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Assessing Credit with Equity: A CEV Model with Jump to Default

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Assessing Credit with Equity: A CEV Model with Jump to Default

Abstract

Unlike in structural and reduced-form models, we use equity as a liquid and observable primitive to analytically value corporate bonds and credit default swaps. Restrictive assumptions on the firm's capital structure are avoided. Default is parsimoniously represented by equity value hitting the zero barrier either diffusively or with a jump, which implies non-zero credit spreads for short maturities. Easy cross-asset hedging is enabled. By means of a tersely specified pricing kernel, we also make analytic credit-risk management possible under systematic jump-to-default risk.

JEL-Classification: G12, G33.

Keywords: Equity, Corporate Bonds, Credit Default Swaps, Constant-Elasticity-of-Variance (CEV) Diffusion, Jump to Default.

1 Introduction

Investors have been showing appetite for models that simultaneously handle credit and equity instruments, which is important in managing a portfolio of these two instruments. Indeed, cross-asset trading of credit risk has been gaining momentum among credit hedge funds and banks. The rise of capital structure arbitrage is a good example (see Yu (2004)). Reduced-form models are not of great help, as they miss the linkage to the firm's capital structure. Structural models are driven by the value evolution in firm's assets. The assets-value evolution is often assumed to be diffusive so that the default can be seen predictably coming by observing changes in the capital structure of the firm (see the seminal papers of Merton (1974) and Black and Cox (1976)). While appealing, structural models suffer when it comes to applications. The underlying (the sum of firm's liabilities and equity) is illiquid and often non-tradable. Obtaining accurate asset volatility forecasts and reliable capital structure leverage data is difficult. Predictability of the default event implies the counterfactual prediction of zero credit spreads for short maturities[1] and, last but not least, arbitrary use of the structural default barrier is often a temptation hard to resist-endogenous barriers are impractical because of the unrealistic capital-structure assumptions under which they are derived.

[1] Zhou (1997) posits assets-value jumps to overcome default predictability. Duffie and Singleton (2001) explain such jumps with the presence of incomplete accounting information.

We propose a parsimonious credit risk model that does look at the firm's balance sheet but avoids the application mishaps of structural models. We take as underlying the most liquid and observable corporate security: Equity. This modelling choice brings in hedging viability and the possibility of reliable model calibration-leverage information from book values can be circumvented. We parsimoniously represent default as equity value hitting the zero barrier either diffusively or with a jump. The presence of an equity-value drop to zero has its credit-risk foundation in the incompleteness of accounting information (see Duffie and Lando (2001)) and rules out default predictability. We assume that the continuous-path part of equity value is a Constant-Elasticity-of-Variance (CEV) diffusion[2], which enables absorption at zero, and that the jump to default is driven by an independent Poisson process. Such distributional assumptions prompt us to obtain closed forms for Corporate Bond (CB) prices and Credit Default Swap (CDS) fees, from which hedge ratios can be easily calculated. Those assumptions and a careful specification of the state-price density also empower analytic credit-risk management-we provide a closed form for the objective default probabili-

[2]The CEV process has been first introduced to finance by Cox (1975). Among others, the CEV-based asset-pricing literature includes the works of Albanese, Campolieti, Carr, and Lipton (2001), Beckers (1980), Boyle and Tian (1999), Cox and Ross (1976), Davydov and Linetsky (2001), Emanuel and MacBeth (1982), Forde (2005), Goldenberg (1991), Leung and Kwok (2005), Lo, Hui, Yuen (2000), Lo, Hui, and Yuen (2001), Lo, Tang, Ku, and Hui (2004), Sbuelz (2004), and Schroder (1989).

ties in the presence of systematic jump-to-default risk. Albanese and Chen (2004) and Campi and Sbuelz (2004) also use a CEV-equity model to price credit instruments but they disregard the default predictability issue. In deriving closed-form values, we build upon a CEV result in Campi and Sbuelz (2004). Brigo and Tarenghi (2004), Naik, Trinh, Balakrishnan, and Sen (2003) and Trinh (2004) introduce a hybrid debt-equity model that considers equity as primitive but that, like structural models, necessitates a free default barrier, which is then left to potentially *ad-hoc* uses-equity value is assumed to be a geometric Brownian motion, except in Brigo and Tarenghi (2004)[3]. Das and Sundaram (2003) have proposed an equity-based model that accounts for default risk, interest risk, and equity risk using a lattice framework. As such, they do not seek hedger-friendly analytic solutions. Numerical credit risk pricing based on equity has also been suggested by the convertible bond literature (see, for example, Andersen and Andreasen (2000), Andersen and Buffum (2003), and Tsiveriotis and Fernandes (1998); McConnell and Schwartz (1986) ignore the possibility of bankruptcy). Linetsky (2005) builds upon the convertible bond literature to assess zero-coupon CB prices within a geometric-Brownian-motion model with jump-like bankruptcy where the hazard rate of bankruptcy is a negative power of the share price. The dependence of the hazard rate on the share price strongly com-

[3]Brigo and Tarenghi (2004) and Hui, Lo, and Tsang (2003) employ a flexible time-varying default barrier. Hui, Lo, and Tsang (2003) do not take equity as the underlying.

plicates the analysis[4].

The rest of the work is organized as follows. Section 2 describes the underlying equity value process. Section 3 provides analytic results for CBs and CDSs. Section 4 specifies a pricing kernel that permits analytic objective default probabilities. After the conclusions (Section 5), an Appendix gathers proofs and technical details.

2 The equity value

Under the equivalent martingale measure $\mathbb{Q}$, the reference entity's share-price process $\{S\}$ has the following pre-default jump-diffusion dynamics:

$$\frac{dS_t}{S_{t-}} = (r-q)\,dt + \sigma S_{t-}^{\rho-1} dz_t - (dN_t - \lambda dt)\,,$$

[4] The valuation formulae in Linetsky (2005) are spectral expansions that embed single integrals with respect to the spectral parameter and calculations imply the use of numerical-integration routines.

$$
\begin{aligned}
S_0 &= S && \text{(current share price)}, \\
S_{t-} &= \lim_{\varepsilon \searrow 0} S_{t-\varepsilon} && \text{(left time limit)}, \\
\rho - 1 &< 0 && \text{(constant elasticity of the diffusive volatility)},
\end{aligned}
$$

$$
\begin{aligned}
N_t &= 1_{\{t \geq \tau\}} && \text{(first-jump-stopped Poisson process)}, \\
\tau &= \inf \{t : N_t = 1\} && \text{(time of jump-like default)}, \\
E^{\mathbb{Q}} \left[1_{\{\tau > T\}}\right] &= \exp(-\lambda T) && \text{(chance of surviving to jump-like default)}, \\
T &> 0 && \text{(finite maturity, in years)}, \\
\lambda &\geq 0 && \text{(jump-to-default intensity)},
\end{aligned}
$$

where r is the constant riskfree rate, q is the constant dividend yield, σ ($\sigma > 0$) is a constant scale factor for the diffusive volatility, and dz is the increment of a Wiener process under $\mathbb{Q}$. The processes $\{z\}$ and $\{N\}$ are independent. According to the boundary classification, an inverse relationship between volatility and share price ($\rho - 1 < 0$) is necessary to have absorption at zero in the absence of jumps. Such an assumption of inverse relationship is unlikely to be counterfactual. The time of absorption at zero in the absence of jumps is ξ,

$$\xi = \inf \{ t : S_t = 0, N_t = 0 \} ,$$

whereas the time of absorption at zero *tout court* is the minimum between τ and ξ, that is

$$\tau \wedge \xi = \inf \{ t : S_t = 0 \} .$$

We take the point 0 to be the absorbing state of the share-price process $\{S\}$, so that, once default has occurred, the share price remains at zero,

$$S_t = 0 , \quad \forall t \geq \tau \wedge \xi .$$

3 Analytic results for CBs and CDSs

Let $V^{\mathbb{Q}}(S,T,y)$ be the T-truncated Laplace transform of $\tau \wedge \xi$'s probability density function under $\mathbb{Q}$ ($\mathbb{Q}$-p.d.f.) with Laplace parameter y ($y \geq 0$),

$$V^{\mathbb{Q}}(S,T,y) = E_0^{\mathbb{Q}}\left[\exp(-y \cdot \tau \wedge \xi)\, 1_{\{\tau\wedge\xi\leq T\}}\right].$$

Such a quantity is of great importance, as it is the building block for the analytic pricing of CBs and CDSs. $V(S,T,r)$ represents the fair present value of 1 unit of currency at the reference entity's default if default occurs within T, while $V^{\mathbb{Q}}(S,T,0)$ represents the risk-neutral probability of default within T.

The next proposition is a neat and useful result stemming from the independence between $\{z\}$ and $\{N\}$. It gives an analytic characterization of $V^{\mathbb{Q}}(S,T,y)$. It states that the quantity of interest is the linear convex combination of the adjusted risk-neutral probability of default within T (with weight $\frac{\lambda}{y+\lambda}$) and of the $(y+\lambda)$-discounted value of 1 unit of currency at the diffusive default within T (with weight $\frac{y}{y+\lambda}$). The latter is the T-truncated

Laplace transform of ξ's $\mathbb{Q}$-p.d.f. with Laplace parameter $y + \lambda$

$$E_0^{\mathbb{Q}}\left[\exp\left(-\left(y+\lambda\right)\xi\right)1_{\{\xi\leq T\}}\right]$$

and its closed form[5] has been recently derived by Campi and Sbuelz (2004). The closed form is provided in the Appendix.

Proposition 1 *Under the above assumptions, the T-truncated Laplace transform of $\tau \wedge \xi$'s $\mathbb{Q}$-p.d.f. with Laplace parameter y is*

$$\begin{aligned} V^{\mathbb{Q}}\left(S,T,y\right) &= \frac{\lambda}{y+\lambda}\left[1-\exp\left(-\left(y+\lambda\right)T\right)\left(1-E_0^{\mathbb{Q}}\left[1_{\{\xi\leq T\}}\right]\right)\right] \\ &\quad + \frac{y}{y+\lambda}E_0^{\mathbb{Q}}\left[\exp\left(-\left(y+\lambda\right)\xi\right)1_{\{\xi\leq T\}}\right]. \end{aligned}$$

[5] Davidov and Linetsky (2001), see pp. 953 and 956, point out that the T-truncated Laplace transform of ξ's $\mathbb{Q}$-p.d.f. with Laplace parameter $y + \lambda$ can be obtained by numerically inverting the closed-form non-truncated Laplace transform

$$\frac{1}{a}E_0^{\mathbb{Q}}\left[\exp\left(-\left(y+\lambda+a\right)\xi\right)\right],$$

where the inversion parameter is $a > 0$.

Proof. See the Appendix. ■

Proposition 1 empowers analytic pricing of CBs and CDSs. Consider a reference entity's CB that has face value F and pays an (annualized) coupon C at regular $\frac{1}{k}$-spaced dates T_j up to its maturity T (k is a positive integer). For the sake of simplifying notation, we take the maturity T to be a rational number of the type $\frac{n}{k}$, $n \in N$. The fair CB price is

$$\begin{aligned} P_{CB}(S,T,r) &= \sum_{j=1}^{kT} \frac{1}{k} \exp(-rT_j) \left[1 - V^{\mathbb{Q}}(S,T_j,0)\right] C \\ &\quad + \left[1 - V^{\mathbb{Q}}(S,T,0)\right] F \\ &\quad + V^{\mathbb{Q}}(S,T,r) \cdot R \cdot F, \end{aligned}$$

where R is the recovery rate at default, which is a fixed historical data input in applications. CB's defaultable part is assessed under the assumption of Recovery of Face Value at Default (RFV), which bears the value $V^{\mathbb{Q}}(S,T,r) \cdot R \cdot F$. Under RFV, CB holders receive the same fractional recovery R of the face value F at default for CBs issued by the reference entity regardless of maturity. Guha and Sbuelz (2003) show that the RFV recovery form is consistent with typical bond indenture language (for example, the claim acceleration clause), defaulted bond price data, and stylised facts that are relevant for interest rate hedging (for example, the low duration of high-yield bonds).

Consider a CDS related to the CB just described. It offers a protection payment of $(1 - R)F$ in exchange for an (annualized) fee f_{CDS} paid at regular $\frac{1}{k}$-spaced dates up to the contract's maturity. The fair CDS fee is

$$f_{CDS}(S,T,r) = \frac{V^{\mathbb{Q}}(S,T,r)(1-R)}{\sum_{j=1}^{kT} \frac{1}{k} \exp(-rT_j)\left[1 - V^{\mathbb{Q}}(S,T_j,0)\right]}.$$

The holder of a CB can achieve total recouping of the face value F at default by being long a CDS. Being short $\frac{\partial}{\partial S} P_{CB}(S,T,r)$ shares Delta-hedges[6] against the pre-default price shocks driven by diffusive news. Recent evidence shows that such equity-based hedges perform reasonably well for high-yield CBs (see Naik, Trinh, Balakrishnan, and Sen (2003)). Our model also states that, in the case of a jump to default ($\tau \wedge \xi = \tau$), Delta hedging recoups a

[6] Parallel shifts of the (flat) term stucture of the interest rates can be hedged by selling a portfolio of default-free bonds that has interest-rate sensitivity equal to $\frac{\partial}{\partial r} P_{CB}(S,T,r)$. Such a hedge ratio can be easily calculated in our model.

fraction

$$\frac{\frac{\partial}{\partial S}P_{CB}\left(S_{\tau-},T-\tau,r\right)S_{\tau-}}{P_{CB}\left(S_{\tau-},T-\tau,r\right)-R\cdot F}$$

of the CB loss suffered at default. CB's analytic Delta-hedge ratio $\frac{\partial}{\partial S}P_{CB}\left(S,T,r\right)$ is provided in the Appendix. Also, given analytic CB prices, an easy and effective measure of the Delta-hedge ratio is

$$\frac{\partial}{\partial S}P_{CB}\left(S,T,r\right) \simeq \frac{P_{CB}\left(S+\varepsilon,T,r\right)-P_{CB}\left(S-\varepsilon,T,r\right)}{2\varepsilon}$$

for a small positive ε.

4 The objective default probability

Our equity-based model contributes also to credit risk management by being conducive to closed forms for the objective default probability, $V^{\mathbb{P}}\left(S,T,0\right)$, with

$$V^{\mathbb{P}}\left(S,T,y\right) = E_0^{\mathbb{P}}\left[\exp\left(-y\cdot\tau\wedge\xi\right)1_{\{\tau\wedge\xi\leq T\}}\right],$$

where $\mathbb{P}$ is the objective probability measure. A parsimonious and closed-form-conducive way of specifying the dynamics of the share price process $\{S\}$ under the objective measure is the following:

$$\frac{dS_t}{S_{t-}} = \mu_{\mathbb{P}} dt + \sigma S_{t-}^{\rho-1} dz_t^{\mathbb{P}} - \left(dN_t^{\mathbb{P}} - \lambda_{\mathbb{P}} dt\right),$$

$$\mu_{\mathbb{P}} = r - q + \theta \cdot \sigma + E^{\mathbb{P}}\left[\left(\exp\left(\zeta\right) - 1\right)\right] \lambda_{\mathbb{P}},$$

$$\theta \cdot \sigma > 0 \qquad \text{(premium for the diffusive risk)},$$

$$E^{\mathbb{P}}\left[\left(\exp\left(\zeta\right) - 1\right)\right] \lambda_{\mathbb{P}} > 0 \qquad \text{(premium for the jump-like default risk)}.$$

Such a terse specification of $\{S\}$'s $\mathbb{P}$-dynamics makes a neat account of systematic jump-like default risk. The risk-neutral jump-to-default intensity λ maintains a simple link to the objective jump-to-default intensity $\lambda_{\mathbb{P}}$ ($\lambda_{\mathbb{P}} > 0$):

$$\lambda = E^{\mathbb{P}}\left[\exp\left(\zeta\right)\right] \lambda_{\mathbb{P}}.$$

If the jump-like default risk disappears ($\lambda_{\mathbb{P}} \searrow 0$), its premium shrinks to zero and the risk-neutral jump-to-default intensity does so as well. In the case of a jump to default ($\tau \wedge \xi = \tau$), the state-price-density process $\{\pi\}$ that backs the measure $\mathbb{Q}$ jumps from $\pi_{\tau-}$ to π_τ,

$$\pi_\tau = \pi_{\tau-} \exp(\zeta).$$

Since π_τ provides the fair present value of 1 unit of currency received at the time of jump-like default per unit probability of such a dislikeable event, it is reasonable to impose the restriction that π_τ must always be at least as much as $\pi_{\tau-}$ is. This implies that ζ, which is a random variable independent from $\{z\}$ and $\{N\}$, must be non-negative. The criterion of parameter parsimony suggests to take for ζ a one-parameter non-negative distribution. One such distribution is the discrete Poisson distribution with parameter ϕ ($\phi > 0$) and with support $\{0, 1, 2, ...\}$, so that the expectation $E^{\mathbb{P}}[\exp(\zeta)]$ admits a concise closed form,

$$
\begin{aligned}
E^{\mathbb{P}}\left[\exp\left(\zeta\right)\right] &= \exp\left(\phi\left(\exp\left(1\right)-1\right)\right) > 1, \\
E^{\mathbb{P}}\left[\zeta\right] &= \phi, \\
Var^{\mathbb{P}}\left[\zeta\right] &= \phi.
\end{aligned}
$$

As long as jump-like default risk is systematic (ϕ is well above 0), the jump-to-default intensity under $\mathbb{Q}$ is always greater than its level under $\mathbb{P}$ ($\lambda > \lambda_{\mathbb{P}}$). If the state-price density does not jump in the case of a jump to default ($\phi \searrow 0$, that is, $\zeta = 0$ $\mathbb{P}$-almost surely), the systematic nature of the jump-like default risk is washed away so that risk-neutral and objective jump-to-default intensities tend to coincide ($\lambda_{\mathbb{P}} \searrow \lambda$).

As far as diffusive risk is concerned, if its premium faints, it is either because such a risk is not priced ($\theta \searrow 0$) or because the risk is dimming ($\sigma \searrow 0$).

The above specification of $\{S\}$'s $\mathbb{P}$-dynamics forces $\{\pi\}$'s $\mathbb{P}$-dynamics to be, for $t < \tau \wedge \xi$,

$$\frac{d\pi_t}{\pi_{t-}} = -rdt$$

$$-\theta S_{t-}^{1-\rho} dz_t^{\mathbb{P}}$$

$$+\big((\exp(\zeta) - 1)\, dN_t^{\mathbb{P}} - [\exp(\phi(\exp(1) - 1)) - 1]\, \lambda_{\mathbb{P}} dt\big),$$

and, for $t \geq \tau \wedge \xi$,

$$\pi_t = \pi_{\tau \wedge \xi} \cdot \exp(-r(t - \tau \wedge \xi)) \quad ,$$

so that, by virtue of Itô's Formula, the π-deflated gain processes generated by holding one share and by holding one unit of currency in the money-market account are local $\mathbb{P}$-martingales[7],

$$E_t^{\mathbb{P}}\left[d\left(\pi_t \cdot S_t \exp\left(qt\right)\right)\right] = 0,$$

$$E_t^{\mathbb{P}}\left[d\left(\pi_t \cdot \exp\left(rt\right)\right)\right] = 0,$$

and, hence, the market is arbitrage-free[8].

Since the objective drift is constant ($E_t^{\mathbb{P}}\left[\frac{dS_t}{S_{t-}}\right] = \mu_{\mathbb{P}}$), arguments similar to those behind Proposition 1 lead to this analytic expression for the quantity $V^{\mathbb{P}}\left(S,T,y\right)$:

$$V^{\mathbb{P}}\left(S,T,y\right) = \frac{\lambda_{\mathbb{P}}}{y+\lambda_{\mathbb{P}}}\left[1-\exp\left(-\left(y+\lambda_{\mathbb{P}}\right)T\right)\left(1-E_0^{\mathbb{P}}\left[1_{\{\xi\leq T\}}\right]\right)\right]$$

$$+\frac{y}{y+\lambda_{\mathbb{P}}}E_0^{\mathbb{P}}\left[\exp\left(-\left(y+\lambda_{\mathbb{P}}\right)\xi\right)1_{\{\xi\leq T\}}\right],$$

[7] The T-time level of the π-deflated gain process generated by holding one unit of currency in the money-market account represents the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$, $\pi_T \cdot \exp\left(rT\right) = \frac{d\mathbb{Q}}{d\mathbb{P}}$.

[8] This indeed rules out arbitrage opportunities involving $S_t \exp\left(qt\right)$ and $\exp\left(rt\right)$, under natural conditions on dynamic trading strategies. See, for example, Appendix B.2 in Pan (2000).

where the T-truncated Laplace transform of ξ's $\mathbb{P}$-p.d.f. with Laplace parameter $y + \lambda_{\mathbb{P}}$ is analytic (see Campi and Sbuelz (2004)). Its closed form is provided in the Appendix.

In summary, we achieve analytic objective default probabilities by augmenting the original parameter set $\{r, q, \sigma, \rho, \lambda\}$ with two preference-based parameters only, θ for the diffusive risk and ϕ for the jump-like default risk.

5 Conclusions

We present an equity-based credit risk model that, by taking as primitive the most liquid and observable part of a firm's capital structure, overcomes many of the problems suffered by structural models in pricing and hedging applications. Our parsimonious model avoids any assumption on the firm's liabilities. It empowers the analytical pricing of CBs and CDSs and it can match non-zero short-maturity spreads. Cross-asset hedging is viable and easy to implement. A careful specification of the state price density enables analytic credit-risk management in the presence of systematic jump-to-default risk.

6 Appendix

Proof of Proposition 1

Proof. The time-s-evaluated $\mathbb{Q}$-p.d.f. of the stopping time $\tau \wedge \xi$ is

$$\begin{aligned} f_{\tau\wedge\xi}(s) &= -\frac{d}{ds} E_0^{\mathbb{Q}}\left[1_{\{\tau\wedge\xi>s\}}\right] \\ &= -\frac{d}{ds} E_0^{\mathbb{Q}}\left[1_{\{\tau>s\}} 1_{\{\xi>s\}}\right] \\ &= -\frac{d}{ds}\left(E_0^{\mathbb{Q}}\left[1_{\{\tau>s\}}\right] E_0^{\mathbb{Q}}\left[1_{\{\xi>s\}}\right]\right) \\ &= f_\tau(s) E_0^{\mathbb{Q}}\left[1_{\{\xi>s\}}\right] + f_\xi(s) E_0^{\mathbb{Q}}\left[1_{\{\tau>s\}}\right] \end{aligned}$$

$$= \lambda \exp(-\lambda s) E_0^{\mathbb{Q}}\left[1_{\{\xi>s\}}\right] + f_\xi(s) \exp(-\lambda s).$$

The T-truncated Laplace transform of $\tau\wedge\xi$'s $\mathbb{Q}$-p.d.f. with Laplace parameter

y is

$$
\begin{aligned}
V^{\mathbb{Q}}(S,T,y) &= E_0^{\mathbb{Q}}\left[\exp\left(-y\cdot\tau\wedge\xi\right)1_{\{\tau\wedge\xi\le T\}}\right] \\
&= \int_0^T \exp\left(-ys\right) f_{\tau\wedge\xi}\left(s\right) ds \\
&= \lambda Y_1 + Y_2,
\end{aligned}
$$

$$
\begin{aligned}
Y_1 &= \int_0^T \exp\left(-\left(y+\lambda\right)s\right) E_0^{\mathbb{Q}}\left[1_{\{\xi>s\}}\right] ds, \\
Y_2 &= \int_0^T \exp\left(-\left(y+\lambda\right)s\right) f_{\xi}\left(s\right) ds.
\end{aligned}
$$

Y_2 is the T-truncated Laplace transform of ξ's $\mathbb{Q}$-p.d.f. with Laplace parameter $y+\lambda$,

$$
Y_2 = E_0^{\mathbb{Q}}\left[\exp\left(-\left(y+\lambda\right)\cdot\xi\right)1_{\{\xi\le T\}}\right].
$$

Its closed form has been derived by Campi and Sbuelz (2004) and it can be found below after this proof. An integration by parts gives

$$
\begin{aligned}
Y_1 &= \left.\frac{-1}{y+\lambda}\exp\left(-\left(y+\lambda\right)s\right)E_0^{\mathbb{Q}}\left[1_{\{\xi>s\}}\right]\right|_0^T \\
&\quad -\int_0^T \frac{-1}{y+\lambda}\exp\left(-\left(y+\lambda\right)s\right)\left(-f_\xi\left(s\right)\right)ds \\
&= \frac{1}{y+\lambda}\left[1-\exp\left(-\left(y+\lambda\right)T\right)E_0^{\mathbb{Q}}\left[1_{\{\xi>T\}}\right]\right]-\frac{1}{y+\lambda}Y_2.
\end{aligned}
$$

This completes the proof. ■

The discounted value of cash at ξ within T

The T-truncated Laplace transform of ξ's $\mathbb{Q}$-p.d.f. with Laplace parameter w ($w \geq 0$) has been shown by Campi and Sbuelz (2004) to be

$$
E_0^{\mathbb{Q}}\left[\exp\left(-w\cdot\xi\right)1_{\{\xi\leq T\}}\right] = \lim_{\epsilon\searrow 0}\sum_{n=0}^{\infty}a_n\left(A,B\right)\left(\frac{x}{2}\right)^n\frac{\Gamma(\nu-n,\frac{x}{2K},\frac{x}{2\epsilon})}{\Gamma(\nu)},
$$

$$\Gamma(\nu) = \int_0^{+\infty} u^{\nu-1} e^{-u} du \qquad \text{(Gamma Function)},$$

$$\Gamma(\nu - n, \frac{x}{2K}, \frac{x}{2\epsilon}) = \int_{\frac{x}{2K}}^{\frac{x}{2\epsilon}} u^{-n} u^{\nu-1} e^{-u} du \qquad \text{(Generalized Incomplete Gamma Function)},$$

$$a_n(A, B) = (-1)^n C(B, n) A^n \ ,$$

$$C(B, n) = \frac{\prod_{k=1}^{n} (B - (k-1))}{n!} 1_{\{n \geq 1\}} + 1_{\{n=0\}} \ ,$$

$$x = S^{2(1-\rho)} \ , \qquad \nu = \tfrac{1}{2(1-\rho)} \ ,$$

$$A = \tfrac{2(r-q+\lambda)}{\sigma^2(1-\rho)} \ , \qquad K = \tfrac{\sigma^2(1-\rho)}{2(r-q+\lambda)} \left(1 - e^{-2T(r-q+\lambda)(1-\rho)}\right) \ ,$$

$$B = \tfrac{w}{2(r-q+\lambda)(1-\rho)} \ .$$

The Generalized Incomplete Gamma Function, the Incomplete Gamma Function, and the Gamma function are built-in routines in many computing software like MATLAB and Mathematica, which makes the above expressions fully viable.

The objective probability of diffusive default within T

The replacement of the risk-neutral intensity-added drift $r-q+\lambda$ with the objective intensity-added drift $\mu_{\mathbb{P}}+\lambda_{\mathbb{P}}$ implies that the T-truncated Laplace transform of ξ's $\mathbb{P}$-p.d.f. with Laplace parameter w ($w \geq 0$) has this analytic expression:

$$E_0^{\mathbb{P}}\left[\exp\left(-w\cdot\xi\right)1_{\{\xi\leq T\}}\right] = \lim_{\epsilon\searrow 0}\sum_{n=0}^{\infty} a_n\left(A_{\mathbb{P}},B_{\mathbb{P}}\right)\left(\frac{x}{2}\right)^n \frac{\Gamma(\nu-n,\frac{x}{2K_{\mathbb{P}}},\frac{x}{2\epsilon})}{\Gamma(\nu)},$$

$$A_{\mathbb{P}} = \tfrac{2(\mu_{\mathbb{P}}+\lambda_{\mathbb{P}})}{\sigma^2(1-\rho)} \quad , \quad K_{\mathbb{P}} = \tfrac{\sigma^2(1-\rho)}{2(\mu_{\mathbb{P}}+\lambda_{\mathbb{P}})}\left(1-e^{-2T(\mu_{\mathbb{P}}+\lambda_{\mathbb{P}})(1-\rho)}\right) \quad ,$$

$$B_{\mathbb{P}} = \tfrac{w}{2(\mu_{\mathbb{P}}+\lambda_{\mathbb{P}})(1-\rho)} \quad .$$

The analytic expression of the objective probability of diffusive default within time T is retrieved by taking $w=0$.

The Delta-hedge ratio for a CB

Simple differentiation gives the following analytic Delta-hedge ratio for a CB:

$$\begin{aligned}
\frac{\partial}{\partial S}P_{CB}\left(S,T,r\right) &= -\sum_{j=1}^{kT}\frac{1}{k}\exp\left(-rT_j\right)\frac{\partial}{\partial S}V^{\mathbb{Q}}\left(S,T_j,0\right)C \\
&\quad -\frac{\partial}{\partial S}V^{\mathbb{Q}}\left(S,T,0\right)F \\
&\quad +\frac{\partial}{\partial S}V^{\mathbb{Q}}\left(S,T,r\right)\omega F,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial S}V^{\mathbb{Q}}\left(S,T,y\right) &= \frac{\lambda}{y+\lambda}\exp\left(-\left(y+\lambda\right)T\right)\frac{\partial}{\partial S}E^{\mathbb{Q}}\left[\exp\left(-0\cdot\xi\right)1_{\{\xi\leq T\}}\mid S\right] \\
&\quad +\frac{y}{y+\lambda}\frac{\partial}{\partial S}E^{\mathbb{Q}}\left[\exp\left(-\left(y+\lambda\right)\xi\right)1_{\{\xi\leq T\}}\mid S\right],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial S}E^{\mathbb{Q}}\left[\exp\left(-w\cdot\xi\right)1_{\{\xi\leq T\}}\mid S\right] &= \lim_{\epsilon\searrow 0}\sum_{n=0}^{\infty}a_n\left(A,B\right)\left(\frac{x}{2}\right)^n\frac{1}{\Gamma(\nu)}Z_n\left(x,\epsilon\right)x', \\
Z_n\left(x,\epsilon\right) &= \left[\begin{array}{c} \frac{n}{2}\left(\frac{x}{2}\right)^{-1}\Gamma(\nu-n,\frac{x}{2K},\frac{x}{2\epsilon}) \\ +\left(g_n(\frac{x}{2\epsilon})\frac{1}{2\epsilon}-g(\frac{x}{2K})\frac{1}{2K}\right)\end{array}\right],
\end{aligned}$$

$$x' = 2(1-\rho)S^{2(1-\rho)-1},$$

$$g_n(u) = u^{-n}u^{\nu-1}e^{-u}.$$

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