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Assignment situations with multiple ownership and their games

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Abstract:

An assignment situation can be considered as a two-sided market consisting of two disjoint sets of objects. A non-negative reward matrix describes the profit if an object of one group is assigned to an object of the other group. Assuming that each object is owned by a different agent, Shapley and Shubik (1972) introduced a class of assignment games arising from these assignment situations.

This paper introduces assignment situations with multiple ownership. In these situations each object can be owned by several agents and each agent can participate in the ownership of more than one object. In this paper we study simple assignment games and relaxations that arise from assignment situations with multiple ownership. First, necessary and sufficient conditions are provided for balanced assignment situations with multiple ownership. An assignment situation with multiple ownership is balanced if for any choice of the reward matrix the corresponding simple assignment game is balanced. Second, balancedness results are obtained for relaxations of simple assignment games.

Keywords: Assignment situations, matchings, assignment games, balancedness.

JEL classification: C71

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1 Introduction

An assignment situation can be considered as a two-sided market consisting of two disjoint sets of objects. An illustration of such a two-sided market is, for example, the housing market. In this situation one set of objects is a collection of houses, where different houses are owned by different agents (the group of sellers). The other set of objects is a collection of agents where each agent wants to buy exactly one house (the group of buyers). Shapley and Shubik (1972) introduced assignment games, a class of cooperative games, in order to analyze these assignment situations. They showed that these games have a non-empty core. Quint (1991) showed that the core of assignment games have a lattice structure. Solymosi and Raghavan (1994, 2001) provided a polynomial algorithm for the nucleolus of assignment games and provide necessary and sufficient conditions for stability of the core. Hamers et al. (2002) proved that all extreme points of the core are marginal vectors, in spite of the fact that assignment games need not be convex. Núñez and Rafels (2003) characterize the extreme points of assignment games by means of the reduced marginal worth vectors.

This paper analyzes assignment situations in which an object can be owned by several agents and where agents can participate in the ownership of more than one object. We will refer to these assignment situations as assignment situations with multiple ownership, AMO for short. An AMO is described by two disjoint sets of agents, two disjoint sets of objects and the description of which agent(s) own(s) which objects. If an object of one set is assigned to an object of the other set there will be some reward. This reward will be described using a reward matrix. Similar to Shapley and Shubik (1972) we define simple assignment games, a class of cooperative games arising from an AMO and a non-negative reward matrix. In a simple assignment game each coalition can only match objects that are completely owned by this coalition. Moreover, if a player is participating in the ownership of more than one object, at most one of these objects can be matched. In the relaxations of simple assignment games this last restriction is relaxed, i.e. if a player is participating in more than one object, at most two (three, ..., etc.) of these objects can be matched.

The house market is still a nice example of this model. The properties of the sellers can be apartment buildings or shopping malls which usually are owned by several agents. Moreover, it can be that some agents are involved in the ownership of several apartment buildings or shopping malls. The buyers can be viewed as a group of investors which are interested in apartment buildings or shopping malls. Obviously, one investor can be member of more than one investor group. The possible restrictions on the number of apartment buildings or shopping malls that can be sold or bought can be imposed by governmental rules, for example, to resist monopolies in city areas of owners of apartments or shopping malls.

We call an AMO balanced if for any choice of the non-negative reward matrix the corresponding simple assignment game is balanced. We will characterize balanced AMO by requiring a structure on the ownership of the objects by the agents. This line of research has been applied by Herer and Penn (1995) and Granot et al. (1999) to characterize submodular and balanced graphs with respect to traveling salesman problems and Chinese postman problems, respectively. Finally, we will study relaxations of simple assignment games and provide sufficient conditions for non-emptiness of the core.

This paper is organized as follows. In Section 2 we introduce formally the AMO's and the corresponding simple assignment games. Section 3 is devoted to balanced AMO's. Finally, in Section 4 we provide a sufficient condition for balancedness of the relaxations of simple assignment games by formulating these games as Integer Linear Programming problems.

2 The assignment model and its game

In this section we introduce assignment situations with multiple ownership and its corresponding games.

An assignment situation with multiple ownership $((M_1, M_2), (A_1, A_2), (S_1, S_2))$, for short AMO, consists of two finite and disjoint agent (player) sets M_1 and M_2, two finite and disjoint sets of objects A_1 and A_2 and functions $S_i : A_i \to 2^{M_i}$, $i \in \{1, 2\}$, that describe the set of agents that own some object. Specifically, for object $a \in A_i$ the set $S_i(a)$ is the subset of agents in M_i that own object a. It is assumed that $\cup_{a \in A_i} S_i(a) = M_i$ for $i = 1, 2$ and $S_i(a) \neq \emptyset$ for any $a \in A_i$ and $i = 1, 2$. These assumptions imply that each agent of M_i is involved in the ownership of at least one object of A_i and that each object is owned by at least one agent. Observe that it is not excluded that agents are involved in the ownership of more than one object. If an object of A_1 is matched with an object of A_2, there can be obtained some reward. A non-negative matrix $R \in \mathbb{R}^{A_1 \times A_2}$ expresses this reward. So, if object $a \in A_1$ is assigned to object $b \in A_2$, then the reward is equal to $R_{ab} \geq 0$. Observe that the assignment situations as discussed in Shapley and Shubik (1972) can be seen as a special class of AMO's with a reward matrix.

Now, we follow the same line as in Shapley and Shubik (1972) to define a simple assignment game that arises from an AMO. Let $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ be an AMO. Let $T_i \subset M_i$, $i \in \{1, 2\}$, be two coalitions and let $B_i(T_i) = \{a \in A_i \mid S_i(a) \subset T_i\}$ be the objects that are completely owned by the members of coalition T_i. A matching μ for $T_1 \cup T_2$ consists of pairwise disjoint pairs in $B_1(T_1) \times B_2(T_2)$. Hence, for a coalition $T_1 \cup T_2$ the definition of a matching implies that we can only assign objects that are completely owned by members of coalition $T_1 \cup T_2$ and each of these objects is assigned to at most one other object. A matching μ is called *admissible* if for any two distinct pairs $(a_1, b_1), (a_2, b_2) \in \mu$ it holds $S_1(a_1) \cap S_1(a_2) = \emptyset$ and $S_2(b_1) \cap S_2(b_2) = \emptyset$. The restriction to admissible matchings implies that for each player at most one object is matched where he is participating in. The set of admissible matchings with respect to coalition $T_1 \cup T_2$ is denoted by $\mathcal{A}(T_1 \cup T_2)$.

Let $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ be an AMO and R a non-negative reward matrix. The corresponding simple assignment game $(M_1 \cup M_2, v)$ is defined for all $T_1 \subset M_1, T_2 \subset M_2$ by

$$v(T_1 \cup T_2) = \max\{\sum_{(a,b)\in\mu} R_{ab} : \mu \in \mathcal{A}(T_1 \cup T_2)\}, \tag{1}$$

i.e. the worth of a coalition is the maximum value of an admissible matching for this coalition. Observe that $v(T_1 \cup T_2) = w(B_1(T_1) \cup B_2(T_2))$ where $(A_1 \cup A_2, w)$ is the assignment game as defined in Shapley and Shubik (1972), arising from the situation (A_1, A_2, R) in which the agents are identified with the objects.

The following example illustrates an AMO and its corresponding game. Moveover, this example shows that the core of a simple assignment game can be empty. The *core* of a cooperative game (N, v) is defined by

$$Core(v) = \{x \in I\!R^N \mid x(S) \geq v(S) \text{ for all } S \subset N, x(N) = v(N)\},$$

where $x(S) = \sum_{i \in S} x_i$. Hence, if $x \in Core(v)$, then no coalition in $S \subset N$ has an incentive to split off from the grand coalition if x is the proposed vector of revenue shares. A game (N, v) is *balanced* if it has a non-empty core.

Example 2.1 Figure 1 displays an AMO $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ with reward matrix R. The rows and columns of R represent the objects of A_1 and A_2, respectively. The players owning an object are printed bold behind the rows and above the columns. For example, column 2 represents the second object of A_2 which is owned by players 3 and 4.

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{3} & \mathbf{3,4} & \mathbf{4} \end{array} \\ R = & \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \end{array} \end{array}$$

Figure 1: An AMO with reward matrix R.

Let $(M_1 \cup M_2, v)$ be the corresponding game. An example of an admissible matching of $\{1, 2, 3, 4\}$ is $\{(r_1, c_1), (r_2, c_3)\}$, where $(r_1, c_1)((r_2, c_3))$ reflects the matching of the objects in row 1 (2) and column 1 (3). This matching yields a reward of 2. The matching $\{(r_1, c_1), (r_2, c_2)\}$ is inadmissible because player 3 is participating in the ownership of c_1 and c_2 and we assumed that each player could be matched at most once. It is straightforward to check that $v(\{1, 2, 3, 4\}) = 2$, $v(\{1, 2, 4\}) = 2, v(\{1, 3, 4\}) = 2$ and $v(\{2, 3\}) = 1$. It follows that this simple assignment game has an empty core, because if $(x_1, x_2, x_3, x_4) \in Core(v)$, then

$$\begin{aligned} 4 &= 2(x_1 + x_2 + x_3 + x_4) \\ &= (x_1 + x_2 + x_4) + (x_1 + x_3 + x_4) + (x_2 + x_3) \\ &\geq v(\{1, 2, 4\}) + v(\{1, 3, 4\}) + v(\{2, 3\}) \\ &= 2 + 2 + 1 = 5, \end{aligned}$$

which is a contradiction. □

In the following sections balancedness of simple assignment games and its relaxations will be studied.

3 Balanced assignment situations with multiple ownership

In this section we provide necessary and sufficient conditions for balancedness of AMO's.

An AMO is called *balanced* if for any reward matrix R the corresponding simple assignment game is balanced.

First, we show that for AMO's in which each agent (player) is participating in one object, are balanced.

Proposition 3.1 An AMO in which each agent is participating in precisely one object is balanced.

PROOF: Let $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ be an AMO that satisfies $S_1(a) \cap S_1(\hat{a}) = \emptyset$ for all $a, \hat{a} \in A_1, a \neq \hat{a}$ and $S_1(b) \cap S_1(\hat{b}) = \emptyset$ for all $b, \hat{b} \in A_2, b \neq \hat{b}$. Let R be a non-negative reward matrix. Now, we have to show that the corresponding simple assignment game $(M_1 \cup M_2, v)$ is balanced. Let $(A_1 \cup A_2, w)$ be the assignment game, as defined in Shapley and Shubik (1972), arising from the situation (A_1, A_2, R) in which the agents are identified with the objects. Because assignment games are balanced there exists a vector $(u^{A_1}, v^{A_2}) \in Core(w)$, where u^{A_1} (resp. v^{A_2}) represents the payoff of the players in A_1 (resp. A_2). Note that $u^{A_1} \geq 0$ and $v^{A_2} \geq 0$. Let $x \in I\!R^{M_1 \cup M_2}$ be defined as follows:

$$\text{For all } i \in M_1 : x_i = \frac{u_a^{A_1}}{\mid S_1(a) \mid} \text{ with } a \in A_1 \text{ such that } i \in S_1(a)$$

$$\text{For all } j \in M_2 : x_j = \frac{v_b^{A_2}}{\mid S_2(b) \mid} \text{ with } b \in A_2 \text{ such that } j \in S_2(b)$$

We show that $x \in Core(v)$. Let $T_1 \subset M_1, T_2 \subset M_2$, then

$$\begin{aligned}
x(T_1 \cup T_2) &= \sum_{i \in T_1} \sum_{a \in A_1 : i \in S_1(a)} \frac{u_a^{A_1}}{|S_1(a)|} + \sum_{j \in T_2} \sum_{b \in A_2 : j \in S_2(b)} \frac{v_b^{A_2}}{|S_2(b)|} \\
&= \sum_{a \in A_1} \frac{|S_1(a) \cap T_1| u_a^{A_1}}{|S_1(a)|} + \sum_{b \in A_2} \frac{|S_2(b) \cap T_2| v_b^{A_2}}{|S_2(b)|} \\
&\geq \sum_{a \in A_1 : S_1(a) \subset T_1} u_a^{A_1} + \sum_{b \in A_2 : S_2(b) \subset T_2} v_b^{A_2} \\
&= \sum_{a \in B_1(T_1)} u_a^{A_1} + \sum_{b \in B_2(T_2)} v_b^{A_2} \\
&\geq w(B_1(T_1) \cup B_2(T_2)) \\
&= v(T_1 \cup T_2).
\end{aligned}$$

Observe that the first inequality holds since $u^{A_1} \geq 0$ and $v^{A_2} \geq 0$, the second inequality holds since $(u^{A_1}, v^{A_2}) \in Core(w)$. Moreover, for the grand coalition all inequalities are equalities, i.e., $x(M_1 \cup M_2) = v(M_1 \cup M_2)$, which completes the proof. □

Next, we will define two properties that fully characterize balanced AMO's.

An AMO $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ satisfies the *partitioning property* if for all $i \in \{1, 2\}$ there exists a partition $\{A_i^1, A_i^2, ..., A_i^{k_i}\}$ of A_i that satisfies the following two conditions:

(i) for every $r \in \{1, ..., k_i\}$, it holds that $\bigcap_{a \in A_i^r} S_i(a) \neq \emptyset$,

(ii) for every $r_1, r_2 \in \{1, ..., k_i\}, r_1 \neq r_2$, it holds $\left(\bigcup_{a \in A_i^{r_1}} S_i(a) \right) \cap \left(\bigcup_{a \in A_i^{r_2}} S_i(a) \right) = \emptyset$.

The first condition states that there is at least one player participating in each object of a partition element. The second condition states that player sets corresponding to two different partition elements have no player in common. Observe, that AMO's where each player is participating in precisely one object satisfies the partitioning property. We illustrate an AMO that satisfies the partitioning property in the following example.

Example 3.2 Let $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ be an AMO such that
$M_1 = \{1, 2, 3, 4, 5\}$, $M_2 = \{6, 7, 8, 9\}$, $A_1 = \{a_1, a_2, a_3\}, A_2 = \{b_1, b_2, b_3\}$, $S_1(a_1) = \{1, 2\}$, $S_1(a_2) = \{1, 3\}, S_1(a_3) = \{4, 5\}, S_2(b_1) = \{6, 7\}, S_2(b_2) = \{7, 8\}, S_2(b_3) = \{9\}$. Taking the partitions $A_1^1 = \{a_1, a_2\}, A_1^2 = \{a_3\}$ and $A_2^1 = \{b_1, b_2\}, A_2^2 = \{b_3\}$, it is straightforward to verify that this AMO satisfies the partitioning property. □

An AMO $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ satisfies the *intersection property* if $\bigcap_{a \in A_1} S_1(a) \neq \emptyset$ or $\bigcap_{b \in A_2} S_2(b) \neq \emptyset$. Hence, an AMO satisfies the intersection property if all row objects or all column objects have at least one player in common. The following example illustrates the intersection property.

Example 3.3 Let $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ be an AMO such that
$M_1 = \{1, 2, 3\}$, $M_2 = \{4, 5, 6\}$, $A_1 = \{a_1, a_2, a_3\}, A_2 = \{b_1, b_2\}$, $S_1(a_1) = \{1, 2\}$, $S_1(a_2) = \{1, 3\}, S_1(a_3) = \{2, 3\}, S_2(b_1) = \{4, 5\}, S_2(b_2) = \{5, 6\}$. Obviously, $\bigcap_{b \in A_2} S_2(b) = \{5\} \neq \emptyset$. Hence, this AMO satisfies the intersection property. □

Observe that the AMO of Example 3.2 does not satisfy the intersection property and the AMO of Example 3.3 does not satisfy the partitioning property.

Now, we can formulate the characterization of balanced AMO's.

Theorem 3.4 An AMO is balanced if and only if it satisfies the partitioning property or the intersection property.

PROOF: First observe that if $|A_1| = 1$ or $|A_2| = 1$ then trivially the intersection property holds and for any reward function the corresponding simple assignment game is balanced. Hence, we have to prove the theorem for $|A_1| \geq 2$ and $|A_2| \geq 2$.

We first prove the "if"-part. Let $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ be an AMO and let R be a reward matrix. Let $(M_1 \cup M_2, v)$ be the corresponding simple assignment game. First assume that $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ satisfies the partitioning property. Hence, for $i \in \{1, 2\}$ there exists a partition $\{A_i^1, A_i^2, ..., A_i^{k_i}\}$ that satisfies the conditions in the partitioning property. From this AMO we create a new AMO that satisfies the condition of Proposition 3.1. First, we will merge the objects in one partition element to one object and these merged objects will be owned by the intersection of the players that own the objects in the partition element. Second, the reward between two new objects is the maximum reward that can be achieved between two objects in the corresponding partition elements. Formally, $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ induces an AMO $((\bar{M}_1, \bar{M}_2), (\bar{A}_1, \bar{A}_2), (\bar{S}_1, \bar{S}_2))$, where $\bar{M}_i = \bigcup_{r=1}^{k_i} \bigcap_{a \in A_i^r} S_i(a)$, $\bar{A}_i = \{a_i^1, ..., a_i^{k_i}\}$ and $\bar{S}_i(a_i^r) = \bigcap_{a \in A_i^r} S_i(a)$ for all $i = 1, 2$, $r \in \{1, ..., k_i\}$. Let $\bar{R}$ be defined by $\bar{R}_{a_1^{r_1} a_2^{r_2}} = \max_{a \in A_1^{r_1}, b \in A_2^{r_2}} R_{ab}$ for all $r_1 \in \{1, ..., k_1\}, r_2 \in \{1, ..., k_2\}$. Let $(\bar{M}_1 \cup \bar{M}_2, w)$ be the game corresponding to the induced AMO and $\bar{R}$. Because the induced AMO satisfies the condition of Proposition 3.1, we have that $Core(w) \neq \emptyset$. Let $\bar{x} \in Core(w)$ and define $x \in I\!R^{M_1 \cup M_2}$ as $x_i = \bar{x}_i$ if $i \in \bar{M}_1 \cup \bar{M}_2$ and $x_i = 0$ otherwise. We show that $x \in Core(v)$. First observe that $w(S) = v(S \cup \bar{M}_1' \cup \bar{M}_2')$ for all $S \subset \bar{M}_1 \cup \bar{M}_2$ with $\bar{M}_i' = M_i \backslash \bar{M}_i, i \in \{1, 2\}$. Then

$$
\begin{aligned}
x(M_1 \cup M_2) &= \bar{x}(\bar{M}_1 \cup \bar{M}_2) \\
&= w(\bar{M}_1 \cup \bar{M}_2) \\
&= v(M_1 \cup M_2)
\end{aligned}
$$

and

$$
\begin{aligned}
x(S) &= \bar{x}(S \cap (\bar{M}_1 \cup \bar{M}_2)) \\
&\geq w(S \cap (\bar{M}_1 \cup \bar{M}_2)) \\
&= v((S \cap (\bar{M}_1 \cup \bar{M}_2)) \cup \bar{M}_1' \cup \bar{M}_2') \\
&\geq v(S)
\end{aligned}
$$

for all $S \subset M_1 \cup M_2$. The first inequality holds because $\bar{x} \in Core(w)$ and the second because $S \subset (S \cap (\bar{M}_1 \cup \bar{M}_2)) \cup \bar{M}_1' \cup \bar{M}_2'$. Hence, $x \in Core(v)$.

Secondly, assume that $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ satisfies the intersection property. Without loss of generality we assume that

$$\bigcap_{a \in A_1} S_1(a) = B \neq \emptyset.$$

Then $v(S) = 0$ if $B \nsubseteq S$. Also $v(S) \leq \max_{a \in A_1, b \in A_2}\{R_{ab}\}$ if $B \subseteq S$ and $v(M_1 \cup M_2) = \max\limits_{a \in A_1, b \in A_2} R_{ab}$. Define the vector $x \in I\!R^{M_1 \cup M_2}$ by $x_i = v(M_1 \cup M_2)$ for some $i \in B$ and $x_j = 0$, $j \neq i$, otherwise. It is straightforward to check that $x \in Core(v)$.

Second we prove the "only if"-part. We show that if an AMO does not satisfy the partition property and the intersection property, then it is not balanced. Assume that $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ does not satisfy the intersection property. This implies that $\bigcap\limits_{a \in A_1} S_1(a) = \emptyset$ and $\bigcap\limits_{b \in A_2} S_2(b) = \emptyset$. We distinguish between two cases:

(i) For all $a_1, a_2 \in A_1$ holds $S_1(a_1) \cap S_1(a_2) \neq \emptyset$, or
for all $b_1, b_2 \in A_2$ holds $S_2(b_1) \cap S_2(b_2) \neq \emptyset$,

(ii) There exists $a_1, a_2 \in A_1$ with $S_1(a_1) \cap S_1(a_2) = \emptyset$, and
there exists $b_1, b_2 \in A_2$ with $S_2(b_1) \cap S_2(b_2) = \emptyset$.

Assume that (i) holds. Without loss of generality assume that $S_1(a_1) \cap S_1(a_2) \neq \emptyset$ for all $a_1, a_2 \in A_1$. Then take the reward matrix R in which every entry is equal to 1. Let $(M_1 \cup M_2, v)$ be the simple assignment game corresponding to $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ and R. Since the intersection of the player sets of any two objects in A_1 is not empty, at most one object of A_1 can be assigned to an object of A_2. Hence, it follows that $v(M_1 \cup M_2) = 1$ and, due to the fact that the intersection property is not satisfied, $v(M_1 \cup M_2 \backslash \{i\}) = 1$ for all $i \in M_1 \cup M_2$. Obviously, $Core(v) = \emptyset$. So, $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ is not balanced. Assume (ii) holds. Let $G = (V, E)$ be the graph with $V = A_1$ and the edge set E defined by $(a,b) \in E$ if $a,b \in V, S_1(a) \cap S_1(b) \neq \emptyset$. The graph G consists of several connected components that partition A_1 in a natural way. Let $\{B_1, B_2, ..., B_k\}$ be this partition. Hence, if $a,b \in B_j$ for some $j \in \{1,...,k\}$ then there exists a path from a to b. A similar partition can be constructed for A_2. Since $((M_1,M_2),(A_1,A_2),(S_1,S_2))$ does not satisfy the partitioning property, we can assume (without loss of generality) that there exists a $j \in \{1,...,k\}$ such that $\bigcap_{a \in B_j} S_1(a) = \emptyset$. Now, one of the following two cases is satisfied:

(α) for all $a_1, a_2 \in B_j$,$a_1 \neq a_2$, it holds $S_1(a_1) \cap S_1(a_2) \neq \emptyset$,

(β) there exists $a_1, a_2 \in B_j$,$a_1 \neq a_2$, with $S_1(a_1) \cap S_1(a_2) = \emptyset$.

If case (α) holds, we define the reward matrix R by $R_{ab} = 1$ for all $a \in B_j, b \in A_2$ and $R_{ab} = 0$ otherwise. Since $\bigcap_{a \in B_j} S_1(a) = \emptyset$, B_j contains at least two objects. Hence, we can conclude, similarly to case (i), that $v(M_1 \cup M_2) = v((M_1 \cup M_2)\backslash\{i\}) = 1$ for all $i \in M_1 \cup M_2$. Hence, $Core(v) = \emptyset$. So, $((M_1,M_2),(A_1,A_2),(S_1,S_2))$ is not balanced.

If case (β) holds, observe that there exists $c_1, c_2, c_3 \in B_j$ such that $S_1(c_1) \cap S_1(c_2) = \emptyset, S_1(c_1) \cap S_1(c_3) \neq \emptyset$ and $S_1(c_2) \cap S_1(c_3) \neq \emptyset$. This observation holds by the following argument. Recall that B_j is a connected component of the graph G. Because $S_1(a_1) \cap S_1(a_2) = \emptyset$ for some $a_1, a_2 \in B_j, a_1 \neq a_2$, the graph is not complete. Hence, there is a path of length two in G. The vertices on this path satisfy the required property. According to assumption (ii) there are $b_1, b_2 \in A_2$ with $S_2(b_1) \cap S_2(b_2) = \emptyset$. Now, we define the matrix R by $R_{ab} = 1$ if $(a,b) \in \{(c_1,b_1),(c_2,b_1),(c_3,b_2)\}$ and $R_{ab} = 0$ otherwise. Obviously, $v(M_1 \cup M_2) = 1$ because $S_1(c_1) \cap S_1(c_3) \neq \emptyset$ and $S_1(c_2) \cap S_1(c_3) \neq \emptyset$. Because $S_2(b_1) \cap S_2(b_2) = \emptyset$ and $S_1(c_1) \cap S_1(c_2) = \emptyset$ we also have for all $i \in M_1 \cup M_2$ that $v((M_1 \cup M_2)\backslash\{i\}) = 1$. Hence, we conclude that $Core(v) = \emptyset$. So, $((M_1,M_2),(A_1,A_2),(S_1,S_2))$ is not balanced. □

4 k-AMO games

In this section we discuss k-AMO games, relaxations of simple assignment games. We provide sufficient conditions for balancedness of k-AMO games. First we will introduce the class of k-AMO games.

Let $((M_1,M_2),(A_1,A_2),(S_1,S_2))$ be an AMO. A matching μ in $B_1(T_1) \times B_2(T_2)$ is called k-*admissible* for coalition $T_1 \cup T_2$ if for any $k+1$ pairwise distinct pairs $(a_1,b_1),...,(a_{k+1},b_{k+1}) \in$

μ it holds that $\bigcap_{i=1}^{k+1} S(a_i) = \emptyset$ and $\bigcap_{i=1}^{k+1} S(b_i) = \emptyset$. Observe that the restriction to k-admissible matchings implies that a player that owns more than k objects can assign at most k objects.

The set of k-admissible matchings with respect to a coalition $T_1 \cup T_2$ is denoted by $\mathcal{A}_k(T_1 \cup T_2)$. Note that $\mathcal{A}_k(T_1 \cup T_2) \subset \mathcal{A}_{k+1}(T_1 \cup T_2)$, for all $k \in I\!N$ and for all $T_1 \cup T_2$. Obviously, $\mathcal{A}_1(T_1 \cup T_2) = \mathcal{A}(T_1 \cup T_2)$ for all $T_1 \cup T_2 \subset M_1 \cup M_2$.

Let $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ be an AMO and let R be a non-negative reward matrix. The corresponding k-AMO game $(M_1 \cup M_2, v_k)$ is defined for all $T_1 \cup T_2$, with $T_1 \subset M_1, T_2 \subset M_2$ by

$$v_k(T_1 \cup T_2) = \max\{ \sum_{(a,b)\in\mu} R_{ab} : \mu \in \mathcal{A}_k(T_1 \cup T_2)\}. \tag{2}$$

Observe that a simple assignment game coincides with a 1-AMO game. The following example shows a 2-AMO game.

Example 4.1 Consider the AMO of Example 2.1. Let $(M_1 \cup M_2, v_2)$ be the corresponding 2-AMO game and consider coalition $\{1, 2, 3, 4\}$. In contrast to the simple assignment game in Example 2.1 in the 2-AMO game the matching $\{(r_1, c_1), (r_2, c_2)\}$ is admissible. It is readily verified that $v_2(\{1, 2, 3, 4\}) = 4$ and $(2, 2, 0, 0) \in Core(v_2)$. $\square$

Next, we will formulate (2) as an Integer Linear Programming problem. For every $i \in M_1 \cup M_2$, define the vector $\mathbf{e_i} \in I\!R^{A_1 \times A_2}$ by $\mathbf{e_i}_{ab} = 1$ if $i \in S_1(a) \cup S_2(b)$, $a \in A_1, b \in A_2$, and $\mathbf{e_i}_{ab} = 0$ otherwise. For every $c \in A_1 \cup A_2$, define the vector $\mathbf{f_c} \in I\!R^{A_1 \times A_2}$ by $\mathbf{f_c}_{ab} = 1$ if $c = a$ or $c = b$, $a \in A_1, b \in A_2$, and $\mathbf{f_c}_{ab} = 0$ otherwise. Define the matrix $A \in I\!R^{(M_1 \cup M_2 \cup A_1 \cup A_2) \times (A_1 \times A_2)}$ by $A = \begin{bmatrix} \mathbf{e} \\ \mathbf{f} \end{bmatrix}$, where $\mathbf{e}$ consists of all vectors $\mathbf{e_i}$, $\mathbf{i} \in M_1 \cup M_2$ and $\mathbf{f}$ consists of all vectors $\mathbf{f_c}$, $\mathbf{c} \in \mathbf{A_1} \cup \mathbf{A_2}$. Let $T_1 \subset M_1, T_2 \subset M_2$. Define the vector $\mathbf{p^{T_1 \cup T_2}(k)} \in I\!R^{M_1 \cup M_2}$ by $p_i^{T_1 \cup T_2}(k) = k$ if $i \in T_1 \cup T_2$ and $p_i^{T_1 \cup T_2}(k) = 0$ otherwise, and define $\mathbf{q^{T_1 \cup T_2}} \in I\!R^{A_1 \cup A_2}$ by $q_a^{T_1 \cup T_2} = 1$ if $S_i(a) \subset T_1 \cup T_2$ for some $i \in \{1, 2\}$ and $q_a^{T_1 \cup T_2} = 0$ otherwise. We define the vector $\mathbf{u^{T_1 \cup T_2}(k)} \in I\!R^{M_1 \cup M_2 \cup A_1 \cup A_2}$ by $\mathbf{u^{T_1 \cup T_2}(k)} = \begin{bmatrix} \mathbf{p^{T_1 \cup T_2}(k)} \\ \mathbf{q^{T_1 \cup T_2}} \end{bmatrix}$. So, the j-th row of $\mathbf{u^{T_1 \cup T_2}(k)}$ corresponds to the j-th row of the matrix A, i.e., if the j-th row of $\mathbf{u^{T_1 \cup T_2}(k)}$ represents player i or object a, then also the j-th row of the matrix A represents player i

or object a. Now, it is straightforward to verify that (2) is equivalent to the following ILP problem:

$$v_k(T_1 \cup T_2) = \max \sum_{a \in A_1, b \in A_2} R_{ab} y_{ab} \tag{3}$$

$$\text{subject to} \quad Ay \leq \mathbf{u}^{\mathbf{T_1 \cup T_2}}(\mathbf{k}),$$

$$y_{ab} \in \{0,1\} \text{ for all } a \in A_1, b \in A_2.$$

Deng et al. (1999) introduced the class of combinatorial games. Let N be the finite set of players and M be a finite set with cardinality n and m, respectively. A combinatorial game (N, v_{com}) is defined for all $S \subset N$ by

$$v_{com}(S) = \max\{c^T y | By \leq 1^S, y \in \{0,1\}^m\}, \tag{4}$$

where B is a $n \times m$ $\{0,1\}$-matrix and $1_i^S = 1$ if $i \in S$ and $1_i^S = 0$ otherwise. Deng et al. (1999) proved the following theorem.

Theorem 4.2 *(Deng et al. (1999))*
A combinatorial game (N, v_{com}) is balanced if and only if the optimum of the ILP-problem (4) of the grand coalition coincides with the optimum of its LP-relaxation.

It follows immediately that for $k = 1$ the ILP formulation (3) is a special case of the ILP formulation of (4). Hence, we have the following corollary.

Corollary 4.3 A simple assignment game is balanced if and only if the solution of the ILP-problem (3) of the grand coalition coincides with the solution of its LP-relaxation.

Obviously, the result of Deng et al. (1999) can not be applied to k-AMO games for $k \geq 2$ because the right-hand side vector in (3) is not a $\{0,1\}$-vector. Indeed, the following example illustrates that the core of a k-AMO game can be non-empty although the ILP optimum does not coincide with the optimum of its LP relaxation.

Example 4.4 Figure 2 displays an AMO $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ with reward matrix R given by

$$
\begin{array}{c|c}
 & \begin{array}{cccc} \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} \end{array} \\
\begin{array}{c} \mathbf{1,2,3} \\ \mathbf{1,2,4} \\ \mathbf{1,3,4} \\ \mathbf{2,3,4} \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{array}
$$

Figure 2: An AMO with reward matrix R.

Then, for the corresponding 2-AMO game $(M_1 \cup M_2, v_2)$ it holds $v_2(M_1 \cup M_2) = 2$, which is the optimal solution of the ILP-problem (3) of the grand coalition. But the optimal solution of the corresponding LP-relaxation is $\frac{8}{3}$ which is larger than the worth of the grand coalition. However, it is straightforward to check that $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0) \in Core(v_2)$.

The following theorem shows that the coincidence of the optimum of the ILP-problem and its relaxation is still a sufficient condition to have a balanced k-AMO game.

Theorem 4.5 Let $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ be an AMO and let R be a reward matrix. Let $(M_1 \cup M_2, v_k)$ be the corresponding k-AMO game. If the optimum of the ILP-problem (3) of the grand coalition coincides with the optimum of its LP-relaxation, then the corresponding k-AMO game is balanced.

PROOF: The LP relaxation of (3) is equal to

$$\max \sum_{a \in A_1, b \in A_2} R_{ab} y_{ab} \tag{5}$$

$$
\begin{aligned}
\text{subject to} \quad & Ay \leq \mathbf{u}^{\mathbf{T_1} \cup \mathbf{T_2}}(\mathbf{k}), \\
& y \geq 0.
\end{aligned}
$$

The dual of (5) is equal to

$$\min z^T \cdot \mathbf{u}^{\mathbf{T_1} \cup \mathbf{T_2}}(k) \tag{6}$$

$$
\begin{aligned}
\text{subject to} \quad & A^T z \geq \mathbf{R}, \\
& z \geq 0,
\end{aligned}
$$

where $\mathbf{R} = (R_{ab})_{a\in A_1, b\in A_2}$. Let $(M_1 \cup M_2, v_k)$ be the corresponding k-AMO game. By the assumption in the theorem and the duality theorem of LP there exists an optimal solution z^* of (6) such that $v_k(M_1 \cup M_2) = z^{*T} \cdot \mathbf{u}^{\mathbf{M_1 \cup M_2}}(k)$. Now, we define the $|M_1| + |M_2|$ dimensional vector $\mathbf{x}$ by $x_i = kz_i^* + \sum_{a\in A_1: i\in S_1(a)} \frac{z_a^*}{|S_1(a)|}$ for all $i \in M_1$ and $x_j = kz_j^* + \sum_{b\in A_2: j\in S_2(b)} \frac{z_b^*}{|S_2(b)|}$ for all $j \in M_2$. We prove that x is in the core of v_k. Let $T_1 \subset M_1, T_2 \subset M_2$. Then

$$\begin{aligned}
x(T_1 \cup T_2) &= \sum_{i\in T_1} kz_i^* + \sum_{i\in T_1} \sum_{a\in A_1: i\in S_1(a)} \frac{z_a^*}{|S_1(a)|} + \sum_{j\in T_2} kz_j^* + \sum_{j\in T_2} \sum_{b\in A_2: j\in S_2(b)} \frac{z_b^*}{|S_2(b)|} \\
&\geq \sum_{i\in T_1} kz_i^* + \sum_{a\in A_1: S_1(a)\subseteq T_1} z_a^* + \sum_{j\in T_2} kz_j^* + \sum_{b\in A_2: S_2(b)\subseteq T_2} z_b^* \\
&= z^* \cdot \mathbf{u}^{\mathbf{T_1 \cup T_2}}(k) \\
&\geq v_k(T_1 \cup T_2),
\end{aligned}$$

where the last inequality holds because for any coalition the feasible region in the dual problem (6) is identical. Hence, z^* is a feasible solution (6) for $T_1 \cup T_2$. Evidently, for $M_1 \cup M_2$ all inequalities become equalities, which completes the proof. □

We conclude this section with two examples. They show that there is no relation between the cores of consecutive k-AMO games.

Example 4.6 Figure 3 displays an AMO $((M_1, M_2), (A_1, A_2), (S_1, S_2))$ with reward matrix R given by

$$\begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{1,2} \\ \mathbf{1,3} \\ \mathbf{2,3} \end{array}
\begin{array}{c}
\begin{array}{cccccc} \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{4,5} & \mathbf{4,6} & \mathbf{5,6} \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}$$

Figure 3: An AMO with reward matrix.

Let $(M_1 \cup M_2, v_1)$ be the corresponding simple assignment game. Then $v_1(M_1 \cup M_2) = 3$ and the core is non-empty since for instance, $(1, 1, 1, 0, 0, 0) \in Core(v_1)$. We now show that $Core(v_2) = \emptyset$.

Let $(M_1 \cup M_2, v_2)$ be the corresponding 2-AMO game. Then $v_2(\{1, 2, 4, 5\}) = 3$, $v_2(\{1, 3, 4, 6\}) = 3$, $v_2(\{2, 3, 5, 6\}) = 3$ and $v_2(M_1 \cup M_2) = 4$. Suppose $(x_1, x_2, x_3, x_4, x_5, x_6) \in$

$Core(v_2)$, then $8 = 2(x_1 + x_2 + ... + x_6) = (x_1 + x_2 + x_4 + x_5) + (x_1 + x_3 + x_4 + x_6) + (x_2 + x_3 + x_5 + x_6) \geq v_2(\{1, 2, 4, 5\}) + v_2(\{1, 3, 4, 6\}) + v_2(\{2, 3, 5, 6\}) = 9$, which is a contradiction. Therefore, $Core(v_2) = \emptyset$.

Example 4.7 Recall that the AMO of Example 2.1 and Example 4.1 are identical. In Example 2.1 it is shown that $Core(v_1) = \emptyset$, whereas in Example 4.1 it is shown that $Core(v_2) \neq \emptyset$.

Finally, we remark that k-AMO games, where $k \geq \max\{|A_1|, |A_2|\}$, are balanced, because these games can be considered as a subgame of the assignment games of Shapley and Shubik (1972) in which the objects are the players.

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