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Bouis, R.; Huisman, K.J.M.; Kort, P.M.

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By Romain Bouis, Kuno J.M. Huisman, Peter M. Kort

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Investment in Oligopoly under Uncertainty: The Accordion Effect*

Romain Bouis[1], Kuno J.M. Huisman[2,3], and Peter M. Kort[2,4,†]

[1]*SDFi, Paris Dauphine University,*

Place du Maréchal de Lattre de Tassigny 75775 Paris cedex 16, France

[2]*Department of Econometrics and Operations Research and CentER, Tilburg University,*

Post Office Box 90153, 5000 LE Tilburg, The Netherlands

[3]*Centre for Quantitative Methods CQM B.V.,*

Post Office Box 414, 5600 AK Eindhoven, The Netherlands

[4]*Department of Economics, University of Antwerp,*

Prinsstraat 13, 2000 Antwerp 1, Belgium

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Abstract

In the strategic investment under uncertainty literature the trade off between the value of waiting known from single decision maker models and the incentive to preempt competitors is mainly studied in duopoly models. This paper aims at studying competitive investments in new markets where more than two (potential) competitors are present.

In case of three firms an accordion effect in terms of investment thresholds is detected in the sense that an exogenous demand shock results in a change of the wedge between the investment thresholds of the first and second investors that is qualitatively different from the change of the wedge between the second and third investment threshold. This result extends to the n firm case. We show that a direct implication of the accordion effect is that there are two types of equilibria in the three firm case. In the first type all firms invest sequentially and in the second type the first two investors invest simultaneously and the third investor invests at a later moment.

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†Corresponding author. Tel: +31-13-4662062; Fax: +31-13-4663280; E-mail: kort@uvt.nl

If we consider sequential equilibria and compare entry times of the first investors for different potential market sizes, it turns out that in the two firm case the first investor invests earlier than in the monopoly case, in the three firm case the investment timing lies in between the one and the two firm case, the four firm case lies in between the two and the three firm case, and so on and so forth. Hence, a policy maker interested in an early start up should hope for an even number of competitors, although for n large the investment times of the first investors are almost equal.

JEL-codes: C73, D92, L13

Keywords: Investment, Real options, Oligopoly

1 Introduction

The theory of investment under uncertainty, also known as real option theory, prescribes that uncertainty and irreversibility creates a value of waiting with undertaking capital investments. The reason is that over time more information becomes available that enables the decision maker to take better investment decisions at a later date. This result especially holds in single decision maker problems of firms where strategic interactions are not present or simply ignored. Standard references on real options are Dixit and Pindyck (1994) and Trigeorgis (1996). However, in the western economies the extensive process of deregulation, combined with a waive of mergers and acquisitions, has resulted in an oligopolistic structure of a large number of sectors. This implies that there is a strong need to combine the real option approach with a multiple decision maker framework, as to include competition.

By now the recent literature contains lots of contributions that take up this challenge by studying investment under uncertainty in duopoly models, see e.g. the surveys by Grenadier (2000) and Huisman *et al.* (2004). In such a framework still the *value of waiting* incentive exists, but it is combined now with strategic considerations that can make it imperative to *act quickly* to preempt investment by the competitor. For example, a firm that enters a new market as first leaves less market share to the other firm, which reduces the profitability of entering this market for this firm.

Of course, in practice only a few markets exist where the number of firms exactly equals two. For this reason the purpose of this paper is to study implications of extending the number of (potential) competitors to three and later on to n. We carry out this research in a so-called new market model where firms have to invest first before they have access to any revenue.

The study of investment under uncertainty with three firms reveals the presence of a new mechanism in the strategic real option literature that we denote as the *accordion effect*. In case all firms invest sequentially and there is an exogenous shock in demand, the wedge between the investment thresholds of the first and second investor changes in an opposite direction compared to the change of the wedge between the second and the third investment threshold. The reason is that if entry of the third firm is delayed[1], the second

[1] For simplicitly we assume in the text below that a lower (higher) threshold implies earlier (later) investment and vice versa. This assumption might not hold if the change of the threshold is caused by a change in the volatility of the underlying stochastic

firm has an incentive to invest earlier because this firm can enjoy the duopoly market structure (rather than having three firms in the market) for a longer time. Hence, the first firm faces earlier entry of the second one and thus a shorter period in which it can enjoy monopoly profits. This reduces the investment incentive for the first firm and thus it invests later. If the exogenous demand change is sufficiently large, the length of this *shorter period* converges to zero. We can thus conclude that a direct consequence of the accordion effect is that there exist two possibilities concerning the order of investment of the firms. In the first case all firms invest sequentially, whereas in the second case two firms start out investing together followed later on by the other firm. The latter case occurs more often in more uncertain environments, thus where the preemption incentive of being the first to invest is counterbalanced by a considerable value of waiting effect. The sequential case especially takes place in case a market with two active firms results in much less revenue per firm than when only one firm is active in this market.

We further compare the entry time of the first investor in case the number of firms equals one, two and three. Here we build on Nielsen (2002) who already found that the first investor in the duopoly case invests earlier than the monopolist in the one firm case. In this paper we find that the first investor in the three firm case invests at a time that lies in between the monopoly and duopoly first investor time in case of sequential investment. However, the first investment in the three firm case can take place earlier than in the two firm case if the equilibrium is of the simultaneous type.

Finally, this paper takes up the challenge to consider a n-firm framework. First, we find that the *accordion effect* also holds here, which means that exogenous demand changes affect the timing of entry of the first, third, fifth,..., investor in the same qualitative way, while the entry time of the second, fourth, sixth,..., investor is affected in exactly the opposite qualitative way. In other words, if a delay is observed for the "odd" investors, then the "even" investors will invest sooner.

A numerical example is set up to check the timing of the first entrant for different potential market sizes and sequential equilibria. It turns out that in the case of four firms the first investor enters at a time that lies in between the entry times of the first investor in the two firm and the three firm case, respectively. In the same way the first investor in the five firm case invests at a time in between the three and four firm case, and so on and so forth. We conclude that the result that relates to up to three firms extends in an analogous manner every time we add a new firm. A second conclusion is that if a social planner is interested in an early start up of the market, it is beneficial to have an even number of potential competitors, although it has to be remarked that for n large the entry times of the first investor are almost equal.

Grenadier (2002) also considers a real option framework with more than two firms. The main difference with our setting is that in Grenadier (2002) every investment increases output with an infinitesimal increment, leading to a path of output that is continuous over time. Contrastingly, our framework has investments in lumps with the result that output jumps every time an investment is undertaken. Grenadier's main result is that the impact of competition drastically erodes the value of the option to wait and leads to investment

process (see Sarkar (2000)).

at very near the zero net present value threshold. This result is confirmed in our numerical example of the n-firm framework, where the value of waiting of the first investor quickly converges to zero as the number of firms increase.

This paper is organized as follows. In the next section the oligopoly model with three firms is presented. Section 3 analyzes the n firm case, while Section 4 concludes. The proofs of the propositions and theorems are given in Appendix A. In Appendix B the equilibrium concept is presented. Appendix C gives some of the derivations of the example that is presented in Sections 2 and 3.

2 Oligopoly with three firms

This section treats the oligopoly model with three firms. Section 2.1 presents the model. Then the solution is obtained in Section 2.2. This section also defines the accordion effect and contains some illustrations of this effect based on comparative statics results with regard to the uncertainty parameter and the curvature of the demand function. Section 2.3 compares the obtained results with those of the duopoly case. Finally, a specific example is provided in Section 2.4.

2.1 Model

The first paper dealing with a multiple decision maker model in a real option context is Smets (1991). It considers an international duopoly with identical firms producing a single homogeneous good. Both firms can increase their revenue stream by investing. Like in the deterministic analysis in Fudenberg and Tirole (1985) two equilibria arise: a preemption equilibrium, where one of the firms invests early, and a simultaneous one, where both firms delay their investment considerably. A simplified version was discussed in Dixit and Pindyck (1994) in the sense that the firms are not active before the investment is undertaken. The resulting new market model only has the preemption equilibrium. The model in the present section builds on Dixit and Pindyck (1994) and extends it by considering three firms instead of two.

We thus consider an oligopolistic industry comprising of three identical firms producing a single, homogeneous good. Each firm has the opportunity to invest once with sunk costs $I > 0$. Undertaking the investment gives access to a profit flow

$$Y(t)\, D_k,$$

where k is the number of active firms and $Y(t)$ follows a geometric Brownian motion with drift parameter μ and volatility parameter σ:

$$\begin{aligned} dY(t) &= \mu Y(t)\, dt + \sigma Y(t)\, d\omega(t)\,, \qquad (1)\\ Y(0) &= y. \qquad (2) \end{aligned}$$

The effect of competition on profits is reflected in D_k. If k goes up the market has to be shared by more firms, which implies that individual firm profits will be lower. For this reason it holds that D_k is strictly decreasing in k as long as $k > 0$, i.e.

$$D_1 > D_2 > D_3 > D_\infty = 0.$$

Such a general formulation embraces, for instance, Cournot quantity competition, as will be illustrated later on in Section 2.4.

Furthermore, the firms are risk neutral, value maximizing, discount with constant rate r, and variable costs of production are absent. Assume that $r > \mu$ in order to ensure convergence.

2.2 Solution

Given the stochastic process $(Y_t)_{t \geq 0}$ we can define the value functions of the firms for the different possible strategies. By $V_{ij}(Y)$ we denote the value of a firm that invests at Y given that j firms are already active in the market and i firms jointly invest. Furthermore, let $W_{ij}(Y)$ be the value of a firm that does not invest when i other firms invest at Y given that j firms were active in the market before the investment (here "W" denotes "waiting"). The optimal investment trigger for the i firms that invest simultaneously when there are j firms already active in the market is denoted by Y_{ij}.

As in the standard approach used to solve dynamic games, we analyze the problem backward in time. First, we derive the optimal strategy of the third investor, who takes the strategy of the other two firms as given. Subsequently, we analyze the decisions of the second and the first investor, respectively.

So, we start out by considering the investment decision of the third investor in a situation where the two other firms have already invested. After the first two investments are undertaken, the only decision left to take is when the third firm should invest. From this it follows that the problem that the third firm faces is a purely decision theoretic one, i.e. strategic considerations are absent. Essentially, the investment problem of the third investor can be analyzed by employing the standard real option model presented in Dixit and Pindyck (1994, Chapter 6). The third investor's value function, V_{12}, is equal to

$$V_{12}(Y) = \begin{cases} \left(\frac{Y}{Y_{12}}\right)^{\beta} \left(\frac{Y_{12} D_3}{r-\mu} - I\right) & \text{if } Y < Y_{12}, \\ \frac{Y D_3}{r-\mu} - I & \text{if } Y \geq Y_{12}, \end{cases} \tag{3}$$

where β is the positive solution of the so-called fundamental quadratic, i.e. $\frac{1}{2}\sigma^2\beta^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\beta - r = 0$. The investment trigger Y_{12} is given by

$$Y_{12} = \frac{\beta}{\beta - 1} \frac{(r-\mu) I}{D_3}. \tag{4}$$

The first row of (3) is the present value of profits when the third firm does not invest immediately. This term equals the value of the option to invest, which is the net present value of the third firm's investment discounted back from the (random) time of reaching the third firm's threshold Y_{12}. Consequently, $\left(\frac{Y}{Y_{12}}\right)^{\beta}$ is interpreted as a stochastic discount factor, and it can in fact be shown that (see, e.g., Dixit and Pindyck (1994))

$$E\left(e^{-rT}\right) = \left(\frac{Y}{Y_{12}}\right)^{\beta},$$

where T is the random first time the process reaches a fixed level Y_{12} starting from the initial position Y.

The optimal investment strategy for the third investor is to invest at the first moment that $Y(t)$ is larger than or equal to Y_{12}. The following proposition is directly obtained from (4).

Proposition 1 *The investment trigger of the third investor, Y_{12}, is decreasing in D_3 and increasing in σ.*

Next, we analyze the investment decision of the second investor in a situation where the first investor has already invested. After the first investor has made its investment, the two remaining firms face the duopoly investment game that is analyzed in Huisman (2001, Chapter 7). Therefore, the value function of the second investor is equal to the leader value function in the duopoly investment game, i.e.

$$V_{11}(Y) = \begin{cases} \frac{YD_2}{r-\mu} - I + \left(\frac{Y}{Y_{12}}\right)^{\beta} \frac{Y_{12}(D_3 - D_2)}{r-\mu} & \text{if } Y < Y_{12}, \\ \frac{YD_3}{r-\mu} - I & \text{if } Y \geq Y_{12}. \end{cases} \tag{5}$$

In the first row of (5) it is taken into account that at Y_{12} the third firm invests, implying that for the second investor the profit flow YD_2 is replaced by YD_3. Huisman (2001, Chapter 7) shows that the expected value of a firm in the duopoly investment game before the first firm invested is equal to the follower value. This result implies for the current model that before the second investment is undertaken the expected value of the second investor equals the value of the third investor, i.e. $W_{11}(Y) = V_{12}(Y)$. Furthermore, Huisman (2001, Chapter 7) shows that there always exists a unique preemption threshold in the duopoly investment game at which the leader, which is here the second investor, will invest, given that the initial value of Y is lower than this preemption threshold. This investment trigger Y_{11} is defined as follows:

$$Y_{11} = \inf \left(Y \in (0, Y_{12}) \,|\, V_{11}(Y) = W_{11}(Y) \right). \tag{6}$$

In the interval (Y_{11}, Y_{12}) the value of being second investor strictly exceeds the value of being third investor. Hence, both firms prefer to invest rather than wait and be third investor. Nevertheless, in case two firms already have invested and it holds that $Y < Y_{12}$, the third investor's value of waiting with investment until Y_{12} is reached, strictly exceeds the value obtained from investing immediately as a third firm. Hence, if the second firm already has invested, the third firm prefers to wait.

The optimal investment strategy for the second investor is to invest at the first moment that $Y(t)$ is larger than or equal to Y_{11} (otherwise, the second investor will be preempted by the third firm). The following proposition reports how the value of Y_{11} can be obtained and how Y_{11} depends on several model parameters.

Proposition 2 *The investment trigger of the second investor, Y_{11}, is implicitly defined by*

$$\frac{Y_{11}D_2}{r-\mu} - I = \left(\frac{Y_{11}}{Y_{12}}\right)^{\beta} \left(\frac{Y_{12}D_2}{r-\mu} - I\right). \tag{7}$$

The threshold Y_{11} is increasing in D_3, decreasing in D_2, and increasing in σ.

Economically, equation (7) states that the threshold Y_{11} is defined as the value at which the firm is indifferent between being the second investor and being the third investor.

Finally, we analyze the investment decision of the first investor. Departing from the optimal investment decisions of the other firms, we can derive the value function for the first investor:

$$V_{10}(Y) = \begin{cases} \frac{YD_1}{r-\mu} - I + \left(\frac{Y}{Y_{11}}\right)^{\beta} \frac{Y_{11}(D_2-D_1)}{r-\mu} + \left(\frac{Y}{Y_{12}}\right)^{\beta} \frac{Y_{12}(D_3-D_2)}{r-\mu} & \text{if } Y < Y_{11}, \\ \frac{YD_2}{r-\mu} - I + \left(\frac{Y}{Y_{12}}\right)^{\beta} \frac{Y_{12}(D_3-D_2)}{r-\mu} & \text{if } Y_{11} \leq Y < Y_{12}, \\ \frac{YD_3}{r-\mu} - I & \text{if } Y \geq Y_{12}. \end{cases} \tag{8}$$

In the first row of (8) it is taken into account that at Y_{11} the second investor invests, implying that for the first investor the profit flow YD_1 is replaced by YD_2. Subsequently, at Y_{12} the third investor enters the market so that then the profit flow changes from YD_2 into YD_3.

Since we are analyzing a new market model, i.e. a firm makes no profit before the investment is made, the value of a firm that invests simultaneously with another firm as first, V_{20}, equals the value of the second investor, V_{11}. Thus we have $V_{20}(Y) = V_{11}(Y)$. Furthermore, if three firms invest simultaneously, each firm's value is equal to the third investor's value, so that $V_{30}(Y) = V_{12}(Y)$.

Let us now focus on the value functions of the firms that do not invest as first. First, consider the case that there is only one firm that invests as first. After this first investment is undertaken, the other two firms end up in a duopoly investment game. From this game we know that the expected value of both firms equals the follower value in that game, so that $W_{10}(Y) = V_{12}(Y)$. Next, assume that there are two firms investing simultaneously as first. Then the value of the firm that did not invest also equals the third investor's value $W_{20}(Y) = V_{12}(Y)$. In other words we have that

$$W_{11}(Y) = W_{10}(Y) = W_{20}(Y) = V_{12}(Y). \tag{9}$$

Provided that it exists (see the next proposition), the investment trigger for the first firm Y_{10} is equal to

$$Y_{10} = \inf\left(Y \in (0, Y_{11}) \,|\, V_{10}(Y) = W_{10}(Y)\right). \tag{10}$$

In the interval (Y_{10}, Y_{11}) the value of being first investor exceeds the value of being second investor. However, once being the second investor, that firm prefers to wait until $Y(t)$ reaches Y_{11} above investing immediately. This implies that if the first investor already has invested, the second investor will wait. Since it has to prevent preemption by one of the other firms, the first investor will choose to invest at the first moment that $Y(t)$ is larger than or equal to Y_{10}.

However, it turns out that existence of Y_{10} is not assured. Once it does not exist, it will be optimal for two firms to invest simultaneously as first. Because of the new market structure the corresponding trigger is then Y_{11}. The following proposition states the necessary and sufficient condition for the existence of the investment trigger of the first investor.

Proposition 3 *There exists a unique value for Y, Y_{10}, such that*

$$V_{10}(Y) = W_{10}(Y) \text{ and } 0 < Y_{10} < Y_{11}, \tag{11}$$

if and only if

$$Y_{11} > Y_1^M, \tag{12}$$

where

$$Y_1^M = \frac{\beta}{\beta - 1} \frac{(r-\mu) I}{D_1}. \tag{13}$$

The threshold Y_{10} is implicitly defined by

$$\frac{Y_{10} D_1}{r - \mu} - I = \left(\frac{Y_{10}}{Y_{11}} \right)^{\beta} \left(\frac{Y_{11} D_1}{r - \mu} - I \right). \tag{14}$$

The investment trigger Y_{10} is equal to that value for which a firm is indifferent between being the first investor and being the second investor. The next proposition shows how this trigger depends on some relevant parameter values.

Proposition 4 *Whenever the investment trigger Y_{10} exists, it is decreasing in D_3, increasing in D_2, decreasing in D_1, and increasing in σ.*

2.2.1 The two equilibria: sequential and simultaneous

Depending on the existence of Y_{10}, either one of two types of equilibria can occur. In the first type the first two investors make their investments sequentially and in the second type the first two investors invest simultaneously. In Figure 1 the value functions are plotted for the sequential equilibrium case and in Figure 2 the same is done for the simultaneous equilibrium case.

Consider Figure 1 first. In the interval $(0, Y_{10})$ the waiting curve is the highest one, so all firms benefit from refraining from investment. The reason is that the profit flow YD_1 is too low to counterbalance the sunk investment costs. In the interval (Y_{10}, Y_{11}) the first investor has the highest payoff, which provides the incentive for the first investor to preempt the two other firms at Y_{10}. If the initial value of the process Y is such that it is in between Y_{10} and Y_{11}, with probability one at least one of the firms will invest immediately. In fact, by applying a mixed strategy concept (for details, see Huisman (2001) or Thijssen (2004)) Appendix B shows that with positive probability either one, two or three firms will invest there[2].

In the interval (Y_{11}, Y_{12}) it holds that $V_{20}(Y) = V_{11}(Y)$ gives the highest payoff, implying that at Y_{11} the second investor wants to preempt the third one. If the initial value falls in this interval, with probability one two firms will invest immediately[3]. Finally, in the interval (Y_{12}, ∞) the market is big enough for all firms to invest at once.

Next, consider Figure 2. We see that here no $Y-$interval exists where $V_{10}(Y)$ has the highest value. For this reason the first two investors will invest simultaneously. This will happen in the interval (Y_{11}, Y_{12})

[2] Of course, firms do not want to end up in an outcome where more than one firm invests, since this leaves them with a low payoff $V_{20}(Y)$ or $V_{30}(Y)$. However, such an outcome can occur as the result of a coordination failure.

[3] Analogous to the interval (Y_{10}, Y_{11}), also here it holds that with positive probability a coordination failure can occur in the sense that three firms will invest immediately, leaving all the firms with a very low payoff.

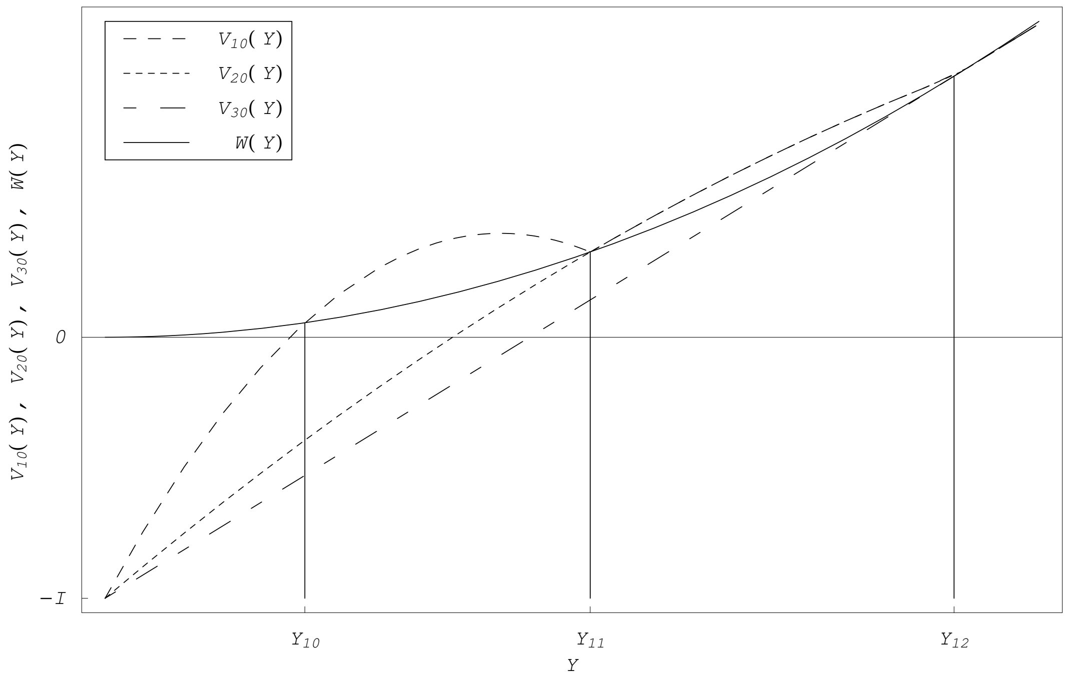

Figure 1: Value functions in the sequential equilibrium case.

(at which precise value of $Y \in (Y_{11}, Y_{12})$ the firms will invest, depends of course on the initial value of Y). Furthermore in the interval $(0, Y_{11})$ the waiting curve gives the highest value, so investing is not optimal, while in (Y_{12}, ∞) all firms will again invest immediately.

Below we show that the position of D_2 on the (D_3, D_1) interval determines the type of the equilibrium that prevails. For relatively low values of D_2, that is, for D_2 close to D_3, the equilibrium will be of the sequential type. The reason is that the first mover advantage is relatively high as D_1 is large compared to D_2, so that the firms have an incentive to become the first investor and earn the relatively high monopoly profits until the second firm has invested. On the other hand, if D_2 is relatively high (close to D_1) the first mover advantage is relatively small, so that firms do not really care to become the first or second investor as long as they are not the third investor. Mathematically it can be shown that there is a unique value of D_2, denoted by D_2^* such that the equilibrium is of the sequential type if D_2 is smaller than D_2^* and the simultaneous equilibrium prevails for D_2 larger than or equal to D_2^*. This boundary D_2^* is dependent on the level of uncertainty. If uncertainty rises, the bound D_2^* decreases as firms become more hesitant to invest quickly. Consequently, the incentive to preempt reduces and the area of simultaneous investment rises. Below we prove that there is a lower bound for D_2^* which is larger than D_3. This implies that there exist values for D_2 (relatively low values) that will always lead to a sequential equilibrium no matter what the level of uncertainty is. Lastly, it turns out that there does not exist a D_2 region for which the equilibrium is always of the simultaneous type no matter the level of uncertainty.

The following theorem formalizes the above statements, where it is important to note that β is decreasing

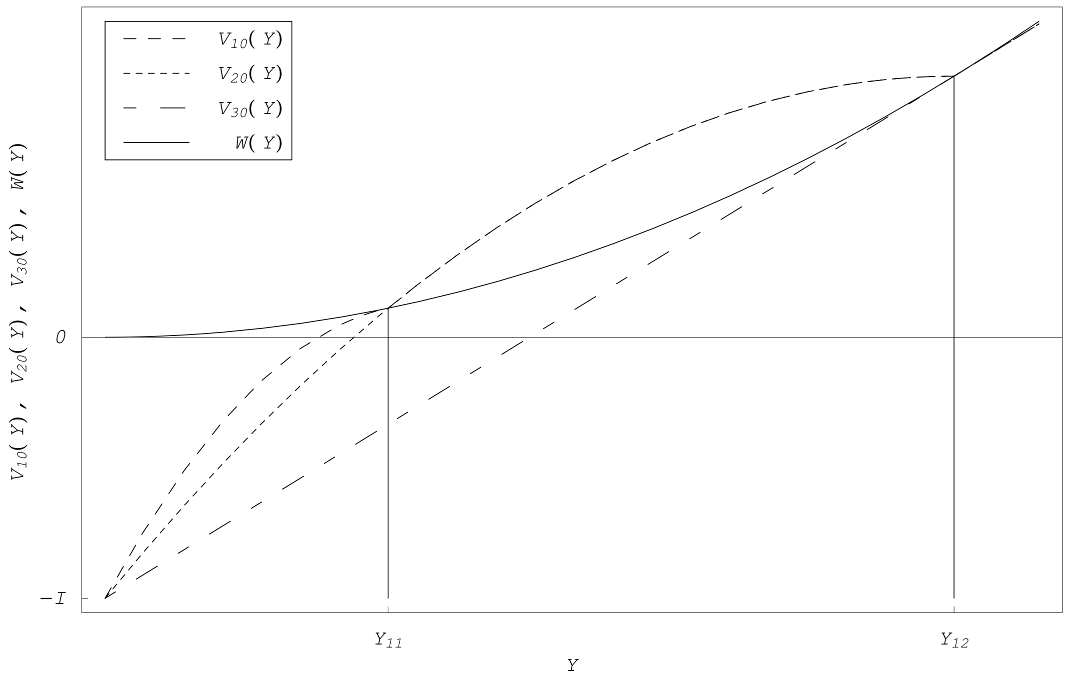

Figure 2: Value functions in the simultaneous equilibrium case.

in uncertainty and that $\beta \rightarrow 1$ for $\sigma \rightarrow \infty$.

Theorem 1 *The first two investors invest sequentially if* $D_2 < D_2^*(\beta)$ *and the first two investors invest simultaneously if* $D_2 \geq D_2^*(\beta)$, *where*

$$D_2^*(\beta) = \frac{\beta - 1}{\beta} \frac{D_1^\beta - D_3^\beta}{D_1^{\beta-1} - D_3^{\beta-1}}. \tag{15}$$

It holds that

$$\frac{\partial D_2^*(\beta)}{\partial \beta} > 0, \tag{16}$$

$$D_2^*(1) = \frac{D_1 - D_3}{\log(D_1) - \log(D_3)} > D_3, \tag{17}$$

$$\lim_{\beta \rightarrow \infty} D_2^*(\beta) = D_1, \tag{18}$$

which implies that the equilibrium is always of the sequential type if $D_2 \in (D_3, D_2^*(1))$ *no matter the level of uncertainty.*

2.2.2 The accordion effect

Propositions 1, 2 and 4 state that a decrease in D_3 leads to an increase in the third firm's threshold, a decrease in the second firm's threshold, and an increase in the first firm's threshold. Hence, the wedge between the first and the second threshold decreases and the wedge between the second and the third threshold increases.

The opposite directions of the changes in these wedges is what we call the *accordion effect*. This effect also arises due to changes in uncertainty, σ, drift, μ, discount rate, r, and the number of competitors (see Section 3).

The next section presents the accordion effect in case of a change in uncertainty. It is shown that this feature especially influences the strategic effect of uncertainty. The section after that explains in detail how a change in the demand function curvature (i.e. changing the value of one of the D_i's) also leads to the occurrence of the accordion effect.

2.2.3 Change in uncertainty

Here we consider the impact of an increase in uncertainty on the investment threshold values. Two effects can be detected that work in opposite directions. The first is the waiting effect and the second the strategic effect. The waiting effect leads to an increase in the investment thresholds. The reason is that uncertainty raises the value of waiting with investment, which is the standard real options result.

Now, let us analyze the strategic effect. First, we observe that, since the other two firms have already taken their decisions in that their investments already took place, the investment trigger of the third investor is not affected by a strategic effect. This implies that only the waiting effect plays a role, so that the third investor will invest later, i.e. Y_{12} goes up with uncertainty (cf. Proposition 1). This in turn implies that the second investor stays in a duopoly (rather than in a market with three firms) for a longer time. Therefore, the investment of the second investor becomes more profitable, so that the second investor has an incentive to invest earlier. Hence, the strategic effect on the second investor's trigger is negative. However, taking the waiting effect and the strategic effect together, it turns out that the waiting effect dominates so that also firm 2's trigger, Y_{11}, increases with uncertainty (see Proposition 2).

So, given that the second investor invests later, also for the first investor it holds that investing becomes more profitable and thus that the strategic effect results in a lower trigger. But still also here the total effect of uncertainty is that the first investor's trigger, Y_{10}, will be increasing when uncertainty goes up, as is reported in Proposition 4.

As a result of the above effects, and especially the strategic effect, it holds that, whenever $D_2 > D_2^*(1)$, an increase in σ reduces the wedge $Y_{11} - Y_{10}$, while it raises the wedge $Y_{21} - Y_{11}$. In other words, as uncertainty goes up the thresholds of the first and second investor approach each other, while the thresholds of the second and third firm diverge. This is again the *accordion effect*.

Section 2.2.1 learned that for $D_2 > D_2^*(1)$ we have the sequential equilibrium for low uncertainty levels and the simultaneous equilibrium for high levels of uncertainty. Then, as long as the sequential equilibrium prevails and uncertainty goes up, the investment triggers Y_{10} and Y_{11} get closer, while they converge at the moment that σ is large enough for the simultaneous equilibrium to occur. This is shown in the left panel of Figure 3, where the wedge between the investment triggers Y_{10} and Y_{11}, i.e. $Y_{11} - Y_{10}$, is plotted as a function of the degree of uncertainty σ. We conclude that the existence of two qualitatively different equilibria is a direct implication of the *accordion effect*.

In the previous section we also detected that for low values of D_2 the incentive to preempt in order to become the first investor is high, so that we always have the sequential equilibrium. This happens when $D_2 < D_2^*(1)$. Then it turns out that the wedge between Y_{10} and Y_{11} does not have to be a monotonous function in σ, as is shown in the right panel of Figure 3.

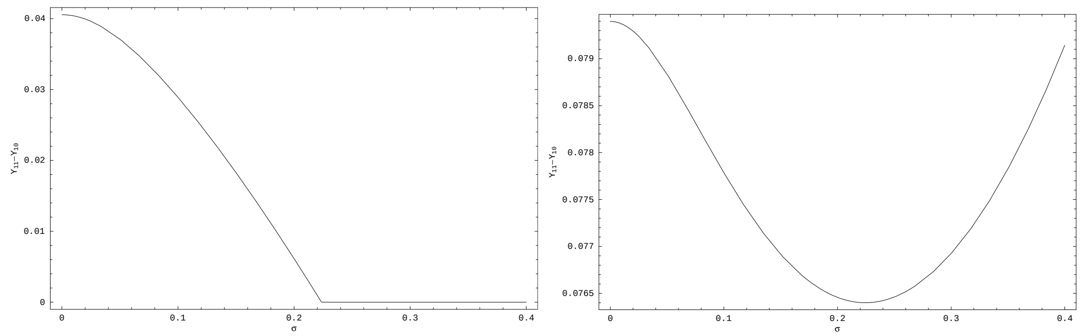

Figure 3: Wedge between the investment triggers Y_{10} and Y_{11} as function of σ for $D_1 = 10$, $D_3 = 2$, $r = 0.1$, $\mu = 0.025$, and $I = 10$. In the left panel $D_2 = 6$ and in the right panel $D_2 = 4.8$. For these parameters it holds that $D_2^*(1) = 4.97$.

2.2.4 Change in the demand function curvature

This section checks how changes in the demand parameters D_1, D_2 and D_3 affect the investment decisions of the firms. If industry output goes up with the number of firms, which is generally true in case of Cournot competition, studying the effects of separate changes of the D_i's provides information on how investment timing depends on the curvature of the demand function.

Let us first analyze what happens when D_3 increases. From Proposition 1 we know that an increase in D_3 leads the third firm to invest earlier. Regarding the investment timing decision of the second firm, a variation in D_3 has two opposing effects. First, an increase in D_3 makes investment by the second firm less attractive as this firm receives the temporary "duopolistic" profit for a shorter period of time, because the third firm enters the market earlier. Second, an increase in D_3 may lead to earlier investment of the second firm since the negative effect of competition on profits is less severe in the sense that the drop in the profit flow due to the entry of the third firm, being equal to $(D_2 - D_3) Y$, is smaller. According to Proposition 2 the first effect always dominates the second one so that the second firm always invests later as D_3 increases.

By a domino effect, an increase in D_3 makes investment by the first firm more attractive (cf. Proposition 4). Indeed, the temporary "monopolistic" profit accruing to this firm lasts for a longer period of time when D_3 is large, as the second firm enters the market later. Hence, an increase in D_3 has a positive effect on the entry decision of the first firm. In other words, an increase in D_3 implies that the investment thresholds of the first and the second investor diverge, while the thresholds of the second and third investor approach each other. We conclude that changing D_3 gives rise to the *accordion effect.*

If D_2 increases the third investor is not affected, because it is simply not active during the time period that the profit flow is dependent on D_2. Furthermore an increase in D_2 straightforwardly leads to earlier investment of the second investor, and, since the monopoly period lasts less long, to later investment by the first firm.

By now it will be clear that an increase in D_1 will accelerate the investment timing of the first investor, while it does not influence the investment decisions of both other firms. The effects of an increase in D_1, D_2 or D_3 on the investment policy of each firm are summarized in Table 1.

	increase in		
	D_3	D_2	D_1
firm 3 invests	earlier	no effect	no effect
firm 2 invests	later	earlier	no effect
firm 1 invests	earlier	later	earlier

Table 1: Effects of D_1, D_2, and D_3 on the investment policies.

It is interesting to note that when D_3 or D_2 varies, the first and the second firm always move in the opposite direction. This explains why joint investment can occur: the first firm invests later (Y_{10} increases) and the second firm invests earlier (Y_{11} decreases) as D_2 increases (D_3 decreases) and for D_2 sufficiently large (or D_3 sufficiently low), both firms will invest at the same time. Indeed, from Theorem 1 we know that the joint investment equilibrium will occur for a larger interval of the uncertainty parameter values when D_2 increases.

2.3 Comparison with Duopoly

Let us now compare the three firms' model with the duopoly model. For the duopoly model we denote the threshold of the first investor by Y_P (the subscript "P" relates to preemption) and Y_F (with "F" of follower) is the threshold of the second investor. Then, from Huisman (2001, Chapter 7) we know that Y_P is implicitly given by

$$\frac{Y_P D_1}{r-\mu} - I = \left(\frac{Y_P}{Y_F}\right)^{\beta}\left(\frac{Y_F D_1}{r-\mu} - I\right),$$

while Y_F satisfies

$$Y_F = \frac{\beta}{\beta-1}\frac{(r-\mu)\,I}{D_2}.$$

Nielsen (2002) has derived that the monopoly investment trigger, given by

$$Y_M = \frac{\beta}{\beta-1}\frac{(r-\mu)\,I}{D_1},$$

is larger than Y_P. The reason is that in the duopoly model the first investor has an incentive to act quickly to achieve the (temporary) monopoly position.

The following proposition shows that, if the equilibrium in the three firm case is of the sequential type, the first investor invests at a time that lies in between the monopoly and duopoly first investor time.

Proposition 5 *Consider the sequential equilibrium in the three firm case. Then the investment threshold of the first firm to invest in the three firm case, Y_{10}, is strictly larger than the investment threshold of the first investor in the duopoly case, Y_P, but smaller than the monopoly investment trigger, Y_M.*

The economic interpretation of this result is as follows. Analogous to the comparison of the first investors in duopoly and monopoly case, it holds that in the three firm case the second investor invests earlier than the second investor in the duopoly case, because in the three firm case the second investor must act quickly in order to enter the market earlier than the third investor. Therefore, in the three firm case the monopoly period of the first investor lasts less long, which reduces the first investor's incentive to invest. Consequently, the first investor will invest later than in the duopoly case.

Next, consider the simultaneous equilibrium in the three firm case. Contrary to Proposition 5, in this case it can happen that the first investment in the three firm case can be earlier than in the duopoly case, i.e. $Y_{20} = Y_{11} < Y_P$. To show this, first observe that in the duopoly case joint investment is only possible if $D_2 = D_1$. Then both the trigger value of the follower and the leader are equal to the trigger value of a monopolist, i.e.

$$D_2 = D_1 \Rightarrow Y_F = Y_P = Y_M. \tag{19}$$

In the case with three firms, we know that sequential investment only occurs when (cf. (12))

$$Y_{20} = Y_{11} > Y_1^M.$$

Hence, in the complementary case,

$$Y_{20} = Y_{11} \leq Y_1^M, \tag{20}$$

the outcome where the first two firms invest at the same time, is the unique equilibrium outcome. After comparing (19) with (20), it can be expected that in that equilibrium the first investment trigger is lower than Y_P whenever D_2 is close to D_1.[4] Economically, this result can be explained by observing that the preemption incentive in the duopoly case is rather low when D_2 is close to D_1, so therefore Y_P will be large in such a case. At the same time the preemption incentive of the first two firms in the three firm case can be large, which is the case when D_3 is low.

The reason why this will not hold in the sequential investment case is the following. Theorem 1 learned us that occurrence of the sequential equilibrium requires a low D_2, implying that in such a case there is a large incentive to preempt in the duopoly model. Consequently, Y_P will be small leading to the result of Proposition 5.

[4]This expectation is confirmed by numerical experiments not reported here.

2.4 Example

This section presents an example that will help to illustrate the results. In this example we start out from Cournot competition with a specific inverse demand function. From thereon we derive the profit flows $Y(t) D_k$ and the resulting optimal investment triggers of the three firms. We do this generally in the sense that we consider n firms rather than just three. This is the reason why we also employ this example in the next section where we analyze the n firm case.

The price of a unit of output, $P(t)$, fluctuates stochastically over time so as to clear the market:

$$P(t) = D(X(t), Q(t)),$$

where D is the inverse demand function, $X(t)$ is an exogenous shock process to demand and $Q(t)$ is total output, i.e.

$$Q(t) = \sum_{i=1}^{n} q_i.$$

In this particular example we employ the following specifications for the inverse demand curve $D(X,Q)$ and the shock process $X(t)$. Assume that the market inverse demand function is of a constant elasticity form:

$$P(t) = X(t)(Q(t))^{-\frac{1}{\gamma}}, \tag{21}$$

where $\gamma\,(> 1)$ is the elasticity parameter. $X(t)$ represents a multiplicative demand shock, and evolves as a geometric Brownian motion:

$$dX(t) = \mu_X X(t)\,dt + \sigma_X X(t)\,dz.$$

Furthermore, let the marginal costs of production be the same for all firms and equal to c.

In Appendix C we show that whenever there are n firms active, the profit flow of firm i is equal to

$$D_n Y(t),$$

where

$$D_n = \frac{1}{n}\left(\frac{c}{n\gamma - 1}\right)^{1-\gamma}(n\gamma)^{-\gamma},$$

and $Y(t)$ follows a geometric Brownian motion with parameters μ and σ that are given by

$$\begin{aligned}\mu &= \gamma\mu_X + \frac{1}{2}\gamma(\gamma-1)\sigma_X^2,\\ \sigma &= \gamma\sigma_X.\end{aligned}$$

As a numerical illustration, consider three firms all having the possibility to enter the market after undertaking a sunk cost investment of $I = 10$. As for the other parameters, we set $\mu_X = 0.025$, $\sigma_X = 0.1$, $I = 10$, $c = 1$, and $r = 0.1$. Table 2 illustrates how the solution depends on the parameter γ. Among other things, the table shows the investment threshold values, Y_{10}, Y_{11}, and Y_{12}, for the first, second and third investor, respectively, where $\gamma = 1.25, 1.5$ and 2.

γ	1.25	1.5	2
D_1	0.534992	0.3849	0.25
D_2	0.176022	0.136083	0.09375
D_3	0.082257	0.0653272	0.0462963
μ	0.0328125	0.04125	0.06
σ	0.125	0.15	0.2
Y_{10}	1.51158	2.1546	3.54222
Y_{11}	4.55069	5.8353	8.45832
Y_{12}	14.3597	18.6358	27.8618
$V_{10}(Y_{10})$	0.0409444	0.165737	1.11924

Table 2: Example with three firms with the settings $\mu_X = 0.025$, $\sigma_X = 0.1$, $I = 10$, $c = 1$, and $r = 0.1$.

Observe that in all three cases the triggers are different, so that all equilibria are of the sequential type. The triggers go up with γ implying that the firms will invest later when the demand elasticity is higher. One of the reasons is that the output price reacts stronger to the shock process $X(t)$, if the demand elasticity is large (cf. (21)). Hence, as is also shown in the table, σ goes up with γ. Therefore, also the value of waiting with investment goes up, which results in higher trigger values. For the first investor this higher value of waiting is also illustrated by the fact that $V_{10}(Y_{10})$ increases with γ.

Another reason is that we know from microeconomic theory that markets get more competitive under a higher demand elasticity, as is reflected in a lower Lerner index (see, e.g., Tirole (1988)). Consequently, profits are smaller, which in the table gives smaller D_n values, and thus the firms will wait longer with investment.

3 Extension to n firms

In this section we analyze the n firm case. We first present some analytical results in Section 3.1. After that, in Section 3.2 we present a numerical example.

3.1 General analysis

The model is the same as in Section 2.1, but now the number of firms is n instead of three. Then for the deterministic part of the profit flow it holds that

$$D_1 > D_2 > ... > D_{n-1} > D_n > D_\infty = 0.$$

The following theorem gives (implicit) equations for the trigger values and provides an existence condition for each particular trigger. In case a trigger does not exist, then, analogous to the three firm case, simultaneous investment will occur.

Theorem 2 *Consider a new market in which n firms may enter. Let $k \in \{1, \ldots, n\}$. Define Y_{1k} to be the optimal investment trigger for the firm that invests when there are already k firms active in the market. The investment trigger Y_{1k} $(k \in \{0, \ldots, n-2\})$ is the solution of the equation*

$$\frac{Y_{1k}D_{k+1}}{r-\mu} - I = \left(\frac{Y_{1k}}{Y_{1k+1}}\right)^{\beta}\left(\frac{Y_{1k+1}D_{k+1}}{r-\mu} - I\right), \tag{22}$$

and

$$Y_{1n-1} = \frac{\beta}{\beta-1}\frac{(r-\mu)\,I}{D_n}. \tag{23}$$

Define Y_k^M to be the monopolistic investment trigger in a market where the profit multiplication factor equals D_k, i.e.

$$Y_k^M = \frac{\beta}{\beta-1}\frac{(r-\mu)\,I}{D_k}. \tag{24}$$

The investment trigger Y_{1k} exists if and only if it holds that

$$Y_{1k+1} > Y_{k+1}^M. \tag{25}$$

Equation (22) implies that at the threshold Y_{1k} a firm is indifferent between investing now and investing at the next trigger. A direct result of Theorem 2 is that the thresholds in the n firm case can be derived backwards. First, equation (23) determines Y_{1n-1}, having this value we can calculate Y_{1n-2} via (22), and so on and so forth until Y_{10}. In each step the existence of the threshold can be checked with equation (25). If the trigger Y_{1k} does not exist, i.e. equation (25) does not hold, we have that $Y_{1k} = Y_{1k+1}$.

Proposition 6 contains comparative statics results for some relevant parameters.

Proposition 6 *Let $k \in \{0, \ldots, n-1\}$. Whenever the investment trigger Y_{1k} exists, it is increasing in σ and decreasing in D_{k+1+2j} with $j \in \left\{0, 1, \ldots, \left\lfloor\frac{n-k-1}{2}\right\rfloor\right\}$. Furthermore, the threshold Y_{1k} with $k \leq n-2$ is increasing in D_{k+2+2j} with $j \in \left\{0, 1, \ldots, \left\lfloor\frac{n-k-2}{2}\right\rfloor\right\}$.*

First, this proposition shows that all investments are delayed if uncertainty increases. Second, this proposition says that an exogenous demand change has the same qualitative effect on the investment timing of the odd firms, while the direction is opposite for the even firms. So, if investments for the odd firms are delayed, then investments for the even firms are accelerated. This implies that the *accordion effect* is extended to the n firm case.

3.2 Example

In this section we continue the example of Section 2.4. Table 3 shows the different triggers for markets with one up to ten firms for the parameter values $\gamma = 1.25$, $\mu_X = 0.025$, $\sigma_X = 0.1$, $I = 10$, $c = 1$, and $r = 0.1$. Some of the highlights of this table are depicted in Figure 4 and Figure 5. In Figure 4 the trigger of the first investor is plotted as a function of the number of firms. The first three "dots" confirm Proposition 5 in that the trigger of the first investor in the three firm case lies in between the ones of the monopoly and the duopoly case. Moreover, we see that this pattern extends as the number of firms goes up in the sense that the trigger of the first investor in the $n+2$ firm case always lies in between the first investment triggers of the n and the $n+1$ firm case. This results in a sawtooth pattern with reducing length of the sawtooth as n increases. Ultimately, for n large enough the timing of the first investor becomes insensitive to the value of n.

Another implication is that first investor entry takes place earlier if the number of firms is even. Since the size of the "sawtooth" decreases as n goes up, this observation is particularly relevant in case the number of firms is small. Furthermore, Table 3 shows that the sawtooth pattern also applies if we consider the second, third or n'th investor instead of the first one, since for each k it holds that Y_{1k} first decreases, then increases, and so on and so forth, as the number of firms goes up.

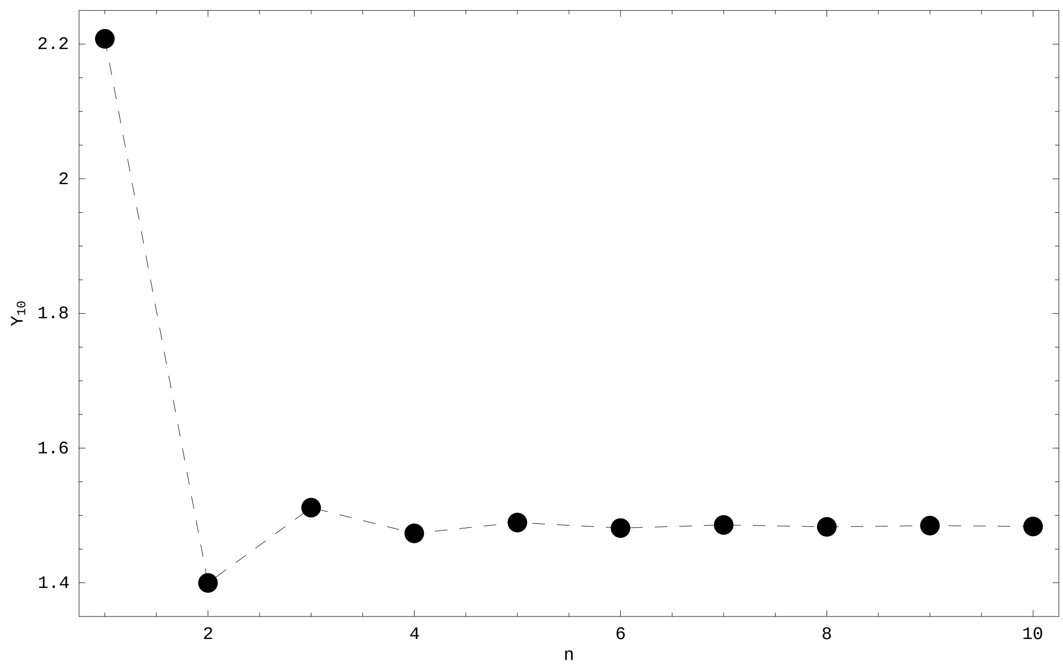

Figure 4: Trigger of the first investor as function of the number of firms.

Figure 5 shows the value of the first investor at its investment threshold as a function of the number of firms. We see that the value of the first investor at the moment of the investment converges fastly to zero as the number of firms increase. This confirms Grenadier (2002)'s result that competition drastically erodes the value of the option to wait.

n	$V_{10}\,(Y_{10})$	Y_{10}	Y_{11}	Y_{12}	Y_{13}	Y_{14}	Y_{15}	Y_{16}	Y_{17}	Y_{18}	Y_{19}
1	7.58045	2.20786									
2	0.200007	1.39978	6.71043								
3	0.0409444	1.51158	4.55069	14.3597							
4	0.0106862	1.4734	5.0604	10.3435	24.979						
5	0.00400427	1.48945	4.82759	11.7157	18.8259	38.5566					
6	0.0017281	1.48127	4.94254	10.9729	21.4652	30.0826	55.0894				
7	0.000862315	1.4859	4.87655	11.3813	19.874	34.3132	44.1668	74.5763			
8	0.000466411	1.48305	4.91693	11.1261	20.8208	31.5666	50.2397	61.1126	97.017		
9	0.00027281	1.48491	4.89045	11.2914	20.1908	33.304	46.078	69.2214	80.943	122.411	
10	0.000167957	1.48363	4.90856	11.1775	20.6179	32.0914	48.8425	63.43	91.2377	103.674	150.758

Table 3: Value of first investor and triggers for different number of firms in the market and the settings $\gamma = 1.25$, $\mu_X = 0.025$, $\sigma_X = 0.1$, $I = 10$, $c = 1$, and $r = 0.1$.

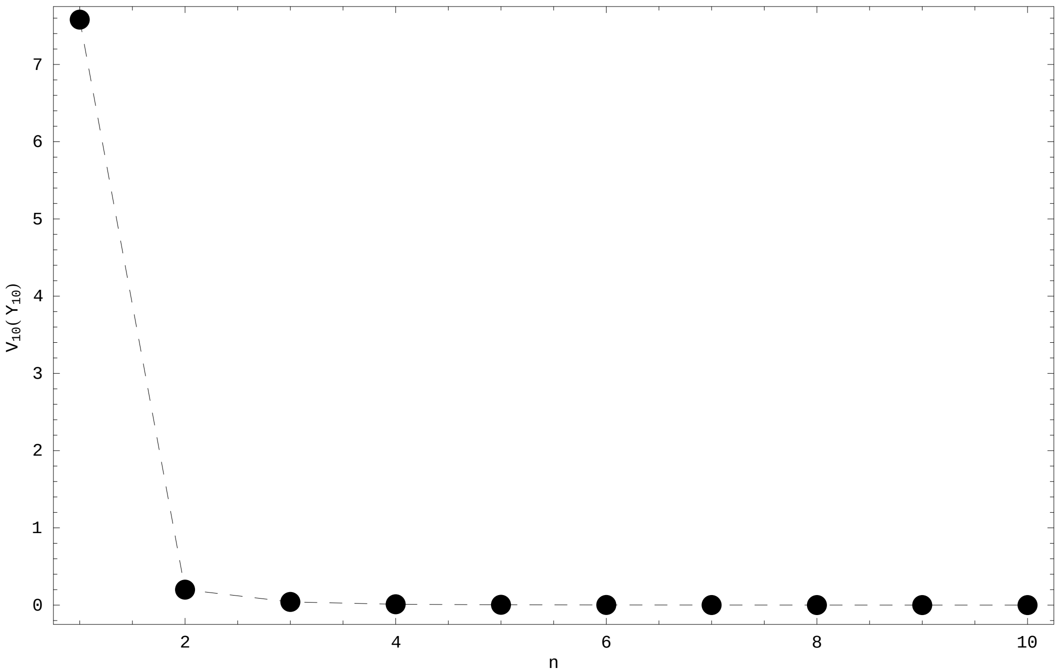

Figure 5: Value of the first investor as function of the number of firms.

4 Conclusion

The study of investment under uncertainty in an oligopolistic market structure generates contradictory effects. On the one hand it holds that an uncertain economic environment implies that there is an incentive to wait with investing. The reason is that over time more price and cost realizations become available so that investment decisions at later dates are based on more information. On the other hand we have that competition provides the incentive for a firm to invest quickly in order to prevent that another firm grabs the market share the firm could have obtained by quickly investing itself.

Until now, in the literature the above trade-off was mainly studied in a duopoly framework. The present paper extends this literature by analyzing investment under uncertainty and competition for a varying number of firms. To do so we adopted a framework where firms have the opportunity to invest just once. Upon investment the firm enters a market after which it receives some profit stream. Profit stochastically fluctuates over time and is negatively affected by entry of other firms.

This research led to the following new insights. First, in the three firm case either the firms invest sequentially over time, or the first two firms invest simultaneously, which is later on followed by entry of the third firm. Second, the effect of an exogenous demand change on investment timing leads to adverse effects on the wedges between subsequent investment thresholds. In other words, if the change in demand leads to a decreasing wedge between the first and second threshold, then the same holds for the wedge between the third and fourth, the fifth and sixth, ..., threshold, while the wedge between the second and third threshold, fourth and fifth,..., threshold will increase. This is what we call the *accordion effect.*

Third, in equilibria where firms invest sequentially, the timing of the first investor in case of $n+2$ firms always lies in between the timing of the n and $n+1$ firm case. This implies that the function describing the dependence of the timing of the first investor on the number of firms has a sawtooth form, where the length of the sawtooth reduces as the number of firms increases. One implication is that market entry occurs most early when the number of potential market entrants is small and even.

A Proofs

Proof of Proposition 1 See the proof of Proposition 6 as this is a special case. □

Proof of Proposition 2 See the proof of Proposition 6 and the proof of Theorem 2 as this is a special case. □

Proof of Proposition 3 See the proof of Proposition 6 and the proof of Theorem 2 as this is a special case. □

Proof of Proposition 4 See the proof of Proposition 6 as this is a special case. □

Proof of Theorem 1 The equilibrium is of the sequential type whenever Y_{10} exists. From Proposition 3 we know that Y_{10} exists if and only if $Y_{11} > Y_1^M$. From the proof of Theorem 2 we know that $\Delta_{11}(Y)$ is a concave function that is negative on the interval $(0, Y_{11})$ and positive on the interval (Y_{11}, Y_{12}). Furthermore, we know that $Y_1^M < Y_{12}$ as $D_1 > D_3$. This implies that the condition $Y_{11} > Y_1^M$ can be written as $\Delta_{11}\left(Y_1^M\right) < 0$. Substitution of Y_1^M in Δ_{11} (56) gives

$$\begin{aligned}
\Delta_{11}\left(Y_1^M\right) &= \frac{\frac{\beta}{\beta-1}\frac{(r-\mu)I}{D_1}D_2}{r-\mu} - I + \left(\frac{\frac{\beta}{\beta-1}\frac{(r-\mu)I}{D_1}}{\frac{\beta}{\beta-1}\frac{(r-\mu)I}{D_3}}\right)^{\beta}\left(I - \frac{\frac{\beta}{\beta-1}\frac{(r-\mu)I}{D_3}D_2}{r-\mu}\right) \\
&= \frac{\beta}{\beta-1}I\frac{D_2}{D_1} - I + \left(\frac{D_3}{D_1}\right)^{\beta}\left(I - \frac{\beta}{\beta-1}I\frac{D_2}{D_3}\right) \\
&= I\left(\frac{\beta}{\beta-1}\frac{D_2}{D_1} - 1 + \left(\frac{D_3}{D_1}\right)^{\beta}\left(1 - \frac{\beta}{\beta-1}\frac{D_2}{D_3}\right)\right) \\
&= I\left(\frac{\beta}{\beta-1}\frac{D_2}{D_1}\left(1 - \left(\frac{D_3}{D_1}\right)^{\beta-1}\right) + \left(\frac{D_3}{D_1}\right)^{\beta} - 1\right).
\end{aligned} \tag{26}$$

Thus we have that $\Delta_{11}\left(Y_1^M\right) < 0$ if and only if

$$\frac{\beta}{\beta-1}\frac{D_2}{D_1}\left(1 - \left(\frac{D_3}{D_1}\right)^{\beta-1}\right) + \left(\frac{D_3}{D_1}\right)^{\beta} - 1 < 0, \tag{27}$$

rewriting gives

$$D_2 < D_1\frac{\beta-1}{\beta}\frac{1-\left(\frac{D_3}{D_1}\right)^{\beta}}{1-\left(\frac{D_3}{D_1}\right)^{\beta-1}} = \frac{\beta-1}{\beta}\frac{D_1^{\beta}-D_3^{\beta}}{D_1^{\beta-1}-D_3^{\beta-1}} = D_2^*(\beta). \tag{28}$$

Define the function $f(\beta)$ as follows

$$f(\beta)=\frac{\beta-1}{\beta}\frac{1-y^{\beta}}{1-y^{\beta-1}}, \tag{29}$$

then we have that for $y=\frac{D_3}{D_1}$

$$D_2^*(\beta)=D_1 f(\beta), \tag{30}$$

so that if we prove that f is increasing in β we have that D_2^* is increasing β. Differentiating f with respect to β gives

$$\frac{\partial f(\beta)}{\partial\beta}=\frac{\left(1-y^{\beta-1}\right)\left(1-y^{\beta}\right)+\beta(\beta-1)y^{\beta-1}(1-y)\log(y)}{\beta^2\left(1-y^{\beta-1}\right)^2}. \tag{31}$$

Define $\phi(\beta)$ as follows

$$\phi(\beta)=\left(1-y^{\beta-1}\right)\left(1-y^{\beta}\right)+\beta(\beta-1)y^{\beta-1}(1-y)\log(y), \tag{32}$$

then it holds that $\phi(1)=0$. Furthermore,

$$\begin{aligned}\frac{\partial\phi(\beta)}{\partial\beta} &= -y^{\beta}\log(y)\left(1-y^{\beta-1}\right)-y^{\beta-1}\log(y)\left(1-y^{\beta}\right)+\beta(\beta-1)y^{\beta-1}\log(y)(1-y)\log(y)\\ &\quad+(2\beta-1)y^{\beta-1}(1-y)\log(y)\\ &= y^{\beta-1}\log(y)\left(2y^{\beta}+(1-y)\log(y)\beta^2+(1-y)(2-\log(y))\beta-2\right).\end{aligned} \tag{33}$$

Let us define $\varphi(\beta)$ as follows

$$\varphi(\beta)=2y^{\beta}+(1-y)\log(y)\beta^2+(1-y)(2-\log(y))\beta-2, \tag{34}$$

then $\varphi(1)=0$ and

$$\frac{\partial\varphi(\beta)}{\partial\beta}=2y^{\beta}\log(y)+2(1-y)\log(y)\beta+(1-y)(2-\log(y)). \tag{35}$$

Thus

$$\begin{aligned}\left.\frac{\partial\varphi(\beta)}{\partial\beta}\right|_{\beta=1} &= 2y\log(y)+2(1-y)\log(y)+(1-y)(2-\log(y))\\ &= 2-2y+\log(y)(y+1).\end{aligned} \tag{36}$$

Define the function $\xi(y)=2-2y+\log(y)(y+1)$, then $\xi(1)=0$, and

$$\begin{aligned}\frac{\partial\xi(y)}{\partial y} &= -2+\frac{1}{y}(y+1)+\log(y)\\ &= -1+\frac{1}{y}+\log(y),\end{aligned} \tag{37}$$

so that

$$\left.\frac{\partial\xi(y)}{\partial y}\right|_{y=1}=0, \tag{38}$$

and

$$\begin{aligned}\frac{\partial^2\xi(y)}{\partial y^2} &= -\frac{1}{y^2}+\frac{1}{y}\\ &= -\frac{1}{y}\left(\frac{1}{y}-1\right)\\ &< 0,\end{aligned} \tag{39}$$

since $y \in (0,1)$. This implies that $\frac{\partial \xi(y)}{\partial y}$ is strictly decreasing on $(0,1)$. Together with (38) this results in the fact that $\frac{\partial \xi(y)}{\partial y}$ is strictly positive on $(0,1)$, i.e.

$$\frac{\partial \xi(y)}{\partial y} > 0. \tag{40}$$

Therefore, $\xi(y)$ is strictly increasing on $(0,1)$ and together with $\xi(1) = 0$ we know that $\xi(y) < 0$ on $(0,1)$. This leads to

$$\left.\frac{\partial \varphi(\beta)}{\partial \beta}\right|_{\beta=1} < 0. \tag{41}$$

Furthermore,

$$\frac{\partial^2 \varphi(\beta)}{\partial \beta^2} = 2y^\beta \log^2(y) + 2(1-y)\log(y). \tag{42}$$

Thus

$$\begin{aligned} \left.\frac{\partial^2 \varphi(\beta)}{\partial \beta^2}\right|_{\beta=1} &= 2y\log^2(y) + 2(1-y)\log(y) \\ &= 2\log(y)(y\log(y) - y + 1). \end{aligned} \tag{43}$$

Define the function $\psi(y) = y\log(y) - y + 1$, then $\psi(1) = 0$, and

$$\frac{\partial \psi(y)}{\partial y} = 1 + \log(y) - 1 = \log(y) < 0. \tag{44}$$

From equation (44) and $\psi(1) = 0$ we derive that $\psi(y) > 0$ for $y \in (0,1)$. This result together with equation (43) gives

$$\left.\frac{\partial^2 \varphi(\beta)}{\partial \beta^2}\right|_{\beta=1} < 0. \tag{45}$$

Furthermore, it holds that

$$\frac{\partial^3 \varphi(\beta)}{\partial \beta^3} = 2y^\beta \log^3(y) < 0, \tag{46}$$

so that $\frac{\partial^2 \varphi(\beta)}{\partial \beta^2} < 0$ for $\beta \in (1,\infty)$. Therefore, together with equation (41) we have that $\frac{\partial \varphi(\beta)}{\partial \beta} < 0$ for $\beta \in (1,\infty)$ and this leads with $\varphi(1) = 0$ to $\varphi(\beta) < 0$ for $\beta \in (1,\infty)$. Finally, combining this result with equation (33) gives

$$\frac{\partial \phi(\beta)}{\partial \beta} = y^{\beta-1}\log(y)\varphi(\beta) > 0, \tag{47}$$

and therefore

$$\frac{\partial f(\beta)}{\partial \beta} = \frac{y^{\beta-1}\log(y)\varphi(\beta)}{\beta^2\left(1-y^{\beta-1}\right)^2} > 0. \tag{48}$$

Furthermore, it holds that

$$\begin{aligned} D_2^*(1) &= \lim_{\beta\to 1} \frac{\beta-1}{\beta}\frac{D_1^\beta - D_3^\beta}{D_1^{\beta-1} - D_3^{\beta-1}} \\ &= \lim_{\beta\to 1} \frac{(\beta-1)\left(D_1^\beta \log(D_1) - D_3^\beta \log(D_3)\right) + D_1^\beta - D_3^\beta}{\beta\left(D_1^{\beta-1}\log(D_1) - D_3^{\beta-1}\log(D_3)\right) + D_1^{\beta-1} - D_3^{\beta-1}} \\ &= \frac{D_1 - D_3}{\log(D_1) - \log(D_3)}, \end{aligned} \tag{49}$$

and

$$\begin{aligned} & D_2^*(1) > D_3, && (50)\\ \iff & \frac{D_1 - D_3}{\log(D_1) - \log(D_3)} > D_3,\\ \iff & D_1 - D_3 > D_3(\log(D_1) - \log(D_3)),\\ \iff & D_1 + D_3\left(\log\left(\frac{D_3}{D_1}\right) - 1\right) > 0,\\ \iff & 1 + \frac{D_3}{D_1}\left(\log\left(\frac{D_3}{D_1}\right) - 1\right) > 0,\end{aligned}$$

which holds if and only if $\phi(y) > 0$ for $y \in (0,1)$, with $\phi(y) = 1 + y(\log(y) - 1)$. We have that $\phi(1) = 0$ and

$$\frac{d\phi(y)}{dy} = \log(y) - 1 + y\frac{1}{y} = \log(y) < 0, \tag{51}$$

for $y \in (0,1)$, so that indeed equation (50) holds. The last thing to show is the limit of $D_2^*(\beta)$ when β goes to infinity:

$$\begin{aligned}\lim_{\beta\to\infty} D_2^*(\beta) &= \lim_{\beta\to\infty} \frac{\beta-1}{\beta}\frac{D_1^\beta - D_3^\beta}{D_1^{\beta-1} - D_3^{\beta-1}}\\ &= \lim_{\beta\to\infty} D_1\frac{\beta-1}{\beta}\frac{1-\left(\frac{D_3}{D_1}\right)^\beta}{1-\left(\frac{D_3}{D_1}\right)^{\beta-1}}\\ &= D_1, \end{aligned} \tag{52}$$

which finishes the proof. □

Proof of Theorem 2 The threshold for the firm that invests as n-th firm is equal to

$$Y_{1n-1} = \frac{\beta}{\beta-1}\frac{(r-\mu)I}{D_n}. \tag{53}$$

The threshold Y_{1k} (for $k \in \{0, 1, \dots, n-2\}$) is the solution of the following equation (the left hand side is the value of the firm that invests when there are already k firms active, and the right hand side is equal to the value of the firm if the firm waits with investing, which is equal to the value of the firm that invests as n-th and last firm, since the rent equalization principle holds)

$$\frac{Y_{1k}D_{k+1}}{r-\mu} - I + \sum_{j=k+1}^{n-1}\left(\frac{Y_{1k}}{Y_{1j}}\right)^\beta \frac{Y_{1j}(D_{j+1} - D_j)}{r-\mu} = \left(\frac{Y_{1k}}{Y_{1n-1}}\right)^\beta\left(\frac{Y_{1n-1}D_{n-1}}{r-\mu} - I\right). \tag{54}$$

In the same fashion we can write down the equation for Y_{1k+1}

$$\frac{Y_{1k+1}D_{k+2}}{r-\mu} - I + \sum_{j=k+2}^{n-1}\left(\frac{Y_{1k+1}}{Y_{1j}}\right)^\beta \frac{Y_{1j}(D_{j+1} - D_j)}{r-\mu} = \left(\frac{Y_{1k+1}}{Y_{1n-1}}\right)^\beta\left(\frac{Y_{1n-1}D_{n-1}}{r-\mu} - I\right). \tag{55}$$

Multiplying (55) by $\left(\frac{Y_{1k}}{Y_{1k+1}}\right)^\beta$ and substitution of the result in equation (54) gives equation (22). Define $\Delta_{1k}(Y)$ as follows

$$\Delta_{1k}(Y) = \frac{YD_{k+1}}{r-\mu} - I + \left(\frac{Y}{Y_{1k+1}}\right)^\beta\left(I - \frac{Y_{1k+1}D_{k+1}}{r-\mu}\right). \tag{56}$$

Then the investment trigger Y_{1k} is the smallest positive zero point of the function $\Delta_{1k}(Y)$. Furthermore, it holds that $Y_{1k} < Y_{1k+1}$ if we show that

$$\Delta_{1k}(0) < 0, \tag{57}$$

$$\Delta_{1k}(Y_{1k+1}) = 0, \tag{58}$$

$$\left.\frac{\partial \Delta_{1k}(Y)}{\partial Y}\right|_{Y=Y_{1k+1}} < 0, \tag{59}$$

$$\frac{\partial^2 \Delta_{1k}(Y)}{\partial Y^2} < 0,\ \forall Y \geq 0. \tag{60}$$

The first and the second statements are straight forward to prove, as

$$\Delta_{1k}(0) = -I < 0, \tag{61}$$

and

$$\begin{aligned}\Delta_{1k}(Y_{1k+1}) &= \frac{Y_{1k+1}D_{k+1}}{r-\mu} - I + \left(\frac{Y_{1k+1}}{Y_{1k+1}}\right)^{\beta}\left(I - \frac{Y_{1k+1}D_{k+1}}{r-\mu}\right)\\ &= \frac{Y_{1k+1}D_{k+1}}{r-\mu} - I + I - \frac{Y_{1k+1}D_{k+1}}{r-\mu}\\ &= 0.\end{aligned} \tag{62}$$

The derivative of Δ_{1k} to Y is given by

$$\frac{\partial \Delta_{1k}(Y)}{\partial Y} = \frac{D_{k+1}}{r-\mu} + \beta Y^{\beta-1}Y_{k+1}^{-\beta}\left(I - \frac{Y_{1k+1}D_{k+1}}{r-\mu}\right), \tag{63}$$

thus

$$\begin{aligned}\left.\frac{\partial \Delta_{1k}(Y)}{\partial Y}\right|_{Y=Y_{1k+1}} &= \frac{D_{k+1}}{r-\mu} + \beta Y_{k+1}^{\beta-1}Y_{k+1}^{-\beta}\left(I - \frac{Y_{1k+1}D_{k+1}}{r-\mu}\right)\\ &= \frac{D_{k+1}}{r-\mu} + \frac{\beta}{Y_{1k+1}}\left(I - \frac{Y_{1k+1}D_{k+1}}{r-\mu}\right)\\ &= \frac{\beta}{Y_{1k+1}}\left(\frac{1-\beta}{\beta}\frac{Y_{1k+1}D_{k+1}}{r-\mu} + I\right).\end{aligned} \tag{64}$$

Therefore it holds that

$$\left.\frac{\partial \Delta_{1k}(Y)}{\partial Y}\right|_{Y=Y_{1k+1}} < 0 \tag{65}$$

if and only if

$$\frac{1-\beta}{\beta}\frac{Y_{1k+1}D_{k+1}}{r-\mu} + I < 0, \tag{66}$$

or

$$Y_{1k+1} > \frac{\beta}{\beta-1}\frac{(r-\mu)I}{D_{k+1}} = Y_{k+1}^{M}. \tag{67}$$

Furthermore, we have that

$$\frac{\partial^2 \Delta_{1k}(Y)}{\partial Y^2} = \beta(\beta-1)Y^{\beta-2}Y_{k+1}^{-\beta}\left(I - \frac{Y_{1k+1}D_{k+1}}{r-\mu}\right) < 0, \tag{68}$$

since $Y_{1k+1} > \frac{(r-\mu)I}{D_{k+1}}$. $\square$

Proof of Proposition 5 For the demonstration that $Y_{10} < Y_M$, see the proof of Proposition 6 as this is a special case. We know that $Y_{10} > Y_P$ if $\Delta_{10}(Y_P) < 0$, where Y_P verifies

$$\frac{Y_P D_1}{r-\mu} - I = \left(\frac{Y_P}{Y_F}\right)^{\beta}\left(\frac{Y_F D_1}{r-\mu} - I\right). \tag{69}$$

From equation (56),

$$\begin{aligned}\Delta_{10}(Y_P) &= \frac{Y_P D_1}{r-\mu} - I + \left(\frac{Y_P}{Y_{11}}\right)^{\beta}\left(I - \frac{Y_{11} D_1}{r-\mu}\right)\\ &= \left(\frac{Y_P}{Y_F}\right)^{\beta}\left(\frac{Y_F D_1}{r-\mu} - I\right) + \left(\frac{Y_P}{Y_{11}}\right)^{\beta}\left(I - \frac{Y_{11} D_1}{r-\mu}\right)\\ &= Y_P^{\beta}\left(Y_F^{-\beta}\left(\frac{Y_F D_1}{r-\mu} - I\right) - Y_{11}^{-\beta}\left(\frac{Y_{11} D_1}{r-\mu} - I\right)\right).\end{aligned} \tag{70}$$

Define the function ζ as follows for $Y_{11} \in (Y_M, Y_{12})$

$$\zeta(Y_{11}) = Y_P^{\beta}\left(Y_F^{-\beta}\left(\frac{Y_F D_1}{r-\mu} - I\right) - Y_{11}^{-\beta}\left(\frac{Y_{11} D_1}{r-\mu} - I\right)\right). \tag{71}$$

It holds that $\zeta(Y_F) = 0$ and

$$\begin{aligned}\frac{\partial \zeta(Y_{11})}{\partial Y_{11}} &= Y_P^{\beta}\left(\beta Y_{11}^{-\beta-1}\left(\frac{Y_{11} D_1}{r-\mu} - I\right) - Y_{11}^{-\beta}\frac{D_1}{r-\mu}\right)\\ &= Y_P^{\beta}\left(Y_{11}^{-\beta}\left((\beta-1)\frac{D_1}{r-\mu} - \frac{\beta I}{Y_{11}}\right)\right) > 0,\end{aligned} \tag{72}$$

since we are in the case that $Y_{11} > Y_M$. Besides, $Y_{11} < Y_F$ if we show that $\Delta_{11}(Y_F) > 0$.

$$\begin{aligned}\Delta_{11}(Y_F) &= \frac{Y_F D_2}{r-\mu} - I + \left(\frac{Y_F}{Y_{12}}\right)^{\beta}\left(I - \frac{Y_{12} D_2}{r-\mu}\right)\\ &= \frac{\beta}{\beta-1}I - I + \left(\frac{D_3}{D_2}\right)^{\beta}\left(I - \frac{\beta}{\beta-1}\frac{D_2}{D_3}I\right)\\ &= I\left[\left(\frac{\beta}{\beta-1} - 1\right) - \left(\frac{D_3}{D_2}\right)^{\beta-1}\left(\frac{\beta}{\beta-1} - \frac{D_3}{D_2}\right)\right]\\ &= I\left[\left(\frac{\beta}{\beta-1} - 1\right) - z^{\beta-1}\left(\frac{\beta}{\beta-1} - z\right)\right],\end{aligned} \tag{73}$$

where $z = \dfrac{D_3}{D_2} \in (0,1)$ and

$$\begin{aligned}\frac{\partial \Delta_{11}(Y_F)}{\partial z} &= -I\left[(\beta-1)z^{\beta-2}\left(\frac{\beta}{\beta-1} - z\right) - z^{\beta-1}\right]\\ &= -I\beta z^{\beta-1}\left(\frac{1}{z} - 1\right) < 0.\end{aligned} \tag{74}$$

As $z \to 1$ ($D_3 \to D_2$), $\Delta_{11}(Y_F) \to 0^+$ hence $\Delta_{11}(Y_F)$ is always strictly positive and $Y_{11} < Y_F$. $\zeta(Y_F) = 0$, $\frac{\partial \zeta(Y_{11})}{\partial Y_{11}} > 0$ for $Y_{11} \in (Y_M, Y_{12})$ and $Y_{11} < Y_F$ imply $\zeta(Y_{11}) < 0$ and $\Delta_{10}(Y_P) < 0$, so that $Y_{10} > Y_P$. $\square$

Proof of Proposition 6 Let the function $\Delta_{1k}(Y)$ be defined by equation (56), so that

$$\Delta_{1k}(Y_{1k}) = 0. \tag{75}$$

Differentiating to D_j, with $j \in \{k+1, \ldots, n\}$ gives

$$\left.\frac{\partial \Delta_{1k}(Y)}{\partial Y}\right|_{Y=Y_{1k}} \frac{\partial Y_{1k}}{\partial D_j} + \left.\frac{\partial \Delta_{1k}(Y)}{\partial D_j}\right|_{Y=Y_{1k}} = 0. \tag{76}$$

Rewriting gives

$$\frac{\partial Y_{1k}}{\partial D_j} = -\frac{\left.\frac{\partial \Delta_{1k}(Y)}{\partial D_j}\right|_{Y=Y_{1k}}}{\left.\frac{\partial \Delta_{1k}(Y)}{\partial Y}\right|_{Y=Y_{1k}}}. \tag{77}$$

Let us first derive the sign of the derivative of Δ_{1k} with respect to Y at $Y = Y_{1k}$

$$\begin{aligned}\left.\frac{\partial \Delta_{1k}(Y)}{\partial Y}\right|_{Y=Y_{1k}} &= \frac{D_{k+1}}{r-\mu} + \beta Y_{1k}^{\beta-1} Y_{1k+1}^{-\beta}\left(I - \frac{Y_{1k+1}D_{k+1}}{r-\mu}\right) \\ &= \frac{\beta}{Y_{1k}}\left(\frac{1}{\beta}\frac{Y_{1k}D_{k+1}}{r-\mu} + \left(\frac{Y_{1k}}{Y_{1k+1}}\right)^{\beta}\left(I - \frac{Y_{1k+1}D_{k+1}}{r-\mu}\right)\right),\end{aligned} \tag{78}$$

substitution of (22) gives

$$\begin{aligned}\left.\frac{\partial \Delta_{1k}(Y)}{\partial Y}\right|_{Y=Y_{1k}} &= \frac{\beta}{Y_{1k}}\left(\frac{1}{\beta}\frac{Y_{1k}D_{k+1}}{r-\mu} + I - \frac{Y_{1k}D_{k+1}}{r-\mu}\right) \\ &= \frac{\beta}{Y_{1k}}\left(\frac{1-\beta}{\beta}\frac{Y_{1k}D_{k+1}}{r-\mu} + I\right)\end{aligned} \tag{79}$$

so that it holds that

$$\left.\frac{\partial \Delta_{1k}(Y)}{\partial Y}\right|_{Y=Y_{1k}} > 0, \tag{80}$$

if and only if

$$\frac{1-\beta}{\beta}\frac{Y_{1k}D_{k+1}}{r-\mu} + I > 0. \tag{81}$$

Rewriting (81) gives

$$Y_{1k} < \frac{\beta}{\beta-1}\frac{(r-\mu)\,I}{D_{k+1}} = Y_{k+1}^{M}. \tag{82}$$

Equation (82) holds if we can show that $\Delta_{1k}\left(Y_{k+1}^{M}\right) > 0$,

$$\begin{aligned}\Delta_{1k}\left(Y_{k+1}^{M}\right) &= \frac{\frac{\beta}{\beta-1}\frac{(r-\mu)I}{D_{k+1}}D_{k+1}}{r-\mu} - I + \left(\frac{Y_{k+1}^{M}}{Y_{1k+1}}\right)^{\beta}\left(I - \frac{Y_{1k+1}D_{k+1}}{r-\mu}\right) \\ &= \frac{I}{\beta-1} + \left(Y_{k+1}^{M}\right)^{\beta}\left(Y_{1k+1}^{-\beta}I - Y_{1k+1}^{1-\beta}\frac{D_{k+1}}{r-\mu}\right).\end{aligned} \tag{83}$$

Define the function ψ as follows for $Y \in \left(Y_{k+1}^{M}, \infty\right)$

$$\psi(Y) = \frac{I}{\beta-1} + \left(Y_{k+1}^{M}\right)^{\beta}\left(Y^{-\beta}I - Y^{1-\beta}\frac{D_{k+1}}{r-\mu}\right). \tag{84}$$

It holds that $\psi\left(Y_{k+1}^{M}\right) = 0$ and

$$\begin{aligned}\frac{\partial \psi(Y)}{\partial Y} &= \left(Y_{k+1}^{M}\right)^{\beta}\left(-\beta Y^{-\beta-1}I - (1-\beta)\,Y^{-\beta}\frac{D_{k+1}}{r-\mu}\right) \\ &= \left(Y_{k+1}^{M}\right)^{\beta}Y^{-\beta}\left(-\frac{\beta I}{Y} - (1-\beta)\frac{D_{k+1}}{r-\mu}\right) > 0,\end{aligned} \tag{85}$$

since we are in the case that $Y > Y_{k+1}^M$. The last two observations imply that $\psi(Y) > 0$ for $Y \in \left(Y_{k+1}^M, \infty\right)$. Thus $\Delta_{1k}\left(Y_{k+1}^M\right) > 0$, so that $Y_{1k} < Y_{k+1}^M$ and $\left.\frac{\partial \Delta_{1k}(Y)}{\partial Y}\right|_{Y=Y_{1k}} > 0$. The derivative of Δ_{1k} with respect to D_{k+1} is equal to

$$\left.\frac{\partial \Delta_{1k}(Y)}{\partial D_{k+1}}\right|_{Y=Y_{1k}} = \frac{Y_{1k}}{r-\mu} - \left(\frac{Y_{1k}}{Y_{1k+1}}\right)^{\beta} \frac{Y_{1k+1}}{r-\mu}$$

$$= \frac{1}{D_{k+1}} \left(\frac{Y_{1k} D_{k+1}}{r-\mu} - \left(\frac{Y_{1k}}{Y_{1k+1}}\right)^{\beta} \frac{Y_{1k+1} D_{k+1}}{r-\mu}\right). \tag{86}$$

Substitution of equation (22) gives

$$\left.\frac{\partial \Delta_{1k}(Y)}{\partial D_{k+1}}\right|_{Y=Y_{1k}} = \frac{I}{D_{k+1}} \left(1 - \left(\frac{Y_{1k}}{Y_{1k+1}}\right)^{\beta}\right) > 0, \tag{87}$$

which implies that Y_{1k} is decreasing in D_{k+1}. Next, we take the derivative of Δ_{1k} with respect to D_j for $j > k+1$

$$\left.\frac{\partial \Delta_{1k}(Y)}{\partial D_j}\right|_{Y=Y_{1k}} = -\beta Y_{1k}^{\beta} Y_{1k+1}^{-\beta-1} I \frac{\partial Y_{1k+1}}{\partial D_j} - (1-\beta) Y_{1k}^{\beta} Y_{1k+1}^{-\beta} \frac{D_{k+1}}{r-\mu} \frac{\partial Y_{1k+1}}{\partial D_j}$$

$$= Y_{1k}^{\beta} Y_{1k+1}^{-\beta} \frac{\partial Y_{1k+1}}{\partial D_j} \left(-\frac{\beta I}{Y_{1k+1}} - (1-\beta) \frac{D_{k+1}}{r-\mu}\right). \tag{88}$$

Since Y_{1k} exists by assumption, we have that $Y_{1k+1} > Y_{k+1}^M$, so that

$$sign\left(\left.\frac{\partial \Delta_{1k}(Y)}{\partial D_j}\right|_{Y=Y_{1k}}\right) = sign\left(\frac{\partial Y_{1k+1}}{\partial D_j}\right). \tag{89}$$

Therefore the investment trigger Y_{1k} is increasing in D_j if and only if Y_{1k+1} is decreasing in D_j. The last part of this proof deals with the effect of σ on Y_{1k}. We have that

$$\left.\frac{\partial \Delta_{1k}(Y)}{\partial \beta}\right|_{Y=Y_{1k}} = -\left(\frac{Y_{1k}}{Y_{1k+1}}\right)^{\beta} \frac{D_{k+1}}{r-\mu} \frac{\partial Y_{1k+1}}{\partial \beta}$$

$$+ \left(\frac{Y_{1k}}{Y_{1k+1}}\right)^{\beta} \left(I - \frac{Y_{1k+1} D_{k+1}}{r-\mu}\right) \left(\log\left(\frac{Y_{1k}}{Y_{1k+1}}\right) - \frac{\beta}{Y_{1k+1}} \frac{\partial Y_{1k+1}}{\partial \beta}\right)$$

$$= \left(\frac{Y_{1k}}{Y_{1k+1}}\right)^{\beta} \left(\frac{\partial Y_{1k+1}}{\partial \beta} \left(-\frac{\beta I}{Y_{1k+1}} - (1-\beta) \frac{D_{k+1}}{r-\mu}\right)\right.$$

$$\left. + \left(I - \frac{Y_{1k+1} D_{k+1}}{r-\mu}\right) \log\left(\frac{Y_{1k}}{Y_{1k+1}}\right)\right), \tag{90}$$

so that $\left.\frac{\partial \Delta_{1k}(Y)}{\partial \beta}\right|_{Y=Y_{1k}}$ is positive if and only if $\eta(Y) > 0$ for $Y \in (Y_{1k}, Y_{1k+1})$ with

$$\eta(Y) = \frac{\partial Y_{1k+1}}{\partial \beta} \left(-\frac{\beta I}{Y_{1k+1}} - (1-\beta) \frac{D_{k+1}}{r-\mu}\right) + \left(I - \frac{Y_{1k+1} D_{k+1}}{r-\mu}\right) \log\left(\frac{Y}{Y_{1k+1}}\right). \tag{91}$$

Whenever Y_{1k} equals Y_{1k+1} we have that $Y_{1k} = Y_{1k+1} = Y_{k+1}^M$, so that

$$\eta(Y_{1k+1}) = \frac{\partial Y_{1k+1}}{\partial \beta} \left(-\frac{\beta I}{Y_{k+1}^M} - (1-\beta) \frac{D_{k+1}}{r-\mu}\right) + \left(I - \frac{Y_{1k+1} D_1}{r-\mu}\right) \log\left(\frac{Y_{1k+1}}{Y_{1k+1}}\right)$$

$$= 0. \tag{92}$$

Furthermore, we have that

$$\frac{\partial \eta (Y)}{\partial Y} = \left(I - \frac{Y_{1k+1} D_{k+1}}{r-\mu} \right) \frac{1}{Y} < 0, \tag{93}$$

which implies together with (92) that $\eta(Y_{1k}) > 0$ for $Y_{1k} \in (Y_{1k}, Y_{1k+1})$. This results in the fact that

$$\left. \frac{\partial \Delta_{1k} (Y)}{\partial \beta} \right|_{Y=Y_{1k}} > 0, \tag{94}$$

which leads to the Y_{1k} being decreasing in β and increasing in σ, since it holds that $\frac{\partial \beta}{\partial \sigma} < 0$ (see Huisman (2001, Chapter 7) for a proof). □

B Equilibrium Strategies

In this appendix we show that the equilibrium concept for stochastic timing games as described in Thijssen (2004, Chapter 4), Huisman (2001, Chapter 7), and Thijssen *et al.* (2002) can be extended from 2 firms to n firms. Please note that the proposed extension is only valid for new market models. Fudenberg and Tirole (1985) state (and show) in their Section 5 that the possibility of preemption need not enforce rent equalization when there are more than two firms in case of an existing market model. In this paper we analyze a new market model, and that is why it is possible for us to derive symmetric equilibrium strategies for the investment game with more than two firms and that the rent equalization principle still holds for more than two firms.

Whenever the initial value of the geometric Brownian motion in the investment timing game is lower than the lowest investment threshold, firms will invest at those times at which the geometric Brownian motion hits a threshold for the first time. For example in the three firm sequential case, the first investment will take place at time $T_{10} = \inf (t \,|Y(t) \geq Y_{10})$, the second investment at time $T_{11} = \inf (t \,|Y(t) \geq Y_{11})$, and the third investment at time $T_{12} = \inf (t \,|Y(t) \geq Y_{12})$. Each firm becomes first, second or third investor with probability $\frac{1}{3}$, and the probability of a mistake, i.e. two firms investing at the same time, equals zero.

From the existing literature on stochastic timing games we know that the most important thing is to derive the intensity functions α_i if the initial value of the geometric Brownian motion is larger than the lowest investment threshold. Below we will derive the equation that defines these intensity functions for the n firm case. Furthermore, we will use this equation to derive the intensity functions in the case with two firms and in the case with three firms.

Let us assume that there are n firms in the market and that none of them is active yet. Firm i invests with probability α_i. Define $p_n(k,i)$ as the probability that out of the $(n-1)$ firms besides firm i, k firms invest. So it holds that

$$p_n (0,i) = \prod_{j \neq i} (1-\alpha_j), \tag{95}$$

and

$$p_n (n-1,i) = \prod_{j \neq i} \alpha_j. \tag{96}$$

Consider the situation that no firm has invested yet.[5] The value of firm i, denoted by Ψ_i, equals (we skip the dependence of Y for the moment)

$$\begin{aligned}\Psi_i &= \alpha_i \sum_{k=0}^{n-1} p_n(k,i) V_{k+10} + (1-\alpha_i) \sum_{k=1}^{n-1} p_n(k,i) W \\ &\quad + (1-\alpha_i) p_n(0,i) \Psi_i,\end{aligned} \tag{97}$$

where $V_{(k+1)0}$ and W are defined as in Section 2. Thus W denotes the value of the firm if the firm does not invest in this investment round and ends up in the next investment game. Note that W is independent of k as it is equal to the expected value of the firm in the investment game that is played after this investment game is finished. This expected value is equal to the value of the firm that invests as last firm, thus

$$W(Y) = \left(\frac{Y}{Y_{1(n-1)}}\right)^{\beta} \left(\frac{Y_{1(n-1)} D_n}{r-\mu} - I\right). \tag{98}$$

Rewriting (97) gives

$$\Psi_i = \frac{\alpha_i \sum_{k=0}^{n-1} p_n(k,i) V_{(k+1)0} + (1-\alpha_i) \sum_{k=1}^{n-1} p_n(k,i) W}{1-(1-\alpha_i) p_n(0,i)}. \tag{99}$$

Taking the derivative with respect to α_i gives

$$\begin{aligned}\frac{\partial \Psi_i}{\partial \alpha_i} &= \frac{(1-(1-\alpha_i) p_n(0,i)) \left(\sum_{k=0}^{n-1} p_n(k,i) V_{(k+1)0} - \sum_{k=1}^{n-1} p_n(k,i) W\right)}{(1-(1-\alpha_i) p_n(0,i))^2} \\ &\quad - \frac{\left(\alpha_i \sum_{k=0}^{n-1} p_n(k,i) V_{(k+1)0} + (1-\alpha_i) \sum_{k=1}^{n-1} p_n(k,i) W\right) p_n(0,i)}{(1-(1-\alpha_i) p_n(0,i))^2} \\ &= \frac{(1-p_n(0,i)) \sum_{k=0}^{n-1} p_n(k,i) V_{(k+1)0} - \sum_{k=1}^{n-1} p_n(k,i) W}{(1-(1-\alpha_i) p_n(0,i))^2}.\end{aligned} \tag{100}$$

Since the firms are symmetric we let the firms play symmetric equilibrium strategies, i.e. $\alpha_i = \alpha_n$. Next we enforce that the derivative (100) is equal to zero. This implies that we have an equilibrium as firm i can not improve by deviating from playing the strategy α. Then we have that

$$p_n(k,i) = \binom{n-1}{k} \alpha_n^k (1-\alpha_n)^{n-k-1}, \tag{101}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{102}$$

Thus, solving the following equation gives α_n :

$$\left(1-(1-\alpha_n)^{n-1}\right) \sum_{k=0}^{n-1} \binom{n-1}{k} \alpha_n^k (1-\alpha_n)^{n-k-1} V_{(k+1)0} - \sum_{k=1}^{n-1} \binom{n-1}{k} \alpha_n^k (1-\alpha_n)^{n-k-1} W = 0. \tag{103}$$

[5] This is without loss of generality. If for example m firms have already invested then n should be replaced by $n-m$ in the analysis and the value functions $V_{(k+1)0}$ by $V_{(k+1)m}$.

B.1 Two firms

For $n = 2$ equation (103) becomes

$$\begin{aligned} & \left(1-\left(1-\alpha_2\right)\right) \sum_{k=0}^{1} \binom{1}{k} \alpha_2^k \left(1-\alpha_2\right)^{1-k} V_{(k+1)0} - \alpha_2 W \\ = \; & \alpha_2 \left(\left(1-\alpha_2\right) V_{10} + \alpha_2 V_{20} - W\right) = 0, \end{aligned} \tag{104}$$

so that $\alpha_2 = 0$ or

$$\alpha_2 = \frac{V_{10} - W}{V_{10} - V_{20}}. \tag{105}$$

Since we only consider symmetric equilibria, $\alpha_2 = 0$ is not a candidate. Equation (105) is equivalent to the probability that was found in Huisman (2001, Chapter 7). The probability that the two firms invest simultaneously is equal to (each firm invests with probability α_2 and with probability $\left(1-\alpha_2\right)^2$ the game is repeated)

$$\frac{\alpha_2^2}{1-\left(1-\alpha_2\right)^2} = \frac{\alpha_2}{2-\alpha_2}, \tag{106}$$

and the probability that only one firm invests is given by

$$2\frac{\alpha_2\left(1-\alpha_2\right)}{1-\left(1-\alpha_2\right)^2} = \frac{2-2\alpha_2}{2-\alpha_2}. \tag{107}$$

B.2 Three firms

For $n = 3$ equation (103) becomes

$$\begin{aligned} & \left(1-\left(1-\alpha_3\right)^2\right) \sum_{k=0}^{2} \binom{2}{k} \alpha_3^k \left(1-\alpha_3\right)^{2-k} V_{(k+1)0} - \sum_{k=1}^{2} \binom{2}{k} \alpha_3^k \left(1-\alpha_3\right)^{2-k} W \\ = \; & \alpha_3\left(2-\alpha_3\right)\left(\left(1-\alpha_3\right)^2 V_{10} + 2\alpha_3\left(1-\alpha_3\right) V_{20} + \alpha_3^2 V_{30}\right) - 2\alpha_3\left(1-\alpha_3\right) W - \alpha_3^2 W \\ = \; & \alpha_3\left(2-\alpha_3\right)\left(\left(1-\alpha_3\right)^2 V_{10} + 2\alpha_3\left(1-\alpha_3\right) V_{20} + \alpha_3^2 V_{30}\right) - \alpha_3\left(2-\alpha_3\right) W \\ = \; & \alpha_3\left(2-\alpha_3\right)\left(\left(1-\alpha_3\right)^2 V_{10} + 2\alpha_3\left(1-\alpha_3\right) V_{20} + \alpha_3^2 V_{30} - W\right) \\ = \; & \alpha_3\left(2-\alpha_3\right)\left(\left(V_{10} - 2V_{20} + V_{30}\right)\alpha_3^2 - 2\left(V_{10} - V_{20}\right)\alpha_3 + V_{10} - W\right) = 0. \end{aligned} \tag{108}$$

So that $\alpha_3 = 0$, $\alpha_3 = 2$, or

$$\alpha_3 = \frac{V_{10} - V_{20} \pm \sqrt{\left(V_{10} - V_{20}\right)^2 - \left(V_{10} - 2V_{20} + V_{30}\right)\left(V_{10} - W\right)}}{V_{10} - 2V_{20} + V_{30}}.$$

Since we only consider symmetric equilibria and α_3 must be in the interval $[0, 1]$ the solutions $\alpha_3 = 0$ and $\alpha_3 = 2$ drop out. Question is which of the other two roots we have to take. Therefore, we make the following definitions

$$\begin{aligned} a &= V_{10} - V_{20} \geq 0, \\ b &= V_{20} - V_{30} \geq 0, \\ c &= V_{10} - W \geq 0, \end{aligned}$$

so that

$$\begin{aligned}\alpha_3^+ &= \frac{a+\sqrt{a^2-(a-b)\,c}}{a-b},\\ \alpha_3^- &= \frac{a-\sqrt{a^2-(a-b)\,c}}{a-b}.\end{aligned}$$

The sign of $a-b$ depends on the values of the parameters. Assume that $a-b$ is negative, then $\alpha_3^+<0$ and $\alpha_3^-\in[0,1]$, as

$$\begin{aligned}& a-\sqrt{a^2-(a-b)\,c}\le 0\\ \Longleftrightarrow\quad & \sqrt{a^2-(a-b)\,c}\ge a\\ \Longleftrightarrow\quad & a^2-(a-b)\,c\ge a^2\\ \Longleftrightarrow\quad & -(a-b)\,c\ge 0,\end{aligned}$$

and

$$\begin{aligned}& \frac{a-\sqrt{a^2-(a-b)\,c}}{a-b}\le 1,\\ \Longleftrightarrow\quad & a-\sqrt{a^2-(a-b)\,c}\ge a-b,\\ \Longleftrightarrow\quad & \sqrt{a^2-(a-b)\,c}\le b,\\ \Longleftrightarrow\quad & a^2-(a-b)\,c\le b^2,\\ \Longleftrightarrow\quad & (a-b)\,(a+b-c)\le 0,\end{aligned}$$

which holds since $a+b-c=W-V_{30}\ge 0$ and $a-b<0$. If $a-b=0$ we have that (using L'Hôpital for α_3^-)

$$\begin{aligned}\alpha_3^+ &\to \infty,\\ \alpha_3^- &= -\frac{1}{2}\left(a^2-(a-b)\,c\right)^{-\frac{1}{2}}\cdot -c=\frac{c}{2a}>0.\end{aligned}$$

Furthermore, we have that $\alpha_3^-\le 1$, since

$$\begin{aligned}& c\le 2a,\\ \Longleftrightarrow\quad & V_{10}-W\le 2V_{10}-2V_{20},\\ \Longleftrightarrow\quad & V_{20}-V_{10}+V_{20}-W\le 0.\end{aligned}$$

Last we consider the case that $a-b>0$. Then the root exists as for $Y\in[Y_{10},Y_{11}]$ we have that $a-c=W-V_{20}\ge 0$ so that $a^2-(a-b)\,c=$

$$\begin{aligned}& a^2-(a-b)\,c\\ =\quad & a\,(a-c)+bc\ge 0,\end{aligned}$$

and for $Y\in[Y_{11},Y_{12}]$ we have that $V_{10}=V_{20}$, so that $a=0$ and

$$\begin{aligned}& a^2-(a-b)\,c\\ =\quad & bc\ge 0.\end{aligned}$$

Then $\alpha_3^+ > 1$ as

$$\begin{aligned} & \frac{a+\sqrt{a^2-(a-b)\,c}}{a-b} > 1, \\ \iff\quad & a+\sqrt{a^2-(a-b)\,c} > a-b, \\ \iff\quad & \sqrt{a^2-(a-b)\,c} > -b, \end{aligned}$$

and $\alpha_3^- \in [0,1]$ as it holds that

$$\begin{aligned} & a-\sqrt{a^2-(a-b)\,c} \geq 0, \\ \iff\quad & \sqrt{a^2-(a-b)\,c} \leq a, \\ \iff\quad & a^2-(a-b)\,c \leq a^2, \\ \iff\quad & -(a-b)\,c \leq 0, \end{aligned}$$

and

$$\begin{aligned} & \frac{a-\sqrt{a^2-(a-b)\,c}}{a-b} \leq 1, \\ \iff\quad & a-\sqrt{a^2-(a-b)\,c} \leq a-b, \\ \iff\quad & \sqrt{a^2-(a-b)\,c} \geq b, \\ \iff\quad & a^2-(a-b)\,c \geq b^2, \\ \iff\quad & (a-b)\,(a+b-c) \geq 0. \end{aligned}$$

Conclusion is that α_3 is equal to α_3^-, i.e.

$$\alpha_3 = \frac{V_{10}-V_{20}-\sqrt{(V_{10}-V_{20})^2-(V_{10}-2V_{20}+V_{30})\,(V_{10}-W)}}{V_{10}-2V_{20}+V_{30}}. \tag{109}$$

Equation (109) is important if $y > Y_{10}$. One can easily verify that $\alpha_3\,(Y_{10}) = 0$ (at Y_{10} we have that $V_{10} = W$) and $\alpha_3\,(Y_{11}) = 0$ (at Y_{11} we have that $V_{10} = V_{20}$ and $V_{10} = W$). The economic interpretation for this is that at these two thresholds the waiting curve coincides with the first investor curve, so that no firm is willing to set a strict positive intensity as it would then risk ending up in the situation were there are three simultaneous investments. In Figure 6 the investment intensity α_3 is plotted as function of y.

Let us consider the case $y \in (Y_{10}, Y_{11})$. The probability that all three firms invest at the same time is equal to

$$\frac{\alpha_3^3}{1-(1-\alpha_3)^3} = \frac{\alpha_3^2}{3-3\alpha_3+\alpha_3^2}, \tag{110}$$

the probability that two firms invest at the same time is

$$3\frac{\alpha_3^2\,(1-\alpha_3)}{1-(1-\alpha_3)^3} = \frac{3\alpha_3\,(1-\alpha_3)}{3-3\alpha_3+\alpha_3^2}, \tag{111}$$

and the probability that there is only one firm that invests equals

$$3\frac{\alpha_3\,(1-\alpha_3)^2}{1-(1-\alpha_3)^3} = \frac{3\,(1-\alpha_3)^2}{3-3\alpha_3+\alpha_3^2}. \tag{112}$$

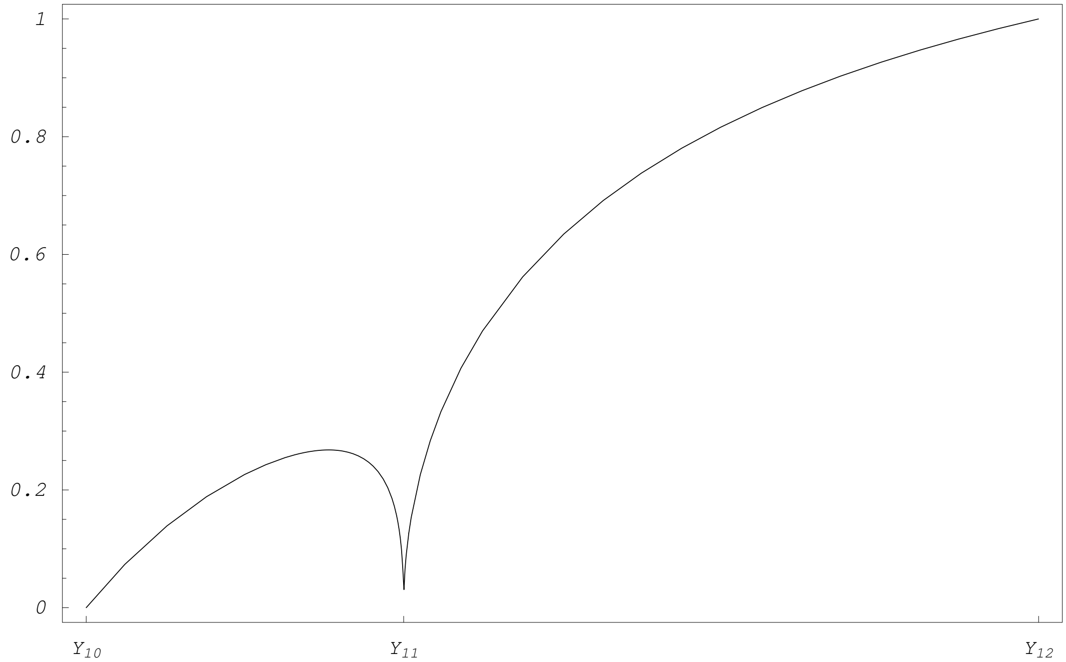

Figure 6: The investment intensity α_3 as function of y in the three firm case.

Next, we consider the case $y \in (Y_{11}, Y_{12})$. We know that if only one firm would have won the investment race, the other two will immediately engage each other in a new investment race. Thus the probability that only one firm invests is equal to zero and the probability that all three firms invest at the same time is equal to

$$\frac{\alpha_3^2}{3-3\alpha_3+\alpha_3^2}+\frac{3\left(1-\alpha_3\right)^2}{3-3\alpha_3+\alpha_3^2}\left(\frac{\alpha_2}{2-\alpha_2}\right), \tag{113}$$

and the probability that two firms invest at the same time is equal to

$$\frac{3\alpha_3\left(1-\alpha_3\right)}{3-3\alpha_3+\alpha_3^2}+\frac{3\left(1-\alpha_3\right)^2}{3-3\alpha_3+\alpha_3^2}\left(\frac{2-2\alpha_2}{2-\alpha_2}\right). \tag{114}$$

In Figure 7 the probabilities of each scenario (one firm, two firms, and three firms) are plotted as function of the starting value y.

C Cournot Competition

The profit flow of firm i is equal to

$$\pi_i\left(t\right)=\left(P\left(t\right)-c\right)q_i\left(t\right). \tag{115}$$

In order to derive the optimal output quantity we write down the first order condition for optimality

$$-\frac{1}{\gamma}X\left(t\right)\left(\sum_{j=1}^{n}q_j\left(t\right)\right)^{-\frac{1}{\gamma}-1}q_i\left(t\right)+X\left(t\right)\left(\sum_{j=1}^{n}q_j\left(t\right)\right)^{-\frac{1}{\gamma}}-c=0, \tag{116}$$

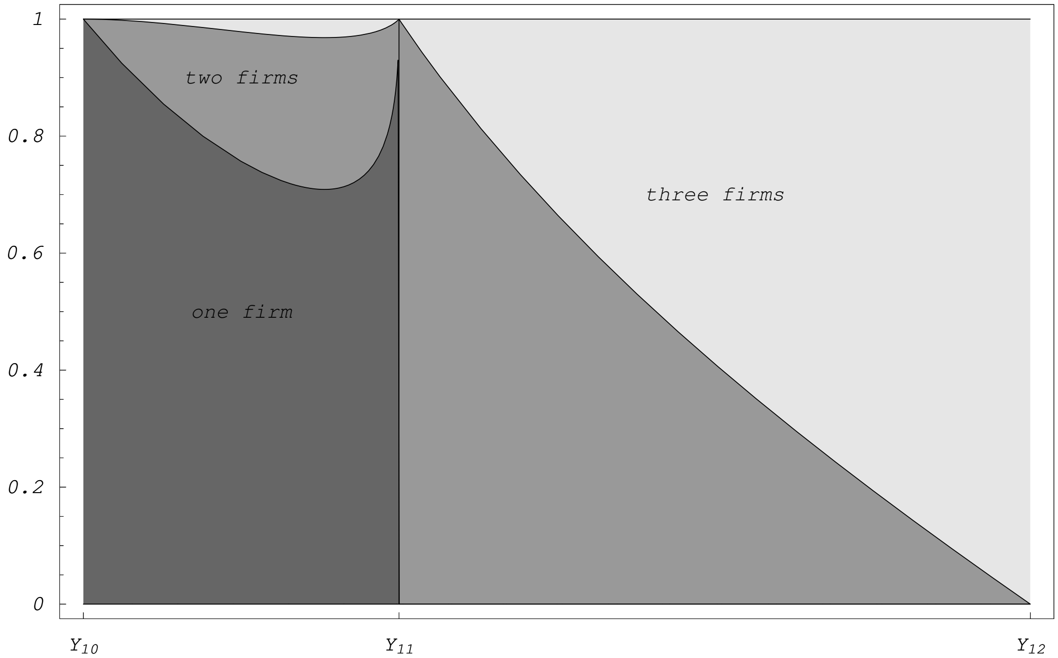

Figure 7: The probability of each possible scenario, i.e. one firm, two firms, or three firms, in the three firm case as function of the starting value y.

since we are looking for a symmetric equilibrium we make the substitution $q_i(t) = q^*(t)$:

$$-\frac{1}{\gamma}X(t)\left(\sum_{j=1}^{n} q^*(t)\right)^{-\frac{1}{\gamma}-1} q^*(t) + X(t)\left(\sum_{j=1}^{n} q^*(t)\right)^{-\frac{1}{\gamma}} - c = 0. \tag{117}$$

Rewriting gives

$$\left(1 - \frac{1}{n\gamma}\right) n^{-\frac{1}{\gamma}} X(t) \left(q^*(t)\right)^{-\frac{1}{\gamma}} = c. \tag{118}$$

Thus

$$q^*(t) = \frac{1}{n}\left(\frac{c}{X(t)}\frac{n\gamma}{n\gamma - 1}\right)^{-\gamma}. \tag{119}$$

To ensure that this is the maximum we check the second order condition for optimality

$$
\begin{aligned}
& \left(-\frac{1}{\gamma}-1\right)-\frac{1}{\gamma}X(t)\left(\sum_{j=1}^{n}q_j(t)\right)^{-\frac{1}{\gamma}-2}q_i(t)-\frac{1}{\gamma}X(t)\left(\sum_{j=1}^{n}q_j(t)\right)^{-\frac{1}{\gamma}-1}-\frac{1}{\gamma}X(t)\left(\sum_{j=1}^{n}q_j(t)\right)^{-\frac{1}{\gamma}-1} \\
= & \left(\frac{1}{\gamma^2}+\frac{1}{\gamma}\right)X(t)\left(\sum_{j=1}^{n}q_j(t)\right)^{-\frac{1}{\gamma}-2}q_i(t)-\frac{2}{\gamma}X(t)\left(\sum_{j=1}^{n}q_j(t)\right)^{-\frac{1}{\gamma}-1} \\
= & \left(\left(\frac{1}{\gamma^2}+\frac{1}{\gamma}\right)q_i(t)-\frac{2}{\gamma}\sum_{j=1}^{n}q_j(t)\right)X(t)\left(\sum_{j=1}^{n}q_j(t)\right)^{-\frac{1}{\gamma}-2} \\
= & \left(\left(\frac{1}{\gamma^2}-\frac{1}{\gamma}\right)q_i(t)-\frac{2}{\gamma}\sum_{j\neq i}q_j(t)\right)X(t)\left(\sum_{j=1}^{n}q_j(t)\right)^{-\frac{1}{\gamma}-2} \\
= & \left(\frac{1-\gamma}{\gamma^2}q_i(t)-\frac{2}{\gamma}\sum_{j\neq i}q_j(t)\right)X(t)\left(\sum_{j=1}^{n}q_j(t)\right)^{-\frac{1}{\gamma}-2} \\
< & \ 0, \qquad (120)
\end{aligned}
$$

since we made the assumption that $\gamma > 1$. Substitution in the profit flow function gives

$$
\begin{aligned}
\pi_i(t) &= \left(X(t)\left(\sum_{j=1}^{n}\frac{1}{n}\left(\frac{c}{X(t)}\frac{n\gamma}{n\gamma-1}\right)^{-\gamma}\right)^{-\frac{1}{\gamma}}-c\right)\frac{1}{n}\left(\frac{c}{X(t)}\frac{n\gamma}{n\gamma-1}\right)^{-\gamma} \\
&= \left(X(t)\left(\left(\frac{c}{X(t)}\frac{n\gamma}{n\gamma-1}\right)^{-\gamma}\right)^{-\frac{1}{\gamma}}-c\right)\frac{1}{n}\left(\frac{c}{X(t)}\frac{n\gamma}{n\gamma-1}\right)^{-\gamma} \\
&= c\left(\frac{n\gamma}{n\gamma-1}-1\right)\frac{1}{n}\left(\frac{c}{X(t)}\frac{n\gamma}{n\gamma-1}\right)^{-\gamma} \\
&= \frac{1}{n}\left(\frac{c}{n\gamma-1}\right)^{1-\gamma}\left(\frac{X(t)}{n\gamma}\right)^{\gamma} \\
&= D_nY(t), \qquad (121)
\end{aligned}
$$

where

$$
D_n=\frac{1}{n}\left(\frac{c}{n\gamma-1}\right)^{1-\gamma}(n\gamma)^{-\gamma}, \qquad (122)
$$

and $Y(t)$ follows a geometric Brownian motion with parameters μ and σ that are equal to

$$
\begin{aligned}
\mu &= \gamma\mu_X+\frac{1}{2}\gamma(\gamma-1)\sigma_X^2, \qquad (123) \\
\sigma &= \gamma\sigma_X. \qquad (124)
\end{aligned}
$$

Let $X(t)$ follow a geometric Brownian motion with parameters μ_X and σ_X, and define $F(X(t)) = (X(t))^\gamma$. Then Ito's lemma gives that

$$
\begin{aligned}
dF(X(t)) &= \frac{\partial F(X(t))}{\partial X(t)}(dX(t)) + \frac{1}{2}\frac{\partial^2 F(X(t))}{\partial X(t)^2}(dX(t))^2 \\
&= \gamma (X(t))^{\gamma-1}(\mu_X X(t)\,dt + \sigma_X X(t)\,d\omega(t)) + \frac{1}{2}\gamma(\gamma-1)(X(t))^{\gamma-2}\sigma_X^2 (X(t))^2\,dt \\
&= \left(\gamma\mu_X + \frac{1}{2}\gamma(\gamma-1)\sigma_X^2\right)(X(t))^\gamma\,dt + \gamma\sigma_X (X(t))^\gamma\,d\omega(t) \\
&= \left(\gamma\mu_X + \frac{1}{2}\gamma(\gamma-1)\sigma_X^2\right)F(X(t))\,dt + \gamma\sigma_X F(X(t))\,d\omega(t) \\
&= \mu F(X(t))\,dt + \sigma F(X(t))\,d\omega(t). \qquad (125)
\end{aligned}
$$

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