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Koshevoy, G.A.; Talman, A.J.J.

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By Gleb A. Koshevoy, Dolf Talman

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Competitive Equilibria in Economies with Multiple Divisible and Indivisible Commodities and No Money [1]

Gleb A. Koshevoy [2]
Dolf Talman[3]

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[2]G.A. Koshevoy, Central Institute of Mathematics and Economics, Russian Academy of Sciences, 117418 Moscow, Russia, e-mail: koshevoy@cemi.rssi.ru

[3]A.J.J. Talman, Department of Econometrics & Operations Research and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands, e-mail: talman@uvt.nl

Abstract

A general equilibrium model is considered with multiple divisible and multiple indivisible commodities. In models with indivisibles it is always assumed that an indivisible commodity, called money, is present that is used to transfer the value of certain amounts of indivisible goods. For these economies with a finite number of divisible and indivisible goods and money and without producers it is well understood that a general equilibrium exists if the individual demands and supplies for the indivisibele goods belong to a same class of discrete convexity.

In this paper we a model with multiple divisible and multiple undivisible commodities, in which none of the divisible goods may serve as money. Moreover, there are a finite number of producers owning a non-increasing returns to scale technology. One of the producesrs is assumed to have a linear production technology in order to produce divisible goods. Individual endowments being sufficienly large for production and discrete convexity guarantees the existence of a competitive equilibrium.

Key words: indivisible commodities, divisible commodities, discrete convexity, competitive equilibrium

JEL-code: D2, D4, D5, D6.

1 Introduction

Indivisible commodities have constituted a prominently important part of commercial commodities in most of the markets. Typical indivisible commodities are, to name a few, houses, cars, employees, airplanes, ships, trains, computers, machinery, and arts. Nowadays, even many divisible commodities are sold in indivisible quantities such as oil being sold in barrel as its smallest unit. Modelling economies with indivisibilities is therefore meaningful and realistic. However, studying such discrete economies stands in general a daunting challenge; see for example Koopmans and Beckman [13], Debreu [6], Henry [10], Kelso and Crawford [12], Gale[7], Quinzii [18], Shapley and Scarf [22], and Scarf [19, 20, 21], and more recently Kaneko and Yamamoto [11], Yamamoto [24], Shell and Wright [23], Garratt [8], Garratt and Qin [9], Ma [17], Bevia et al. [1], Bikhchandani and Mamer [2], van der Laan et al. [15], Yang [26]. In Danilov et al. [5] it was shown that discrete convex analysis is an appropriate tool to deal with indivisibles. Specifically, economies with indivisibles, money and no other perfectly divisible goods can be studied as continuous economies with divisible goods when individual demands and supplies for the indivisible goods belong to a same class of discrete convexity. Van der Laan et al. [16] consider economies with multiple divisible and indivisible goods and money. In their model the divisible goods are being produced from money by a unique linear production technology, while there are no other producers. Koshevoy and Talman [14] consider a model with multiple indivisible and divisible goods and money but without production.

In this paper we consider a general equilibrium model with multiple indivisibles and multiple divisible goods without money. Instead of money there is at least one producer with a production technology being linear for the divisible goods. Initial endowments should be large enough for production and the divisible goods are all desirable. Preferences and production sets are pseudoconvex and the individual demands and supply for the indivisibles should all belong to a same class of discrete convexity. The former again guarantees that the convexified economy has a competitive equilibrium and the latter that this equilibrium induces a competitive equilibrium of the discrete economy.

The plan of the paper is as follows. In Section 2 the concept of discrete convexity is reviewed. Section 3 the economic model with multiple divisible and indivisible goods without money is introduced. The existence proofs are given in Section 4.

2 Discrete convexity

In this section a survey of the results by Danilov and Koshevoy [3] about discrete convexity is given. A first idea on convexity of discrete sets is to consider the convex hull $\mathrm{co}(X)$ of a subset $X \subset \mathbb{Z}^K$, and require that $X = \mathrm{co}(X) \cap \mathbb{Z}^K$. Such sets are called *pseudoconvex*. The reason, why such sets are called pseudoconvex and not convex, is that they may not satisfy the separation property, the cornerstone of Convex Analysis (and therefore, of Equilibrium Analysis). Consider the following example.

Example 1. Consider the two two-points pseudoconvex sets $A = \{(0,0),(1,1)\}$ and $B = \{(0,1),(1,0)\}$. These sets do not intersect, but their convex hulls intersect at the interior point $(1/2, 1/2)$. Thus the sets can not be separated by a linear functional on $\mathbb{R}^2$. $\square$

The discrete convexity theory is constituted of classes of subsets of $\mathbb{Z}^K$ that are closed under Minkowski summation. The Minkowski sum of two subsets A and B in $\mathbb{R}^K$ is given by $A + B = \{a + b | a \in A,\ b \in B\}$.

Definition 2.1 *A class $\mathcal{D}$ of subsets of $\mathbb{Z}^K$ is a* class of discrete convex sets *if the following properties hold:*

DC1. For any $A \in \mathcal{D}$ it holds that A is pseudoconvex, $-A \in \mathcal{D}$, and co(A) is a polyhedron;

DC2. For any A and $B \in \mathcal{D}$ it holds that $A + B \in \mathcal{D}$.

One can easily check that sets of a class of discrete convexity $\mathcal{D}$ are well behaved with respect to the separation property. In fact, let $A,\ B \in \mathcal{D}$ and $A \cap B = \emptyset$. Then $0^K \notin A + (-B)$, $A + (-B) \in \mathcal{D}$, and so $0^K \notin \mathrm{co}(A + (-B))$. Since the convex hull commutes with the Minkowski sum, we have $0^K \notin \mathrm{co}(A) + \mathrm{co}(-B)$. Hence, $\mathrm{co}(A)$ and $\mathrm{co}(B)$ can be separated and so A and B.

In the previous example, with $A = \{(0,0),(1,1)\}$ and $B = \{(0,1),(1,0)\}$, we have $0^K \notin A + (-B)$, but $A + (-B)$ is not a pseudoconvex set, and so the convex hulls $\mathrm{co}(A)$ and $\mathrm{co}(B)$ can not be separated. Therefore, there does not exist a class of discrete convexity which contains both sets.

Classes of discrete convexity are constructed as integer points of integral polyhedra. A polyhedron $P \subset \mathbb{R}^K$ is said to be an *integral polyhedron* if $P = \mathrm{co}(P \cap \mathbb{Z}^K)$.

Let $\mathcal{P}$ be a class of polyhedra with the following properties:

DCP1. Any polyhedron $P \in \mathcal{P}$ is integral.

DCP2. For any polyhedra $P,\ Q \in \mathcal{P}$, we have $P \pm Q \in \mathcal{P}$ and

$$(P \pm Q) \cap \mathbb{Z}^K = (P \cap \mathbb{Z}^K) \pm (Q \cap \mathbb{Z}^K). \tag{1}$$

A class of polyhedra $\mathcal{P}$ with properties DCP1 and DCP2 is said to be a *class of discrete convexity.* Because taking the convex hull commutates with adding up and substracting sets and the sum of polyhedra is again a polyhedron, for any class $\mathcal{P}$ of discrete convex polyhedra it holds that the class $\mathcal{D}$ of subsets of $\mathbb{Z}^K$ of the form $P \cap \mathbb{Z}^K$, $P \in \mathcal{P}$, satisfies DC1 and DC2.

When $|K| = 1$, the class of integral polyhedra, being segments with integral endpoints, is the only class of discrete convexity. This is, of course, not the case in higher dimensions.

Example 2. *Hexagons.* Consider a class $\mathcal{H}$ of polyhedra in $\mathbb{R}^2$, which consists of hexagons defined by inequalities $a_1 \leq x_1 \leq b_1$, $a_2 \leq x_2 \leq b_2$, $c \leq x_1 + x_2 \leq d$ with integral a_1, a_2, b_1, b_2, c and d (such a hexagon can be degenerated to a polyhedron with less than six vertices). It is easy to check that the vertices of such a hexagon are integral. Because the intersection of hexagons is again a hexagon, we conclude that $\mathcal{H}$ is a class of discrete convexity. $\square$

Observe, that the edges of the hexagons in Example 2 are parallel to the vectors e_1, e_2 or $e_1 - e_2$. These vectors have the following property: any pair of these vectors form a basis of the lattice $\mathbb{Z}^2$. As we have seen in Example 1, if a class of integral polyhedra in $\mathbb{R}^2$ contains polyhedra having edges being parallel to $e_1 - e_2$ and to $e_1 + e_2$, such a class fails to be a class of discrete convexity. The reason is that the pair of vectors $e_1 - e_2$ and $e_1 + e_2$ does not form a basis of $\mathbb{Z}^2$. For example, points of the form $(2n+1)e_1$, $n \in \mathbb{Z}$, can not be obtained as combinations of vectors $e_1 - e_2$ and $e_1 + e_2$ with integer coefficients. The property that every set of $|K|$ linearly independent primitive vectors being parallel edges of polytopes of some class of polyhedra forms a basis of the abelian group (lattice) $\mathbb{Z}^K$ is the decisive property for a class of polyhedra to be a class of discrete convexity.

A collection $\mathcal{R}$ of vectors of $\mathbb{R}^K$ is said to be a *unimodular system* if, for any subset $R \subset \mathcal{R}$, the abelian group $\mathbb{Z}(R) = \{\sum_i a_i r_i \,|\, r_i \in R,\, a_i \in \mathbb{Z}\}$ coincides with the lattice $\mathbb{R}(R) \cap \mathbb{Z}^K$, where $\mathbb{R}(R) = \{\sum_i a_i r_i \,|\, r_i \in R,\, a_i \in \mathbb{R}\}$. Now we have the following result (see Danilov and Koshevoy [3]).

Theorem 2.2 *Let $\mathcal{P}$ be a collection of pointed integral polyhedra of $\mathbb{R}^K$. Let $\mathcal{R}(\mathcal{P})$ denote the set of vectors in $\mathbb{Z}^K$ being parallel to edges of polyhedra of $\mathcal{P}$[1)]. Then $\mathcal{P}$ is a class of discrete convexity if and only if $\mathcal{R}(\mathcal{P})$ is a unimodular system.*

The next example is a well-known unimodular system.

Example 3. The set $\mathbb{A}_K := \{\pm e_i, e_i - e_j,\, i,\, j \in K\}$ of vectors of $\mathbb{Z}^K$ is a unimodular system. Because $\mathbb{A}_K$ contains the standard basis, we need to show that any $|K|$ linear independent vectors of $\mathbb{A}_K$ form a basis of $\mathbb{Z}^K$. Let $B \subset \mathbb{A}_K$ be a basis of $\mathbb{R}^K$. Check that B is a basis of $\mathbb{Z}^K$. One of $\pm e_i$, $i \in K$, belongs to B, otherwise B is a subset of the hyperplane $\sum_{i \in K} x_i = 0$, and, hence, B cannot be a basis of $\mathbb{R}^K$. Let $e_1 \in B$. If none of the vectors $\pm(e_i - e_1)$ belongs to B, then the set $B \setminus \{e_1\}$ is a subspace of the hyperplane $\{x \in \mathbb{R}^K, |\, x_1 = 0\}$. By induction $B \setminus \{e_1\}$ forms a basis of $\mathbb{Z}^{K \setminus \{1\}}$. Hence B is a basis of

[1] A vector r belongs to $\mathcal{R}(\mathcal{P})$ if and only if there is a polyhedron $P \in \mathcal{P}$ which has an edge of the form $[x, x + ar]$ for some $a \in \mathbb{N}$ or $\{y | y = x + br, b \in \mathbb{R}\}$ for some $x \in \mathbb{Z}^K$.

$\mathbb{Z}^K$. If $e_j - e_1$ belongs to B for some $j \neq 1$, then, changing $e_j - e_1$ to $e_j = e_1 + (e_j - e_1)$, we obtain a new basis B'. Obviously, B and B' are either both bases or both not bases of $\mathbb{Z}^K$. Repeating the same argument, we may assume that none of the vectors $\pm(e_i - e_1)$ belongs to B'. Therefore, B' is a basis of $\mathbb{Z}^K$, and, hence, so is B. $\square$

The discrete convexity corresponding to the unimodular system of Example 3 is called *polymatroidal* discrete convexity. It is interesting to note here, that nearly all known existence results with indivisibles fit into the polymatroidal discrete convexity (see Danilov et al. [4]).

3 The model

In this paper we deal with the problem of the existence of a competitive equilibrium in an exchange economy $\mathcal{E}$ with consumption and production and with multiple divisible and multiple indivisible commodities. There is a finite set K of k discrete (indivisible) commodities and a finite set L of l perfectly divisible commodities. Bundles of commodities are denoted by elements of the set $\mathbb{Z}^K \times \mathbb{R}^L$. The set J denotes the finite set of producers and H denotes the finite set of consumers. A producer $j \in J$ is described by its input-output production set $C_j \subset \mathbb{Z}^K \times \mathbb{R}^L$. A vector $(Y, y) \in C_j$ means that producer j, $j \in J$, is able to produce the output vector $(Y, y)^+$, being the positive part of (Y, y), from the input vector $-(Y, y)^-$, being minus the negative part of (Y, y). Standard assumptions on C_j are $C_j \cap \mathbb{Z}^K_+ \times \mathbb{R}^L_+ = \{0^{K+L}\}$, $C_j = C_j - (\mathbb{Z}^K_+ \times \mathbb{R}^L_+)$ and C_j is a closed set, for all $j \in J$.

The preferences of consumer h, $h \in H$, are described by a preference relation $\preceq_h$, being a monotone, continuous weak order on the consumption set $\mathbb{Z}^K_+ \times \mathbb{R}^L_+$. Consumer $h \in H$ has a vector of initial endowments $\omega_h = (W_h, w_h) \in \mathbb{Z}^K_+ \times \mathbb{R}^L_+$ and is endowed with shares in the production: $\theta_{jh} \geq 0$, $j \in J$, is consumer h's share in the production of producer j, where $\sum_{h \in H} \theta_{jh} = 1$ for all $j \in J$.

Agents are assumed to be price takers. Given a price vector p, being a linear functional on $\mathbb{R}^K \times \mathbb{R}^L$, producer $j \in J$ solves the following maximization program:

$$\max_{(Y,y) \in C_j} p(Y, y). \tag{2}$$

The number $\pi_j(p) = \max_{(Y,y) \in C_j} p(Y, y)$ is the profit of producer j and

$$S_j(p) = \mathrm{Argmax}_{(Y,y) \in C_j} p(Y, y)$$

is producer j's supply at price p. Consumer $h \in H$ seeks a best element with respect to his preference $\preceq_h$ in the budget set

$$B_h(p) = \{(X, x) \in \mathbb{Z}^K_+ \times \mathbb{R}^L_+ \,|\, p(X, x) \leq \beta_h(p)\},$$

where at price vector p consumer h's income, $\beta_h(p)$, is defined by

$$\beta_h(p) = p(W_h, w_h) + \sum_{j \in J} \theta_{jh} \pi_j(p).$$

The demand of consumer h, $h \in H$, is the set $D_h(p)$ of best elements in the set $B_h(p)$ with respect to the preference $\preceq_h$.

Definition 3.1 *An equilibrium is a tuple $(p, (X_h, x_h)_{h \in H}, (Y_j, y_j)_{j \in J})$ of a price vector p, individual demands $(X_h, x_h) \in D_h(p)$, $h \in H$, and individual supplies $(Y_j, y_j) \in S_j(p)$, $j \in J$, such that all markets clear:*

$$\sum_{h \in H} (X_h, x_h) = \sum_{j \in J} (Y_j, y_j) + \sum_{h \in H} (W_h, w_h).$$

To guarantee the existence of an equilibrium we assume that there at least one of the producers owns a production technology being linear in the divisible goods.

Assumption T1. There is one production technology being linear in the divisible part, i.e. there exists a producer, say $j = 1$, such that for any $p \in \mathbb{R}^L_+$, $S_1(p) = S_1^{ind}(p) \times T$, where $T \subset \mathbb{R}^L$ is a linear subspace of codimension 1. □

In the model of van der Laan et al. [16] it is assumed that there is also money in the model and that there is only one producer and this producer produces the divisible non-money goods using money as an input.

Because of Assumption T1 the equilibrium prices of the divisible goods are completely determined by the rule $p^{div}(x) = 0$ for any $x \in T$. Because of our assumptions it holds that $p^{div} \in \mathbb{R}^L_+$. Therefore, only the appropriate prices of indivisible goods can equilibrate demands and supplies. Let us normalize the prices of the divisible goods such that $p^{div}(1^L) = 1$.

The preferences of the consumers are such that the divisible goods are more desirable than the indivisible goods.

Assumption T2. For each $(X, x) \in \mathbb{Z}^K_+ \times \mathbb{R}^L_+$ and $h \in H$ there exists $x_h \in \mathbb{R}^L$ such that $(X, x) \preceq_h (0^K, x_h)$. □

Furthermore, we assume that all production sets and preferences are pseudoconvex and that production sets have no asymptotes.

Assumption T3. For every $h \in H$ and any tuple of bundles $(X, x) \sim_h (X_1, x_1) \sim_h \ldots \sim_h (X_r, x_r)$ in $\mathbb{Z}^K_+ \times \mathbb{R}^L_+$ such that $X = \sum_i \alpha_i X_i \in \mathbb{Z}^K_+$, $\sum_i \alpha_i = 1$, $\alpha_i \geq 0$, $i = 1, \ldots, r$, it holds that $(X, x) \succeq_h (X, \sum_i \alpha_i x_i)$. For every $j \in J$ and any tuple of bundles $(Y_1, y_1), \ldots, (Y_r, y_r)$ in C_j and $Y \in \mathbb{Z}^K$ such that $Y = \sum_i \alpha_i Y_i$, $\sum_i \alpha_i = 1$, $\alpha_i \geq 0$, $i = 1, \ldots, r$, there exists $y \in \mathbb{R}^L$ such that $(Y, y) \in C_j$ and $p^{div}(y) \geq \sum_i \alpha_i p^{div}(y_i)$. Moreover, the production sets coC_j, $j \in J$, have no asymptotes (in all codimensions). □

The next assumption requires that total endowment is strictly positive and that each consumer has enough initial endowment.

Assumption T4. The total endowment is strictly positive: $\sum_{h\in H}(W_h, w_h) > (1^K, 1^L)$. For every h, $h \in H$, it is possible to produce from the initial endowment (W_h, w_h) a vector of goods which is strictly preferred by consumer h to any vector without divisible goods. $\square$

The convexified economy co$(\mathcal{E})$ of $\mathcal{E}$ is obtained by replacing demands and supplies of $\mathcal{E}$ by their convex hulls. In Section 3 it will be shown that under the Assumptions T1–T4 a competitive equilibrium in the convexified economy exists.

Proposition 3.2 *Let $\mathcal{E}$ be a discrete economy and let the Assumptions T1–T4 hold, then there exists a competitive equilibrium in the convexified economy co$(\mathcal{E})$.*

To guarantee that the discrete economy $\mathcal{E}$ itself has a competitive equilibrium we have to assume that the individual demands and supplies for the indivisibles belong to a same class of discrete convexity.

Assumption T5. The sets $D_h^{ind}(p)$, $h \in H$, and $S_j^{ind}(p)$, $j \in J$, belong for every $p \in \mathbb{R}_+^K \times \mathbb{R}_+^L$ all to the same class of discrete convexity $\mathcal{D}$. $\square$

For a price system $p \in \mathbb{R}^K \times \mathbb{R}^L$, let $S_j^{ind}(p) = \{Y \in \mathbb{Z}^K \,|\, \exists\, y \in \mathbb{R}^L \,:\, (Y, y) \in S_j(p)\}$ be the projection of producer j's supply $S_j(p)$ along the divisible goods coordinates, $j \in J$. Similarly, let $D_h^{ind}(p) = \{X \in \mathbb{Z}_+^K \,|\, \exists\, x \in \mathbb{R}_+^L \,:\, (X, x) \in D_h(p)\}$ be the projection of consumer h's demand $D_h(p)$ along the divisible goods coordinates, $h \in H$.

Theorem 3.3 *Let Assumptions T1–T5 be satisfied. Then there exists a competitive equilibrium in the economy $\mathcal{E}$.*

Example 4. Suppose the preferences of consumer h, $h \in H$, can be represented by a utility function u^h satisfying $u^h(X, x) = u_1^h(X) + u_2^h(x)$, where $u_1^h(\cdot)$ satisfies stepwise gross-substitutability and $u_2^h(\cdot)$ is a concave function. The production set of firm j, $j \in J$, is specified by the cost function $c^j(Y, y) = c_1^j(Y) + c_2^j(y)$, where $-c_1^j(\cdot)$ satisfies stepwise gross-substitutability and $c_2^j(\cdot)$ is a convex function. In this case the demand functions and the supply functions belong to the same class of discrete convexity with unimodular system of Example 3 and therefore Assumption T5 is satisfied. For the definition of stepwise gross-substitutability and for the proof of this claim see Danilov et al. (2003).

In the next section the proposition and theorem of this section are proved.

4 Proof of Existence

In this section we prove Proposition 3.2 and Theorem 3.3.

4.1 Proof of Proposition 3.2

First we construct an auxiliary economy. Because of Assumption T4, the production set $\sum_j C_j$ of the aggregate producer is a closed convex set.[2] Now, we explain how to aggregate consumers. Pick some price $p \in \mathbb{R}^K_+$. For each $h \in H$, we consider an indifference level "touching" the budget set $B_h(p, p^{div})$. Denote by $I_h(p)$ this indifference level. First we set the preference $\tilde{\preceq}_h$ of the hth consumer such that the indifference levels are parallel translations of the "touching" level by the vector $\lambda(0^K, 1^L)$, $\lambda \in [\lambda_h, +\infty)$, where λ_h is such that the translation of the indifference level by the vector $\lambda_h(0^K, 1^L)$ passes through the endowment vector (W_h, w_h). Note that $\lambda_h \leq 0$. Now set indifference levels of a preference $\preceq (p)$ of the aggregate consumer, endowed with the aggregate vector $(W, w) = \sum_h (W_h, w_h)$, by the rule

$$\sum_h (I_h(p) - \lambda_h t(0^K, 1^L)), \text{ if } t \in [-1, 0],$$

and

$$\sum_h (I_h(p) + t(0^K, 1^L)), \text{ if } t \geq 0.$$

Because there exists an indifference level of $\preceq (p)$ which is passing through (W, w), this list of indifference levels suffices to set up the preference due to individual rationality. Note also that any indifference level is well defined since all $I_h(p)$ belong to the cone $\mathbb{R}^K_+ \times \mathbb{R}^L_+$. We define $P(p)$ as the set of equilibrium prices in the economy $\mathcal{E}(p)$ with one producer with production set $C = \sum_j C_j$ and one consumer with preference relation $\preceq (p)$. The equilibrium prices come of the form of the separating functionals between the set C and a translation on the vector $-(W, w)$ of the set being the sum of the indifference level of $\preceq (p)$ passing through the point $(W, w) + y(p)$ and the positive orthant $\mathbb{R}^K_+ \times \mathbb{R}^L_+$, where $y(p) \in \operatorname{Argmax}_{y \in C}(p, p^{div})(y)$, i.e., we translate the set with respect to vectors of the form $a(0^K, 1^L)$, $a \geq -1$, such that the production set and the translated set touch each other.
In order to get a fixed point of P, we take a cube $Q = \{p \in \mathbb{R}^K \,|\, 0 \leq p_k \leq M\}$ for some $M > 0$ such that P maps every $p \in Q$ to a subset of Q. The number M is determined as follows. Given the initial endowments, there exist bounds for the maximal production of each good due to Assumptions T4 (we may exclude the linear producer, having fixed p^{div}). Let $(B, b) \in \mathbb{R}^K_+ \times \mathbb{R}^L_+$ be a vector which is in every coordinate larger than the maximal production of the good corresponding to this coordinate, and for $h \in H$ let T_h be the cost $p^{div}(x_h)$ of producing at price p^{div} the vector $(0^K, x_h) \in \mathbb{R}^K_+ \times \mathbb{R}^L_+$ satisfying $(0^K, x_h) \sim_h (W_h + B, w_h + b)$. Then we take M equal to $\sum_h T_h$.
Because any $p' \in P(p)$ is a separating functional, we have that $M \geq p'(W)$, and since $W \geq 1^K$, we obtain $p'_k \leq M$ for every $k \in K$. Clearly, P has compact convex images and is

[2]In general, the sum of convex closed sets might not be closed, but because of our assumptions the sum $\sum_j C_j$ is a closed set.

a closed mapping. Therefore, by Kakutani fixed point theorem, P has a fixed point. Since due to Walras' law at a fixed point p^* of P the vector p^* supports the indifference level $\sum_h I_h(p^*)$, a fixed point of P yields an equilibrium of the convexified economy. Q.E.D.

4.2 Proof of Theorem 3.3

In Proposition 3.2 we proved the existence of an equilibrium in the convexified economy. Now let us assume we have an equilibrium in $\mathrm{co}(\mathcal{E})$, that is a tuple of prices $p^* : \mathbb{R}_+^L \to \mathbb{R}$, supplies $(z_j^*, y_j^*) \in \mathrm{co}(S_j(p^*))$, $j \in J$, and demands $(t_h^*, x_h^*) \in \mathrm{co}(D_h(p^*))$, $h \in H$, satisfying $\sum_h t_h^* + \sum_j z_j^* = \sum_h W_h$ and $\sum_h x_h^* + \sum_j y_j^* = \sum_h w_h$. Therefore, we have

$$\sum_{h \in H} W_h \in \sum_{h \in H} \mathrm{co}(D_h^{ind}(p^*)) + \sum_{j \in J} \mathrm{co}(S_j^{ind}(p^*)).$$

By Assumption T5, there exist $T_h^* \in D_h^{ind}(p^*))$, $h \in H$, and $Z_j^* \in S_j^{ind}(p^*)$, $j \in J$, satisfying $\sum_h T_h^* + \sum_j Z_j^* = \sum_h W_h$. Let x_h, $h \in H$, and y_j, $j \in J$, be such that $(T_h^*, x_h) \in D_h(p^*)$, $h \in H$, and $(Z_j^*, y_j) \in S_j(p^*)$, $j \in J$.
By Walras' law we have

$$\sum_{h \in H} (p^*(T_h^*) + p^{div}(x_h)) = \sum_{h \in H} p(W_h, w_h) + \sum_{j \in J} (p^*(Z_j^*) + p^{div}(y_j)).$$

Because of the balance of the indivisible goods, $\sum_h T_h^* + \sum_j Z_j^* = \sum_h W_h$, we have $p^{div}(\sum_h x_h - \sum_j y_j) = 0$. Define the new production plan of the producer 1 as (Z_1^*, y_1'), where $y_1' := \sum_h (x_h - w_h) - \sum_{j \neq 1} y_j$. By Assumption T3, (Z_1^*, y_1') belongs to $S_1(p^*)$, and with this modification for the first producer, we obtain a competitive equilibrium of the economy $\mathcal{E}$.
Q.E.D.

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