrelations at all lags, so we focus on a limited set of autocorrelations. In particular, we focus on lag lengths of 1 to 10, 20, 30, 40, 50, and 60 periods. To see how well the model class $\mathcal{M}_{MF}^{f}$ is able to match the autocorrelations r_t, r_t^2 and $|r_t|$ separately, we calibrate the model to the autocorrelations for each quantity separately, resulting in $\widehat{m}_{MF,r_t}$, $\widehat{m}_{MF,r_t^2}$, and $\widehat{m}_{MF,|r_t|}$, respectively. In addition, we try to find the best model in $\mathcal{M}_{MF}^{f}$ for the autocorrelations of all three quantities by calibrating the model to the total set of autocorrelations simultaneously. The resulting MF-model is denoted by $\widehat{m}_{MF}$.

TABLE 8. Original and estimated MF models

	α	γ	a_1	a_2	μ	m	δ	b	σ_ε	σ_δ
m_{MF}	0.1	0.3	0.8	0.8	2	0	0.85	1	0.013	1
$\widehat{m}_{MF,r_t}$	1.000	0.754	0.400	0.713	2.829	-0.249	0.797	1.188	0.021	0.750
$\widehat{m}_{MF,r_t^2}$	0.998	3.185	0.382	0.578	4.995	-0.063	0.282	0.155	0.020	0.096
$\widehat{m}_{MF,\|r_t\|}$	0.046	1.429	0.572	0.198	1.671	0.496	0.977	0.766	0.012	0.531
$\widehat{m}_{MF}$	0.999	0.754	0.400	0.650	2.829	-0.250	0.797	1.438	0.021	0.750

Table 8 contains the parameter values of the original model m_{MF}, as used by He and Li (2005a, b) and the calibrated models $\widehat{m}_{MF,r_t}$, $\widehat{m}_{MF,r_t^2}$, $\widehat{m}_{MF,|r_t|}$, and $\widehat{m}_{MF}$. For the calibrated models, apart from $\hat{m}_{MF,|r_t|}$, the fundamentalists are more confident about their estimate of the fundamental price (since α is close to 1) and less risk averse

[9]The set $\mathcal{M}_{MF}$ is chosen by making a few steps in a generalized simplex algorithm, where the parameters are chosen to lie in the following ranges: $\alpha \in [0,1]$, $\gamma \in [0.05, 5.5]$, $a_1, a_2 \in [0.05, 1.35]$, $\mu \in [0.1, 5]$, $m \in [-1, 1]$, $\delta \in [0,1]$, $b \in [0.05, 1.5]$, $\sigma_\varepsilon \in [0.1, 1.5]$, and $\sigma_\delta \in [0.05, 1.5]$. $\overline{P} = 100$, and $q = r^2 = 0.05^2$ are kept fixed.

[10]Each simulation run consisted of 6,306 time periods. The first 1,000 periods were discarded to wash out potential initial condition effects.

compared to the trend followers. In addition, as revealed by the values of m, there are more trend followers than fundamentalists in the market. In case of $\hat{m}_{MF,|r_t|}$, there are about 75% ($m = 0.496$) fundamentalists who have almost no confidence (α is close to 0) in their fundamental price and are more risk averse. The memory decay rate δ is close to 1, indicating that the trend followers put more weight on the historical trend rather than the current price. Based on the stability and bifurcation analysis of the underlying deterministic system in He and Li (2005a), among the five cases, the case m_{MF} is the only one where the steady state of the underlying deterministic system is locally asymptotically stable. Among the four unstable cases, the case $\hat{m}_{MF,|r_t|}$ is the only one where $\mu = 1.671$ exceeds the boundary of the flip bifurcation (which is 0.355), and for the other three cases, the Hopf bifurcations occur (where the boundaries of μ for $\widehat{m}_{MF,r_t}$, $\widehat{m}_{MF,r_t^2}$, and $\widehat{m}_{MF}$ are 0.610, 3.488 and 1.517, respectively). When the steady state is unstable, the prices will eventually flip around the steady state (in case of flip bifurcation) or fluctuate (quasi) periodically around the steady state (in case of Hopf bifurcation).

The implication of the stable (unstable) steady state for the statistical stationarity of the returns is an important question. At present, the mathematical theory does not yet seem to be able to give a clear answer to it, see, for instance, the discussions in Arnold (1998). To investigate whether the assumption of stationarity and 'asymptotic independence' makes sense, we ran 1000 independent simulations for each of the MF model over 100,000 periods. Table 9 reports the averaged means and variances (where we divided the whole sample equally into 10 sub-samples) over 1000 independent simulations. Although the averaged means and variances seem to fluctuate over time, there does not seem to be a systematic trend, for instance, going up or going down. We take this as indication that the assumption of stationarity is not too bad an approximation. We also estimate the long distance covariances, i.e., the covariance between

the returns in the first and final 10,000 observations. According to these estimations the long distance correlation seems to be zero in most cases, indicating that also the assumption of 'asymptotic independence' might not be too bad an approximation.

TABLE 9. Means and variances ($\times 10^3$) of the MF models

	Period-1	Period-2	Period-3	Period-4	Period-5	Period-6	Period-7	Period-8	Period-9	Period-10
m_{MF}	0.016	0.002	-0.002	-0.001	0.005	-0.003	0.001	0.002	-0.005	-0.012
	1.907	1.839	1.823	1.802	1.808	1.794	1.836	1.806	1.805	1.917
$\widehat{m}_{MF,r_t}$	0.072	0.021	-0.006	0.005	0.003	0.001	-0.007	0.008	-0.007	-0.029
	0.770	0.695	0.688	0.693	0.673	0.677	0.678	0.689	0.683	0.697
$\widehat{m}_{MF,r_t^2}$	0.212	0.086	0.039	0.028	0.015	-0.017	-0.014	-0.020	-0.064	-0.108
	0.537	0.468	0.459	0.452	0.450	0.449	0.453	0.460	0.463	0.488
$\widehat{m}_{MF,\lvert r_t\rvert}$	0.035	0.047	0.026	0.034	0.017	-0.009	0.005	-0.010	-0.030	-0.057
	0.236	0.274	0.207	0.165	0.142	0.140	0.141	0.154	0.172	0.285
$\widehat{m}_{MF}$	0.081	0.031	-0.007	0.002	0.019	-0.008	-0.012	0.002	-0.006	-0.036
	0.708	0.668	0.664	0.671	0.650	0.645	0.661	0.659	0.663	0.679

Note: In each row of the table, the first sub-row reports the mean and the second sub-row reports the variance. For instance, 0.016 is the averaged mean of the first 10,000 periods, and 1.907 is the averaged variance of the first 10,000 periods over 1000 independent simulations of the m_{MF} model; 0.002 and 1.839 are those for the second 10,000 periods; so on and so forth.

Comparison with actual data via autocorrelations—For the purpose of comparison with real data, we combine in Figure 4 the data based autocorrelations with their confidence intervals and the autocorrelations resulting from the first three calibrated MF-models. In general, the calibrated autocorrelations fit their actual counterpart well. For example, the calibrated autocorrelations of the squared returns fit those of the data relatively well in Figure 4(b), but there are large differences for the autocorrelations of the returns and the absolute returns. Hence, the $\mathcal{M}_{MF}^{f}$ class of models is able to fit the autocorrelations of r_t, r_t^2 and $|r_t|$ individually rather well.

For the overall calibration $\widehat{m}_{MF}$, Figure 5 plots the corresponding autocorrelations. Here we see that the calibrated MF-model is able to fit the actual autocorrelations of the log returns and their absolute values quite well. The calibrated MF-model, however, has difficulties in fitting the squared log returns simultaneously.

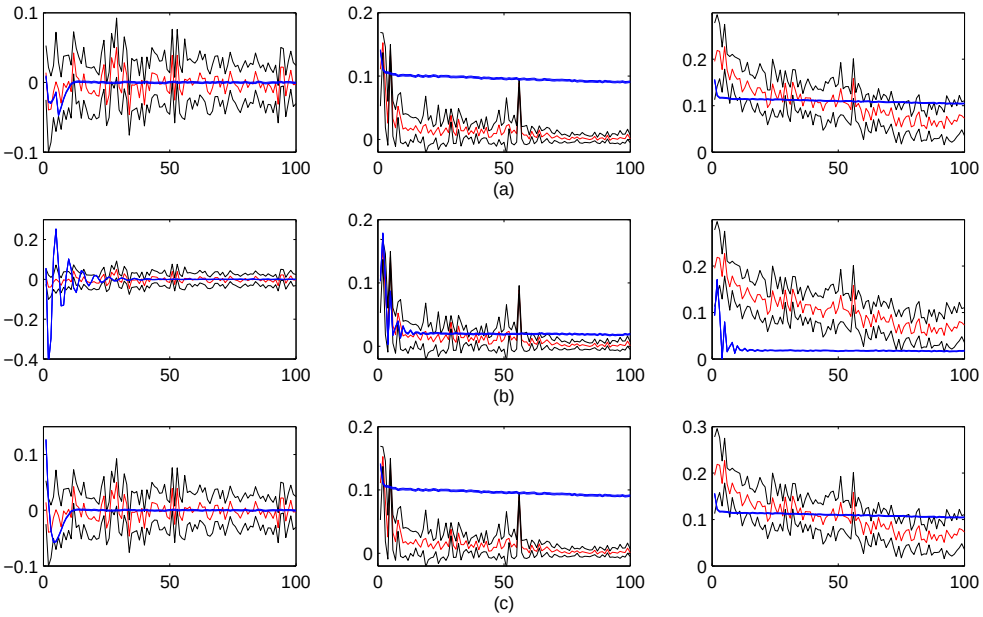

FIGURE 4. Autocorrelations of returns, squared returns and absolute returns of the S&P 500 (with confidence intervals) and the calibrated MF model $\widehat{m}_{MF,r_t}$ (a), $\widehat{m}_{MF,r_t^2}$ (b), and $\widehat{m}_{MF,|r_t|}$ (c).

To further illustrate the comparison methodology for the market fraction models, we will focus on $\widehat{m}_{MF}$, as this MS model best fits all three autocorrelation series (by construction).

Comparison with actual data via ARFIMA—For the daily return, absolute return, and squared return of the S&P 500, we estimate the ARFIMA$(0, d, 0)$ model; the estimates of parameter d are summarized in Table 10.

TABLE 10. Estimates of d for the S&P 500

	d	Std.	P-value	95% CI
r	-0.0192	0.0112	0.086	[-0.0410, 0.0027]
r^2	0.1233	0.0102	0.000	[0.1033, 0.1433]
$\|r\|$	0.1762	0.0085	0.000	[0.1594, 0.1931]

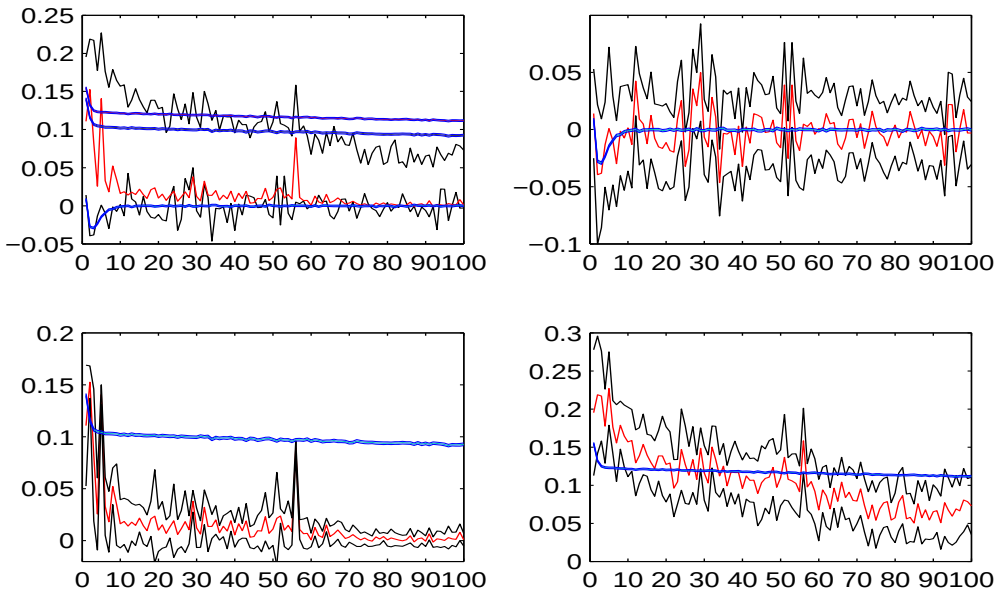

FIGURE 5. Autocorrelations of returns, squared returns and absolute returns for the MF model (bold lines) and the S&P 500 (top left). Autocorrelations of returns for the MF model and the S&P 500 (top right). The bold line refers to the MF model while the confidence intervals are constructed for the S&P 500, this also holds to squared returns (low left), and absolute returns (low right).

We see that the results do provide evidence of long run persistence for squared returns and absolute returns, while it seems that the estimated d is not significant for the daily returns: the persistence in absolute returns seems to be much stronger than that in squared returns. These results coincide with the well-established findings in the empirical finance literature.

The corresponding estimates of the calibrated MF-model are reported in Table 11, where the last column indicates the percentage of the estimates that are significantly different from zero at the 5% level, for 1,000 independent simulations.

The average value of d according to the calibrated MF-model does not lie in the 95%-confidence interval of the corresponding estimate of d based on the actual data in

TABLE 11. Estimates of d for the MF model

	d	Std.	p-value	95% CI	Sig%
r	-0.0525	0.0112	0.0225	[-0.0532, -0.0518]	93.9
r^2	0.1252	0.0088	0.000	[0.1247,0.1257]	100
$\|r\|$	0.1286	0.0086	0.000	[0.1281, 0.1291]	100

case of the log returns and in case of the absolute returns, while it fits inside the 95% confidence interval for the squared log returns.

Thus, the calibrated MF model is able to give partial descriptions of the actual data, but not a full description. In terms of the log returns and absolute log returns, the autocorrelation structure can be described, but not the long memory parameter, while in terms of the squared log returns the situation is the other way around: the memory parameter can be described, but not the autocorrelation structure. This indicates that the MF model class might require adaptations, if the aim is to describe aspects of actual data as the ones considered here.

Comparison with actual data via (FI)GARCH—Table 12 reports the estimates of the GARCH $(1, 1)$ model applied to the actual data, where the mean process follows an AR(1) structure, cf. *Example 3*. Based on the estimates, one can see that a

TABLE 12. GARCH $(1, 1)$ Parameter Estimates for the S&P 500

$a \times 10^3$	b	$\alpha_0 \times 10^5$	α_1	β_1
0.608	0.0359	0.113	0.0783	0.9145
(0.125)	(0.014)	(0.059)	(0.0304)	(0.0305)

Note: Standard errors are in parentheses. This is also true for other tables in this section

small influence of the most recent innovation ($\alpha_1 < 0.1$) is accompanied by a strong persistence of the variance coefficient ($\beta_1 > 0.9$). It is also interesting to observe that the sum of the coefficients $\alpha_1 + \beta_1$ is close to one, i.e., the process is close to an integrated GARCH (IGARCH) process. Such parameter estimates are rather common

when considering returns from high frequency daily financial data of both share and foreign exchange markets (Pagan, 1996).

Table 13 reports the S&P 500-estimates of the FIGARCH $(1, d, 1)$ model, where the mean process follows an AR(1) model. The estimate for the fractional differencing parameter d is positive and significantly different from zero. This is consistent with the well known findings that the shocks to the conditional variance dies out at a slow hyperbolic rate.

TABLE 13. FIGARCH $(1, d, 1)$ Parameter Estimates for the S&P 500

a	b	$\alpha_0 \times 10^4$	d	ϕ_1	β
-0.0258	0.0166	0.000017	0.3933	0.1012	0.7968
(0.00039)	(0.0083)	(0.1930)	(0.0091)	(0.0116)	(0.0035)

Next, we turn to look at the GARCH estimates and the FIGARCH estimates for the calibrated MF-model $\widehat{m}_{MF}$. We report the estimates of the GARCH and FIGARCH model in Table 14 and Table 15, respectively. Again, these estimates are obtained from the estimation for each run of the simulation model and then averaged over independent simulations.

TABLE 14. GARCH $(1, 1)$ Parameter Estimates for the MF Model

$a \times 10^3$	b	$\alpha_0 \times 10^4$	α_1	β
-0.1396	0.0620	0.0510	0.0194	0.9605
(0.3546)	(0.0139)	(0.1421)	(0.0036)	(0.0766)
1.5	88.7	12.9	95.7	96.8

Note: The last row presents the percentage of the simulations for which the parameter estimates are significantly different from zero.

The resulting Wald statistic for testing equality of the S&P 500 and calibrated MF-model parameters is 38.29, which suggests that the null hypothesis is strongly rejected and hence the GARCH$(1, 1)$ model in case of the MF model and that of the S&P 500 are significantly different. Similarly, for the FIGARCH$(1, 1)$ estimates, the resulting

TABLE 15. FIGARCH $(1, d, 1)$ Parameter Estimates for the MF Model

a	b	$\alpha_0 \times 10^4$	d	ϕ_1	β
0.0348	0.0733	0.2594	0.4182	0.2398	0.7928
(0.0014)	(0.0189)	(0.9175)	(0.0366)	(0.0255)	(0.0199)
55.4	75	11	91.4	76.7	98.4

Note: The last row presents the percentage of the simulations for which the parameter estimates are significantly different from zero.

Wald statistics is 4568, which is far beyond the critical value at any conventional significant level. So, the FIGARCH$(1, d, 1)$ model of the calibrated MF model seems to be significantly different from that of the S&P 500, confirming the conclusion of the previous subsection that the MF model class might need to be adapted to better match the features of the actual data we investigated.

5. CONCLUSIONS

Microscopic Simulation (MS-)models are a promising way to study financial markets, since they allow for the possibility to include all kinds of realistic and complex behavior of interacting economic agents, without having to worry about analytical tractability. However, in many cases judgements of the outcomes of MS models seem to be based solely on visual inspection.

In this paper we propose to investigate Microscopic Simulation (MS) models using statistical and econometric techniques. Such techniques can be used to study the impact of changes in the initial parameter settings and initial conditions on the simulated time series behavior of the relevant quantities. But also different MS economies can be compared using these techniques, in order to find out whether particular adaptations are crucial or not. We also develop the methodology to compare real life data with the MS economies.

We consider two applications. In the first one, we compare the original LLS-model, developed by Levy *et al.* (2000), with variations by changing the inputs. This reveals which input changes turn out to be significant and which ones insignificant. We also extend the LLS model by including an additional type of investor. The extended model does not seem to deviate much from the original model: in the end the extra agents are competed away by the originally included economic agents. Finally, we compare the LLS model with actual data, by comparing various distributional characteristics. When the actual data are described by means of an ARFIMA$(0, d, 0)$-model, the LLS-model seems to be able to provide a good description; however, in terms of an ARFIMA$(1, d, 1)$-model the LLS-model is unable to provide an accurate description. However, the sensitivity analysis suggests that when calibrating the LLS model class using a flexible specification of the dividend process, the LLS model might be able to provide appropriate descriptions of actual distributional characteristics.

The second application is the market fraction model introduced by He and Li (2005a, b). Here, we first calibrate the model class to actual data from the S&P 500 by minimizing the distance between the actual and model based autocorrelations of returns, squared returns, and absolute returns. Then we investigate how well the calibrated model is able to describe the actual data in terms of ARFIMA and (FI)GARCH models. The MF-model class seems to have some difficulties in providing an accurate description of all actual distributional characteristics that we investigated at the same time.

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