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Nieuwenhuis, G.

Publication date:
2005

No. 2005–07

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By Gert Nieuwenhuis

January 2005

ISSN 0924-7815

Dominated families of shifted Palm distributions

Gert Nieuwenhuis

Tilburg University

Department of Econometrics and Operations Research

P.O. Box 90153, 5000 LE Tilburg, The Netherlands

January 2005

Abstract

In stationary point process theory, the concept 'Palm distribution' plays an important role. Many important results (like for instance Little's law, so important in many fields) arise from it. However, in the non-stationary case a whole family of local Palm distributions (PD's) has to be considered and the concept seems to loose its importance.

The present paper mainly considers non-stationary point processes, and studies relations between the distribution P of a point process, the family $\{P^x\}$ of PD's, and the family $\{P^{0,x}\}$ of shifted PD's. Here $P^{0,x}$ is the probability distribution that is experienced **from** an occurrence (arrival, point, transaction) at x. It is attempted to regain some of the glance of the concept 'Palm distribution' by considering generalizations of results that are basic for stationary point processes. Starting point is a refined version of Campbell's equation, which expresses the general relationship between the distribution P of the point process and the family $\{P^x\}$ of PD's. It is used to generalize the inversion formula, well known from stationary point process theory. This generalization is basic; it leads to several relations regarding the above distributions.

In the second part of the research domination assumptions are imposed: either the null-sets of a time-stationary distribution are also null-sets of P or the null-sets of one event-stationary distribution are also null-sets of 'almost all' shifted PD's. Under such domination regulations, $P^{0,x}$ can explicitly be expressed in terms of P and several stationary-case long-run properties can be generalized. The relationship between the two types of domination assumptions is carefully studied.

Keywords: point processes, non-stationarity, family of Palm distributions, domination
JEL classification: C49

1 Introduction

Many results in **stationary** point process theory originate from the relationships between the time-stationary distribution of the point process and the accompanying (unique) event-stationary Palm distribution. Simple equations explicitly express the Palm distribution P^0 in the time-stationary distribution P, and vice versa. In **non-stationary** settings, the distribution P of the point process has to make contact with a whole family of Palm distributions (PD's), that are not event-stationary anymore. The relationship between P and this family of PD's is less explicitly embodied in the refined Campbell equation. As a consequence, the concept 'Palm distribution' seems to loose its importance; see Daley and Vere-Jones (1988; p. 456). We attempt to make this relationship more transparent by generalizing well-known theoretical results for stationary point processes to more general non-stationary point process settings.

The present research especially aims at the theoretical interrelationship between the distribution P of a non-stationary point process and its local Palm distributions P^x or the shifted Palm distributions $P^{0,x}$. Here P^x can heuristically be considered as the probability distribution of the point process if it is already given that there is an occurrence at x, and $P^{0,x}$ as the probability mechanism experienced **from** an occurrence at x. It is our intention to make the general point process theory better accessible by pointing at resemblances with results that are well known for stationary point processes. Since we are especially interested in the validity of strong laws under non-stationary circumstances, we assume in the larger part of this paper that either P is dominated by a time-stationary distribution or the whole family $\{P^{0,x}\}$ of shifted PD's is dominated by one event-stationary distribution. That is, either the null-sets of P include the null-sets of a time-stationary distribution or the null-sets of 'almost every' member of the shifted PD-family include the null-sets of one event-stationary distribution. When considering these two types of domination assumptions, the question arises how the assumptions themselves are related and how the respective Radon-Nikodym densities are related. This problem will be studied in detail.

We first rewrite the refined Campbell equation into equations that more directly express P in terms of the family $\{P^x\}$ or the family $\{P^{0,x}\}$ and that generalize nice inversion formulae from stationary point process theory. These results are essential for the major part of this research. In view of long-run limit results, we compare the null-sets under P^x and under $P^{0,x}$ with null-sets under P. Regarding the above mentioned domination assumption of $\{P^{0,x}\}$ by one event-stationary distribution it turns out that one has to be careful because this family of shifted PD's is only uniquely defined in the a.e. (almost everywhere) sense with respect to the intensity measure of the point process.

For the study of stationary point processes, the so-called intermediate probability distributions P_n that arise from P by shifting the origin to the n^{th} occurrence were easy tools to facilitate jumping from the event-stationary distribution to the time-stationary one (and vice versa); see Nieuwenhuis (1989, 1994, 1998). We generalize P_n, the probability mechanism experienced from the n^{th} occurrence (if finite), to the non-stationary setting and study the relationships with $P^{0,x}$, the probability mechanism experienced from an occurrence at x.

Below, in the second part of the present Section 1, we first mention some notations and conventions that are used in this paper. Next, we briefly review the results from stationary point process theory that are important for the present research. Special attention is paid to the inversion formulae that express P in terms of the event-stationary distribution P^0 and to the way these distributions are related to the intermediate probability distributions P_n. We also repeat a lemma from Nieuwenhuis (1994) that ensures that the invariant σ-fields of the time-shifts and the point-shifts coincide. Since we are interested in strong laws (long-run properties), this lemma will be used frequently.

In Section 2 we first define the Palm distributions P^x via the well-known refined Campbell equation; the shifted PD's $P^{0,x}$ are also introduced and studied. Next, we derive a generalization of the inversion formula that has proven to be so important for stationary point process theory. As immediate corollaries, the distribution of the k^{th} occurrence (if finite) under P turns out to be dominated by the intensity measure ν and a local characterization result for the shifted PD's can be formulated. Some general long-run results are considered. In an example, the shifted PD's of the **event**-stationary point process are characterized.

Section 3 is about the (generalized) intermediate probability measures P_n, distributions under the condition that the n^{th} occurrences are finite. They can be expressed in terms of the shifted PD's, in a way that nicely generalizes the stationary case. A null-set under P is a null-set under P^x for ν-almost all x, and vice versa; a null-set under all P_n (if defined) is a null-set under $P^{0,x}$ for ν-almost all x, and vice versa The intermediate position of the P_n between P and $\{P^{0,x}\}$ is nicely illustrated by a corollary that relates strong laws under these three types of distributions. In an example it is demonstrated that the above results regarding null-sets not necessarily imply domination results regarding P and $\{P^x\}$ or regarding $\{P_n\}$ and $\{P^{0,x}\}$. This is caused by the fact that the family of PD's is only unique in the ν-almost everywhere sense.

In Sections 4 - 6 we bring our point process framework closer to stationarity by assuming a domination property. We either assume that P is dominated by a time-stationary distribution or that the $P^{0,x}$ are dominated by one event-stationary distribution for ν-almost all x. In Section 4 we adopt the second domination assumption, which not necessarily implies the first. A theorem that characterizes the second type of domination is proved. As a corollary, the sequence $\{P_n\}$ of intermediate distributions is dominated by the same event-stationary distribution that also dominates $\{P^{0,x}\}$. The sequence of Radon-Nikodym derivatives turns out to be stationary. It also follows that it is now possible to express $P^{0,x}$ explicitly in terms of P provided that a weak additional assumption holds. Many long-run properties that are valid under the event-stationary distribution and/or under the accompanying time-stationary distribution turn out to be valid under P and $P^{0,x}$ too. Furthermore, from the assumed domination criterion it follows that the point process is asymptotically event-stationary and time-stationary. The relationship between the stationary limit distributions is studied.

Section 5 considers some immediate consequences of the first type of domination, with Radon-Nikodym density denoted as σ. The intensity measure is then dominated by Lebesgue measure. The case that P is also time-stationary itself is studied in detail, to demonstrate the relationship with σ being invariant under all point- and event-shifts. In an example, we choose σ such that event-stationarity is experienced from all occurrences.

Section 6 studies the relationships between the two types of domination assumptions. It is proved that the validity of the first type of domination mentioned above is equivalent to the joint validity of the second domination and the domination of ν by Lebesgue measure. This result remains valid if all three domination properties also hold in the reversed direction. It follows that the equivalence of P and a time-stationary distribution P_{st} in the sense of domination allows to express $P^{0,x}$ explicitly in terms of P without any additional assumptions. Some long-run properties are reconsidered and interrelated. In an example, we explicitly express $P^{0,x}$ in P when P is dominated by P_{st} and σ satisfies some invariance condition.

In the present research, $\mathbb{R}$ is the set of reals and Bor($\mathbb{R}$) is the set of Borel-sets in $\mathbb{R}$. We use the symbol $\mathbb{Z}$ to denote the set of all integers in $\mathbb{R}$. For $k \in \mathbb{Z}$, the set $\mathbb{R}_k$ is defined as the positive half-line $(0,\infty)$ if $k > 0$ and as the non-positive half-line $(-\infty, 0]$ if $k \leq 0$.The notation $:=$ means is *by definition equal to.* Lebesgue-measure is denoted by Leb.

For two measures Q_1 and Q_2 (with Q_2 σ-finite) on the same measurable space, it is said that Q_1 is *dominated by* Q_2 (notation $Q_1 \ll Q_2$) if the Q_2-null-sets are also Q_1-null-sets. A Radon-Nikodym density (or derivative) of Q_1 with respect to Q_2 will be denoted by $\mathrm{d}Q_1/\mathrm{d}Q_2$ or shortly by RN. If $Q_1 \ll Q_2$ and $Q_2 \ll Q_1$, we say that the two measures are *equivalent* (notation $Q_1 \sim Q_2$). If Q is a probability measure and a certain eventuality B has Q-probability 1, we say that B holds Q-a.s. (almost surely).

Next suppose that Q_1 and Q_2 are both probability measures, both dominated by a σ-finite measure ω having RN-densities h_1 and h_2 respectively. The *total variation distance* d between Q_1 and Q_2 is defined by

$$d(Q_1, Q_2) := \int |h_1 - h_2| d\omega.$$

It is well-known that

$$d(Q_1, Q_2) = 2 \sup_A |Q_1(A) - Q_2(A)|.$$

If μ is a measure on Bor($\mathbb{R}$) and $\mu(A^c) = 0$ for a certain Borel-set A, we say 'A holds μ-a.e.' (A holds μ-almost everywhere) or 'A holds for μ-a.e. $x \in \mathbb{R}$' (A holds for μ-almost every x in $\mathbb{R}$).

Preliminary definitions and notations

In the following, we will give several definitions and notations from point process theory. A point process on $\mathbb{R}$ is a measurable mapping Φ from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the set N of all integer-valued measures φ on $\mathbb{R}$ for which

$$\varphi(B) < \infty \text{ for all bounded } B \in \mathrm{Bor}(\mathbb{R}).$$

Here N is endowed with the σ-field generated by the sets $[\varphi(B) = k] := \{\varphi \in N : \varphi(B) = k\}$, for all integers k and $B \in \mathrm{Bor}(\mathbb{R})$. Set

$$M := \{\varphi \in N : \varphi(\mathbb{R}) > 0;\ \varphi\{s\} \leq 1 \text{ for all } s \in \mathbb{R}\}$$

with σ-field $\bar{M} := M \cap \bar{N}$. Denote the probability distribution of Φ by P and the corresponding expectation operator by E. We will only consider point processes with single occurrences (also called *simple* point processes). That is, we will always assume that

(i) $P(M) = 1$.

The atoms (called *points, events, occurrences, arrivals, transaction times*) of $\varphi \in M$ are denoted by $T_n(\varphi)$ under the convention that

$$\ldots < T_{-1}(\varphi) < T_0(\varphi) \leq 0 < T_1(\varphi) < T_2(\varphi) < \ldots,$$

provided that they are finite. Furthermore, we write $\alpha_n(\varphi) := T_{n+1}(\varphi) - T_n(\varphi)$, $n \in \mathbb{Z}$, for the *interval lengths* between finite occurrences. Measurable sets in $\bar{M}$ will be called *eventualities*. Eventualities like the set of the $\varphi \in M$ for which $\alpha_n(\varphi) < 3$ will be written as $[\alpha_n(\varphi) < 3]$ and also as $[\alpha_n < 3]$. We also need notations for special subsets of M. Set

$$F_n := \{\varphi \in M : |T_n(\varphi)| < \infty\} \text{ and } \bar{F}_n := F_n \cap \bar{M},$$

$$M_x := \{\varphi \in M : \varphi\{x\} = 1\} \text{ and } \bar{M}_x := M_x \cap \bar{M},$$

$$M^\infty := \{\varphi \in M : \varphi(-\infty, 0] = \varphi(0, \infty) = \infty\} \text{ and } \bar{M}^\infty := M^\infty \cap \bar{M},$$

$$M^0 := \{\varphi \in M^\infty : \varphi\{0\} = 1\} \text{ and } \bar{M}^0 := M^0 \cap \bar{M};\ x \in \mathbb{R}.$$

(Note the difference between M^0 and M_0.) The family $\{\theta_t : t \in \mathbb{R}\}$ of *time-shifts* $\theta_t : N \to N$ defined by $\theta_t(\varphi) := \theta_t\varphi := \varphi(t + \cdot)$ will play an important role. The same holds for the family $\{\eta_n : n \in \mathbb{Z}\}$ of *event-shifts* $\eta_n : F_n \to N$ with $\eta_n(\varphi) := \eta_n\varphi := \varphi(T_n(\varphi) + \cdot)$. Note that $\theta_t\varphi$ has occurrences in $T_n(\varphi) - t$ (if finite) and it arises from φ by shifting the origin to t, while $\eta_n\varphi$ has occurrences in $T_k(\varphi) - T_n(\varphi)$ (if finite) and it arises from φ by shifting the origin to the n^{th} occurrence. Regarding these shifts, the following notations are adopted:

$$\theta_t^{-1}A := \{\varphi \in N : \theta_t\varphi \in A\}, \quad t \in \mathbb{R} \text{ and } A \in \bar{N},$$

$$\eta_n^{-1}A := \{\varphi \in F_n : \eta_n\varphi \in A\}, \quad n \in \mathbb{Z} \text{ and } A \in \bar{N}.$$

Also consider the invariant σ-fields

$$\mathcal{I}' := \{A \in \bar{M}^\infty : \theta_t^{-1}A = A \text{ for all } t \in \mathbb{R}\},$$

$$\mathcal{I} := \{A \in \bar{M}^\infty : \eta_1^{-1}A = A\}.$$

The following lemma is important for the present research; see Nieuwenhuis (1994; Lemma 2).

Lemma 1.1

(a) $\mathcal{I} = \mathcal{I}'$;

(b) If $f : M^\infty \to \mathbb{R}$ is $\mathcal{I}$-measurable, then $f \circ \theta_t = f$ and $f \circ \eta_n = f$ for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$.

For measurable functions $f : M \to \mathbb{R}$ we use each of the notations $\mathbb{E}f(\Phi)$, $E(f)$, and $Ef(\Phi)$ to denote the expectation of $f(\Phi)$. When adopting the last notation, we implicitly assume that Φ is the canonical version of the point process on $(M, \bar{M}, P)$. Furthermore, the distribution of a measurable mapping α on $(M, \bar{M}, P)$ into some measurable space is denoted by P_α.

Stationary point processes

In this subsection, we will review some important properties of the distribution P and its Palm distribution P^0 in the case that the point process is time-stationary. See Franken et al. (1982) or Daley and Vere-Jones (1988) for an overview. We especially aim at long-run properties under P and P^0. In forthcoming sections we will generalize many of the results for non-stationary settings.

Assume that

(ii) $\quad P(M^\infty) = 1 \quad \text{and} \quad \lambda := \mathbb{E}\Phi(0,1] < \infty,$

and that the point process (and its distribution P) is *time-stationary:*

(iii) $\quad P\theta_t^{-1}(A) := P(\theta_t^{-1}A) = P(A) \quad \text{for all } t \in \mathbb{R} \text{ and } A \in \bar{M}^\infty.$

Hence, the probability distribution of the point process is the same seen from all positions t in the set of reals. As a consequence, it can be shown that λ is positive and that for all $t > 0$ the definition below gives one probability measure P^0 on $(M^\infty, \bar{M}^\infty)$, the so-called *Palm distribution* (PD) of Φ :

$$P^0(A) := \frac{1}{\lambda t}\mathbb{E}\Big(\sum_{i=1}^{\Phi(0,t]} 1_A \circ \eta_i(\Phi)\Big) = \frac{1}{\lambda t}\mathbb{E}\Big(\int_{(0,t]} 1_A(\theta_x\Phi)d\Phi(x)\Big); \tag{1.1}$$

$A \in \bar{M}^\infty$. We denote P^0-expectation by E^0. The PD has the following properties:

$$P^0(M^0) = 1 \quad \text{and} \quad P^0\eta_n^{-1} = P^0 \quad \text{for all } n \in Z; \quad P^0 = P \text{ on } \mathcal{I}; \tag{1.2}$$

$$\lambda = 1/E^0(\alpha_0) = E(1/\alpha_0). \tag{1.3}$$

(The last result in (1.2) follows from Lemma 1.1.) That is, under P^0 there is an occurrence in the origin and the probability mechanisms observed from all occurrences are the same. Because of the second property in (1.2), the PD is called *event-stationary*. As a consequence, the sequence $\{\alpha_n\}$ of interval lengths is stationary under P^0.

The PD is often informally described as the 'conditional' distribution of the point process if there is an occurrence in the origin. This originates from the following more formal result:

$$P[\eta_1\varphi \in A | T_1(\varphi) \leq \frac{1}{n}] \to P^0(A) \text{ as } n \to \infty, \tag{1.4}$$

which holds uniformly over $A \in \bar{M}^\infty$; see Nieuwenhuis (1994).

Note that the definition in (1.1) expresses P^0 in terms of P. The following so-called *inversion formulae* work the other way round:

$$\begin{aligned} P(A) &= \lambda \int\limits_{\mathbb{R}_k} P^0[\varphi(-x+\cdot) \in A \text{ and } T_{-k}(\varphi) \leq -x < T_{-k+1}(\varphi)]dx \\ &= \lambda E^0(\int\limits_{-T_{-k+1}}^{-T_{-k}} 1_A \circ \theta_{-x} dx) = \lambda E^0(\int\limits_{T_{-k}}^{T_{-k+1}} 1_A \circ \theta_x dx) \end{aligned} \tag{1.5}$$

for all $A \in \bar{M}^\infty$ and $k \in \mathbb{Z}$. (Usually, $k = 0$ is used to express the inversion.)

In Palm theory more probability measures are important. For $n \in \mathbb{Z}$, let P_n be the so-called *intermediate distribution* that arises from P by shifting the origin to the n^{th} occurrence:

$$P_n(A) = P[\eta_n\varphi \in A], \quad A \in \bar{M}^\infty.$$

In a queuing framework, P_n can be considered as the probability mechanism observed by the n^{th} customer if P is the ruling underlying distribution. Note the difference between P_0 and P^0. In Nieuwenhuis (1989) it was proved that

$$P_n \sim P^0 \quad \text{and} \quad dP_n/dP^0 = \lambda\alpha_{-n}\ P^0 - \text{a.s.}, \quad n \in \mathbb{Z}, \tag{1.6}$$

which explains the characterization of P_n as being intermediate (between P and P^0). Consequently,

$$P^0(A) = \frac{1}{\lambda} E(\frac{1}{\alpha_0} 1_A \circ \eta_n) \tag{1.7}$$

for all $A \in \bar{M}^\infty$ and $n \in \mathbb{Z}$, yielding other ways to express P^0 in terms of P and vice versa.

Long-run properties of stationary point processes

By the stationarity property of P^0, Birkhoff's ergodic theorem, Lemma 1.1, and (1.6) it follows that

$$\frac{1}{n}\sum_{i=1}^{n} 1_A \circ \eta_i \to E^0(1_A|\mathcal{I}) \text{ as } n \to \infty \quad P^0\text{- and } P\text{-a.s.,} \tag{1.8}$$

$$\frac{1}{n}\sum_{i=1}^{n} P_i(A) \to Q^0(A) := E(E^0(1_A|\mathcal{I})), \tag{1.9}$$

$$\frac{1}{t}\int_0^t 1_A \circ \theta_x dx \to E(1_A|\mathcal{I}) \text{ as } t \to \infty \quad P\text{- and } P^0\text{-a.s.,} \tag{1.10}$$

$$\frac{1}{t}\int_0^t P^0[\theta_x\varphi \in A]dx \to Q(A) := E^0(E(1_A|\mathcal{I})), \tag{1.11}$$

for all $A \in \bar{M}^\infty$. (Note that n runs through the set of positive integers and t through the positive half-line.) Heuristically, the **event-stationary** probability measure Q^0 on $(M^\infty,\ \bar{M}^\infty)$ can be considered as the probability mechanism observed from an 'arbitrarily chosen' occurrence. The **time-stationary** probability measure Q on $(M^\infty,\ \bar{M}^\infty)$ is the probability mechanism observed from an 'arbitrarily chosen' time-point in $(0, \infty)$. Because of Lemma 1.1 and (1.6), we obtain by conditioning on $\mathcal{I}$ that

$$Q^0(A) = E(E^0(1_A|\mathcal{I}) \circ \eta_0) = \lambda E^0(\bar{\alpha} \cdot 1_A),$$

where $\bar{\alpha} := E^0(\alpha_0|\mathcal{I})$. Since $P^0[\bar{\alpha} > 0] = 1$, the probability measures Q^0 and P^0 are equivalent; we have

$$dQ^0/dP^0 = \lambda\bar{\alpha} \ P^0\text{-a.s.} \tag{1.12}$$

Note that $\bar{\alpha}$ is just the long-run average of the P^0-stationary sequence of interval lengths:

$$\frac{1}{n}\sum_{i=1}^{n} \alpha_i \to \bar{\alpha} \ P^0\text{-a.s.} \tag{1.13}$$

Since $P^0 \sim P_0$, relation (1.13) also holds P_0-a.s. and hence P-a.s. On the other hand, $\bar{\Phi} := \mathbb{E}(\Phi(0,1]|\mathcal{I})$ is the long-run average number of occurrences on unit intervals, since

$$\Phi(0,t]/t \to \bar{\Phi} \text{ as } t \to \infty, \tag{1.14}$$

not only P-a.s. but also P^0-a.s.; see Nieuwenhuis (1994). In Nieuwenhuis (1998) it was proved that $\bar{\alpha}$ and $\bar{\Phi}$ are both positive and finite (P^0- and P-a.s.) and that

$$\bar{\Phi} = 1/\bar{\alpha} = E(\frac{1}{\alpha_0}|\mathcal{I}) \quad P^0\text{-a.s. and } P\text{-a.s.,} \tag{1.15}$$

which is a conditional version of (1.3). Consequently, the probability distribution Q in (1.11) satisfies

$$Q(A) = \frac{1}{\lambda} E(\frac{1}{\bar{\alpha}} 1_A) \text{ and } Q \sim P \text{ with } dQ/dP = \frac{1}{\lambda}\frac{1}{\bar{\alpha}}. \tag{1.16}$$

Although the notation may suggest it, Q^0 is **not** the PD of Q. See Nieuwenhuis (1998) for more details.

A stationary point process, its distribution, and its PD are called *pseudo-ergodic* if $P^0[\bar{\alpha} = 1/\lambda] = 1$, that is: if $Q^0 = P^0$. It turns out that pseudo-ergodicity is weaker than ergodicity; see Nieuwenhuis (1994) for details. Consequently, A 'randomly chosen customer' experiences the Palm distribution iff this distribution is pseudo-ergodic.

We end this section with definitions concerning asymptotic stationarity; see Sigman (1995) for similar concepts on the half-line.

Suppose that Φ is a non-stationary point process. The point process (and its distribution P) is called *asymptotically time-stationary* (ATS) if a probability distribution Q on $(M, \bar{M})$ exists such that

$$\frac{1}{t} \int_0^t P[\theta_s \varphi \in A] ds \to Q(A) \text{ as } t \to \infty, \text{ for all } A \in \bar{M}. \tag{1.17}$$

A point process (and its distribution P) with $P(M^\infty) = 1$ is called *asymptotically event-stationary* (AES) if a probability distribution Q^0 on $(M^\infty, \bar{M}^\infty)$ exists such that

$$\frac{1}{n} \sum_{i=1}^{n} P[\eta_i \varphi \in A] \to Q^0(A) \text{ as } n \to \infty, \text{ for all } A \in \bar{M}^\infty. \tag{1.18}$$

By Lemma 1.1, Q^0 is indeed event-stationary and Q is time-stationary. Q^0 (Q) can intuitively be interpreted as the probability mechanism observed from a randomly chosen occurrence (time-point). Note - again by Lemma 1.1 - that Q^0, P, and Q coincide on $\mathcal{I}$.

2 Non-stationary point processes

For time-stationary point processes the relationship between the distribution P and its Palm distribution P^0 is nice and simple; see Section 1. To generalize things for non-stationary point processes, we need to relate P with a whole family $\{P^x\}$ of Palm distributions. The relationship between P and $\{P^x\}$ is described by the well-known refined Campbell equation, which at first sight looks complicated. In the second and third subsections below, we use the refined Campbell equation to derive relations between P

and $\{P^x\}$ that are less repulsive or resemble nice relations of Section 1. The results in the first subsection below come from Jagers (1973); see also Kallenberg (1983/86), and Daley and Vere-Jones (1988).

We assume that the point process satisfies (i). That is, we study simple point processes on $\mathbb{R}$ with at least one occurrence. The *intensity measure* ν on Bor($\mathbb{R}$) with $\nu(A) := \mathbb{E}(\Phi(A))$ for $A \in \text{Bor}(\mathbb{R})$, will play an important role; we assume that it exists and is locally finite. The set function

$$C_1(A \times B) := \mathbb{E}(\Phi(A) 1_B(\Phi)),\ \ A \in \text{Bor}(\mathbb{R}) \text{ and } B \in \bar{M}, \tag{2.1}$$

uniquely extends to a σ-finite measure C_1, the so-called *Campbell measure*, on the product σ-field Bor($\mathbb{R}$)$\times\bar{M}$. Especially, note that - for all $B \in \bar{M}$ - the locally finite measure ν_B defined by

$$\nu_B(A) := \mathbb{E}(\Phi(A) 1_B(\Phi)),\ \ A \in \text{Bor}(\mathbb{R}), \tag{2.2}$$

is dominated by ν. Let the function $x \to P^x(B)$ be a Radon-Nikodym density. Then we have:

$$\nu_B(A) = \int_A P^x(B) d\nu(x),\ \ A \in \text{Bor}(\mathbb{R}). \tag{2.3}$$

The basic result now is that the family $\{P^x(B) : x \in \mathbb{R} \text{ and } B \in \bar{M}\}$ can be chosen such that

(a) the function $x \to P^x(B)$ is measurable for all $B \in \bar{M}$,

(b) P^x is a probability measure on $\bar{M}$ for all $x \in \mathbb{R}$,

(c) the following holds for all Bor($\mathbb{R}$) $\times\ \bar{M}$-measurable functions f on $\mathbb{R} \times M$ that are either nonnegative or satisfy $\mathbb{E}(\int_{\mathbb{R}} f(x, \Phi) d\Phi(x)) < \infty$:

$$\begin{aligned} \int_M \int_{\mathbb{R}} f(x,\varphi) d\varphi(x) dP(\varphi) &= \iint_{\mathbb{R}\times M} f(x,\varphi) C_1(dx \times d\varphi) \\ &= \int_{\mathbb{R}} \int_M f(x,\varphi) dP^x(\varphi) d\nu(x). \end{aligned} \tag{2.4}$$

Furthermore, the family $\{P^x\}$ of probability distributions turns out to be **uniquely defined** by (2.4) - even by (2.3) - apart from a Borel-set in $\mathbb{R}$ with ν-measure 0. Note that the choice $f(x, \varphi) = 1_{A\times B}(x, \varphi)$ returns (2.3). In the sequel, we will always assume that the family $\{P^x\}$ in (2.4) also satisfies (a) and (b).

The probability measures P^x, $x \in \mathbb{R}$, are called *Palm distributions* (PD's). According to the above, the family $\{P^x\}$ of PD's belonging to the distribution P of a point process is uniquely defined by (2.4) - and even by (2.3) - in the ν-a.e. sense. On the other hand, if for a distribution P of a point process the family $\{P^x\}$ of PD's and the intensity measure ν is given, then the distribution P follows from (2.3) or (2.4). It can be proved that $P^x(M_x) = 1$ for ν-a.e. $x \in \mathbb{R}$. By letting A in (2.3) shrink to $\{x\}$, we obtain the intuitive meaning for $P^x(B)$ as the probability that $\Phi \in B$ under the condition that $\Phi\{x\} = 1$:

$$\lim_{h \to 0} \frac{\mathbb{E}(\Phi(x-h, x+h) 1_B(\Phi))}{\mathbb{E}(\Phi(x-h, x+h))} = P^x(B) \quad \text{for } \nu\text{-a.e. } x \in \mathbb{R}. \tag{2.5}$$

Shifted Palm distributions

We will especially be interested in $\{P^{0,x}\}$, the family of shifted PD's defined by

$$P^{0,x} := P^x \theta_x^{-1}.$$

Note that $P^{0,x}$ satisfies $P^{0,x}(M_0) = 1$, and that (in queuing terms, with T_n as arriving times) it can be considered as the probability mechanism observed (experienced) by a customer arriving at time x. For time-stationary P satisfying (ii) we have $P^{0,x} = P^0$ for ν-a.e. $x \in \mathbb{R}$, where P^0 is defined as in (1.1); cf. Proposition 7 in Jagers (1973).

With (2.4), the choice $f(x, \varphi) = 1_B(\theta_x \varphi) 1_A(x)$ yields that

$$C_2(A \times B) := \mu_B(A) := \mathbb{E}\Big(\sum_{\{i: T_i \in A\}} 1_B \circ \eta_i\Big) = \int_A P^{0,x}(B) d\nu(x), \tag{2.6}$$

$A \in \text{Bor}(\mathbb{R})$ and $B \in \bar{M}$. This set function C_2 can also uniquely be extended to a σ-finite measure C_2 on $\text{Bor}(\mathbb{R}) \times \bar{M}$. Also note that, for all $B \in \bar{M}$, $\mu_B(A)$ is the (under P) expected number of the occurrences in A from which B is observed, and that μ_B is an other locally finite measure on $\text{Bor}(\mathbb{R})$. It is dominated by ν and $d\mu_B/d\nu$, a function of $x \in \mathbb{R}$, can be chosen as $P^{0,x}(B)$. This also is heuristically obvious: by letting A in (2.6) shrink to $\{x\}$ we obtain the intuitive meaning for $P^{0,x}(B)$ as the probability that $\theta_x \Phi \in B$ under the condition that $\Phi\{x\} = 1$:

$$\lim_{h \to 0} \frac{E(\Phi(x-h, x+h) 1_B(\theta_x \Phi))}{E(\Phi(x-h, x+h))} = P^{0,x}(B) \;\; \nu\text{-a.e.};$$

cf. (2.2), (2.3) and (2.5).

We next derive a generalization of (1.5). For this, recall the definitions of F_k and $\mathbb{R}_k$ in Section 1 and set $I_x := (0, x]$ if $x > 0$, $I_x := (x, 0]$ if $x \leq 0$, and choose f in (2.4) as follows:

$$f(x, \varphi) = 1_B(\varphi) 1_{\{|k|\}}(\varphi(I_x)) 1_{\mathbb{R}_k}(x),$$

where $B \in \bar{M}$ and $k \in \mathbb{Z}$. We obtain, for $B \in \bar{M}$ and $k \in \mathbb{Z}$:

$$\begin{aligned} P(B \cap F_k) &= \int_{\mathbb{R}_k} P^x(B \cap [\varphi(I_x) = |k|]) d\nu(x) \\ &= \int_{\mathbb{R}_k} P^x(B \cap [T_k = x]) d\nu(x) \\ &= \int_{\mathbb{R}_k} P^{0,x}([\theta_{-x}\varphi \in B] \cap [T_{-k} \leq -x < T_{-k+1}]) d\nu(x). \end{aligned} \tag{2.7}$$

(The second equality follows by intersection with M_x.) Note that in the second and third equality it is allowed to replace $\mathbb{R}_k$ by $\mathbb{R}$. Also note that the last equality obviously generalizes (1.5).

Equation (2.7) can be used to derive other, very useful, results. For $A \in \mathrm{Bor}(\mathbb{R})$, substitute $B \cap [T_k \in A]$ for B in the last equality. It follows that

$$P(B \cap [T_k \in A]) = \int_A P^{0,x}([\theta_{-x}\varphi \in B] \cap [T_{-k} \leq -x < T_{-k+1}]) d\nu(x) \tag{2.8}$$

for all $k \in \mathbb{Z}$, $A \in \mathrm{Bor}(\mathbb{R})$ and $B \in \bar{M}$. By taking $\sum_{k \in \mathbb{Z}}$, the left-hand side becomes equal to $\mathbb{E}[\Phi(A) 1_B(\Phi)]$ and we get (2.3) back. When B in (2.8) is replaced by $[\eta_k \varphi \in B]$, we obtain

$$P([\eta_k \varphi \in B] \cap [T_k \in A]) = \int_A P^{0,x}(B \cap [T_{-k} \leq -x < T_{-k+1}]) d\nu(x) \tag{2.9}$$

for all $k \in \mathbb{Z}$, $A \in \mathrm{Bor}(\mathbb{R})$ and $B \in \bar{M}$. Note that we get (2.6) back by taking $\sum_{k \in \mathbb{Z}}$. The choice $B = M$ in (2.9) ensures that, if $P(F_k) > 0$, the conditional distribution $P([T_k \in \cdot] | F_k)$ of T_k is dominated by ν and that the function $x \to P^{0,x}[T_{-k} \leq -x < T_{-k+1}] / P(F_k)$ is a RN-density.

Since the Borel-set $\{x \in \mathbb{R} : \nu(x - h, x + h) > 0$ for some $h > 0\}$ has ν-measure 0 in many relevant cases, we obtain - by writing the conditional probability on the right-hand side below as a ratio of unconditional probabilities and dividing both the numerator and the denumerator by $\nu(x - h, x + h)$, and letting h tend to 0 from above - that, under some additional regularity conditions,

$$P^{0,x}(B | [T_k(\theta_{-x}\varphi) = x]) = \lim_{h \to 0} P([\eta_k \varphi \in B] | [T_k(\varphi) \in (x - h, x + h)]) \tag{2.10}$$

for ν-a.e. $x \in \mathbb{R}$ and all $B \in \bar{M}$. This result describes the shifted Palm distributions locally; cf. (1.4).

Long-run properties

By (2.3) it is (because of generalized bounded convergence) an easy exercise to show that

$$\frac{1}{\nu(0,t]} \int_{(0,t]} P^x(B)d\nu(x) \to P(B) \text{ as } t \to \infty \quad \text{for all } B \in \bar{M}, \tag{2.11}$$

provided that $P[\lim_{t\to\infty}(\varphi(0,t]/\nu(0,t]) = 1] = 1$. This result becomes clear if the intuitive meaning of $P^x(B)$, outlined in (2.5) is used. The long-run property (2.11) specializes into

$$\frac{1}{\lambda t} \int_{(0,t]} P^x(B)d\nu(x) \to P(B) \text{ as } t \to \infty \quad \text{for all } B \in \bar{M},$$

if it is given that $P[\theta(0,t]/t \to \lambda] = 1$ and $E\Phi(0,t]/t \to \lambda$ (as $t \to \infty$) for some positive constant λ.

Suppose additionally that $P(M^\infty) = 1$ and that Φ is AES; see (1.18). Let Q^0 be the event-stationary limit distribution. Since $Q^0 = P$ on $\mathcal{I}$, we obtain that

$$\frac{\varphi(0,t]}{\nu(0,t]} \frac{1}{\varphi(0,t]} \sum_{i=1}^{\varphi(0,t]} 1_A \circ \eta_i \to 1 \cdot E_{Q_0}(1_A|\mathcal{I}) \text{ as } t \to \infty \quad P\text{-a.s.}$$

if $P[\lim_{t\to\infty}(\varphi(0,t]/\nu(0,t]) = 1] = 1$. (Here E_{Q_0} denotes expectation under Q^0.) By taking expectation under P we obtain from (2.6) that

$$\frac{1}{\nu(0,t]} \int_{(0,t]} P^{0,x}(A)d\nu(x) \to Q^0(A) \text{ as } t \to \infty \quad \text{for all } A \in \bar{M}^\infty. \tag{2.12}$$

The probability measure Q^0 will be studied again in Sections 4 and 6.

Example 2.1.

Let Φ_{ST} and Φ respectively denote a time-stationary Poisson process and an accompanying (event-stationary) Palm version. (So, Φ has an occurrence in the origin.) Denote their distributions by P_{st} and P, respectively. We will write $\nu_{st}(A) := \mathbb{E}(\Phi_{\text{ST}}(A)) = \lambda_{st}\,\text{Leb}(A)$ and $\nu(A) := \mathbb{E}(\Phi(A))$. Below, we will characterize the shifted Palm-distributions $P^{0,x}$ of P.

First note that the distributions P_{st} and P are related as P and P^0 are in (1.1) and (1.5). Proposition 5 of Jagers (1973) implies a characterization of the Palm distributions

$(P_{st})^x$ of P_{st} : it follows that $\Phi_{\rm ST} + e_x$ has distribution $(P_{st})^x$, where $e_x(C) := 1_C(x)$ for $C \in \mathrm{Bor}(\mathbb{R})$. Hence,

$$\mathbb{E}[\Phi_{\rm ST}(A) \cdot 1_C(\Phi_{\rm ST})] = \lambda_{st} \int_A \mathbb{P}[\Phi_{\rm ST} + e_x \in C] dx \tag{2.13}$$

for $A \in \mathrm{Bor}(\mathbb{R})$ and $C \in \bar{M}$; cf. (2.3). Consequently, $\Phi_{\rm ST} + e_0$ has distribution $(P_{st})^{0,x}$(which equals P), so Φ and $\Phi_{\rm ST} + e_0$ have the same distribution. Regarding the PD's of P, we will show that $P^x = P^{0,x} = P$ for $x = 0$, and that for $x \neq 0$ the point processes $\Phi + e_x$ and $\Phi + e_{-x}$ have distributions P^x and $P^{0,x}$, respectively. That is,

$$P^x(B) = \mathbb{P}[\Phi + e_x \in B] \text{ and } P^{0,x}(B) = \mathbb{P}[\Phi + e_{-x} \in B] \tag{2.14}$$

for $B \in \bar{M}$ and for ν-a.e. $x \neq 0$.

First note that $\Phi + e_x$ and $\Phi_{\rm ST} + e_0 + e_x$ are equally distributed as long as $x \neq 0$. Using (2.3) it will be demonstrated that

$$\mathbb{E}[\Phi(A) \cdot 1_B(\Phi)] = \int_A Q^x(B) d\nu(x) \tag{2.15}$$

in case $B = [\varphi(C) = k]; A,\ C \in \mathrm{Bor}(\mathbb{R})$, k a non-negative integer, and $Q^x(B) := \mathbb{P}[\Phi_{\rm ST} + e_0 \in B]$ if $x = 0$ and $Q^x(B) := \mathbb{P}[\Phi_{\rm ST} + e_0 + e_x \in B]$ if $x \neq 0$. Write A_1 for $A \backslash \{0\}$ and C' for the complement of C. Then (by (2.13)) the left-hand side (LHS) of (2.15) equals

$$\begin{aligned}
\mathrm{LHS} &= \mathbb{E}[(\Phi_{\rm ST} + e_0)(A)) \cdot\ 1_{[\varphi(C)=k]}(\Phi_{\rm ST} + e_0)] \\
&= \mathbb{E}[(\Phi_{\rm ST}(A) + 1_A(0)) \cdot\ (1_{[\varphi(C)=k]}(\Phi_{\rm ST}) \cdot 1_{C'}(0) + 1_{[\varphi(C)=k-1]}(\Phi_{\rm ST}) \cdot 1_C(0))] \\
&= \lambda_{st} \int_A \mathbb{P}[(\Phi_{\rm ST} + e_x)(C) = k] dx \cdot 1_{C'}(0) + \\
&\quad + \lambda_{st} \int_A \mathbb{P}[(\Phi_{\rm ST} + e_x)(C) = k-1] dx \cdot\ 1_C(0) + \\
&\quad + 1_A(0) \cdot 1_{C'}(0) \cdot \mathbb{P}[\Phi_{\rm ST}(C) = k] + 1_A(0) \cdot 1_C(0) \cdot \mathbb{P}[\Phi_{\rm ST}(C) = k-1].
\end{aligned}$$

The right-hand side (RHS) of (2.15) equals

$$\begin{aligned}
\mathrm{RHS} &= \mathbb{P}[(\Phi_{\rm ST} + e_0)(C) = k] \cdot 1_A(0) + \lambda_{st} \int_{A_1} \mathbb{P}[(\Phi_{\rm ST} + e_0 + e_x)(C) = k] dx \\
&= \mathbb{P}[(\Phi_{\rm ST} + e_0)(C) = k] \cdot 1_A(0) + \lambda_{st} \int_{A_1} \mathbb{P}[(\Phi_{\rm ST} + e_x)(C) = k] dx \cdot 1_{C'}(0) + \\
&\quad + \lambda_{st} \int_{A_1} \mathbb{P}[(\Phi_{\rm ST} + e_x)(C) = k-1] dx \cdot 1_C(0).
\end{aligned}$$

Since the summation of the terms 3 and 4 in LHS is equal to term 1 in RHS, (2.15) follows for $B = [\varphi(C) = k]$. If B is of the form $[\varphi(C_i) = k_i$ for $i = 1, \ldots, m]$, (2.15) follows from similar arguments. Hence, (2.15) is valid for general B and the same holds for the left-hand sight of (2.14) because of the uniqueness of the family of PD's in the ν-a.e. sense; see below (2.4). The right-hand side of (2.14) follows immediately, since $\theta_x(\Phi_{\mathrm{ST}} + e_0 + e_x)$ and $\Phi_{\mathrm{ST}} + e_{-x} + e_0$ have the same distribution. □

3 Intermediate probabilities

In this section we generalize the concept 'intermediate probability distribution' P_n to the non-stationary case. It is also shown that a null-set under P is also a null-set under P^x for ν-a.e. $x \in \mathbb{R}$ (and vice versa). However, this does not imply that the family $\{P^x\}$ is dominated by P.

For $n \in \mathbb{Z}$ with $P(F_n) > 0$, we define the *intermediate probability measures* P_n by:

$$P_n(A) := P([\eta_n\varphi \in A]|F_n), \; A \in \bar{M}, \tag{3.1}$$

which generalizes the corresponding definition in Section 1. By (2.9) we have, for all $B \in \bar{M}$ and $k \in \mathbb{Z}$ with $P(F_k) > 0$,

$$P_k(B) = (P(F_k))^{-1} \int_{\mathbb{R}_k} P^{0,x}(B \cap [T_{-k} \leq -x < T_{-k+1}]) d\nu(x). \tag{3.2}$$

(Note that it is allowed to replace $\mathbb{R}_k$ by $\mathbb{R}$.) This result is basic for the following theorem.

Theorem 3.1. Let Φ be a simple point process on $\mathbb{R}$ with at least one occurrence. assume that the intensity measure ν exists and is locally finite. Then, for all $B \in \bar{M}$ the following holds:

(1) $P(B) = 0 \;\Leftrightarrow\; P^x(B) = 0$ for ν-a.e. $x \in \mathbb{R}$,

(2) $P^{0,x}(B) = 0$ for ν-a.e. $x \in \mathbb{R} \;\Leftrightarrow\; P_n(B) = 0$ for all $n \in \mathbb{Z}$ with $P(F_n) > 0$.

Proof. Let $B \in \bar{M}$. Suppose that $P(B) = 0$. Then $\mathrm{P}(B \cap F_n) = 0$ and - by the second equality in (2.7) and the remark thereafter - $P^x(B \cap [T_n = x]) = 0$ for ν-a.e. $x \in \mathbb{R}$ and all integers n. Hence,

$$P^x(B) = \sum_{m \in \mathbb{Z}} P^x(B \cap [T_m = x]) = 0 \quad \text{for } \nu\text{-a.e. } x \in \mathbb{R}.$$

The other implication of (1) also follows from the second equation of (2.7), since $P(B) = P(B \cap [\varphi(\mathbb{R}) > 0]) \leq P(B \cap F_0) + P(B \cap F_1)$.

For (2), the implication $\Rightarrow$ follows from (3.2). For the reversed implication, suppose that $P_n(B) = 0$ for all $n \in \mathbb{Z}$ with $P(F_n) > 0$. Because of (3.2),

$$P^{0,x}(B \cap [T_{-n} \leq -x < T_{-n+1}]) = 0 \quad \nu\text{-a.e.}$$

for all such $n \in \mathbb{Z}$. For all $n \in \mathbb{Z}$ with $P(F_n) = 0$ we have:

$$P^{0,x}(B \cap [T_{-n} \leq -x < T_{-n+1}]) \leq P^{0,x}[T_{-n} \leq -x < T_{-n+1}] = 0 \quad \nu\text{-a.e.},$$

which follows by choosing $B = M$ in the last equation of (2.7). Consequently,

$$P^{0,x}(B) = \sum_{n \in \mathbb{Z}} P^{0,x}([T_{-n} \leq -x < T_{-n+1}] \cap B) = 0 \quad \text{for } \nu\text{-a.e. } x \in \mathbb{R}. \qquad \square$$

Remark. Part (1) does not automatically imply that P dominates P^x for ν-a.e. $x \in \mathbb{R}$; see Example 3.3 below.

The intermediate position of the probability measures $\{P_n\}$ between P and the shifted PD's $\{P^{0,x}\}$ is illustrated nicely by the following corollary. It is about strong laws holding equivalently under P and under $\{P^{0,x}\}$. Its proof is straightforward and makes use of Lemma 1.1.

Corollary 3.2 Let Φ be a simple point process on $\mathbb{R}$ with $P(M^\infty) = 1$. Assume that ν exists and is locally finite. Then, for any $\mathcal{I}$-measurable functions V and U on M^∞ and all $B \in \bar{M}$,

$$\begin{aligned} P[\lim_{t\to\infty} \frac{1}{t}\varphi(0,t] &= V(\varphi)] = 1 \\ &\Leftrightarrow P_n[\lim_{t\to\infty} \frac{1}{t}\varphi(0,t] = V(\varphi)] = 1 \quad \text{for all } n \in \mathbb{Z} \\ &\Leftrightarrow P^{0,x}[\lim_{t\to\infty} \frac{1}{t}\varphi(0,t] = V(\varphi)] = 1 \quad \nu\text{-a.e. } x \in \mathbb{R}. \end{aligned}$$

and

$$\begin{aligned} P^{0,x}[\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} 1_B \circ \eta_i &= U] = 1 \quad \nu\text{-a.e. } x \in \mathbb{R} \\ &\Leftrightarrow P_n[\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} 1_B \circ \eta_i = U] = 1 \text{ for all } n \in \mathbb{Z} \\ &\Leftrightarrow P[\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} 1_B \circ \eta_i = U] = 1. \end{aligned}$$

In the following sections we will consider domination of the family $\{P^{0,x}\}$ by one event-stationary distribution P^0_{st} and/or domination of P by a time-stationary distribution P_{st}. By the first type of domination (notation: $P^{0,x} \ll P^0_{st}$ for ν-a.e. $x \in \mathbb{R}$) we mean that for ν-a.e. $x \in \mathbb{R}$ and all $B \in \bar{M}$, the implication $P^0_{st}(B) = 0 \Rightarrow P^{0,x}(B) = 0$ is valid. The first type of domination **not** necessarily implies the second, as follows from the example below. This example also shows that Theorem 3.1 does **not** imply domination.

Example 3.3. Suppose that P itself is **event**-stationary. Then all intermediate probability distributions are equal to P. By part (2) of Theorem 3.1 it follows for all $B \in \bar{M}$ that: $P(B) = 0$ implies $P^{0,x}(B) = 0$ for ν-a.e. $x \in \mathbb{R}$. Since $P[\varphi\{0\} = 1] = 1$, it follows from part (1) that $P^x[\varphi\{0\} = 1] = 1$ and hence $P^{0,x}[\varphi\{0\} = 1$ and $\varphi\{-x\} = 1] = 1$ for ν-a.e. $x \in \mathbb{R}$.

If P is the distribution of the event-stationary deterministic point process with interval lengths 1, then $P\{\varphi_0\} = 1$ where φ_0 is the counting measure with atoms in all integers. Note that P is **not** dominated by any time-stationary distribution P_{st}, since $P[\varphi\{0\} = 1] = 1$ while $P_{st}[\varphi\{0\} = 1]$ has to be 0 because of time-stationarity. Furthermore, ν is **not** dominated by Leb, since $\text{Leb}\{0\} = 0$ while $\nu\{0\} = 1$. However, $P^x = P$ for ν-a.e. $x \in \mathbb{R}$ since

$$\mathbb{E}[\Phi(A)1_B(\Phi)] = \varphi_0(A) \cdot 1_B(\varphi_0) = \int_A P(B) d\nu(x)$$

for all $A \in \text{Bor}(\mathbb{R})$ and all $B \in \bar{M}$. Consequently, $P^{0,x} = P$ and $P^{0,x}$ **is** dominated by an event-stationary distribution (P itself) for ν-a.e. $x \in \mathbb{R}$; see the remark about uniqueness following (2.4).

If P is the distribution of an ordinary renewal process with exponentially distributed interval lengths, then $P_n = P$ for all $n \in \mathbb{Z}$. Recall that the eventualities $A_x := [\varphi\{-x\} = 1]$ have $P^{0,x}$-probability 1. Although they have P-probability 0 as long as $x \neq 0$, this does not contradict part (2) of Theorem 3.1 because of the ν-a.e. inclusion. But it does show that $P^{0,x}$ is not for ν-a.e. $x \in \mathbb{R}$ dominated by $P_n = P$ since the ν-measure of the set of $x \in \mathbb{R}$ for which there exists an eventuality B with $P(B) = 0$ and $P^{0,x}(B) \neq 0$, is unequal to 0. In a similar way, the eventualities $B_x := [\varphi\{x\} = 1]$ have P-probability 0 and P^x-probability 1 as long as $x \neq 0$. Hence, in spite of part (1) of the theorem, $\{P^x\}$ is not dominated by P. □

4 Domination of $\{P^{0,x}\}$ by an event-stationary distribution

It is well-known that the strength of the concept 'Palm distribution' appears best under stationary circumstances; cf. Daley and Vere-Jones (1988; p. 456). In this section we will not assume that the point process itself is stationary, but that its family of shifted Palm distributions is dominated by one event-stationary Palm distribution P^0_{st}. For several properties valid for stationary point processes (see Section 1), we derive generalizations valid under the new circumstances.

Assume that $P^{0,x} \ll P^0_{st}$ for ν-a.e. $x \in \mathbb{R}$; notation: $\{P^{0,x}\} \ll P^0_{st}$. Here P^0_{st} is assumed to be an event-stationary distribution with $E^0_{st}(\alpha_0) < \infty$. ($E^0_{st}$ denotes expectation under P^0_{st}.) Hence, for ν-a.e. $x \in \mathbb{R}$, the null-sets under P^0_{st} are also null-sets under $P^{0,x}$. It is well-known that P^0_{st} can be considered as the Palm distribution of a time-stationary distribution P_{st} satisfying (ii); see Theorem 1.3.1 in Franken et al. (1982). So, P^0_{st} and P_{st} are related like P^0 and P in Section 1. Let ρ_x be the Radon-Nikodym density (defined for ν-a.e x) of $P^{0,x}$ with respect to P^0_{st}. That is, for ν-a.e. $x \in \mathbb{R}$ it holds that

$$P^{0,x}(B) = E^0_{st}(\rho_x \cdot 1_B) \quad \text{for all } B \in \bar{M}, \tag{4.1}$$

which expresses $P^{0,x}$ explicitly in terms of P^0_{st}. By applying (1.1) or (1.7), we could get two equations expressing $P^{0,x}$ in terms of P_{st}.

The domination assumption also yields that $P^{0,x}(M^\infty) = 1$ for ν-a.e. $x \in \mathbb{R}$. Hence, $P(M^\infty) = 1$ as follows from part (1) of Theorem 3.1. We can rewrite (2.7) and express P in terms of P^0_{st} (or, if you like, in P_{st}):

$$P(B) = E^0_{st}\Big(\int_{(-T_{-k+1}, -T_{-k}]} \rho_y \cdot 1_B \circ \theta_{-y} d\nu(y)\Big);\ B \in \bar{M}^\infty \text{ and } k \in \mathbb{Z}. \tag{4.2}$$

Recall that the domination of $\{P^{0,x}\}$ by an event-stationary distribution not necessarily implies that P is dominated by a time-stationary distribution; see Example 3.3. However, all intermediate probability distributions P_k of P **are** dominated by P^0_{st}. This is a consequence of the following theorem that characterizes the domination-assumption of $\{P^{0,x}\}$.

Theorem 4.1. Let P^0_{st} be an event-stationary distribution on $(M^\infty, \bar{M}^\infty)$ for which $E^0_{st}(\alpha_0) < \infty$. Then the following holds:

The family $\{P^{0,x}\}$ of shifted PD's of P on $(M^\infty, \bar{M}^\infty)$ is dominated by P^0_{st} iff for all $k \in \mathbb{Z}$ the P-probability distribution of (η_k, T_k) on $(M^\infty \times \mathbb{R}, \bar{M}^\infty \times \mathrm{Bor}(\mathbb{R}))$ is dominated by the product measure $P^0_{st} \times \nu$. The RN-derivatives

$$\frac{dP^{0,x}}{dP^0_{st}}(\varphi, x) =: \rho_x(\varphi) \text{ and } \frac{dP_{(\eta_k,T_k)}}{d(P^0_{st} \times \nu)}(\varphi, x) =: \tau_k(\varphi, x),$$

$\varphi \in M^\infty$ and $x \in \mathbb{R}$, are related as

(i) $\tau_k(\varphi, x) = \rho_x(\varphi) \cdot 1_{[T_{-k} \le -x < T_{-k+1}]}$ $(P^0_{st} \times \nu)$-a.e.,

(ii) $\rho_x(\varphi) = \sum_{k\in\mathbb{Z}} \tau_k(\varphi, x)$ P^0_{st}-a.s. for ν-a.e. $x \in \mathbb{R}$.

Proof. The only-if part is an immediate consequence of (2.9) and (4.1); relation (i) follows too. For the if part, suppose that for all $k \in \mathbb{Z}$ the probability distribution of (η_k, T_k) under P is dominated by the product measure $P^0_{st} \times \nu$ with RN-derivative $\tau_k(\varphi, x)$. By applying (2.9) and taking $\sum_{k\in\mathbb{Z}}$ we obtain that, for all $A \in \mathrm{Bor}(\mathbb{R})$ and $B \in \bar{M}^\infty$,

$$\mu_B(A) = \int_A P^{0,x}(B) d\nu(x) = \int_A Q^{0,x}(B) d\nu(x) \tag{4.3}$$

with $Q^{0,x}(B) := \int_B (\sum_{k\in\mathbb{Z}} \tau_k(\varphi, x)) dP^0_{st}(\varphi)$ for $x \in \mathbb{R}$. Consequently, for all $x \in \mathbb{R}$, the measures $Q^{0,x}$ are dominated by P^0_{st} and

$$dQ^{0,x}/dP^0_{st} = \sum_{k\in\mathbb{Z}} \tau_k(\varphi, x) \quad P^0_{st}\text{-a.s.}$$

Especially, the $Q^{0,x}$ are **probability** measures. Recall that $\{P^{0,x}\}$, the shifted family of PD's, is unique in the ν-a.e. sense. So we can conclude that, for ν-a.e. $x \in \mathbb{R}$, we have $P^{0,x} = Q^{0,x}$. The if-part and relation (ii) follow from this observation. □

Corollary 4.2. Suppose that $\{P^{0,x}\} \ll P^0_{st}$; set $\{\rho_x\}$ for the RN-derivatives. Then, for $k \in \mathbb{Z}$, the intermediate probability distributions P_k are also dominated by P^0_{st} with RN-derivatives δ_{-k} (on M^∞) equal to

$$\delta_{-k} = \int_{(-T_{-k+1}, -T_{-k}]} \rho_x d\nu(x).$$

For all $n \in \mathbb{Z}$ it holds that $P^0_{st}[\delta_{n+1} = \delta_n \circ \eta_1] = 1$.

Proof. The domination result and the equation for δ_{-k} follow from Theorem 4.1 and Fubini's theorem. As a consequence, note that for all $A \in \bar{M}^\infty$:

$$\begin{aligned} E^0_{st}(1_A \cdot \delta_{-(k+1)}) &= P_{k+1}(A) = P_k[\eta_1 \varphi \in A] \\ &= E^0_{st}(1_A \circ \eta_1 \cdot \delta_{-k}) = E^0_{st}(1_A \cdot \delta_{-k} \circ \eta_{-1}). \end{aligned}$$

So, $P^0_{st}[\delta_{-(k+1)} = \delta_{-k} \circ \eta_{-1}] = 1$ for all $k \in \mathbb{Z}$. The result follows by replacing $(k+1)$ by $-n$ and applying the event-stationarity of P^0_{st} once again. $\square$

As a consequence, the sequence $\{\delta_n\}$ is stationary under P^0_{st}. If P is time-stationary itself with finite intensity, then the PD's $P^{0,x}$ are equal to the accompanying event-stationary PD P^0. So, we can take $P^0_{st} = P^0$ and $\rho_x \equiv 1$. Since $\nu = \lambda \cdot \mathrm{Leb}$, it follows immediately that $\delta_{-k} = \lambda \alpha_{-k}$ P^0_{st}-a.s. Consequently, Corollary 4.2 generalizes (1.6).

Under additional assumptions, $P^{0,x}$ can explicitly be expressed in terms of P.

Corollary 4.3. Suppose that $P^{0,x} \sim P^0_{st}$ with $\rho_x = dP^{0,x}/dP^0_{st}$ for ν-a.e. $x \in \mathbb{R}$. Assume additionally that $m \in \mathbb{Z}$ exists such that $P^0_{st}[\nu(-T_{m+1}(\varphi), -T_m(\varphi)] > 0] = 1$. Then we have for ν-a.e. $x \in \mathbb{R}$ and all $k \in \mathbb{Z}$ that $P_k \sim P^0_{st}$, that $P^{0,x} \sim P_k$, and that

$$P^{0,x}(B) = \mathbb{E}(\frac{1}{\delta_0 \circ \eta_0} \cdot \rho_x \circ \eta_k \cdot 1_B \circ \eta_k); \quad B \in \bar{M}^\infty.$$

Proof. We first prove that $P^0_{st}[\delta_m > 0] = 1$. Note that $P^0_{st}[\rho_x(\varphi) > 0] = 1$ for ν-a.e. $x \in \mathbb{R}$. Also note that the function $U : M^\infty \to [0, \infty]$ defined by $U(\varphi) := \nu\{x \in \mathbb{R} : \rho_x(\varphi) = 0\}$ is measurable with respect to $\bar{M}^\infty$ and $\mathrm{Bor}([0,\infty])$ and that the set $S := \{(x, \varphi) \in \mathbb{R} \times M^\infty : \rho_x(\varphi) = 0\}$ belongs to the product σ-field $\mathrm{Bor}(\mathbb{R}) \times \bar{M}^\infty$ on the product space $\mathbb{R} \times M^\infty$. With these considerations and Fubini's theorem in mind, the following equivalences are immediate:

$$P^0_{st}[U = 0] = 1 \Leftrightarrow E^0_{st}(U) = 0 \Leftrightarrow \int_M \int_{\mathbb{R}} 1_S(s, \psi) d\nu(s) dP^0_{st}(\psi) = 0$$

$$\Leftrightarrow \int_{\mathbb{R}} \int_M 1_S(s, \psi) dP^0_{st}(\psi) d\nu(s) = 0$$

$$\Leftrightarrow \int_{\mathbb{R}} P^0_{st}[\rho_s(\varphi) = 0] d\nu(s) = 0$$

$$\Leftrightarrow P^0_{st}[\rho_x(\varphi) = 0] = 0 \quad \nu\text{-a.e. } x \in \mathbb{R}$$

As a consequence we obtain that $P^0_{st}[\rho_x(\varphi) > 0$ for ν-a.e. $x \in \mathbb{R}] = 1$. Since $P^0_{st}[\nu(-T_{m+1}(\varphi), -T_m(\varphi)] > 0] = 1$, it follows from the definition of δ_m that $P^0_{st}[\delta_m > 0] = 1$. Consequently, $P^0_{st}[\delta_k > 0] = 1$ and $P_k \sim P^0_{st}$ for all $k \in \mathbb{Z}$. Furthermore, with E_k denoting expectation under P_k, we obtain for ν-a.e. $x \in \mathbb{R}$ that

$$\begin{aligned} P^{0,x}(B) &= E_k(\frac{\rho_x}{\delta_{-k}} 1_B) = E_0(\frac{\rho_x \circ \eta_k}{\delta_0} 1_B \circ \eta_k) \\ &= E(\frac{\rho_x \circ \eta_k}{\delta_0 \circ \eta_0} 1_B \circ \eta_k) \quad \text{for all } B \in \bar{M}^\infty, \end{aligned}$$

since $1 = P^0_{st}[\delta_{-k} \circ \eta_k = \delta_0] = P_0[\delta_{-k} \circ \eta_k = \delta_0]$. □

Long-run properties

Here are immediate long-run consequences (under P) of the domination assumption. Results (4.4) and (4.5) follow from Birkhoff's ergodic theorem, and from part (2) of Theorem 3.1 or Corollary 3.2.

$$\frac{1}{n}\sum_{i=1}^{n} \alpha_i \to E^0_{st}(\alpha_0|\mathcal{I}) =: \bar{\alpha} \text{ as } n \to \infty \quad P\text{-a.s.}; \tag{4.4}$$

$$\frac{1}{n}\sum_{i=1}^{n} 1_A \circ \eta_i \to E^0_{st}(1_A|\mathcal{I}) \text{ as } n \to \infty \quad P\text{-a.s.}; \quad A \in \bar{M}^\infty; \tag{4.5}$$

$$\frac{1}{n}\sum_{i=1}^{n} P[\eta_i(\varphi) \in A] \to E(E^0_{st}(1_A|\mathcal{I})) =: Q^0(A) \text{ as } n \to \infty; \quad A \in \bar{M}^\infty. \tag{4.6}$$

(Note that the convergences in (4.4) and (4.5) also hold $P^{0,x}$-a.s. for ν-a.e. $x \in \mathbb{R}$.) Since $Q^0(A) = E(E^0_{st}(1_A|\mathcal{I}) \circ \eta_0)$, we obtain by Corollary 4.2 that

$$Q^0(A) = E^0_{st}(\delta_0 \cdot E^0_{st}(1_A|\mathcal{I})) = E^0_{st}(\bar{\delta} \cdot 1_A), \quad A \in \bar{M}^\infty, \tag{4.7}$$

where $\bar{\delta} = E^0_{st}(\delta_0|\mathcal{I})$, the long-run average (under P^0_{st}) of the sequence $\{\delta_n\}$. Consequently, $Q^0 \ll P^0_{st}$ and $dQ^0/dP^0_{st} = \bar{\delta}$. Furthermore, $Q^0 = P^0_{st}$ if P^0_{st} is ergodic.

It can easily be proved that the convergence in (4.6) is uniform in A, again by using Corollary 4.2. Set

$$\hat{P}_n(A, \varphi) := \frac{1}{n}\sum_{i=1}^{n} 1_A(\eta_i \varphi); \quad A \in \bar{M}^\infty \text{ and } n \in \mathbb{N}.$$

Then

$$2 \sup_{A \in \bar{M}^\infty} \left|\frac{1}{n}\sum_{i=1}^{n} P_i(A) - Q^0(A)\right| = d(E\hat{P}_n, Q^0) = E^0_{st}\left|\frac{1}{n}\sum_{i=1}^{n} \delta_{-i} - \bar{\delta}\right|,$$

which tends to 0 as $n \to \infty$ since the i.d. sequence $\{\delta_{-i}\}$ - and hence the sequence $\{\sum_{i=1}^n \delta_{-i}/n\}$ too - is uniformly P^0_{st}-integrable.

We next consider long-run properties in continuous time. Let P_{st} be the time-stationary probability distribution (and E_{st} the corresponding expectation operator) that has P^0_{st} as Palm distribution. First, note that $P[\lim_{t\to\infty} \varphi(0,t] = \infty] = 1$ and that

$$P[\lim_{t\to\infty} (T_{\varphi(0,t]}(\varphi)/\varphi(0,t]) = \bar{\alpha}(\varphi)] = 1$$

by (4.4). Consequently, using (1.15),

$$\frac{\Phi(0,t]}{t} \to E_{st}(\Phi(0,1]|\mathcal{I}) =: \bar{\Phi} \text{ as } t \to \infty \quad P\text{-a.s.} \tag{4.8}$$

Furthermore,

$$\frac{1}{t}\int_0^t 1_A \circ \theta_s ds \to E_{st}(1_A|\mathcal{I}) \text{ as } t \to \infty \quad P\text{-a.s.,} \tag{4.9}$$

since the convergence holds P_{st}-a.s. because of Birkhoff's ergodic theorem, P^0_{st}-a.s. because of the right-hand side of (1.2), and P_1-a.s. by part (2) of Theorem 3.1. (Note that the convergences in (4.8) and (4.9) also hold $P^{0,x}$-a.s. for ν-a.e $x \in \mathbb{R}$.)

By taking P-expectation we obtain that

$$\frac{1}{t}\int_0^t P[\theta_s\varphi \in A]ds \to E(E_{st}(1_A|\mathcal{I})) =: Q(A) \text{ as } t \to \infty;\ A \in \bar{M}^\infty. \tag{4.10}$$

Note that Q is a time-stationary probability distribution. By Corollary 4.2, (1.7) and (1.15) we obtain that

$$\begin{aligned} Q(A) &= E(E_{st}(1_A|\mathcal{I}) \circ \eta_0) = E^0_{st}(\bar{\delta} \cdot E_{st}(1_A|\mathcal{I})) \\ &= \frac{1}{\lambda_{st}} E_{st}(\frac{\bar{\delta}}{\alpha_0} \cdot E_{st}(1_A|\mathcal{I})) = \frac{1}{\lambda_{st}} E_{st}(\bar{\Phi}\bar{\delta} \cdot 1_A);\ A \in \bar{M}^\infty. \end{aligned} \tag{4.11}$$

Also note that

$$E_Q(\Phi(0,1]) = E^0_{st}(\bar{\Phi}\bar{\delta}) = E^0_{st}(\delta_0/\bar{\alpha}).$$

We conclude that $Q \ll P_{st}$ with $dQ/dP_{st} = \bar{\Phi}\bar{\delta}/\lambda_{st}$, and that Q satisfies (ii) (and hence the PD of Q exists) if $E^0_{st}(\delta_0/\bar{\alpha}) < \infty$. Furthermore, $Q = P_{st}$ if P_{st} is ergodic.

The following theorem summarizes some of the results of this subsection.

Theorem 4.4. Suppose that the family $\{P^{0,x}\}$ of the shifted PD's of a point process with distribution P is dominated by an event-stationary distribution P^0_{st}. Then the point process is AES and ATS. The event-stationary limit distribution Q^0 and the time-stationary limit distribution Q are related as follows:

$$Q^0(A) = E_Q(\frac{\bar{\alpha}}{\alpha_0} \cdot 1_A \circ \eta_0) \quad \text{and} \quad Q[\eta_0\varphi \in A] = E_{Q^0}(\frac{\alpha_0}{\bar{\alpha}} \cdot 1_A),\ A \in \bar{M}^\infty;$$

Q^0 is the PD of Q if P^0_{st} is pseudo-ergodic.

Proof. The results on AES and ATS were proved above. In (4.11) we expressed Q in terms of P_{st}. By (1.6) we can do the same for Q^0 :

$$Q^0(A) = E^0_{st}(\bar{\delta}1_A) = \frac{1}{\lambda_{st}} E_{st}(\frac{\bar{\delta}}{\alpha_0} \cdot 1_A \circ \eta_0);\ A \in \bar{M}.$$

The relations between Q^0 and Q follow immediately.

Suppose that P^0_{st} is pseudo-ergodic. First note that

$$E_Q(\Phi(0,1]) = E^0_{st}(\delta_0/\bar{\alpha}) = \lambda_{st} E^0_{st}(\delta_0) = \lambda_{st}$$

since δ_0 is RN-density of P_0 w.r.t P^0_{st}. Consequently, Q satisfies (ii), has intensity λ_{st}, and the PD of Q is well defined. Since $\bar{\alpha} = 1/\lambda_{st}$ holds P^0_{st}-a.s., it also holds $P^{0,x}$-a.s. for ν-a.e. $x \in \mathbb{R}$ because of the domination assumption of $\{P^{0,x}\}$. It holds P_1-a.s. by Theorem 3.1, and P-a.s. by Lemma 1.1. Since P and Q coincide on $\mathcal{I}$ it also holds Q-a.s. So,

$$Q^0(A) = \frac{1}{\lambda_{st}} E_Q(\frac{1}{\alpha_0} \cdot 1_A \circ \eta_0);\ A \in \bar{M}.$$

Hence, by (1.7), Q^0 is the PD of Q. □

5 Domination of P by a time-stationary distribution

Some immediate consequences of domination of P by a time-stationary distribution P_{st} are considered. Especially the implication $P \ll P_{st} \Rightarrow \nu \ll$Leb is derived. Furthermore, we investigate the relation between time-stationarity of P itself and $\mathcal{I}$-measurability of dP/dP_{st}.

Assume that P is dominated by a time-stationary distribution P_{st} that satisfies (ii). Let λ_{st} denote the intensity under P_{st}. We will write E for expectation under P, E_{st} for expectation under P_{st}, and E^0_{st} for expectation under the event-stationary Palm distribution P^0_{st} that - according to (1.1) - belongs to P_{st}. Let σ be a Radon-Nikodym density; that is,

$$P(B) = E_{st}(\sigma \cdot 1_B) \quad \text{for all } B \in \bar{M}. \tag{5.1}$$

First notice that $P(M^\infty) = 1$ since $P_{st}(M^\infty) = 1$. Here are other results that can easily be proved from (5.1) and relations in Section 1:

$$\begin{aligned} P(B) &= \lambda_{st} E^0_{st}(\int_{T_{k-1}}^{T_k} \sigma \circ \theta_u \cdot\ 1_B \circ \theta_u du), \\ &= \lambda_{st} E^0_{st}(\int_{(-T_{-k+1}, -T_{-k}]} \sigma \circ \theta_{-y} \cdot 1_B \circ \theta_{-y} dy) \end{aligned} \tag{5.2}$$

for all $B \in \bar{M}^\infty$ and $k \in \mathbb{Z}$;

$$P\theta_t^{-1} \ll P_{st} \text{ and RN} = \sigma \circ \theta_{-t} \quad \text{for all } t \in \mathbb{R}. \tag{5.3}$$

Lemma 5.1. Let P_{st} be a time-stationary distribution that satisfies (ii). Then:

$$P \ll P_{st} \Rightarrow \nu \ll \text{Leb}.$$

Proof. Suppose that $P \ll P_{st}$. If $A \in \text{Bor}(\mathbb{R})$ satisfies $\text{Leb}(A) = 0$, then $E_{st}(\Phi(A)) = \lambda_{st} \cdot \text{Leb}(A) = 0$ and $1 = P_{st}[\varphi(A) = 0] = P[\varphi(A) = 0]$, and hence $\nu(A) = 0$. Consequently, $\nu \ll$Leb. □

If P ***is time-stationary too*** ...

If $P \ll P_{st}$ and P is time-stationary too, what are the consequences? The following results are straightforward:

$$P \text{ is time-stationary too} \Leftrightarrow P_{st}[\sigma \circ \theta_t = \sigma] = 1 \text{ for all } t \in \mathbb{R}; \tag{5.4}$$

$$\sigma \text{ is } \mathcal{I}\text{-measurable} \Rightarrow P \text{ is time-stationary.} \tag{5.5}$$

Regarding a reversed version of (5.5) we have to be careful:

Lemma 5.2. Suppose that $P \ll P_{st}$, and that P_{st} is time-stationary and satisfies (ii). Then:

$$P \text{ is time-stationary} \Rightarrow P_{st}[\sigma \circ \theta_t = \sigma \text{ for Leb-a.e } t \in \mathbb{R}] = 1.$$

Proof. Suppose that P is time-stationary. Note that the function $U : M^\infty \to [0, \infty]$ defined by $U(\varphi) :=$Leb$\{t \in \mathbb{R} : \sigma(\theta_t \varphi) \neq \sigma(\varphi)\}$ is measurable with respect to $\bar{M}^\infty$ and $\text{Bor}([0, \infty])$. Also note that the set $S := \{(t, \varphi) \in \mathbb{R} \times M^\infty : \sigma(\theta_t \varphi) \neq \sigma(\varphi)\}$ belongs to the product σ-field $\text{Bor}(\mathbb{R}) \times \bar{M}^\infty$ on the product space $\mathbb{R} \times M^\infty$. Similar to the proof of Corollary 4.3 we have:

$$P_{st}[U = 0] = 1 \Leftrightarrow P_{st}[\sigma(\theta_t \varphi) = \sigma(\varphi)] = 1 \quad \text{Leb-a.e. } t \in \mathbb{R}.$$

By (5.4) the last statement is valid. So, the implication of the lemma follows immediately. □

Suppose that $\lambda := E(\Phi(0, 1]) < \infty$, so P satisfies (ii). If P is time-stationary too and P_{st} is pseudo-ergodic, then $E(\Phi(0, 1]|\mathcal{I})$ and λ_{st} are both P-a.s. limits of $\Phi(0, t]/t$ as $t \to \infty$. So, $P[E(\Phi(0, 1]|\mathcal{I}) = \lambda_{st}] = 1$, P is pseudo-ergodic too, and $\lambda = \lambda_{st}$. However, for general time-stationary P with $P \ll P_{st}$ the two intensities are not necessarily equal. Both $\lambda < \lambda_{st}$ and $\lambda > \lambda_{st}$ is possible, as in the following example.

Example 5.3. Take $\sigma(\varphi) := E_{st}(\Phi(0,1]|\mathcal{I})/\lambda_{st} =: \bar{\Phi}/\lambda_{st}$. Then, $E_{st}(\sigma) = 1$ and P (defined by (5.1)) is time-stationary because of (5.5). Assume that σ is not degenerated; cf. Nieuwenhuis(1994; p. 53/54). By conditioning on $\mathcal{I}$, we have

$$\lambda = E(\Phi(0,1]) = E_{st}(\bar{\Phi} \cdot \Phi(0,1])/\lambda_{st} = E_{st}(\bar{\Phi}^2)/\lambda_{st},$$

which is larger than λ_{st} since the random variable σ is non-degenerated.

Suppose that $\bar{\alpha} := E^0_{st}(\alpha_0|\mathcal{I})$ is non-degenerated. The definition $\sigma(\varphi) := \bar{\alpha}/E_{st}(\bar{\alpha})$ leads to a time-stationary distribution P dominated by P_{st} for which $\lambda < \lambda_{st}$:

$$\begin{aligned}\lambda &= E(\Phi(0,1]) = E_{st}(\bar{\alpha} \cdot \ E_{st}(\Phi(0,1]|\mathcal{I}))/E_{st}(\bar{\alpha}) \\ &= 1/E_{st}(\bar{\alpha}) = 1/(\lambda_{st} \cdot \ E^0_{st}(\bar{\alpha}^2)) < \lambda_{st}.\end{aligned}$$

(In the third and fourth equality, we used (1.15) and (1.6), respectively. Regarding the inequality we used that $\bar{\alpha}$ is non-degenerated.) □

To illustrate things, we ask ourselves the following question: Starting with a time-stationary distribution P_{st}, is it possible to adopt a model P that is dominated by it and that (to say it in a queuing kind of way) allows from all occurrences the experience of an event-stationary distribution? As we will see, the answer is yes.

Example 5.4. For $\varphi \in M^\infty$ and $B \in \bar{M}^\infty$, set $\sigma(\varphi) := 1/(\lambda_{st} \cdot \alpha_0)$ and $P(B) := E_{st}(\sigma \cdot 1_B)$. By (1.3) it follows that $E_{st}(\sigma) = 1$, as it should. By (1.7) we have, for all $n \in \mathbb{Z}$,

$$P[\eta_n \varphi \in B] = E_{st}(\frac{1}{\alpha_0} 1_B \circ \eta_n)/\lambda_{st} = P^0_{st}(B);$$

$B \in \bar{M}^\infty$. All intermediate probabilities are the same and equal to the event-stationary distribution that belongs to P_{st}. Under the model P, event-stationarity is observed from all occurrences. □

6 Equivalence of domination properties

Domination of P by a time-stationary distribution P_{st} is equivalent to the joint validity of the domination properties $\{P^{0,x}\} \ll P^0_{st}$ and $\nu \ll \text{Leb}$. Under such domination circumstances, the shifted PD's $P^{0,x}$ can explicitly be expressed in terms of the distribution P of the point process. Some consequences are derived.

The theorem below describes the tight relationship between the two kinds of domination we considered before.

Theorem 6.1. Let P be the distribution of a point process, P_{st} a time-stationary distribution satisfying (ii) and P^0_{st} the accompanying Palm distribution. Then:

$$P \ll P_{st} \;\Leftrightarrow\; \nu \ll \text{Leb and } \{P^{0,x}\} \ll P^0_{st}.$$

The RN-derivatives $\sigma = dP/dP_{st}$, $\lambda(\cdot) := d\nu/d\text{Leb}$, and $\rho_x = dP^{0,x}/dP^0_{st}$ are related as follows:

(i) $\rho_y \cdot \lambda(y) = \lambda_{st} \cdot \sigma \circ \theta_{-y}$ for ν-a.e. y in $\mathbb{R}$ $\quad P^0_{st}$-a.s.;

(ii) $\lambda(x) = \lambda_{st} \cdot\; E^0_{st}(\sigma \circ \theta_{-x})$ for Leb-a.e. x in $\mathbb{R}$;

(iii) $\lambda_{st} \cdot \sigma = \rho_{T_k} \circ \eta_k \cdot\; \lambda(T_k) \quad P_{st}$-a.s. for all $k \in \mathbb{Z}$.

Proof. Suppose that $P \ll P_{st}$, with RN-density σ. First note that $\nu \ll$ Leb by Lemma 5.1. Write $\lambda(\cdot)$ for an RN-density and note that $\lambda(x) \neq 0$ for ν-a.e. $x \in \mathbb{R}$.

Let $B \in \bar{M}^\infty$ and let A be a Borel-set in $\mathbb{R}$. We will use (2.3) and the remark about uniqueness in the paragraph thereafter. By (5.2) we have:

$$\begin{aligned}
E(\Phi(A) \cdot 1_B(\Phi)) &= \sum_{k\in\mathbb{Z}} P([T_k \in A] \cap B) \\
&= \sum_{k\in\mathbb{Z}} \lambda_{st} E^0_{st}\Big(\int_A \sigma \circ \theta_{-x} \cdot 1_B \circ \theta_{-x} \cdot 1_{[T_{-k} \le -x < T_{-k+1}]} dx\Big) \\
&= \lambda_{st} E^0_{st}\Big(\int_A \sigma \circ \theta_{-x} \cdot 1_B \circ \theta_{-x} dx\Big) \\
&= \int_A E^0_{st}(\lambda_{st} \cdot \sigma \circ \theta_{-x} \cdot 1_B \circ \theta_{-x}/\lambda(x)) d\nu(x).
\end{aligned}$$

Note that the choice $B = M^\infty$ yields

$$\int_A 1 d\nu(x) = \int_A E^0_{st}(\lambda_{st} \cdot \sigma \circ \theta_{-x}/\lambda(x)) d\nu(x)$$

for all Borel-subsets A of $\mathbb{R}$. Hence, $Q^{0,x}(C) := \lambda_{st} E^0_{st}(\sigma \circ \theta_{-x} \cdot 1_C)/\lambda(x)$, $C \in \bar{M}^\infty$, defines a probability measure for ν-a.e. $x \in \mathbb{R}$, and it follows that

$$E(\Phi(A) \cdot 1_B(\Phi)) = \int_A Q^{0,x}[\theta_{-x}\varphi \in B] d\nu(x).$$

From (2.3) and the remark on uniqueness in the paragraph thereafter it follows that $P^{0,x} = Q^{0,x}$ for ν-a.e $x \in \mathbb{R}$. We conclude that - for ν-a.e. $x \in \mathbb{R}$ - $P^{0,x}$ is dominated by P^0_{st} and that the RN's ρ_x satisfy (i). Note that, since $1 = P^{0,x}(M^\infty)$, the above also yields that $\nu(A)$ equals

$$\int_A P^{0,x}(M^\infty)\lambda(x) dx = \int_A E^0_{st}(\rho_x \cdot \lambda(x)) dx = \lambda_{st} \int_A E^0_{st}(\sigma \circ \theta_{-x}) dx$$

for all $A \in \text{Bor}(\mathbb{R})$. So, (ii) follows.

Suppose that $\nu \ll \text{Leb}$ and $P^{0,x} \ll P^0_{st}$ for ν-a.e. $x \in \mathbb{R}$. By applying the last equality of (2.7) with $\mathbb{R}_k$ replaced by $\mathbb{R}$, the domination property of $\{P^{0,x}\}$, (1.7) with n replaced by k, Fubini's theorem, and substituting y for $-x + T_k(\varphi)$, we obtain that $P(B)$ equals

$$\int_{-\infty}^{\infty} E_{st}(\frac{1}{\alpha_0} \cdot \rho_{T_k - y}(\eta_k) \cdot 1_B \circ \theta_y \cdot \lambda(T_k - y) \cdot 1_{[T_0 \le y < T_1]})dy/\lambda_{st}.$$

Recall the definition of I_x in the paragraph preceding (2.7). Note that for fixed $y \in \mathbb{R}$,

$$T_0(\psi) \le y < T_1(\psi) \Leftrightarrow \psi(I_y) = 0 \Leftrightarrow \theta_y \psi \in [\varphi(-y + I_y) = 0]$$

for all $\psi \in M^\infty$. Note also that all $\psi \in M^\infty$ with $T_0(\psi) \le y < T_1(\psi)$ satisfy

$$T_k(\psi) - y = T_k(\theta_y \psi);\ \eta_k(\psi) = \eta_k(\theta_y \psi);\ \alpha_0(\psi) = \alpha_0(\theta_y \psi).$$

By using the stationarity property (iii) of P_{st} it follows that

$$\begin{aligned} P(B) &= \int_{-\infty}^{+\infty} E_{st}(\frac{1}{\alpha_0} \cdot \rho_{T_k}(\eta_k) \cdot 1_B \cdot \lambda(T_k) \cdot 1_{[\varphi(-y+I_y)=0]})dy/\lambda_{st} \\ &= E_{st}(1_B \cdot \int_{(-T_1,-T_0]} \frac{1}{\alpha_0} \cdot \rho_{T_k}(\eta_k) \cdot \lambda(T_k))dy/\lambda_{st} \\ &= E_{st}(1_B \cdot \rho_{T_k}(\eta_k) \cdot \lambda(T_k))/\lambda_{st} \end{aligned}$$

for all $k \in \mathbb{Z}$. This yields: $P \ll P_{st}$ and (iii). $\square$

Remarks. As an immediate consequence of (i) and (ii), note that it holds P^0_{st}-a.s. that

$$\rho_y = \frac{\sigma \circ \theta_{-y}}{E^0_{st}(\sigma \circ \theta_{-y})} \quad \text{for } \nu\text{-a.e. } y \in \mathbb{R}.$$

Starting with some preliminary, **time**-stationary model P_{st} (for instance a stationary Poisson process), we can use any measurable function $\sigma : M^\infty \to \mathbb{R}$ with $E_{st}(\sigma) = 1$ to transform P_{st} into a new model P (possibly non-stationary) via $P(B) = E_{st}(\sigma \cdot 1_B), B \in \bar{M}^\infty$. The accompanying family $\{P^{0,x}\}$ of shifted Palm distributions is then dominated by the (event-stationary) Palm distribution that belongs to the preliminary model. The family of RN-densities $\{\rho_x\}$ is determined by (i) with $\lambda(\cdot)$ as in (ii).

However, starting with some preliminary **event**-stationary model P^0_{st}, we cannot freely choose the RN-densities $\rho_x : M^0 \to \mathbb{R}$ with $E^0_{st}(\rho_x) = 1$. Relation (i) puts a stamp on the family $\{\rho_x\}$ of RN-densities that can be used to transform an event-stationary model P^0_{st} into a family of more general Palm distributions. Obviously, $\rho_x(\varphi)$ has (for $\varphi \in M^0$) to satisfy $\rho_x(\varphi) = f(\theta_{-x}(\varphi))g(x)$ for suitable functions $f : M^\infty \to [0,\infty)$ and $g : \mathbb{R} \to (0,\infty)$ satisfying $(E^0_{st} f \circ \theta_{-x}) \cdot g(x) = 1$ for Leb-a.e. x.

Having knowledge of the relationship described in (iii), we can, very elegantly, deduce it from (i). Start with the second equality for $P_{st}(A)$ in (1.5) while taking $A := [\lambda_{st} \cdot \ \sigma = \rho_{T_k} \circ \eta_k \cdot \lambda(T_k)]$. The result (iii) follows from (1.3). □

The following lemma gives equivalent expressions for relation (i). In the lemma, Leb $\times$ P^0_{st} is a product measure on the product space $\mathbb{R} \times M^0$.

Lemma 6.2. Relation (i) can be rewritten in three equivalent ways:

$$P^0_{st}[\nu\{y \in \mathbb{R} : \lambda_{st} \cdot \sigma(\theta_{-y}\varphi) \neq \lambda(y) \cdot \rho_y(\varphi)\} = 0] = 1$$

$$\Leftrightarrow$$

$$P^0_{st}[\lambda_{st} \cdot \sigma(\theta_{-y}\varphi) = \lambda(y) \cdot \rho_y(\varphi)] = 1 \text{ for } \nu\text{-a.e } y \in \mathbb{R}$$

$$\Leftrightarrow$$

$$\lambda_{st} \cdot \sigma(\theta_{-y}\varphi) = \lambda(y) \cdot \rho_y(\varphi) \text{ for } (\nu \times P^0_{st})\text{-a.e. pair } (y,\varphi) \in \mathbb{R} \times M^0.$$

Proof. The proof of the first equivalence is similar to the proof of Corollary 4.3; replace the definitions of the function $U : M^\infty \to [0,\infty]$ and the set $S \in \text{Bor}(\mathbb{R}) \times \bar{M}^\infty$ by

$$U(\varphi) := \nu\{y \in \mathbb{R} : \lambda_{st} \cdot \sigma(\theta_{-y}\varphi) \neq \lambda(y) \cdot \rho_y(\varphi)\} \text{ for } \varphi \in M^\infty,$$

$$S := \{(y,\varphi) \in \mathbb{R} \times M^\infty : \lambda_{st} \cdot \sigma(\theta_{-y}\varphi) \neq \lambda(y) \cdot \rho_y(\varphi)\}.$$

For the second equivalence, note that, by Fubini's theorem,

$$\int_{\mathbb{R}} P^0_{st}[\lambda_{st} \cdot \sigma(\theta_{-y}\varphi) \neq \lambda(y) \cdot \rho_y(\varphi)] d\nu(y) = 0$$

$$\Leftrightarrow \iint_{\mathbb{R} \times M} 1_S(y,\psi) d(\nu \times P^0_{st})(y,\psi) = 0 \Leftrightarrow (\nu \times P^0_{st})(S) = 0.$$ □

Theorem 6.3. Let P be the distribution of a point process. Then:

$$P \sim P_{st} \ \Leftrightarrow \ \nu \sim \text{Leb and } P^{0,x} \sim P^0_{st} \text{ for } \nu\text{-a.e. } x \in \mathbb{R}.$$

Proof. Suppose that $P \sim P_{st}$ and let $\sigma, \lambda(\cdot)$ and ρ_x be defined and related as in Theorem 6.1. By this theorem, $\nu \ll$ Leb and $P^{0,x} \ll P^0_{st}$ for ν-a.e. $x \in \mathbb{R}$. By (1.5) we have, for all $k \in \mathbb{Z}$,

$$P_{st}[\sigma > 0] = \lambda_{st} E^0_{st}\left(\int_{-T_k}^{-T_{k-1}} 1_{[\sigma \circ \theta_{-x} > 0]} dx\right).$$

Since $1_{[\sigma\circ\theta_{-x}>0]} = 1 - 1_{[\sigma\circ\theta_{-x}=0]}$ and $P_{st}[\sigma > 0] = 1$, we obtain

$$1 = \lambda_{st} E^0_{st}(\alpha_{k-1}) - \lambda_{st} E^0_{st}(\int_{-T_k}^{-T_{k-1}} 1_{[\rho_x\cdot\lambda(x)=0]}dx).$$

By (1.2) and (1.3),

$$0 = E^0_{st}(\int_{-T_k}^{-T_{k-1}} 1_{[\rho_x\cdot\lambda(x)=0]}dx) \quad \text{for all } k \in \mathbb{Z}.$$

Consequently,

$$P^0_{st}[\rho_x > 0 \text{ and } \lambda(x) > 0 \text{ for Leb-a.e. } x \in \mathbb{R}] = 1.$$

Hence, $\nu \sim$ Leb and $P^{0,x} \sim P^0_{st}$ for ν-a.e. $x \in \mathbb{R}$.

Next, suppose that $\nu \sim$ Leb and $P^{0,x} \sim P^0_{st}$ for ν-a.e. $x \in \mathbb{R}$. Then

$$\lambda(y) > 0 \text{ for Leb-a.e. } y \in \mathbb{R},$$

$$P^0_{st}[\rho_y > 0] = 1 \text{ for Leb-a.e. } y \in \mathbb{R}.$$

By Lemma 6.2 we know that

$$P^0_{st}[\rho_y \cdot \lambda(y) = \lambda_{st}\cdot\sigma\circ\theta_{-y}] = 1 \text{ for Leb-a.e. } y \in \mathbb{R}.$$

Hence, $P^0_{st}[\sigma\circ\theta_{-y} > 0] = 1$ for Leb-a.e. $y \in \mathbb{R}$. Since

$$P_{st}[\sigma > 0] = \lambda_{st} E^0_{st}(\int_{-T_1}^{0} 1_{[\sigma\circ\theta_{-x}>0]}dx) = \lambda_{st} E^0_{st}(\alpha_0) = 1,$$

it follows that $P \sim P_{st}$. □

In Corollary 4.3 we expressed - under an additional assumption - $P^{0,x}(B)$ explicitly in terms of P and $\{\rho_y\}$, in case $\{P^{0,y}\}$ was dominated by P^0_{st}. Now assume that the equivalence $P \sim P_{st}$ holds. Since $\nu \sim$ Leb and $\lambda(x) > 0$ Leb-a.e., the additional assumption is now satisfied. So, the relation (i) of Theorem 6.1 can be used to transform the equation for $P^{0,x}(B)$ in Corollary 4.3 into an explicit expression in terms of P and σ. A less complicated expression can be obtained by subsequently applying (4.1), (i) of Theorem 6.1, (1.7), and Theorem 6.3. We obtain that, for all $k \in \mathbb{Z}$ and ν-a.e. $x \in \mathbb{R}$,

$$P^{0,x}(B) = \frac{1}{\lambda(x)} E(\frac{1}{\sigma\alpha_0}\sigma\circ\theta_{-x}\circ\eta_k \cdot 1_B\circ\eta_k); \quad B \in \bar{M}^\infty. \tag{6.1}$$

This equation yields that, for all $k \in \mathbb{Z}$ and ν-a.e. $x \in \mathbb{R}$,

$$\lambda(x)\cdot E^x(\frac{1}{\sigma}1_B) = E(\frac{1}{\sigma\alpha_0}1_B\circ\theta_{-x}\circ\eta_k), \tag{6.2}$$

$$E^x(\frac{1}{\sigma}) = \lambda_{st}/\lambda(x). \tag{6.3}$$

Long-run properties

We consider some consequences of Theorem 6.1. Suppose that one of the two equivalent domination criteria of that theorem holds.

First recall (4.6) and (4.7), especially Q^0. In Section 4 this event-stationary distribution was heuristically considered as the distribution observed at an arbitrarily chosen occurrence. By applying Lemma 1.1(b) and (1.6) with $n = 0$ (in the fourth equality below), we obtain

$$\begin{aligned} Q^0(A) &= E(E^0_{st}(1_A|\mathcal{I})) = E_{st}(\sigma \cdot E^0_{st}(1_A|\mathcal{I})) = E_{st}(E_{st}(\sigma|\mathcal{I}) \cdot E^0_{st}(1_A|\mathcal{I})) \\ &= \lambda_{st} E^0_{st}(\alpha_0 \cdot E_{st}(\sigma|\mathcal{I}) \cdot E^0_{st}(1_A|\mathcal{I})) = \lambda_{st} E^0_{st}(\bar{\alpha} \cdot \bar{\sigma} \cdot 1_A), \end{aligned}$$

where $\bar{\alpha} = E^0_{st}(\alpha_0|\mathcal{I})$ as before and $\bar{\sigma} := E_{st}(\sigma|\mathcal{I})$. By (4.7) we obtain:

$$\lambda_{st} \cdot \bar{\sigma} = \frac{\bar{\delta}}{\bar{\alpha}} \quad P^0_{st}\text{-a.s.} \tag{6.4}$$

This relation can be considered as a conditional version of (i) in Theorem 6.1.

Next recall the time-stationary distribution Q in (4.11). Since (6.4) also holds P-a.s., we can rewrite (4 11) as follows:

$$Q(A) = E_{st}(\bar{\sigma} 1_A);\ A \in \bar{M}^\infty. \tag{6.5}$$

We next confront our results with (2.12). It is an easy exercise to show that

$$\frac{1}{\nu(0,t]} \int_{(0,t]} P^{0,x}(B) d\nu(x) \to E^0_{st}(\bar{\sigma} 1_B)/E^0_{st}(\bar{\sigma}) \quad \text{as } t \to \infty \tag{6.6}$$

for all $B \in \bar{M}^\infty$. The limits in (6.6) determine an event-stationary probability measure R^0 on $(M^\infty, \bar{M}^\infty)$. Also note that

$$\frac{\Phi(0,t]}{\nu(0,t]} \to \frac{\bar{\Phi}}{E_{st}(\bar{\sigma}\bar{\Phi})} \text{ as } t \to \infty \quad P\text{-a.s.,}$$

and that the limit equals 1 (the proviso in (2.12)) iff P^0_{st} is pseudo-ergodic. However, pseudo-ergodicity of P^0_{st} implies that $R^0 = Q^0$ since, by (6.4) and Corollary 4.2,

$$R^0(B) = \frac{E^0_{st}(\bar{\delta} 1_B/\bar{\alpha})}{E^0_{st}(\bar{\delta}/\bar{\alpha})} = \frac{E^0_{st}(\bar{\delta} 1_B)}{E^0_{st}(\bar{\delta})} = \frac{Q^0(B)}{1}.$$

So, the limit distribution in (2.12) is indeed equal to Q^0.

Example 6.4. Suppose that $P \ll P_{st}$ and that the RN-density σ is such that $\sigma(\varphi) = \sigma(\eta_0\varphi)$ for all $\varphi \in M^\infty$. By Corollary 4.2 it follows that the intermediate distributions P_n are dominated by P^0_{st}, with RN-densities δ_{-n} equal to $\lambda_{st}\alpha_{-n}\sigma \circ \eta_{-n}$; cf. (1.6). Furthermore, for ν-a.e. $x \in \mathbb{R}$ and $B \in \bar{M}^\infty$ we have:

$$P^{0,x}(B) = E^0_{st}(1_B \cdot \sigma \circ \theta_{-x})/E^0_{st}(\sigma \circ \theta_{-x});$$

$$\begin{aligned} E^0_{st}(1_B \cdot \ \sigma \circ \theta_{-x}) &= \sum_{k\in\mathbb{Z}} E^0_{st}(1_B \cdot \sigma \circ \eta_{-k} \cdot 1_{[T_{-k} \le -x < T_{-k+1}]}) \\ &= \sum_{k\in\mathbb{Z}} E^0_{st}(\sigma \cdot \ 1_B \circ \eta_k \cdot 1_{[x+T_0 \le T_k < x+T_1]}). \end{aligned}$$

Writing $\Phi_B[x+T_0, x+T_1)$ for $\sum_{k\in\mathbb{Z}}(1_B \circ \eta_k \cdot 1_{[x+T_0 \le T_k < x+T_1]})$, the number of occurrences in the interval $[x + T_0, x + T_1)$ from which the eventuality B is seen, we obtain that for ν-a.e. x in $\mathbb{R}$:

$$\begin{aligned} P^{0,x}(B) &= \frac{E^0_{st}(\sigma \cdot \Phi_B[x + T_0, x + T_1))}{E^0_{st}(\sigma \cdot \Phi[x + T_0, x + T_1))} \\ &= \frac{E_{st}(\sigma \cdot \Phi_B[x + T_0, x + T_1)/\alpha_0)}{E_{st}(\sigma \cdot \Phi[x + T_0, x + T_1)/\alpha_0)} \\ &= \frac{E(\Phi_B[x + T_0, x + T_1)/\alpha_0)}{E(\Phi[x + T_0, x + T_1)/\alpha_0)}. \end{aligned}$$

The denumerator in this ratio equals $\lambda(x)$; cf. (ii) of Theorem 6.1. Interpret the numerator of the ratio as the local rate at x for the expected number of occurrences from which B is seen (the so-called expected number of B-occurrences). Then we can interpret $P^{0,x}(B)$ as the relative rate at x of expected number of B-occurrences when compared to the rate at x of expected number of M-occurrences.

Notice that the relationship between $P^{0,x}$ and P as expressed by the last equality above is the same for all RN-densities σ satisfying $\sigma(\varphi) = \sigma(\eta_0(\varphi))$ on M^∞. Also note that the distribution P in Example 5.4 (allowing the experience of event-stationarity from all occurrences onwards) has $\sigma(\varphi) := 1/(\lambda_{st} \cdot \alpha_0)$ and hence belongs to the class of distributions considered in the present example. With $\sigma(\varphi) = \sigma_0(\varphi)/E_{st}\sigma_0$ and $\sigma_0(\varphi)$ equal to one of the following functions, we find other examples as well:

$$\varphi(T_0(\varphi), T_0(\varphi) + 1],$$

$\varphi(y+T_0(\varphi),\ y+T_1(\varphi)]$,

$f(\eta_0(\varphi))$,

$\int\limits_{T_0(\varphi)}^{T_1(\varphi)} g(\theta_s(\varphi))ds$;

$y \in \mathbb{R}$ and $\varphi \in M^\infty$, and f and g suitable functions on M^∞. $\square$

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