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Modelling of and empirical studies on portfolio choice, option pricing, and credit risk

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Proefschrift

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To my parents

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Hoofdstuk 1

Introduction

1.1 Summary

The corner stones of the modern theory of finance are Portfolio Choice and Arbitrage Pricing. The modern portfolio choice theory introduced by Markowitz (1952) tries to explain the way individual or institutional investors (should) allocate their wealth among risky financial assets. The arbitrage pricing theory, initially used for option pricing by Black and Scholes (1973) and Merton (1973), further developed by Harrison and Kreps (1979), Harrison and Pliska (1981), and generalized by Delbaen and Schachermayer (1994) and Delbaen and Schachermayer (2005), addresses pricing financial securities by no-arbitrage arguments.[1] This thesis contains four essays in the fields of portfolio choice and arbitrage asset pricing. The relevant literature review is contained in the introduction of every chapter separately.

A portfolio choice process is usually thought of as a tradeoff between return and

[1]Historically, the arbitrage pricing argument is related to the Neo-Walrasian theories of general equilibrium with asset markets (complete and incomplete) developed by Radner (1968) and Hart (1975). Ross (1976) uses no-arbitrage arguments to justify the multi-factor capital asset pricing model. The proof of the well known Modigliani-Miller theorem on irrelevance of corporate financial structure for the value of the firm, see Modigliani and Miller (1958), also employs arbitrage logic.

risk of the portfolio. Investors preferring higher portfolio returns, generally try to avoid too volatile assets. From the perspective of regulatory capital requirements, institutional invertors are often interested to limit their risk exposure as well. Thus, risk management can be seen as a special case of portfolio choice. A traditional approach in the modern portfolio selection was developed by Markowitz (1952) who proposed to use the variance of the portfolio as a measure of risk and expected return as a reward measure. For many years, this approach was the industry standard, mostly due to its computational simplicity. However, from the point of view of risk measurement, the variance is not a satisfactory risk measure. First, being a symmetric measure of risk, the variance regards both losses and gains as equally undesirable. This disadvantage became especially apparent with the development of equity derivatives, such as options, and credit structured products, such as portfolio default swaps and collateralized debt obligations. Second, the variance is inappropriate to describe the risk of low probability extreme events, such as, for example, the default risk. Finally, from a theoretical perspective, the mean-variance approach is not consistent with second-order stochastic dominance and, thus, with the benchmark expected utility approach for portfolio selection.

Alternative models in portfolio selection were suggested, where the reward-risk approach is maintained, but the choice of an alternative risk measure instead of the variance makes the models more appropriate for practical applications. In parallel, an axiomatic approach for the risk measure theory was developed by Artzner *et al.* (1999), who introduced the concept of a coherent measure of risk that satisfies properties desirable from a regulatory perspective. Special attention both from a theoretical and a practical point of view has been paid to expected shortfall, a coherent risk measure consistent with second-order stochastic dominance. Bassett *et al.* (2004) and Portnoy and Koenker (1997) have shown that an in-sample mean-expected shortfall portfolio selection problem can be reformulated as a linear program that can be efficiently solved by well developed simplex and interior point algorithms. As shown by Kusuoka (2001) expected shortfall can be generalized to the class of coherent regular risk (CRR) measures, which maintain the desirable properties of expected shortfall. Chapters 2 and 3 study the statistical and economic properties of mean-CRR portfolios.

Chapter 2 develops a statistical spanning test for mean-coherent regular risk (CRR) efficient frontiers applied in chapter 3. Tests for mean-variance spanning, introduced by Huberman and Kandel (1987), use regression analysis to test whether a mean-variance efficient frontier generated by a particular set of assets statistically coincides with a mean-variance efficient frontier generated by a subset of the assets. Subsequently, different modifications of the test for mean-variance spanning have been proposed. A nice overview is contained in DeRoon and Nijman (2001). As soon as an investor decides to switch from the conventional mean-variance to a mean-CRR portfolio selection, the necessity for similar statistical inferences arises. Indeed, analogously to the mean-variance efficient frontier in the mean-variance approach one can construct mean-CRR efficient frontiers. The test for mean-CRR spanning becomes an important statistical tool to gauge the redundancy of certain subsets of assets from the point of view of mean-CRR efficiency. As chapter 2 shows, similarly in spirit to Huberman and Kandel (1987), this test can be implemented by means of a simple semi-parametric instrumental variable regression, where instruments have a direct link with a stochastic discount factor. The test is based on the relation developed by Tasche (1999), which holds for all assets entering the mean-CRR market portfolio. Applications of the mean-CRR spanning tests for several coherent regular risk measures, including the well known expected shortfall, are illustrated.

Chapter 3 compares the mean-variance and mean-coherent regular risk (CRR) portfolios, both statistically and economically. CRR measures are becoming more popular in empirical applications. However, Bertsimas *et al.* (2004) point out that the variance and a CRR measure should yield the same optimal portfolios for asset returns with elliptically symmetric distributions. As theoretical advantages of a CRR measure over the variance have been shown in numerous studies, the question of the practical significance of the difference between them remains. This is especially the case for typical financial assets, such as stocks, currencies, and market indexes, whose return distributions are often assumed to be close to elliptically symmetric. The comparison in chapter 3 requires the derivation of the asymptotic distributions of optimal portfolio weights obtained from in-sample mean-risk optimization. The results suggest that even for typical assets the outcomes of mean-variance and mean-CRR optimizations can be statistically and economically different. The tests developed in the chapter also demonstrate how to "switch offänd "switch on"the estimation uncertainty caused by the sampling error in mean returns, which is reported to be problematic in portfolio selection context, as reported by Chopra and Ziemba (1993). Finally, spanning tests for mean-CRR efficient frontiers, developed in chapter 2, are applied to several market indexes. The results are compared to their equivalents in the mean-variance framework. It is shown that for conventional classes of assets mean-variance and mean-CRR spanning tests typically yield similar conclusions. However, for assets with asymmetric returns the mean-CRR efficiency of the mean-variance efficient portfolio is rejected. This suggests superiority of the CRR measure for portfolios of non-standard instruments, such as pools of credit instruments and derivatives. For conventional assets, such as equities and currencies the mean-variance and mean-CRR approaches can be used interchangeably.

Chapters 4 and 5 of the thesis study applications of the asset pricing theory to option pricing and credit risk modelling. The asset pricing theory usually deals with no-arbitrage pricing of derivatives written on some basic underlying assets, whose dynamics is statistically modelled. A noble example of this approach is the model developed by Black and Scholes (1973) and Merton (1973), which derives prices of European options written on an underlying asset following a Geometric Wiener Process. With growing organized and over-the-counter markets for derivative instruments, the asset pricing theory became a very important tool for pricing contingent contracts. Option pricing models are widely used in the industry, sometimes with sophisticated assumptions on the dynamics of the underlying assets. Motivated by the empirical evidence on the implied volatility skew, Heston (1993) provides a closed-form solution for a stochastic volatility option pricing model. In this model option prices account for the additional volatility risk factor, which makes the model more realistic by adjusting the distribution of returns for frequently observed excess kurtosis and negative skewness. Duffie *et al.* (2000) generalize Heston's stochastic volatility model to the class of affine-jump diffusions. In parallel with the equity derivative pricing, the asset pricing theory found its way to credit instruments. Merton (1974) applies the no-arbitrage pricing principles for pricing corporate debt, using the leverage ratio as the underlying process and statistically modelling its dynamics. Numerous modifications of Merton's ideas were implemented in the credit risk models used by financial institutions. Merton's model also served as a foundation for the structural-form approach to credit risk modelling in the academic literature.

The main focus of Chapter 4 is the empirical side of the option pricing under Heston's stochastic volatility assumption. Clustering and stochastic dynamics of the return volatility is an empirical fact, which, probably, should be incorporated in realistic statistical models of asset price behavior. Numerous ARCH and GARCH models originated by Engle (1982) and Bollerslev (1986) were suggested to take into account observed heteroskedasticity in asset returns in discrete time models. Nelson (1991) introduces E-GARCH models that, in addition, can model the leverage effect in return distributions.

Apart from modelling the realistic dynamics of asset returns, the empirical literature on option pricing has shown that the Black-Scholes model applied to option prices observed in the market leads to a phenomenon known as the implied volatility smile or skew, which is model inconsistent. This phenomenon was primarily attributed to the leverage effect in asset returns as well as to the fat tails of the empirical return distribution, which are ignored by the Black-Scholes model. Stochastic volatility option pricing models partially correct for both option pricing and equity dynamic inconsistencies. However, it is well known that, in case of stochastic volatility models, financial markets are generally incomplete in terms of the underlying asset, since the stochastic volatility cannot be hedged. This means that the volatility risk premium is not identifiable on the basis of the underlying asset dynamics only. Traded option contracts, on the other hand, can be used to extract the lacking information about the pricing mechanism. In particular, analogously to implied volatilities in the Black-Scholes model, implied prices of volatility risk can be estimated on a daily basis using option data. The price of volatility risk can be interpreted as the market's attitude towards risk. Chapter 4 analyzes the dynamics of the implied prices of volatility risk from this perspective. It investigates the dynamics of the implied prices of volatility risk and shows that modelling their dynamics significantly helps to improve the out-of-sample option pricing performance.

Chapter 5 proposes an alternative way to model the credit risk of companies in distress. Existing structural form credit risk models require the use of infrequent and often noisy information on the firm's capital structure. The resulting pricing performance of these models, especially for companies in distress, is not satisfactory, see Eom *et al.* (2004). At the same time, the equity value of a company in distress can be an informative indicator of the credit risk perceived by the market. Being an imperfect hedge against default, the equity price becomes more informative as the company approaches bankruptcy. Also, from an econometric perspective, modelling the default through the equity price is attractive since better quality and more frequent data is available. Unlike in structural and reduced-form models of credit risk, the model proposed in chapter 5 uses equity as a liquid and observable primitive to analytically value corporate bonds and credit default swaps. In this way, restrictive assumptions on the firm's capital structure are avoided. Default is parsimoniously represented by the equity value hitting the zero barrier either diffusively or with a jump, which implies non-zero credit spreads for short maturities. Easy cross-asset hedging is enabled. By means of a tersely specified Radon-Nikodým derivative, we also make analytic credit-risk management possible under systematic jump-to-default risk.

1.2 Further research

The topics discussed in this thesis contain interesting possibilities for further research. The mean-coherent risk spanning test outlined in Chapter 2 has an alternative interpretation through a stochastic discount factor. Therefore, one could look at returns observed in the market from the point of view of a mean-coherent risk investor. The empirical properties of the mean-coherent risk stochastic discount factor projected on the space of returns can be studied empirically. It can be compared to the conventional discount factor of Fama and French (1995) obtained as an affine function of the market, size, and book-to-market factors. As a result an alternative view on the mean-CRR optimization can be developed.

Additionally, minimization of a coherent risk measure, such as expected shortfall, can find numerous applications in finance. Often investors are not indifferent to the direction of errors they make, since negative returns are avoided while positive returns are welcome. Conventional variance minimizing regression methods treat positive and negative errors symmetrically. As an example, one could consider the problem of tracking a bond or equity index with a portfolio of given instruments. In this situation over-performing means a negative tracking error, which is minimized by the variance. An empirical analysis that quantifies the economic and statistical benefits from a coherent risk measure could be of interest.

Empirical analyses of option prices have recently become a hot topic in finance. Indeed, option contracts can be used to gauge the market future expectations in terms of risk-neutral probabilities. By analyzing risk-neutral probability distributions implicit in option prices, one could find a way to look for market aggregate behavioral phenomena recently found in many field and laboratory experimental studies. Alternatively, it is possible to develop better option pricing models.

The equity-based credit risk model developed in Chapter 5 uses equity as an informative signal about the issuer's credit quality. This model should be especially useful for credit instruments issued by distressed companies due to a high sensitivity of their values to shocks in the equity price of the issuer. An empirical confirmation of this fact as well as an empirical comparison of different credit risk models in an application to the debts of distressed companies could be an interesting topic for further research.

Hoofdstuk 2

Testing for Mean-Coherent Regular Risk Spanning

2.1 Introduction

Introduced by Artzner *et al.* (1999), coherent risk measures received considerable attention in the recent literature. Indeed, coherent risk measures satisfy a set of properties desirable from the perspective of risk management, motivated by regulatory concerns. With additional requirements, making a risk measure among other things empirically identifiable, Kusuoka (2001) introduces the class of coherent *regular* risk (CRR) measures. A particular CRR measure is expected shortfall, which has become especially popular in theoretical and empirical applications due to its computational tractability.[1] In parallel with these developments in the risk measure theory, there is also an increasing understanding that risk measures alternative to the industry- standard variance can (and maybe should be) used in asset allocation decisions. Indeed, the variance as risk measure treats overperformance equally

[1]See, for example, Acerbi and Tasche (2002), Tasche (2002), and Bertsimas *et al.* (2004) for theoretical properties of expected shortfall; and Bassett *et al.* (2004), Kerkhof and Melenberg (2004), and chapter 3 of this thesis for practical applications.

as underperformance. Starting with Markowitz (1952), who suggested the use of the semi-variance instead of the variance, many alternative risk measures, treating underperformance differently from overperformance, have been proposed, see, for example, Pedersen and Satchell (1998). In particular, also CRR measures found their way to the optimal portfolio choice theory by means of expected shortfall. Rockafellar and Uryasev (2000) suggest an efficient numerical method to solve an in-sample analog of the mean-expected shortfall portfolio optimization problem. Bertsimas *et al.* (2004) elaborate on the method further. Bassett *et al.* (2004) show that the mean-expected shortfall optimization problem can be seen as a constrained quantile regression, for which very efficient numerical methods have been developed.[2] They also suggest a point mass approximation for a general CRR measure and show that the mean-CRR optimal portfolio problem with such an approximation can be solved by quantile regression algorithms.

Portfolio choice based on expected utility might be considered as a benchmark to evaluate the choice of risk measure. For instance, the variance as risk measure in a mean-variance portfolio choice corresponds to expected utility with a quadratic utility index or when asset returns jointly follow an elliptically symmetric distribution. But otherwise a mean-variance optimal portfolio is not consistent with second order stochastic dominance. On the other hand, CRR measures, when combined with expected return, turn out to be consistent with second order stochastic dominance. Indeed, De Giorgi (2005) introduces portfolio choice based upon a reward-risk tradeoff, isotonic with respect to second order stochastic dominance. This latter isotonicity requirement means that for the reward one should take the mean return, while risk measures based upon particular Choquet integrals qualify as appropriate risk measures. Expected shortfall and, more generally, CRR measures are such Choquet integral based risk measures. As a consequence, mean-CRR optimal port-

[2]See Barrodale and Roberts (1974), Koenker and D'Orey (1987), and Portnoy and Koenker (1997).

folios are consistent with second order stochastic dominance. For the special case of the mean expected shortfall trade-off this has already been demonstrated by, for example, Ogryczak and Ruszczyński (2002).

As noticed by Bassett *et al.* (2004), an alternative justification for mean-CRR efficient portfolios can be given from the point of view of an investor who maximizes a Choquet expected utility with a linear utility index and a convex distortion of the original probability. This framework is an alternative to the expected utility paradigm developed by Ramsey (1931), von Neumann and Morgenstern (1944), and Savage (1954), see Schmeidler (1989), Yaari (1987), and Quiggin (1982). While in the classical expected utility theory the utility index bears the entire burden of representing the decision maker's attitude towards risk, Choquet expected utility theory introduces the possibility that preferences may require a distortion of the original probability assessments. The cumulative prospect theory, as developed by Tversky and Kahneman (1992) and Wakker and Tversky (1993), is also closely aligned with the Choquet approach.

Mean-CRR efficient portfolios lead to mean-CRR efficient frontiers. For example, Tasche (1999) calculates expected shortfall based risk contributions and discusses a mean-expected shortfall based capital asset pricing theory (CAPM).

Then a natural question to ask is whether analogs of statistical methods, well known in the mean-variance portfolio analysis,[3] can be developed in the mean-CRR case. In this chapter we develop a simple mean-CRR spanning test, which is used to check whether the mean-CRR frontier of a set of assets spans the frontier of a larger set of assets. We show that, analogous to the mean variance spanning test developed by Huberman and Kandel (1987), the mean-CRR spanning test can be performed as a significance test for the intercept coefficient in a simple linear regression model. The difference, however, is that in case of the mean-CRR spanning a semi-parametric instrumental variable (IV) estimation technique should be applied.

[3] See the survey by DeRoon and Nijman (2001).

The instrumental variable has a direct link to the stochastic discount factor. We illustrate applications of this spanning test for several CRR measures, including expected shortfall and the point mass CRR approximation, suggested by Bassett *et al.* (2004), and compare the results to the mean-variance analogs. Though quite different in approach, our analysis is similar in spirit to the analysis of Gourieroux and Monfort (2005), who analyze statistical properties of efficient portfolios in a constrained parametric expected utility optimization setup.

The remainder of this chapter is structured as follows. Section 2.2 briefly describes coherent regular risk (CRR) measures. In section 2.3 we introduce the mean-CRR problem and derive the risk contributions of a CRR measure. Spanning tests and their limit distributions are presented in section 2.4. Section 2.5 discusses the relation between the instrumental variable and the stochastic discount factor. Empirical applications of the mean-CRR spanning test are given in section 2.6. Section 2.7 concludes.

2.2 Coherent regular risk (CRR) measures

Artzner *et al.* (1999) follow the axiomatic approach to define a risk measure coherent from a regulator's point of view. They relate a risk measure to the regulatory capital requirement and deduce four axioms which should be satisfied by a "rational"risk measure. We discuss these axioms below. Let $\mathcal{X} = L_\infty(\Omega, \mathcal{F}, P)$ be a set of (essentially) bounded real valued random variables.[4]

Definition 2.1 *A mapping $\rho : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is called a coherent risk measure if it satisfies the following conditions for all real valued random variables $X, Y \in \mathcal{X}$:*

- *Monotonicity: if $X \leq Y$, then $\rho(X) \geq \rho(Y)$.*

[4] Ω is the set of states, $\mathcal{F}$ denotes the σ-algebra, and P is the probability measure. Delbaen (2000) extends the definition of coherent risk measure to the general probability space $L_0(\Omega, \mathcal{F}, P)$ of all equivalence classes of real valued random variables.

- *Translation Invariance: if* $m \in \mathbb{R}$, *then* $\rho(X+m) = \rho(X) - m$.
- *Positive Homogeneity: if* $\lambda \geq 0$, *then* $\rho(\lambda X) = \lambda \rho(X)$.
- *Subadditivity:* $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

These axioms are natural requirements for any risk measure that reflects a capital requirement for a given risk. The monotonicity property, which, for example, is not satisfied by the variance and other risk measures based on second moments, means that the downside risk of a position is reduced if the payoff profile is increased. Translation invariance is motivated by the interpretation of the risk measure $\rho(X)$ as a capital requirement, i.e., $\rho(X)$ is the amount of the capital which should be added to the position to make X acceptable from the point of view of the regulator. Thus, if the amount m is added to the position, the capital requirement is reduced by the same amount. Positive homogeneity says that riskiness of a financial position grows in a linear way as the size of the position increases. This assumption is not always realistic as the position size can directly influence risk, for example, a position can be large enough that the time required to liquidate it depends on its size. Withdrawing the positive homogeneity axiom leads to a family of convex risk measures, see Föllmer and Schied (2002).[5] The subadditivity property, which is not satisfied by the widely implemented value-at-risk, allows one to decentralize the task of managing the risk arising from a collection of different positions: If separate risk limits are given to different desks, then the risk of the aggregate position is bounded by the sum of the individual risk limits. The subadditivity is also closely related to the concept of risk diversification in a portfolio of risky positions.

Kusuoka (2001) adds another two conditions for coherent risk measures

- *Law Invariance: if* $P[X \leq t] = P[Y \leq t]\ \forall t$, *then* $\rho(X) = \rho(Y)$.
- *Comonotonic Additivity: if* $f, g : \mathbb{R} \to \mathbb{R}$ *are measurable and non-decreasing, then* $\rho(f \circ X + g \circ X) = \rho(f \circ X) + \rho(g \circ X)$.

[5] However, see De Giorgi (2005) on homogenization of risk measures.

The intuition of the two axioms is simple: the Law of Invariance means that financial positions with the same probability distribution should have the same risk. This property allows identification from an empirical point of view. The second condition of Comonotonic Additivity refines slightly the subadditivity property: subadditivity becomes additivity when two positions are comonotone. In fact, Comonotonic Additivity strengthens the concept of "perfect dependence" between two random variables. Indeed, if two random variables are monotonic transformations of the same third random variable, the risk of their combination should be equal to the sum of their separate risks.

A risk measure that is coherent and regular and that has received considerable attention is expected shortfall,[6] defined as

$$s_\alpha(X) = -\alpha^{-1} \int_0^\alpha F^{-1}(t)dt, \tag{2.1}$$

where F stands for the cumulative distribution function of the random variable X. An important characterization result, modifications of which are obtained by Kusuoka (2001) and Tasche (2002), is

Theorem 2.1 *A risk measure $\rho : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ defined on $\mathcal{X} = L_\infty(\Omega, \mathcal{F}, P)$, with P non-atomic, is coherent and regular if and only if it has a representation*

$$\rho(X) = \int_0^1 s_\alpha(X)d\phi(\alpha), \tag{2.2}$$

where ϕ is a probability measure defined on the interval $[0, 1]$.

Notice, that a coherent regular risk measure corresponds to a Choquet expectation over $F^{-1}(t)$ with a concave distortion probability function.[7] Indeed, a Choquet

[6] Here we use the terminology of Acerbi and Tasche (2002). In fact, variants of this risk measure have been suggested under a variety of names, including conditional value-at-risk (CVaR) by Rockafellar and Uryasev (2000) and tail conditional expectation by Artzner *et al.* (1999).

[7] This corresponds to a convex distortion in case the risk measure is defined as Choquet expectation over X, instead of $F^{-1}(t)$, see Bassett *et al.* (2004).

expectation over $F^{-1}(t)$ with a distortion probability ν is

$$\rho(X) = -\int_0^1 F^{-1}(t)d\nu(t).$$

If we substitute expression (2.1) for expected shortfall into equation (2.2) the relation between the distortion probability ν and the probability measure ϕ in (2.2) can be found

$$\nu'(t) = \int_t^1 \alpha^{-1}d\phi(\alpha). \tag{2.3}$$

We call the function $\nu'(t)$ a Choquet distortion probability density function (pdf). Since ϕ is a probability measure it follows that $\nu(t)$ has to be a concave function. Hence the probability distortion ν acts to increase the likelihood of the least favorable outcomes, and to depress the likelihood of the most favorable ones. This is the reason why, for example, Bassett *et al.* (2004) call a CRR measure a pessimistic risk measure. Through the Choquet representation, CRR measures can be related to the family of non-additive, or dual, or rank-dependent uncertainty choice theory formulations of Schmeidler (1989), Yaari (1987), and Quiggin (1982).

A nice way to approximate a CRR measure by a weighted sum of Dirac's point mass functions[8] was suggested by Bassett *et al.* (2004). The point mass function $\delta_\tau(\alpha)$ is defined trough the integral $\int_{-\infty}^{x} \delta_\tau(\alpha)d\alpha = I(x \geq \tau)$. Let $\phi(\alpha) = \sum_{k=1}^{m} \phi_k \delta_{\tau_k}(\alpha)$, with $\phi_k \geq 0$, $\sum \phi_k = 1$, then the CRR measure in (2.2) can be rewritten as

$$\rho(X) = \sum_{k=1}^{m} \phi_k s_{\tau_k}(X). \tag{2.4}$$

Clearly, expected shortfall is a particular case of this approximation. We use this approximation in our empirical applications of the mean-CRR spanning test in section 2.6.

[8]Notice, that such an approximation also corresponds to a piecewise linear approximation of the concave probability distortion function ν in the Choquet expectation.

2.3 Mean-CRR portfolios and risk contributions

In this section we first use the CRR-measures to formulate optimal portfolio choice problems, and then we generalize the risk contribution results for the case of expected shortfall obtained by Bertsimas *et al.* (2004) and Tasche (1999) to general CRR measures.

Consider a portfolio of p assets whose random returns are described by the random vector $R = (R_1, \ldots, R_p)'$ having a joint density with the finite mean $\mu = E[R]$. For simplicity, assume that the joint distribution of R is continuous. Let $\theta = (\theta_1, \ldots, \theta_p)'$ be portfolio weights, so that the total random return on the portfolio is $Z = R'\theta$ with distribution function F_z. This allows us to view a CRR measure of a portfolio as a function of portfolio weights $\rho(\theta) = \rho(R'\theta)$. An optimization problem for a mean-CRR efficient portfolio can now be formulated in full analogy with the mean-variance case

$$\min_{\theta \in \mathbb{R}^p} \rho(R'\theta) \ s.t. \ \mu'\theta = m, \iota'\theta = 1 \tag{2.5}$$

where m is the required expected portfolio return and ι is a $p \times 1$ vector of ones.

The fact that a CRR measure can be written as a Choquet expectation over $F^{-1}(t)$ with a concave distortion function ν (or, equivalently, as a Choquet expectation over Z with a concave distrotion function), means that the optimization problem (2.5) is isotonic with second order degree stochastic dominance, see, for instance, De Giorgi (2005). In combination with the empirical identifiability (due to the law invariance condition), makes optimal mean-CRR portfolio choice attractive, both from a theoretical and an empirical point of view. Moreover, as explained in Bassett *et al.* (2004), a CRR measure can be approximated by a finite sum of expected shortfalls. A sample analog of a mean-CRR problem with this finite sum approximation can be reformulated as a linear program and efficiently solved, see Portnoy and Koenker (1997), Rockafellar and Uryasev (2000), and chapter 3 of this thesis, making mean-CRR optimal portfolio choice also practically feasible. In summary, a CRR measure is a natural choice for a risk measure in case of a portfolio choice based on a mean-risk trade-off. In chapter 3, we also derive the asymptotic distribution of the mean-CRR portfolio weights θ and consider special cases of a point mass approximation of a CRR measure and expected shortfall.

In the remainder of this section, we consider the risk contribution results obtained by Bertsimas *et al.* (2004) and Tasche (1999) for the case of expected shortfall and generalize them to a general CRR measure. This result, being interesting by itself,[9] is needed for the mean-CRR spanning test, which is to follow.

Proposition 2.1 *If the distribution of the returns R has a continuous density, then the CRR contributions of assets in R are given by the gradient vector*

$$\nabla_\theta \rho(\theta) = -E\left[R \int_{F_z(Z)}^{1} \alpha^{-1} d\phi(\alpha)\right]. \tag{2.6}$$

Proof. First, notice that expected shortfall of the portfolio return Z can be expressed as

$$s_\alpha(Z) = -\alpha E\left[Z I(F_z(Z) \leq \alpha)\right],$$

where $I(A)$ is the usual indicator function. This means that a CRR measure of the portfolio Z is

$$\begin{aligned}
\rho(\theta) &= -\int_0^1 \alpha^{-1} E\left[Z I(F_z(Z) \leq \alpha)\right] d\phi(\alpha) \\
&= -E\left[Z \int_0^1 \alpha^{-1} I(F_z(Z) \leq \alpha) d\phi(\alpha)\right] \\
&= -E\left[Z \int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha)\right].
\end{aligned}$$

The distribution function $F_z(\cdot)$ is continuously differentiable with respect to portfolio weights θ since the distribution of the returns R has a continuous density. Therefore, we can calculate the risk contributions of a CRR measure in a straightforward way. Notice, that portfolio $Z = R'\theta$ and its distribution function F_z depend on the

[9] One can interpret risk contributions as an amount of required capital for a particular asset in the portfolio.

portfolio weights θ. Then, applying the chain rule to the expression for a CRR measure $\rho(\theta)$, we obtain

$$\begin{aligned}\nabla_\theta \rho(\theta) &= -\nabla_\theta E\left[Z\int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha)\right] \qquad (2.7)\\ &= -E\left[R\int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha) - Z\frac{\phi'(F_z(Z))}{F_z(Z)}\left(f_z(Z)R + \nabla_\theta\, F_z(s)|_{s=Z}\right)\right].\end{aligned}$$

To finish the derivation we need to calculate the gradient $\nabla_\theta F_z(s)$. It suffices to derive only component j of this vector, the rest being analogous. Denote by θ_j the portfolio weight of asset R_j and by θ_{-j} the vector of portfolio weights of the rest of the assets, which we denote by R_{-j}. Further, let $Z_{-j} = R'_{-j}\theta_{-j}$ be the portfolio of assets R excluding asset j. Denote by $F_{z_{-j}|R_j}$ and $f_{z_{-j}|R_j}$ the conditional probability and density functions of return Z_{-j} conditional on return R_j. Then we can express the cumulative probability function F_z of portfolio Z through the expectation of the conditional probability $F_{z_{-j}|R_j}$

$$\begin{aligned}F_z(s) = E\left[I(R'\theta \le s)\right] &= E\left[E\left[I(R'_{-j}\theta_{-j} \le s - R_j\theta_j)\middle| R_j\right]\right]\\ &= E\left[F_{z_{-j}|R_j}(s - R_j\theta_j)\right].\end{aligned}$$

Now the calculation of the derivative of $F_z(s)$ with respect to weight θ_j is straightforward

$$\begin{aligned}\frac{\partial F_z(s)}{\partial \theta_j} &= -E\left[f_{z_{-j}|R_j}(s - R_j\theta_j)R_j\right] = -E\left[f_{z|R_j}(s)R_j\right]\\ &= -\int_{-\infty}^{+\infty}\frac{f_{z,R_j}(s,R_j)}{f_{R_j}(R_j)}R_j dF_{R_j}(R_j) = -f_z(s)E\left[R_j|Z=s\right],\end{aligned}$$

where $f_{z|R_j}$ is the conditional density function of the portfolio return Z conditional on return R_j of the asset j, and f_{z,R_j} is their joint probability density function. Stacking the components into one vector yields

$$\nabla_\theta F_z(s) = -f_z(s)E\left[R|Z=s\right],$$

Then, substituting this expression into equation (2.7), we obtain the result for the

CRR risk contributions

$$\begin{aligned}
\nabla_\theta \rho(\theta) &= -E\left[R\int_{F_z(Z)}^1 \alpha^{-1}d\phi(\alpha) - Z\frac{\phi'(F_z(Z))}{F_z(Z)}\left(f_z(Z)R - f_z(Z)E\left[R|Z\right]\right)\right] \\
&= -E\left[R\int_{F_z(Z)}^1 \alpha^{-1}d\phi(\alpha)\right],
\end{aligned}$$

which concludes the proof. ■

The second proposition gives the expression for the Hessian of a CRR measure. This result is a generalization of the expression given in Bertsimas *et al.* (2004) for expected shortfall.

Proposition 2.2 *If the distribution of the returns R has a continuous density, then the Hessian of a CRR measure is given by the matrix*

$$\nabla^2_\theta \rho(\theta) = E\left[\frac{\phi'(F_z(Z))f_z(Z)}{F(Z)}\mathrm{Cov}(R|Z)\right], \tag{2.8}$$

where f_z is the probability density function of the portfolio return Z.

Proof. The proof is straightforward

$$\begin{aligned}
\nabla^2_\theta \rho(\theta) &= -\nabla_\theta E\left[R\int_{F_z(Z)}^1 \alpha^{-1}d\phi(\alpha)\right] \\
&= E\left[R\frac{\phi'(F_z(Z))}{F_z(Z)}\left(f_z(Z)R' + \nabla_\theta\, F_z(s)|_{s=Z}\right)\right] \\
&= E\left[\frac{\phi'(F_z(Z))f_z(Z)}{F_z(Z)}\left(RR' - RE[R'|Z]\right)\right] \\
&= E\left[\frac{\phi'(F_z(Z))f_z(Z)}{F_z(Z)}\mathrm{Cov}(R|Z)\right].
\end{aligned}$$

■

Note that (2.8) implies the convexity of a CRR measure $\rho(\theta)$ because the conditional covariance matrix $\mathrm{Cov}(R|Z)$ is positive semi-definite and the other terms are positive. This means that the mean-CRR portfolio optimization problem (2.5) is well defined.

2.4 Mean-CRR spanning test

In this section we present the mean-CRR-spanning test. First, Tasche (1999) shows that an analog of the two fund separation theorem holds for a τ-homogeneous risk measure satisfying certain regularity conditions, see the discussion in Tasche (1999). A risk measure $\rho(X)$ is called τ-homogeneous if for any $t > 0$ it satisfies $\rho(tX) = t^\tau \rho(X)$. The CRR measure is a homogeneous risk measure of degree one. Any τ-homogeneous risk-efficient portfolio can be represented as a linear combination of the risk-free asset (assumed to be present) and a *risk-market portfolio*. The risk-market portfolio $Z = R'\theta^*$ can be characterized by the maximal Sharpe-risk ratio, so that the following relation holds:

$$\mu - \iota r_f = \frac{\mu_z - r_f}{\tau \rho((R-\mu)'\theta^*)} \nabla_\theta \rho \left((R-\mu)'\theta^*\right),$$

where r_f is the risk-free rate, μ is the vector of the expected returns, μ_z is the expected return of the risk-market portfolio Z, and ι is a vector of ones. Notice, that this relation for the risk-efficient portfolio includes the risk contribution vector $\nabla_\theta \rho((R-\mu)'\theta^*)$. Using equation (2.6) we obtain the following expression for risk contributions entering the characterization of the risk-market portfolio.

$$\nabla_\theta \rho \left((R-\mu)'\theta^*\right) = -E\left[(R-\mu)\int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha)\right] = -\text{Cov}\left(R, \nu'(F(Z))\right),$$

where $\nu'(F_z(s)) = \int_{F_z(s)}^1 \alpha^{-1} d\phi(\alpha)$ is the Choquet distortion probability density function. Thus, the characterization of an efficient portfolio for a CRR measure (2.2) becomes

$$\mu - \iota r_f = \frac{\text{Cov}(R, \nu'(F_z(Z)))}{\text{Cov}(Z, \nu'(F_z(Z)))} \left(\mu_z - r_f\right). \tag{2.9}$$

This expression says that the expected excess return on any asset in a CRR market portfolio is proportional to the expected excess return of the CRR market portfolio with the coefficient proportional to the covariance between the asset return and the distorted cumulative distribution function of the risk-market portfolio Z. This characterization can be used for a spanning test. For expositional simplicity we derive the spanning test for a single asset, potentially to be included in the portfolio under consideration. The extension to the multiple asset case is straightforward.

Let Y be a random return of an asset for which we want to perform a spanning test. Denote by μ_y its expected return. Under the spanning hypothesis this asset is redundant for the portfolio, i.e., its weight in the portfolio is zero. This means that under the spanning hypothesis the CRR-market portfolio Z does not change. Clearly, the characterization (2.9) should hold. It is straightforward to see that the relation (2.9) can be reformulated in terms of the semi-parametric instrumental variable (IV) regression

$$Y_i^e = \alpha + \beta Z_i^e + \epsilon_i, \tag{2.10}$$

$$E[\epsilon_i] = 0, \tag{2.11}$$

$$E[V_i \epsilon_i] = 0, \tag{2.12}$$

where $Y_i^e = Y_i - r_f$, $Z_i^e = Z_i - r_f$, and $V_i = \nu'(F_z(Z_i))$ is the semi-parametric instrument, which depends on the distribution F_z of the optimal portfolio return Z. The restriction imposed by the spanning hypothesis on the regression (2.10) is

$$\begin{aligned} &\alpha = 0, \\ &\beta \mathrm{Cov}(Z, V) - \mathrm{Cov}(Y, V) = 0. \end{aligned}$$

Thus, the mean-CRR spanning test is a test on significance of the intercept parameter α in the semi-parametric IV regression (2.10). Denote by $W_i = (1, V_i)'$ the two instruments of (2.10), and by $X_i = (1, Z_i^e)'$ the regressors. Then, the IV estimator is given by

$$\widehat{\gamma} = \begin{bmatrix} \widehat{\alpha} \\ \widehat{\beta} \end{bmatrix} = \left(\frac{1}{n} \sum_{i=1}^{n} \widehat{W}_i X_i' \right)^{-1} \frac{1}{n} \sum_{i=1}^{n} \widehat{W}_i Y_i^e. \tag{2.13}$$

where $\widehat{W}$ stands for a non-parametric estimation of the instrumental variable W, which depends on the Choquet distortion pdf $\nu'(F_z(Z))$. The estimation of this functional is straightforward

$$\nu'(F_n(s)) = \int_{F_n(s)}^{1} \alpha^{-1} d\phi(\alpha),$$

where $F_n(s)$ is a consistent estimator of F_z.[10] Notice, that the methods developed by Newey (1994) to derive the asymptotic variance of a semi-parametric estimator are fully applicable to our semi-parametric IV case. The asymptotic distribution of the parameters can be determined (under appropriate regularity conditions) by

$$\sqrt{n}\,(\widehat{\gamma} - \gamma) = \left(\frac{1}{n}\sum_{i=1}^{n} \widehat{W}_i X_i'\right)^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \widehat{W}_i \epsilon_i + o_p(1), \tag{2.14}$$

We consider two cases. First, we ignore the estimation inaccuracy in the CRR market portfolio weights. This corresponds to the case where we assume a certain traded portfolio to be the CRR market portfolio, for example, the S&P 500 index. Then we consider the case when the estimation inaccuracy in the CRR market portfolio weights is taken into account. This corresponds, for instance, to the case where we want to test whether some chosen portfolio, likely based on estimated mean returns and probably some optimal criterion, is indeed optimal from the point of view of mean-CRR efficiency.

2.4.1 Spanning for a given CRR efficient portfolio

Suppose that the returns of the CRR market portfolio are observable, i.e, we do not need to take into account estimation inaccuracy in the CRR portfolio weights. Applying the Law of Large Numbers and the Central Limit Theorem to (2.14), we obtain

$$\frac{1}{n}\sum_{i=1}^{n} \widehat{W}_i X_i' \to_p E\left[W_i X_i'\right] = G, \tag{2.15}$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \widehat{W}_i \epsilon_i = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \psi(Z_i, \epsilon_i) + o_p(1) \to_d N(0, E[\psi\psi']), \tag{2.16}$$

[10] In principle, usual empirical distribution function $F_n(s) = n^{-1}\sum_{i=1}^{n} I(Z_i \leq s)$ can be used.

where $\psi = (\psi_1(Z,\epsilon), \psi_2(Z,\epsilon))'$ is a 2×1 vector with the components ψ_1 and ψ_2 being the influence functions of the functionals[11]

$$\begin{aligned}\phi_1(F) &= \int \epsilon dF(Z,\epsilon),\\ \phi_2(F) &= \int \epsilon \nu'(F_z(Z)) dF(Z,\epsilon).\end{aligned}$$

The influence function of the first functional $\phi_1(F)$ is obvious. The influence function of the functional $\phi_2(F)$ is derived in the Appendix. The results are

$$\psi_1(Z,\epsilon) = \epsilon, \tag{2.17}$$

$$\psi_2(Z,\epsilon) = \chi(Z,\epsilon) - E\left[\chi(Z,\epsilon)\right], \tag{2.18}$$

where

$$\chi(Z,\epsilon) = \int_{F_z(Z)}^{1} \left(\epsilon - E[\epsilon|Z = F_z^{-1}(\alpha)]\right) \alpha^{-1} d\phi(\alpha). \tag{2.19}$$

Finally, the asymptotic result for the semi-parametric IV estimator in (2.14) is

$$\sqrt{n}\left(\widehat{\gamma} - \gamma\right) \to_d N\left(0, G^{-1} E\left[\psi\psi'\right] G'^{-1}\right),$$

with the components of the influence function ψ given in equations (2.17), (2.18), and (2.19). The asymptotic distribution of the intercept α is

$$\sqrt{n}\left(\widehat{\alpha} - \alpha\right) \to_d N\left(0, \left[G^{-1} E\left[\psi\psi'\right] G'^{-1}\right]_{11}\right), \tag{2.20}$$

where the sub-index 11 stands for the (1,1)-component of the asymptotic covariance matrix of the semi-parametric IV estimator.

The mean-CRR spanning test is equivalent to the significance test of the intercept coefficient. Notice, that this result is close in spirit to the mean-variance spanning test developed by Huberman and Kandel (1987). They propose to test the mean-variance spanning by means of a significance test on the intercept coefficient in an OLS regression similar to (2.10), but with a mean-variance market portfolio excess return Z^e instead of the CRR one.

[11] $W\epsilon = (\epsilon, V\epsilon)'$.

Example: Expected Shortfall

A particular CRR measure which has recently received a lot of attention is expected shortfall $s_\tau(X)$, defined in (2.1). It is well known that a sample analog of a mean-expected shortfall portfolio problem can be reformulated as a linear program and solved efficiently, see Bertsimas *et al.* (2004) and Bassett *et al.* (2004). Our results immediately yield the mean-expected shortfall spanning test. We start with the instrumental variable V, which is used to estimate regression (2.10):

$$V = \Gamma_F(Z) = \tau^{-1} I(F_z(Z) \leq \tau).$$

The function $\chi(Z, \epsilon)$ in (2.19) becomes

$$\chi(Z,\epsilon) = \tau^{-1} \left(\epsilon - E[\epsilon | Z = F_z^{-1}(\tau)]\right) I(F_z(Z) \leq \tau).$$

The result for the mean-expected shortfall spanning test is immediately obtained by means of equation (2.20) with

$$G = E \begin{bmatrix} 1 & Z^e \\ V & Z^e V \end{bmatrix}, \tag{2.21}$$

and

$$E[\psi\psi'] = \begin{bmatrix} \mathrm{var}(\epsilon) & \mathrm{cov}(\epsilon,\chi) \\ \mathrm{cov}(\epsilon,\chi) & \mathrm{var}(\chi) \end{bmatrix}. \tag{2.22}$$

An interesting observation is that in case of expected shortfall the components $\mathrm{var}(\chi)$ and $\mathrm{cov}(\epsilon,\chi)$ are mainly determined by the usual IV part $\tau^{-1}\epsilon I(F_z(Z) \leq \tau)$ of the function χ. This is because the non-parametric adjustment is effectively constant. The shift which appears at the τ quantile brings a negligible correction to the covariance matrix $E[\psi\psi']$. This means that, when performing a usual IV inference without taking into account the non-parametric adjustment, one only makes a very small error.

Example: CRR point mass approximation

As suggested by Bassett *et al.* (2004), one can approximate a CRR measure (2.2) by taking a point mass probability distribution on the interval $[0, 1]$. In this case the exogenous probability $\phi(\alpha)$ in the definition (2.2) becomes

$$\phi(\alpha) = \sum_{k=1}^{m} \phi_k I(\alpha \geq \tau_k),$$

where the weights ϕ_k sum up to one. A point mass approximation (PMA) of a CRR measure becomes a weighted sum of expected shortfalls

$$\rho(Z) = \sum_{k=1}^{m} \phi_k s_{\tau_k}(Z).$$

As shown in chapter 3 of this thesis, a sample analog of a the mean-PMA CRR portfolio problem can be reformulated as a linear program and efficiently solved with existing numerical algorithms. The spanning test results of this section are applicable for the mean-PMA CRR spanning as well. The instrumental variable V of regression (2.10) becomes

$$V = \Gamma_F(Z) = \sum_{k=1}^{m} \phi_k \tau_k^{-1} I(F_z(Z) \leq \tau_k).$$

The function $\chi(Z, \epsilon)$ in expression (2.18) for the influence function of the functional $\phi_2(F) = E[\epsilon V]$ becomes

$$\chi(Z, \epsilon) = \sum_{k=1}^{m} \phi_k \tau_k^{-1} I(F_z(Z) \leq \tau_k) \left(\epsilon - E[\epsilon | Z = F_z^{-1}(\tau_k)]\right).$$

The spanning test, equivalent to the significance test of the intercept in the IV regression (2.10), is performed by means of equation (2.20) with expressions for G and $E[\psi\psi']$ given in (2.21) and (2.22), respectively.

2.4.2 Estimation inaccuracy in market portfolio weights

The Mean-CRR spanning test (2.20) obtained in subsection (2.4.1) ignores the potential estimation inaccuracy in the weights of the CRR market portfolio Z. This is reasonable if one wants to test a CRR version of the Capital Asset Pricing Model (CAPM) with a given market index as a CRR market portfolio. Alternatively, one could form *a priori* believes about the portfolio weights, so that they are not considered as having estimation inaccuracy. In this section we discuss an adjustment required to the limit distribution (2.20) of the intercept coefficient α of the IV regression (2.10) in the case one also wants to take into account the error resulting from the estimation of the market portfolio weights. Our setup is quite general, as we consider an investor who wants to test his/her portfolio for CRR optimality, but whose portfolio is determined by solving some (arbitrary) optimization problem.

In principle, an alternative approach to test for mean-CRR spanning would be a straightforward significance test for the weight of the new asset in the market efficient portfolio. However, to implement this test one needs to re-derive the whole CRR market portfolio with the new asset included. This approach is similar in spirit to the mean-variance spanning test of Britten-Jones (1999). In this chapter, however, we would like to separate the estimation of the market portfolio and the test for mean-CRR spanning for new candidate assets. The advantage is that one does not need to re-derive the market portfolio weights every time a new spanning test needs to be performed. All we need are asset returns and weights of the öld"market portfolio, which need to be derived only once.

Suppose, that the limit distribution of the market efficient portfolio weights $\widehat{\theta}$ resulting from the solution of an optimization problem[12] is characterized by an influence function $\xi(R^e, Z)$, i.e.,

$$\begin{aligned} &\sqrt{n}\left(\widehat{\theta}-\theta\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi(R_i^e, Z_i) + o_p(1), \\ &E\xi = 0, \;\; E\xi\xi' < \infty, \end{aligned}$$

where R^e is a vector of asset returns in excess of the risk free rate r_f. The result (2.16) has to be adjusted in a straightforward way to take into account the estimation

[12] In the Appendix 2.B we consider the case of the mean-CRR portfolio weights.

inaccuracy in the portfolio weights

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} W_i\epsilon_i = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi(Z_i,\epsilon_i) + \nabla_\theta E\begin{bmatrix}\epsilon\\ \epsilon\nu'(F_z(Z))\end{bmatrix}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi(R_i^e, Z_i) + o_p(1).$$

It is straightforward to show that

$$M \equiv \nabla_\theta E\begin{bmatrix}\epsilon\\ \epsilon\nu'(F_z(Z))\end{bmatrix} = -\beta E\begin{bmatrix}R^{e\prime}\\ R^{e\prime}\nu'(F_z(Z))\end{bmatrix}.$$

Given the expressions for the components of the vector $\psi(Z,\epsilon)$ provided in (2.17), (2.18), and (2.19) we obtain

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} W_i\epsilon_i = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\zeta(R_i^e, Z_i,\epsilon_i) + o_p(1) \equiv \begin{bmatrix}I_2 & M\end{bmatrix}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\begin{bmatrix}\psi(Z_i,\epsilon_i)\\ \xi(R_i^e, Z_i)\end{bmatrix} + o_p(1),$$

with limit distribution

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} W_i\epsilon_i \to_d N\left(0, E\left[\zeta\zeta'\right]\right).$$

Finally, the spanning test result (2.20) becomes

$$\sqrt{n}\left(\widehat{\alpha}-\alpha\right) \to_d N\left(0, \left[G^{-1}E\left[\zeta\zeta'\right]G'^{-1}\right]_{11}\right). \tag{2.23}$$

The last step that remains is to find the influence function $\xi(R^e, Z)$ of the estimated market portfolio weights $\widehat{\theta}$. We report the relevant formulas for a mean-CRR market portfolio in the Appendix 2.B, referring for the derivation details to chapter 3. The considered cases are mean-CRR, with as special cases mean-expected shortfall, and mean-PMA CRR.

2.5 Stochastic discount factor, instrumental variables, and performance measurement

In this section we demonstrate that, if considered as a pricing model, system (2.10) implies a linear relation between a stochastic discount factor (that can be used to price the assets) and the instrumental variable V. From the perspective of the mean-CRR portfolio this can be interpreted as a model of general equilibrium where the portfolio choices are based on the mean-CRR optimization. In this case the instrumental variable V is given by the Choquet distortion probability density function $\nu'(F_z(Z))$. Alternatively, there could be an investor who makes his/her portfolio choice according to mean-CRR optimization. In this case, the assets in his/her portfolio should satisfy

$$R^e = \beta Z^e + \epsilon,$$
$$E[\epsilon] = 0,\ E[V\epsilon] = 0,$$

and the stochastic discount factor should be an affine function of the instrumental variable V, which can then be interpreted as the single risk factor. Notice, however, that this single risk factor is not a return on a portfolio. This means that we cannot construct a simple test of a zero intercept in a linear regression equation of the excess return R^e on the (non-existing) excess return "V^e". Instead, our spanning test, based on a linear regression but with an instrumental variable, allows one to perform a zero intercept test.

The general statement regarding the stochastic discount factor and the instrumental variable V is as follows.

Proposition 2.3 *Suppose that the asset excess returns satisfy*

$$R^e = \beta Z^e + \epsilon,$$
$$E[\epsilon] = 0,$$
$$E[V\epsilon] = 0,$$

where Z^e is a global market factor, and V is the global market instrumental variable. Then

$$m = \frac{1}{r_f \mathrm{Cov}(Z^e, V)} \left(E[Z^e V] - E[Z^e]V\right) \tag{2.24}$$

is a valid stochastic discount factor.

Proof. We need to show that for any (relevant) asset return R, the pricing equation $E[mR] = 1$ is satisfied. Notice that from the stated version of the modified CAPM model it follows that $R = r_f + \beta Z^e + \epsilon$. Then, substituting the expression (2.24) for the stochastic discount factor m, we obtain

$$E\left[mR\right] = \frac{r_f}{r_f \mathrm{Cov}(Z^e, V)} \left(E[Z^e V] - E[Z^e]E[V]\right) + \frac{\beta}{r_f \mathrm{Cov}(Z^e, V)} \left(E[Z^e V]E[Z^e] - E[Z^e]E[V Z^e]\right) = 1$$

■

The mean-CRR portfolio model (2.5) implies a specific choice of the instrumental variable V in (2.24), namely

$$V = \int_{F_z(Z)}^{1} \alpha^{-1} d\phi(\alpha).$$

As we have shown, the stochastic discount factor m should be an affine function of this instrument. This means that the proposed spanning test (2.20) can also be viewed as a test for the validity of a model for the stochastic discount factor in (2.24).

Given the SDF in (2.24) valid for returns satisfying (2.24)-(2.24), we can introduce a performance measure, following Chen and Knez (1996), for returns not yet marketed according to this SDF. This performance measure is defined as $kE[m(R - R^{\text{ref}})]$ with R a non-marketed return, R^{ref} an already marketed return, satisfying conditions (2.24)-(2.24), and k some constant. Straightforward calculations show that in case one chooses $k = r_f$ the performance measure equals the intercept α of the IV regression (2.10)-(2.12). This yields an alternative interpretation for the spanning test, comparable to Jensen's α and its relationship with MV-spanning tests.

2.6 Empirical examples

2.6.1 Testing the world capital index for market efficiency

In this subsection we consider an application of the mean-CRR spanning test to capital market indexes of different countries. In particular, we test the Morgan Stanley World Capital Index for mean-expected shortfall and mean-point mass approximated (PMA) CRR market efficiency with respect to inclusion of individual country indexes. This exercise is similar in spirit to Cumby and Glen (1990), who test the world index for mean-variance efficiency using the mean-variance spanning test. The data is available from Thomson Datastream. In our analysis we use the Morgan Stanley World Capital Market Index, individual country indexes denominated in local currencies, and currency exchange rates. The countries in the data set are divided into four groups based on geography and development level: American developing economies, Asian developing economies, European developing economies, and OECD countries. In the category of American developing countries we consider Argentina (ARG), Brazil (BRA), Chile (CHIL), Peru (PER), Mexico (MEX), and Venezuela (VEN). The group of Asian developing economies includes China (CHI), India (INDIA), Indonesia (INDO), Malaysia (MAL), Pakistan (PAK), Philippines (PHIL), Sri-Lanka (SRIL) and Thailand (THAIL). The Czech Republic (CZE), Hungary (HUN), Poland (POL), Romania (ROM), Russia (RUS), and Turkey (TURK) are the European developing economies. Finally, Australia (AU), Canada (CAN), The Euro zone (EU), Japan (JAP), South Korea (KOR), the United Kingdom (UK) and the United States (US) constitute the OECD group. As we want to exclude the effects of the Asian and Russian crisis (August 1998) on the world capital markets, we consider the time period from January 3, 1999 to May 12, 2005. We use daily US dollar index returns for our analysis. The US one-month interbank rate is taken as a risk-free interest rate.

Table 2.1 shows descriptive sample statistics of the country index returns. The empirical return distributions are typically skewed and fat tailed.

Table 2.2 shows the result of the world index (WRLD) efficiency tests. The table reports significance levels of the mean-variance (MV), mean-expected shortfall (ShF), and the mean-PMA CRR (PMA) market efficiency tests with respect to inclusion of individual country indexes. The expected shortfall probability threshold is chosen to be 5%, while probability thresholds for PMA CRR are taken at the levels of 5%, 10%, 15%, 20%, and 25% with equal weights of 20%. Significance levels of joint spanning tests for inclusion of country groups as a whole are reported in the table as well.

We see that the market efficiency tests with different risk measures (variance, expected shortfall, and PMA CRR) lead to similar conclusions. In most cases the market efficiency of the WRLD index cannot be rejected at the usual significance levels. A strong rejection of the efficiency hypothesis is observed for Mexico, Romania, Russia, and Canada (5% significance level). Indeed, Russian and Romanian markets have shown a significant growth over the past decade. The spanning hypothesis is also rejected for Pakistan at the 10% significance level.

The fact that the mean-CRR spanning tests perform at similar significance levels with the mean-variance spanning tests is encouraging. It shows that the mean-CRR spanning tests for country indexes work reasonably well. Moreover, for a moderate levels of skewness and kurtosis in the index return distributions the different risk measures are statistically equivalent and can be used interchangeably. This is in line with findings in chapter 3 that perform a systematic comparison of the mean-variance and the mean-CRR approaches in portfolio management.

2.6.2 Testing for mean-CRR spanning in portfolios of credit instruments

Our second example concerns portfolios of credit instruments. In particular, we consider collateralized debt obligations (CDO) as elementary entries of the portfolio.

This example is chosen for two reasons. First, since CDO return distributions are not symmetric the mean-variance and mean-CRR market efficient portfolios are likely to be different. As a result, the outcomes of the spanning tests might be different as well. Second, CDO tranches are becoming very popular financial instruments among investors, for example, hedge funds, insurance companies, etc. The past several years have seen an increasingly growing market for CDO tranches. This means that the problem of finding an optimal portfolio of CDOs is relevant for practical applications. The mean-variance approach might not be good idea in this case due to significantly asymmetric returns.

A collateralized debt obligation (CDO) is a structure of fixed income securities whose cash flows are linked to the incidence of default in a pool of debt instruments. These debts may include loans, emerging market corporate or sovereign debt, and subordinate debt from structured transactions. The fundamental idea behind a CDO is that one can take a pool of defaultable bonds or loans and issue securities whose cash flows are backed by the payments due on the loans or bonds. Using a rule for prioritizing the cash flow payments to the issued securities, it is possible to redistribute the credit risk of the pool of assets to create securities with a variety of risk profiles. In our example we consider the simplest case of investing in securities linked to the total pool of the underlying debt, while receiving a fixed interest payment in exchange.

In the industry the analysis of CDOs is usually exclusively based on theoretical models. This is due to the fact that historical data on defaults, and especially joint defaults, is very sparse. Another reason is that the specification of the full joint default probabilities is too complex: for example, for a CDO with 50 obligors there are 2^{50} joint default events. CDO models differ in their complexity: while some of them admit analytical solutions for loss distribution functions, others require Monte-Carlo simulation techniques. However, as soon as one wants to construct an optimal mean-risk portfolio from several CDOs, no closed form solution is usually available.

Therefore, a Monte-Carlo simulation is the only alternative. In our example, we use a simple one factor large homogeneous portfolio model to construct the return distributions of the CDOs.[13] Here we briefly outline the model.

The model assumes that a portfolio of loans consists of a large number of credits with the same default probability p. In addition, it is assumed that the default of a firm (obligor) is triggered when the normally distributed value of its assets $V_n(T)$ falls below a certain level K. Without loss of generality we can standardize the developments of the firm values such that $V_n(T) \sim N(0,1)$. In this case the default barrier level is the same for all obligors and equals $K = \Phi^{-1}(p)$. To introduce a default correlation structure it is assumed that the firm values are driven by a factor model

$$V_n(T) = \sqrt{\varrho}Y + \sqrt{1-\varrho}\epsilon_n,$$

where Y is the systematic factor for all obligors in the pool of credits, and ϵ_n is the idiosyncratic risk of a firm. The higher the correlation coefficient ϱ, the higher the probability of a joint default in the pool. Notice that, conditional on the factor Y, defaults are independent. The individual default probability conditional on the realization y of the systematic factor Y is

$$p(y) = \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\varrho}y}{\sqrt{1-\varrho}}\right).$$

Conditional on the realization y of Y, the individual defaults happen independently from each other. Therefore, in a very large portfolio, as we assume to be the case, the law of large numbers ensures that the fraction of obligors that actually defaults is almost surely equal to the individual default probability.

For purposes of our analysis we simulate returns of three CDOs using the described one factor model. The steps that we take are as follows:

[13] We use a simplified form of the firm's value model due to ?. Similar approach is used in Belkin *et al.* (1998) and Finger (1999).

- We simulate 10,000 realizations of three factors (y_{1i}, y_{2i}, y_{3i}) from the three-variate standard normal distribution with the identity correlation matrix.[14]

- From the simulated factors we generate fractions of obligors that actually default in the pool $j = \{1, 2, 3\}$ using the formula

$$x_{ji} = \Phi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\varrho_j}\, y_{ji}}{\sqrt{1-\varrho_j}}\right)$$

  with individual default probabilities $p_j, j = \{1, 2, 3\}$ of 2.5%, 5%, and 7.5%; and default correlations $\varrho_j, j = \{1, 2, 3\}$ of 0.15, 0.1, and 0.05.

- Finally, for each CDO j we obtain the returns R_{ji}

$$R_{ji} = (1 + r_j)(1 - x_{ji}) - 1,$$

  where r_j is the risk premium for holding pool j of defaultable obligors. We choose these risk premiums to be 4%, 10%, and 12%, correspondingly.

Even though the parameter choice in our simulation may seem ad-hoc, there are two reasons which make it plausible for a realistic situation. First, depending on the credit rating and the investment horizon, individual default probabilities can vary in a wide range from 0.00% (for one year default probability of an Aaa rated company) to 44.57% (for ten years default probability of a B rated company), according to Moody's, see Table 2.3. The default probabilities that we choose fall in this range. Second, it is possible to redistribute the credit risk of the pool of assets to create securities with a variety of risk profiles, which makes many possible combinations of parameters justified.

Table 2.4 shows descriptive statistics of the simulated returns of the three CDOs. The distributions of the returns are substantially skewed and fat tailed. The CDO

[14] In principle, it is possible to make returns on the 3 CDOs dependent by introducing positive or negative correlations among the factors.

with the smallest default correlation among obligors is the closest to the normal distribution.

From the simulated credit pool returns we construct three market portfolios:[15] mean-variance (MV), mean-expected shortfall (ShF) and mean-PMA CRR (PMA). In addition, we consider returns of CDO1 hypothesizing its market efficiency. The probability threshold for expected shortfall is chosen to be 5%. The probability thresholds for PMA CRR measure are chosen to be 5%, 10%, 15%, 20%, and 25% with equal weights of 20%. For these four portfolios (CDO1, MV, ShF, and PMA) we perform mean-variance, mean-expected shortfall, and mean-CRR PMA spanning tests with respect to inclusion of CDO2 and CDO3. Table 2.5 reports significance levels of these tests.

The results indicate a statistical difference between mean-variance and mean-CRR market portfolios. For the mean-CRR market portfolios (Mkt. ShF and Mkt. PMA), the mean-variance spanning tests result in strong rejection. At the same time, for the mean-variance market portfolio (Mkt. MV) mean-CRR spanning tests result in rejection as well. The difference between the mean-expected shortfall market portfolio (Mkt. ShF) and the mean-PMA CRR market portfolio (Mkt. PMA) with respect to the inclusion of CDO2 and CDO3 turns out to be significant as well.

In this exercise the mean-variance and the mean-CRR spanning tests do not produce similar results any more. The reason is the asymmetrically distributed returns. Skewness of the returns make variance a bad risk measure from the point of view of a CRR investor. Therefore, the mean-variance optimal portfolio is not recognized as a mean-CRR efficient one by the mean-CRR spanning test. This exercise demonstrates applicability of the mean-CRR spanning test to portfolios of credit instruments or other portfolios with comparable characteristics. It shows that the correct choice of the risk measure becomes increasingly important for assets with asymmetric returns.

[15] We assume a zero risk-free rate.

2.7 Conclusion

In this chapter we consider coherent regular risk (CRR) measures as an alternative to the conventional variance in a mean-risk optimal portfolio problem. Following trends in the recent literature on expected shortfall we derive useful properties of CRR measures. In particular, expressions for risk contributions and the Hessian of a CRR measure are obtained.

Our main contribution is the regression-based test for mean-CRR spanning. We show that this test can be performed in the spirit of Huberman and Kandel (1987) as a significance test of the intercept coefficient in a semi-parametric instrumental variable regression. The instrument in this regression is a functional, depending on a certain choice of the CRR measure.

We derive the limit distribution of the regression intercept coefficient to test for mean-CRR spanning. The resulting asymptotic covariance matrix is the variance of the usual IV estimator with an adjustment for the non-parametric part. In case of mean-expected shortfall or mean-PMA CRR portfolios this adjustment is likely to be negligible so that the non-parametric part can be ignored. Further, we illustrate how the estimation error in the mean-CRR portfolio weights can be incorporated in the spanning test.

The instrumental variable in the semi-parametric IV regression is shown to be related to the stochastic discount factor of a CRR version of the CAPM. In particular, we show that the stochastic discount factor is an affine function of this instrumental variable. This allows for an alternative interpretation of our spanning test in terms of a performance measure similar in spirit to the way the performance measure Jensen's α is related to the mean-variance spanning test.

Finally, as an empirical application, we use the mean-CRR spanning test to test for CRR efficiency of the world capital market index. In particular, we test for mean-expected shortfall and mean-PMA CRR efficiency with respect to the inclusion of individual country indexes. We find that the mean-CRR and mean-variance spanning tests produce similar significance levels. In addition we consider spanning tests for simulated returns of simplistic CDOs. We show that due to asymmetry of the return distributions mean-variance and mean-CRR spanning tests produce statistically different results.

2.A Influence function of semi-parametric IV regressors

As discussed in subsection 2.4.1, the derivation of the limit distribution of parameters in semi-parametric IV regression (2.14) requires the derivation of the influence function of the functional

$$\phi_2(F) = \int \epsilon\nu'(F(Z))dF(z,\epsilon). \tag{2.25}$$

We refer to Van der Vaart (1998) and Newey (1994) for the methodology and the appropriate regularity conditions. The idea is that the functional delta method applies to a functional $\phi_2(\cdot) : D_F \to \mathbb{R}$ satisfying Hadamard differentiability, so that for any square root consistent estimator F_n of the function F

$$\begin{aligned}
\sqrt{n}\left(\phi_2\left(F_n\right) - \phi_2\left(F\right)\right) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi_2\left(z_i,\epsilon_i\right) + o_p\left(1\right), \\
\psi_2\left(z,\epsilon\right) &= \frac{d}{dt}\left[\phi_2\left(\left(1-t\right)F + t\delta_x\right)\right]_{t=0}.
\end{aligned}$$

The function $\psi_2\left(F\right)$ is known as the *influence function* of the functional ϕ_2. We rewrite the functional (2.25) as an expectation:

$$\phi_2(F) = E\left[\epsilon\nu'(F(Z))\right],$$

where the random variable ϵ stands for the error term of the linear model (2.10), Z is the random variable corresponding to the return on the CRR market portfolio, and F is the cumulative distribution function of Z.

Denote by $g(Z)$ the projection of ϵ on Z, i. e., $g(Z) = E[\epsilon|Z]$. Introduce a "misspecified"joint distribution function $F_\theta(z,\epsilon)$ along the path θ, such that F_0 is the true distribution function. Then we can calculate the influence function from the pathwise derivative of the functional, using the pathwise derivative (see Newey (1994)):

$$\frac{dE_\theta\left[g_\theta(Z)\,\nu'(F_\theta)\right]}{d\theta} = \frac{\partial E_\theta\left[g(Z)\,\nu'(F)\right]}{\partial\theta} + \frac{\partial E\left[g_\theta(Z)\,\nu'(F)\right]}{\partial\theta} + \frac{\partial E\left[g(Z)\,\nu'(F_\theta)\right]}{\partial\theta},$$

where we denote F_θ as the "misspecified"marginal distribution function of Z corresponding to the "misspecificationöf the joint distribution function $F_\theta(z,\epsilon)$, $g_\theta(Z)$ as the "misspecified"conditional expectation of ϵ given Z, $E_\theta[\cdot]$ as the expectation under the "misspecified"distribution $F_\theta(z,\epsilon)$. Formally:

$$\begin{aligned} F_\theta(z) &= \int_{-\infty}^{z}\int_{-\infty}^{\infty} dF_\theta(s,t)\,, \\ g_\theta(z) &= \int_{-\infty}^{\infty} t dF_\theta(t|Z=z)\,, \\ E_\theta[\cdot] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdot dF_\theta(s,t)\,. \end{aligned}$$

From the expression for the pathwise derivative we can see that the influence function of the functional (2.25) can be represented as a superposition of three influence functions of the misspecified functionals:

$$\psi_2(z,\epsilon) = \psi_A(z,\epsilon) + \psi_B(z,\epsilon) + \psi_C(z,\epsilon)\,,$$

with

$$\left[\frac{dE_\theta\left[g_\theta(Z)\,\nu'(F_\theta)\right]}{d\theta}\right]_{\theta=0} = E\left[\psi_2(Z,\epsilon)\left(\frac{\partial \ln dF_\theta}{\partial\theta}\right)_{\theta=0}\right].$$

Further, we calculate the separate pathwise derivative and find the influence function of the functional (2.25). The first part of the influence function is easy to find:

$$\frac{\partial E_\theta \left[g\left(Z\right)\nu'\left(F\right)\right]}{\partial\theta} = E\left[g\left(Z\right)\nu'\left(F\right)\frac{\partial \ln dF_\theta}{\partial\theta}\right],$$

so that the first part of the influence function is

$$\psi_A\left(z,\epsilon\right) = g\left(z\right)\nu'\left(F\left(z\right)\right) \tag{2.26}$$

For the second part, we find, using the chain rule and the definition of the projection g_θ:

$$\begin{aligned}
\frac{\partial E\left[g_\theta\left(Z\right)\nu'\left(F\right)\right]}{\partial\theta} &= \frac{\partial E_\theta\left[g_\theta\left(Z\right)\nu'\left(F\right)\right]}{\partial\theta} - \frac{\partial E_\theta\left[g\left(Z\right)\nu'\left(F\right)\right]}{\partial\theta} \\
&= \frac{\partial E_\theta\left[\left(\epsilon - g\left(Z\right)\right)\nu'\left(F\right)\right]}{\partial\theta} \\
&= E\left[\left(\epsilon - g\left(Z\right)\right)\nu'\left(F\right)\frac{\partial \ln dF_\theta}{\partial\theta}\right],
\end{aligned}$$

so that the second part of the influence function is

$$\psi_B\left(z,\epsilon\right) = \left(\epsilon - g\left(z\right)\right)\nu'\left(F\left(z\right)\right). \tag{2.27}$$

To calculate the last part of the influence function we directly apply the definition of the influence function to the functional $E\left[g\left(Z\right)\nu'\left(F_\theta\right)\right]$:

$$\begin{aligned}
\psi_C\left(z,\epsilon\right) &= \frac{d}{dt}\left[\int g\left(s\right)\nu'\left(\left(1-t\right)F + t\delta_z\right)dF\right]_{t=0} = \qquad (2.28)\\
&= \int g\left(s\right)\left(\delta_z - F\right)d\nu'\left(F\right).
\end{aligned}$$

The influence function of the functional (2.25) is the superposition of the three calculated influence functions (2.26), (2.27) and (2.28):

$$\begin{aligned}
\psi\left(z,\epsilon\right) &= \psi_A\left(z,\epsilon\right) + \psi_B\left(z,\epsilon\right) + \psi_C\left(z,\epsilon\right) \\
&= \chi\left(z,\epsilon\right) - E\left[\chi\left(Z,\epsilon\right)\right], \qquad (2.29)\\
\chi\left(z,\epsilon\right) &= \epsilon\nu'\left(F\left(z\right)\right) + \int_z^\infty g\left(s\right)d\nu'\left(F\left(s\right)\right). \qquad (2.30)
\end{aligned}$$

Substituting the expression for Choquet distortion pdf $\nu'(t)$ from (2.3) into (2.30) we obtain the final result

$$\chi(z,\epsilon) = \int_{F(z)}^{1} \left(\epsilon - E[\epsilon|Z = F^{-1}(\alpha)]\right) \alpha^{-1} d\phi(\alpha).$$

2.B Influence function of CRR efficient portfolio weights

Results on the asymptotic distribution of mean-CRR efficient portfolio weights are obtained in chapter 3. Here we briefly restate the results without the derivation details.

Let R^e be a vector of the asset excess returns $(R_1 - r_f, \ldots, R_p - r_f)$, and $Z = r_f + R^{e\prime}\theta$ be a portfolio of these assets. The mean-CRR portfolio problem can be formulated as

$$\min_{\theta \in \mathbb{R}^p} E\left[-Z \int_{F_z(Z)}^{1} \alpha^{-1} d\phi(\alpha)\right] \; s.t. \; E[Z] = m,$$

where m is the expected return on the efficient portfolio. From the econometric perspective this problem is a standard constrained extremum estimation problem, so that the limit distribution of resulting portfolio weights can be found in the usual way, see Gourieroux and Monfort (2005) and chapter 3 of this thesis. The asymptotic distribution results can be equivalently expressed through the estimator influence function. Here we report the final results. The influence function of the mean-CRR optimal portfolio weighs is

$$\xi(R^e, Z) \quad = \quad H^{-1} \begin{bmatrix} bCC'H^{-1} - I_p \\ -bC \end{bmatrix}' \begin{bmatrix} \psi_{\nabla f} - \lambda \psi_{\nabla g} \\ \psi_g \end{bmatrix},$$

where we use notations similar with chapter 3. The vector C stands for the gradient of the constraint function with respect to portfolio weighs $C = E[R^e]$. The scalar λ

is the Lagrange multiplier

$$\lambda = -E\left[\iota' R^e \int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha)\right] \left(\iota' E[R^e]\right)^{-1},$$

where ι stands for a $(p \times 1)$ vector of ones. The matrix H is the Hessian of the objective function with respect to portfolio weights evaluated at the optimum

$$H = E\left[\frac{\phi'(F_z(Z)) f_z(Z)}{F_z(Z)} \mathrm{Cov}(R^e|Z)\right].$$

The functions $\psi_{\nabla f}$ and $\psi_{\nabla g}$ are the influence functions of the objective and constraint function gradient functionals, respectively. The expressions for them are given by

$$\begin{aligned}
\psi_{\nabla f} &= \chi_{\nabla f} - E[\chi_{\nabla f}], \\
\chi_{\nabla f} &= -\int_{F_z(Z)}^1 \left(R^e - E\left[R^e|Z = F^{-1}(\alpha)\right]\right) \alpha^{-1} d\phi(\alpha), \\
\psi_{\nabla g} &= R^e - E[R^e].
\end{aligned}$$

The function ψ_g is the influence function of the constraint functional, $\psi_g = Z - m$. Finally, the scalar b is a notation

$$b = \left(C' H^{-1} C\right)^{-1}.$$

The asymptotic distribution of the mean-CRR optimal portfolio weights is

$$\sqrt{n}\left(\widehat{\theta} - \theta\right) \to_d N\left(0, E[\xi\xi']\right).$$

Expected shortfall

The asymptotic result for the mean-expected shortfall optimal weights is a special case of the mean-CRR weighs considered above with $\phi(\alpha) = I(\alpha \geq \tau)$. Substituting this expression into the corresponding formulas yields

$$\begin{aligned}
\lambda &= -\tau^{-1} E\left[\iota' R^e I(F_z(Z) \leq \tau)\right] \left(\iota' E[R^e]\right)^{-1}, \\
H &= = \frac{f(F_z^{-1}(\tau))}{\tau} \mathrm{Cov}(R^e|Z = F_z^{-1}(\tau)), \\
\chi_f &= -\tau^{-1} I(F_z(Z) \leq \tau) \left(R^e - E[R^e|Z = F_z^{-1}(\tau)]\right).
\end{aligned}$$

The expression for the influence function of the mean-expected shortfall portfolio weights follows immediately.

Point mass approximation (PMA) of a CRR

The point mass approximation of a CRR measure suggested by Bassett *et al.* (2004) takes

$$\phi(\alpha) = \sum_{k=1}^{m} \phi_k I(\alpha \geq \tau_k).$$

This is also a special case of a CRR measure. Therefore, the derived asymptotic results for a mean-CRR portfolio weights still apply. We have

$$\begin{aligned}
\lambda &= -\sum_{k=1}^{m} \phi_k \tau_k^{-1} E\left[\iota' R^e I(F_z(Z) \leq \tau_k)\right] \left(\iota' E[R^e]\right)^{-1}, \\
H &= E\left[\nabla_\theta^2 f\right] = \sum_{k=1}^{m} \phi_k \tau_k^{-1} f(F_z^{-1}(\tau_k)) \mathrm{Cov}(R^e | Z = F_z^{-1}(\tau_k)), \\
\chi_f &= -\sum_{k=1}^{m} \phi_k \tau_k^{-1} I(F_z(Z) \leq \tau_k) \left(R^e - E[R^e | Z = F_z^{-1}(\tau_k)]\right).
\end{aligned}$$

The expression for the influence function of the mean-PMA CRR portfolio weights follows immediately.

2.C Tables

Descriptive Statistics of Country Index Returns					
Indexes	Mean	Median	Kurtosis	Skewness	Volatility
	America				
WRLD	0.6%	9.1%	5.1	0.10	15.2%
ARG	1.2%	0.0%	32.9	-1.43	33.2%
BRA	15.9%	7.2%	7.4	0.13	30.8%
CHIL	11.3%	4.9%	4.5	-0.18	15.2%
PER	9.8%	6.6%	16.4	-0.23	14.3%
MEX	15.5%	19.8%	6.2	0.13	22.7%
VEN	-3.1%	-7.3%	75.0	-2.98	32.4%
	Asia				
CHI	1.7%	0.0%	9.6	1.05	22.0%
INDIA	18.5%	24.7%	7.0	-0.47	27.1%
INDO	10.5%	1.2%	8.8	0.09	35.9%
MAL	13.5%	0.0%	26.5	1.97	19.7%
PAK	22.5%	18.6%	7.2	0.10	28.4%
PHIL	-2.4%	-8.1%	51.5	3.43	22.2%
SHRIL	10.6%	0.0%	51.1	0.99	23.2%
THAIL	11.3%	1.4%	6.6	0.46	29.5%
	Europe				
CZE	14.5%	29.7%	7.2	-0.02	24.0%
HUN	12.9%	11.5%	5.0	0.16	26.0%
POL	23.5%	4.1%	8.8	-0.20	32.5%
ROM	37.7%	30.5%	6.3	-0.09	34.7%
RUS	70.2%	0.0%	22.9	1.78	53.5%
TURK	-2.5%	5.9%	4.2	0.02	19.9%
	OECD				
AU	10.6%	13.5%	5.9	-0.34	16.6%
CAN	12.3%	18.3%	6.2	-0.41	17.3%
EU	5.4%	2.6%	4.4	-0.06	21.4%
JAP	4.8%	0.0%	4.4	-0.10	22.6%
KOR	19.4%	11.5%	5.0	-0.11	36.5%
UK	2.5%	5.1%	4.7	-0.11	18.0%
US	1.5%	0.0%	5.1	0.18	19.5%

Tabel 2.1: Annualized descriptive statistics of country capital index returns.

Indexes	Efficiency of WRLD index MV	ShF	PMA
	America		
ARG	0.948	0.937	0.953
BRA	0.182	0.170	0.172
CHIL	0.095	0.088	0.089
PER	0.231	0.231	0.231
MEX	**0.042**	**0.040**	**0.040**
VEN	0.633	0.634	0.639
Joint	0.291	0.275	0.279
	Asia		
CHI	0.851	0.847	0.842
INDIA	0.133	0.125	0.126
INDO	0.581	0.556	0.561
MAL	0.175	0.175	0.172
PAK	**0.085**	**0.091**	**0.083**
PHIL	0.529	0.533	0.551
SRIL	0.422	0.406	0.422
THAIL	0.438	0.427	0.425
Joint	0.396	0.389	0.387
	Europe		
CZE	0.177	0.162	0.166
HUN	0.277	0.273	0.272
POL	0.112	0.110	0.111
ROM	**0.006**	**0.006**	**0.006**
RUS	**0.001**	**0.001**	**0.001**
TURK	0.497	0.490	0.493
Joint	**0.004**	**0.004**	**0.003**
	OECD		
AU	0.217	0.210	0.211
CAN	**0.033**	**0.031**	**0.031**
EU	0.613	0.588	0.600
JAP	0.776	0.766	0.764
KOR	0.206	0.200	0.198
UK	0.832	0.788	0.818
US	0.697	0.747	0.719
Joint	0.452	0.443	0.437

Tabel 2.2: Efficiency tests of the Morgan Stanley world capital index (WRLD). The table reports p-values of the mean-variance (MV), mean-expected shortfall (ShF) and mean-PMA CRR (PMA) spanning tests. Probability threshold for expected shortfall is 5%. Probability thresholds for PMA CRR are 5%, 10%, 15%, 20%, and 25% with equal weights of 20%.

	Cumulative Default Probability to Year (%)									
Rating	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Aaa	0	0	0	0.04	0.12	0.21	0.31	0.42	0.54	0.67
Aa	0.02	0.04	0.08	0.2	0.31	0.43	0.55	0.67	0.76	0.83
A	0.01	0.05	0.18	0.31	0.45	0.61	0.78	0.96	1.18	1.43
Baa	0.14	0.44	0.83	1.34	1.82	2.33	2.86	3.39	3.97	4.56
Ba	1.27	3.57	6.11	8.65	11.23	13.5	15.32	17.21	19	20.76
B	6.16	12.9	18.76	23.5	27.92	31.89	35.55	38.69	41.51	44.57

Tabel 2.3: Moody's cumulative default probabilities by letter rating from 1-10 years, 1970-2000. Source: Dominic O'Kane, LB Structured Credit Research, Credit Derivatives Explained.

	Simulation Parameters		
Def. Prob.:	2.5%	5%	7.5%
Def. Corr.:	0.15	0.1	0.05
Risk Prem.:	4%	10%	12%
	Sample Return Statistics		
Min.:	-27.00%	-30.80%	-16.80%
1st Qu.:	0.64%	2.78%	1.58%
Median:	2.25%	5.47%	4.20%
Mean:	1.40%	4.52%	3.60%
3rd Qu.:	3.16%	7.27%	6.29%
Max.:	3.99%	9.83%	10.80%
Std. Dev.	2.72%	3.86%	3.68%
Skew.	-2.63	-1.75	-1.03
Kurtos.	13.85	8.35	4.62
CDO1	1.00	0.01	0.00
CDO2	0.01	1.00	0.01
CDO3	0.00	0.01	1.00

Tabel 2.4: Descriptive statistics of the simulated CDO returns. Sample correlation matrix is given at the bottom of the table. Returns are simulated from the one-factor large homogeneous portfolio model.

	Returns	Efficient Portfolios CDO1	Mkt. MV	Mkt. ShF	Mkt. PMA
MV Span	CDO2	0.000	1.000	0.000	0.003
	CDO3	0.000	1.000	0.000	0.000
	All	0.000	1.000	0.000	0.000
ShF Span	CDO2	0.000	0.000	0.995	0.000
	CDO3	0.000	0.000	0.887	0.053
	All	0.000	0.000	0.989	0.000
PMA Span	CDO2	0.000	0.001	0.001	0.871
	CDO3	0.000	0.002	0.004	0.943
	All	0.000	0.001	0.000	0.916

Tabel 2.5: Spanning tests for simulated credit portfolio returns. The table reports p-values of mean-variance (MV Span), mean-expected shortfall (ShF Span), and mean-PMA CRR (PMA Span), spanning tests for assets CDO2 and CDO3. Four market efficient portfolios are considered: CDO1; mean-variance market portfolio (Mkt. MV); mean-expected shortfall market portfolio (Mkt. ShF); and mean-PMA CRR market portfolio (Mkt. PMA). The probability threshold for expected shortfall is 5%. The probability thresholds for PMA CRR are 5%, 10%, 15%, 20%, and 25% with equal weights of 20%.

Hoofdstuk 3

Mean-coherent risk and mean-variance approaches in portfolio selection: an empirical comparison.

3.1 Introduction

There is an ongoing debate in the financial literature on which risk measure to use in risk management and portfolio choice. As some risk measures are more theoretically appealing, others are easier to implement practically. For a long time, the standard deviation has been the predominant measure of risk in asset management. Mean-variance portfolio selection via quadratic optimization, introduced by Markowitz (1952), used to be the industry standard (see, for instance, Tucker *et al.* (1994)). Two justifications for using the standard deviation in portfolio choice can be given. First, an institution can view the standard deviation as a measure of risk, which needs to be minimized to limit the risk exposure. Second, a mean-variance portfolio maximizes expected utility of an investor if the utility index is quadratic or asset returns jointly follow an elliptically symmetric distribution.[1]

Despite the computational advantages, the variance is not a satisfactory risk measure from the risk measurement perspective. First, mean-variance portfolios are not consistent with second-order stochastic dominance (SDD) and, thus, with the benchmark expected utility approach for portfolio selection. Second, but not independently, as a symmetric risk measure, the variance penalizes gains and losses in the same way.

Artzner *et al.* (1999) give an axiomatic foundation for so-called coherent risk measures. They propose that a "rational" risk measure related to capital requirements[2] should be monotonic, subadditive, linearly homogeneous, and translation invariant. Tasche (2002) and Kusuoka (2001) demonstrate that a Choquet expectation with a concave distortion function represents a general class of coherent risk measures. Moreover, with some additional regularity restrictions, as imposed by Kusuoka (2001), the class of coherent risk measures becomes consistent with the second order stochastic dominance principle and thus generates portfolios consistent with the expected utility paradigm, see, for example, Ogryczak and Ruszczyński (2002), De Giorgi (2005), and Leitner (2004).

The class of coherent risk measures generalizes expected shortfall, a coherent risk measure which received a lot of attention in the recent literature due to its easy practical implementability and tractability. Tasche (2002) discusses theoretical properties of expected shortfall and its generalizations. He suggests a general method how to calculate expected shortfall risk contributions of individual assets in a portfolio. At the same time, a literature on how to apply expected shortfall in portfolio optimization appeared. Rockafellar and Uryasev (2000) provide an algorithmic solution to the expected shortfall-based portfolio optimization and hedging. Bertsimas *et al.* (2004) report theoretical properties of expected shortfall and show

[1] See, for instance, Ingersoll (1987).

[2] The capital requirements are relevant for asset management since they are directly applied to financial institutions, see the Basel Accord (1999).

that the mean-expected shortfall optimization problem can be solved efficiently as a convex optimization problem. They also provide some empirical evidence on asset allocation and index tracking applications.

There is also a broad empirical literature on expected shortfall. Bassett *et al.* (2004) show that a sample portfolio choice problem based on expected shortfall is equivalent to a quantile regression. Focusing mainly on the quantitative economic effect, they demonstrate that for certain asymmetric distributions of asset returns the difference between mean-variance and mean-expected shortfall efficient portfolio weights can be substantial. Kerkhof and Melenberg (2004) develop a framework for backtesting expected shortfall using the functional delta method. They show in a simulation study that tests for expected shortfall have better performance than tests for value-at-risk with acceptably low probability thresholds. Bertsimas *et al.* (2004) discuss various properties of expected shortfall. They provide empirical evidence based on asset allocation and tracking index examples that the mean-expected shortfall approach might have advantages over the mean-variance approach. Similarly to Bassett *et al.* (2004), the authors focus mainly on examples with simulated returns.

Even though the literature on coherent risk measures emphasizes the importance of the difference between these and conventional risk measures in asset allocation and risk management, there still seems to be lack of evidence on the statistical and economic significance of this difference in practical applications. The aim of this chapter is to analyze the degree of statistical and economic relevance of the switch from the traditional standard deviation to a coherent risk measure in a typical asset allocation problem, which consists of determining the optimal portfolio weights or of deciding whether particular assets have to be additionally included into the portfolio. Our contribution is twofold. First, we compare portfolios obtained by mean-coherent risk and mean-variance optimizations both statistically and economically. We do this for simulated asset returns as well as for actually traded securities. If the distribution of asset returns and liabilities were elliptically symmetric then any coherent regular risk measure of a portfolio would be proportional to its standard deviation, and, as a result, would lead to the same implications in risk management. In reality, asset returns are likely to be skewed and fat tailed. It is, however, an empirical question whether skewness and excess kurtosis alone are sufficient to generate statistically and economically different efficient portfolios if the variance is replaced by a coherent risk measure in a portfolio optimization problem. Here, we address this question by first deriving the asymptotic distribution of the mean-coherent risk portfolio weights and using these to statistically and economically compare the mean-coherent risk and mean-variance efficient portfolio weights. Additionally, we explain how to reformulate the point mass approximated mean-coherent risk problem as a linear program, which can be efficiently solved by numerical algorithms. The results obtained for simulated and actual portfolios suggest that portfolios based on coherent risk measures are often statistically and economically different from the portfolios based on the standard deviation for a typical portfolio of equities. Our simulation study confirms that for portfolios with asymmetric distributions of returns, such as portfolios of derivatives or credit instruments, an optimization based on a coherent risk measure behaves differently as it accounts mostly for negative returns[3]. As second contribution, we implement spanning tests for the mean-coherent risk efficient frontiers as developed in chapter 2. These tests can be regarded as an analog for the usual mean-variance spanning tests, see DeRoon and Nijman (2001) for a survey of the mean-variance tests. The test statistics are compared to their counterparts in the mean-variance framework. Our mean-variance and mean-coherent risk spanning tests for portfolios of common equities give statistically and economically similar results.

The remainder of the chapter is structured as follows. Section 3.2 describes the methodology, including the statistical comparison of mean-variance and mean-

[3]We do not study actual portfolios with derivatives due to related problems with stationarity.

coherent risk efficient portfolio weights and spanning tests for coherent risk measures. Empirical results on the comparison of the efficient portfolio weights are described in section 3.3. Applications of the coherent risk-spanning test are investigated in section 3.4. Section 3.5 discusses effects of estimation error in expected asset returns. Finally, section 3.6 concludes.

3.2 Methodology

3.2.1 Coherent risk measures and portfolio choice

Consider a probability space $(\Omega, \mathcal{F}, P)$,[4] and let $L_0(\Omega, \mathcal{F}, P)$ be the space of all equivalence classes of real valued random variables $X : \Omega \to \mathbb{R}$. A random variable $X \in L_0(\Omega, \mathcal{F}, P)$ can be seen as a risky financial position (profit or loss) and we call it a *risk*. If we consider the set $\mathcal{X} := L_0(\Omega, \mathcal{F}, P, \mathbb{R})$ of all risks then a *risk measure* ρ defined on $\mathcal{X}$ is a map from $\mathcal{X}$ to $\mathbb{R} \cup \{+\infty\}$, see Delbaen (2000).[5] Intuitively, one can consider a risk measure as measuring the riskiness of the position or cost of risk. The concept of the cost of risk can be formalized by defining the *capital requirement* or amount of reserved capital (”sweetener”) as a function of the risk measure ρ. We consider risk measures defined on general probability spaces $L_0(\Omega, \mathcal{F}, P)$, and probability spaces of bounded random variables $L_\infty(\Omega, \mathcal{F}, P) = \{X \in L_0(\Omega, \mathcal{F}, P) : P[|X| < \infty] = 1\}$. Denote

$$\rho_\infty \quad : \quad L_\infty(\Omega, \mathcal{F}, P, \mathbb{R}) \to \mathbb{R}, \tag{3.1}$$

$$\rho_0 \quad : \quad L_0(\Omega, \mathcal{F}, P, \mathbb{R}) \to \mathbb{R} \cup \{\infty\}. \tag{3.2}$$

For a long time, the standard deviation has served as the common risk measure.[6] Since it measures the ”degree of the deviationöf a random variable from its mean

[4] Ω is the set of states, $\mathcal{F}$ is the σ-algebra, and P is the probability measure.

[5] The range includes ∞ to make *coherent* risk measures on $L_0(\Omega, \mathcal{F}, P)$ possible.

[6] Well defined on the space $L_2(\Omega, \mathcal{F}, P)$ and set equal to $+\infty$ on $L_0(\Omega, \mathcal{F}, P) \backslash L_2(\Omega, \mathcal{F}, P)$, where $L_k(\Omega, \mathcal{F}, P) = \left\{X \in L_0 : \int |X|^k \, dP < \infty\right\}$ for $k > 0$.

it was perceived as a good measure of risk. Moreover, it has some very attractive properties. In particular, the standard deviation is closely related to the measure concept of square integrable random variables. This property leads to some nice theoretical results in mean-variance analysis. The standard deviation is also attractive for its analytical and numerical tractability. Indeed, it is easy to model, estimate, and implement in empirical problems of asset management. The main criticism regarding the standard deviation is related to the fact that it symmetrically measures losses and profits as contributions to riskiness of a financial position. Many different alternatives that concentrate on the downside part of the risk distribution have been proposed. The paper by Pedersen and Satchell (1998) illustrates this effort by providing an overview and classifying common measures of risk.

Artzner *et al.* (1999) follow an axiomatic approach to define a risk measure coherent from a regulator's point of view. They relate a risk measure to the regulatory capital requirement and deduce four axioms which should be satisfied by a "rational" risk measure. Delbaen (2000) extends the definition to general probability spaces $L_0(\Omega, \mathcal{F}, P)$.

Definition 3.1 *A mapping $\rho = \rho_0 : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is called a coherent measure of risk if it satisfies the following conditions for all $X, Y \in \mathcal{X}$.*

- *Monotonicity: if $X \leq Y$, then $\rho(X) \geq \rho(Y)$.*
- *Translation Invariance: if $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.*
- *Positive Homogeneity: if $\lambda \geq 0$, then $\rho(\lambda X) = \lambda \rho(X)$.*
- *Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.*

The financial meaning of monotonicity is clear: The downside risk of a position is reduced if the payoff profile is increased. Translation invariance is motivated by the interpretation of the risk measure $\rho(X)$ as a capital requirement, i.e., $\rho(X)$ is the amount of capital which should be added to the position to make X acceptable from the point of view of the regulator. Thus, if the amount m is added to the position, the capital requirement is reduced by the same amount. Positive homogeneity says that riskiness of a financial position grows in a linear way as the size of the position increases. This assumption is not always realistic. Withdrawing the positive homogeneity axiom leads to a family of convex risk measures, see Föllmer and Schied (2002).[7] The subadditivity property allows one to decentralize the task of managing the risk arising from a collection of different positions: If separate risk limits are given to different desks, then the risk of the aggregate position is bounded by the sum of the individual risk limits. The subadditivity is also closely related to the concept of risk diversification in a portfolio of risky positions.

These axioms rule out many of the conventional measures of risk traditionally used in finance. For instance, the standard deviation and other measures based on second moments are ruled out by the monotonicity requirement. Quantile based measures, such as the value-at-risk (VaR), are ruled out by subadditivity.

Kusuoka (2001) adds another two axioms that further constraint the set of coherent risk measures

- *Law Invariance: if* $P[X \leq t] = P[Y \leq t]$ $\forall t$, *then* $\rho(X) = \rho(Y)$.
- *Comonotonic Additivity: if* $f, g : \mathbb{R} \to \mathbb{R}$ *are measurable and non-decreasing, then* $\rho(f \circ X + g \circ X) = \rho(f \circ X) + \rho(g \circ X)$.

The intuition of the two axioms is simple: the Law of Invariance means that financial positions with the same distribution should have the same risk. It allows empirical identification of the risk measure. The second condition on Comonotonic Additivity refines slightly the subadditivity property: subadditivity becomes additivity when two positions are comonotonic. By comonotonicity we understand that the random variables are monotonic transformations of the same random variable.

[7] See, however, De Giorgi (2005) on homogenizing risk measures.

Suppose that we are given two non-decreasing functions $f, g : \mathbb{R} \to \mathbb{R}$ and a random variable $X \in L_0(\Omega, \mathcal{F}, P)$. Then the random variables $Z = f(X)$ and $Y = g(X)$ are called comonotonic. The following result was shown by Kusuoka (2001), Tasche (2002), and Denneberg (1990):

A risk measure $\rho = \rho_\infty$ defined on $L_\infty(\Omega, \mathcal{F}, P)$, with P non-atomic, is coherent, law invariant, and comonotonic additive if and only if for any random variable X with cumulative distribution function $F_X(\cdot)$ it can be represented as

$$\rho(X) = \int_0^1 s_\alpha(X) d\phi(\alpha), \tag{3.3}$$

where ϕ is a probability measure defined on the interval $[0,1]$, and s_α is the expected shortfall of X

$$s_\alpha(X) = -\alpha^{-1} \int_0^\alpha F_X^{-1}(t) dt.$$

This risk measure defined on the general probability space $L_0(\Omega, \mathcal{F}, P)$ for non-positive random variables X stays coherent, law invariant, and comonotonic additive, see Delbaen (2000). We call a coherent, law invariant, and comonotonic additive measure of risk represented by equation (3.3) a *coherent regular risk* (CRR) measure.

Example 3.1 (Expected Shortfall) *A CRR risk measure that gained a lot of attention in the recent literature is the expected shortfall, given by*

$$s_\tau(X) = -\tau^{-1} \int_0^\tau F_X^{-1}(t)\, dt,$$

which corresponds to $\phi(\alpha) = I(\alpha \geq \tau)$. Being a coherent regular risk measure, it satisfies comonotonic additivity, law invariance and all axioms of a coherent risk measure. Many useful properties of expected shortfall are established, for example, in Tasche (2002) and Bertsimas et al. *(2004).*

Example 3.2 (Point Mass Approximation (PMA) of CRR measure) *Bassett* et al. *(2004) suggested to approximate a CRR measure by a weighted sum of Dirac's* *point mass functions.*[8] *This approximation corresponds to the probability measure* $\phi'(\alpha) = \sum_{k=1}^{m} \phi_k \delta_{\tau_k}(\alpha)$ *in expression (3.3), with* $\phi_k \geq 0$ *and* $\sum \phi_k = 1$. *The PMA CRR measure can be written as*

$$\rho(X) = \sum_{k=1}^{m} \phi_k s_{\tau_k}(X). \tag{3.4}$$

Notice, that the PMA CRR measure is itself a CRR measure, and the term PMA refers to the fact that the integral in expression (3.3) is replaced by a finite weighted sum in (3.4). From the form of the PMA CRR measure it is clear that the expected shortfall is a particular case of this approximation.

A nice property of these two examples is that in both cases the in-sample mean-CRR optimization problem can be reformulated as a linear program, which can be solved efficiently. The mean-expected shortfall optimization is considered, among others, by Rockafellar and Uryasev (2000), Bertsimas *et al.* (2004), and Bassett *et al.* (2004). The mean-PMA CRR optimization is discussed in subsection 3.2.4. Additionally, as special cases of the mean-CRR portfolio selection problem, mean-expected shortfall and mean-PMA CRR optimizations are consistent with second-order stochastic dominance and, thus, fall in the reward-risk theoretical framework developed by De Giorgi (2005).

For a fixed set of random returns $\{R_0, \ldots, R_p\}$, a risk measure $\rho = \rho\left(\sum_{i=0}^{p} w_i R_i\right)$ can be considered as a function of portfolio weights,

$$\rho\left(w_0, \ldots, w_p\right) : \left\{(w_0, \ldots, w_p) \in \mathbb{R}^p : \sum\nolimits_i w_i = 1\right\} \to \mathbb{R}.$$

Denote by $\mu_i = E\left[R_i\right]$ the expected return of asset i (which we assume to exist). Given the required portfolio expected return ν we try to find portfolio weights $\{w_i\}$ that minimize the chosen risk measure. The corresponding optimization problem can be formulated as follows:

[8]The point mass function $\delta_\tau(\alpha)$ is defined through the integral $\int_{-\infty}^{x} \delta_\tau(\alpha) d\alpha = I(x \geq \tau)$.

$$\min_{\{w_0,\ldots,w_p\}} \rho\left(w_0, \ldots, w_p\right) \; s.t. \; \sum_{i=0}^{p} w_i = 1, \; \sum_{i=0}^{p} w_i \mu_i = \nu. \tag{3.5}$$

When solving this problem, we assume that $\rho\left(w_1, \ldots, w_p\right) < \infty$. It is straightforward that the first equality constraint can be eliminated by passing it to the objective function. Denote by $y = R_0$ the return on the benchmark asset R_0. Define by $x = (R_1 - R_0, \ldots, R_p - R_0)'$ the vector of excess returns of the other assets. The mean-risk optimization problem (3.5) can be rewritten as

$$\min_{\theta \in \mathbb{R}^p} \rho(y + x'\theta) \; s.t. \; E[y + x'\theta] = \nu, \tag{3.6}$$

where θ is the $p \times 1$ vector of portfolio weights of assets $1, \ldots, p$. When one chooses the standard deviation as the risk measure ρ in optimization (3.6) the standard mean-variance portfolio problem is obtained. Alternatively, when a CRR measure is chosen, the solution to (3.6) is the vector of mean-CRR portfolio weights. The standard deviation has an advantage over other risk measures in empirical applications since the estimation and optimization parts can be separated from each other. In this case the random returns $(R_0, \ldots, R_p)$ should be square integrable. The expected shortfall portfolio optimization problem is an example of the mean-CRR portfolio that can be solved by convex programming methods as, for example, suggested by Bertsimas *et al.* (2004) and Rockafellar and Uryasev (2000). Bassett *et al.* (2004) show that the mean-expected shortfall efficient portfolio problem is equivalent to a quantile regression with linear constraints. As a result the problem can be solved by well developed standard methods.[9]

3.2.2 Comparison of portfolio weights

The question of the comparison of the efficient portfolio weights for the standard deviation and a CRR risk measure arises naturally. For elliptically symmetric distributions the standard deviation and a CRR measure give the same portfolio weights in

[9] See Portnoy and Koenker (1997).

the mean-risk optimization.[10] For other distributions the efficient portfolio weights will, in general, alter. But the question then is whether this difference is significant, either economically or statistically, or both.

To statistically compare the mean-variance and mean-CRR portfolio weights we need to derive their joint asymptotic distribution. Then, standard statistical procedures can be applied. The asymptotic results on portfolio weights as well as the equality test for mean-CRR and mean-variance portfolio weights are given in Appendixes 3.A, 3.B, and 3.C.

It is well known that portfolio weights are very sensitive to estimation inaccuracy in asset expected returns, see, for example, Chopra and Ziemba (1993). This often leads to insignificance of estimated portfolio weights due to high standard errors and potentially can yield insignificant comparison results for portfolio weights in practical sample sizes. Therefore, we consider two situations. First, we ignore the estimation inaccuracy in asset expected returns, taking the viewpoint of Markowitz (1952) who suggests existence of *a priori* believes about the future expected returns. Then we include the asset expected return estimation inaccuracy into the portfolio weight comparison test.

3.2.3 Mean-variance and mean-CRR spanning tests

By analogy with the mean-variance spanning test, which tests whether two mean-variance frontiers generated by different sets of assets coincide, it is possible to develop a similar test for a CRR measure, see chapter 2 of this thesis. The standard question to be answered is whether the introduction of a new asset to a set of assets forming the optimal portfolio shifts the mean-CRR efficient frontier in a statistical sense.

In the literature spanning tests are usually considered in the mean-variance con-

[10]This fact is a straightforward generalization of proposition 1 in Bertsimas *et al.* (2004) for expected shortfall.

text. A conventional procedure for such a spanning test is suggested by Huberman and Kandel (1987). It is based on the notion that the restrictions on the tangent portfolio weights can be expressed as moment restrictions on excess returns of assets in the portfolio. These moment restrictions can be reformulated in terms of restrictions in an OLS regression, see, for example also, DeRoon and Nijman (2001). In chapter 2 we develop a test similar to Huberman and Kandel (1987) for mean-CRR spanning, expressed in terms of restrictions on IV regression coefficients.

An alternative approach to the spanning test is followed by Britten-Jones (1999), who formulates the spanning hypothesis in the mean-variance framework in terms of restrictions on the tangent portfolio weights. These weights can be found as OLS regression coefficients. Results from the previous subsection can be used to implement this approach in the mean-CRR setup with the restrictions on the OLS regression coefficients in Britten-Jones (1999) replaced by restrictions on the corresponding mean-CRR portfolio weights.

In this chapter we follow the approach developed in chapter 2 for the mean-CRR spanning and in Huberman and Kandel (1987) for the mean-variance spanning. The mean-variance spanning test is based on the notion that the restrictions on the tangent portfolio weights can be expressed as moment restrictions on excess returns of assets in the portfolio. These moment restrictions can be reformulated in terms of restrictions on regression coefficients. In particular, let Y^e be a random return excess of the risk-free rate of an asset for which we want to perform a spanning test. Let Z^e be the excess return of the mean-variance optimal market portfolio. Consider the regression

$$\begin{aligned} Y_i^e = \alpha + \beta Z_i^e + \epsilon_i, \\ E\left[\epsilon_i\right] = 0, \\ E\left[Z_i^e \epsilon_i\right] = 0. \end{aligned}$$

The spanning hypothesis can be reformulated in terms of the restrictions on parameters α and β:

$$\alpha = 0, \tag{3.7}$$

$$\beta \mathrm{Var}(Z^e) - \mathrm{Cov}(Y^e, Z^e) = 0. \tag{3.8}$$

Restriction (3.8) shows that the coefficient β can be consistently estimated by an OLS regression, while restriction (3.7) states that the constant term in the regression (Jensen's α) should be equal to 0.

Chapter 2 of this thesis shows that the test for mean-CRR spanning can be reformulated in terms of restrictions on the instrumental variable (IV) regression

$$\begin{aligned} Y_i^e &= \alpha + \beta Z_i^e + \epsilon_i, \\ E[\epsilon_i] &= 0, \\ E[V_i \epsilon_i] &= 0, \end{aligned}$$

where V is the instrumental variable[11]

$$V = \int_{F_z(Z)}^{1} \alpha^{-1} d\phi(\alpha).$$

This instrumental variable defines a monotonic transformation of the original cumulative probability function F_z of portfolio returns. As a result more probability is assigned to the least favorable outcomes. We call this instrumental variable the *risk instrument* as it also defines the CRR measure. The restrictions imposed by the spanning hypothesis are

$$\alpha = 0, \tag{3.9}$$

$$\beta \mathrm{Cov}(Z^e, V) - \mathrm{Cov}(Y^e, V) = 0. \tag{3.10}$$

It follows from relation (3.10) that under the spanning hypothesis coefficient β can be consistently estimated by the IV regression with the risk instrument V. Restriction

[11]Notice, that in an empirical application the instrumental variable V has to be non-parametrically estimated.

(3.9) can then be checked as a zero-intercept test. Thus, the spanning test in case of the mean-variance portfolio is equivalent to the significance test of the intercept α in OLS regression,[12] and the mean-CRR spanning test is equivalent to the significance test of the intercept α in the IV regression. The asymptotic properties of the IV intercept coefficient are discussed in chapter 2.

3.2.4 Sample mean-CRR optimization

In this section we discuss algorithmic solutions to the sample mean-CRR optimization. A CRR measure can be viewed as a weighted combination of expected shortfalls for the whole range of probability thresholds, see (3.3). In practical applications, however, one would deal with the PMA version of a CRR measure, given in (3.4). Numerical solutions to an in-sample mean-expected shortfall optimization were proposed, among others, by Rockafellar and Uryasev (2000), Bertsimas *et al.* (2004), and Bassett *et al.* (2004). Generally, a sample analog of the mean-expected shortfall optimization can be reformulated as a linear program and solved efficiently with existing numerical algorithms, see Barrodale and Roberts (1974), Koenker and D'Orey (1987), and Portnoy and Koenker (1997). The method can be generalized to a PMA CRR measure, which uses Dirac's point mass functions to approximate an arbitrary CRR measure. This also corresponds to a piecewise linear approximation of the cumulative probability function $\phi(\alpha)$ in (3.3).

Suppose that a PMA approximation of a CRR measure is given by the piecewise linear cumulative distribution function ϕ:

$$\phi(\alpha) = \sum_{k=1}^{m} \phi_k I(\alpha \geq \tau_k).$$

[12]The spanning tests discussed in this subsection takes into account the estimation inaccuracy in the asset expected returns. Alternatively, one can ignore the estimation error in the asset expected returns by following the approach of Britten-Jones (1999). The mean-variance and the mean-CRR spanning tests can be straightforwardly performed by testing the significance of the new asset tangent portfolio weight, using the results derived in the Appendix.

Then the population mean-CRR portfolio problem is

$$\min_{\theta} \sum_{k=1}^{m} \phi_k s_{\tau_k}(v) \ s.t. \ E[v] = \nu, \tag{3.11}$$

where $s_{\tau_k}(\cdot)$ is the expected shortfall with the probability threshold τ_k, $v = y + x'\theta$ is the return of the portfolio, and ν is the required expected return of the portfolio. As noticed by Bassett *et al.* (2004) expected shortfall can be equivalently expressed as

$$s_{\tau_k}(v) = \tau_k^{-1} \min_{\vartheta \in \mathbb{R}} E\varrho_{\tau_k}(v - \vartheta) - \nu,$$

where $\varrho_\alpha(u) = u(\alpha - I(u < 0))$ and ν is the expected return of portfolio v. Using this expression for expected shortfall, the mean-CRR problem (3.11) can be reformulated as

$$\min_{\theta \in \mathbb{R}^p, \vartheta \in \mathbb{R}^m} \sum_{k=1}^{m} \phi_k \tau_k^{-1} E\left[\varrho_{\tau_k}(v - \vartheta_k)\right] \ s.t. \ E[v] = \nu. \tag{3.12}$$

Introduce auxiliary variables $u_k^+ \in \mathbb{R}_+^p$ and $u_k^- \in \mathbb{R}_+^p$ for $k = 1, 2, \ldots, m$. Denote by μ a vector of asset expected excess returns $E[x]$ and by μ_y the expected excess return of the asset y. Then the sample analog of the problem (3.12) can be formulated as the linear program

$$\begin{aligned}
&\min \sum_{k=1}^{m} \phi_k \tau_k^{-1} \left(\tau_k e' u_k^+ + (1 - \tau_k) e' u_k^-\right) \\
&s.t. \\
&Y + X\theta - u_k^+ + u_k^- - e\vartheta_k = 0, \\
&\mu'\theta = \nu - \mu_y, \\
&(u_k^+, u_k^-, \theta, \vartheta_k) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}^p \times \mathbb{R} \text{ for } k = 1, 2, \ldots, m,
\end{aligned}$$

where Y is the $(n \times 1)$-vector of sample returns of the asset y, X is the $(n \times p)$-matrix of sample excess returns of assets x, and e is the $(n \times 1)$-vector of ones. This linear program can be solved very efficiently by classical simplex and interior point methods, see Barrodale and Roberts (1974) and Portnoy and Koenker (1997).

3.3 Statistical comparison of portfolio weights

3.3.1 Simulated returns

First, we compare the mean-variance and the mean-CRR efficient portfolio weights for simulated returns. We focus our attention on the expected shortfall and PMA CRR measure. This exercise emphasizes the fact that the variance and a CRR measure in the portfolio optimization context give different outcomes only in the case when the distribution of returns substantially deviates from the elliptically symmetric case. For expected shortfall similar examples with simulated returns were considered in Bertsimas *et al.* (2004) and Bassett *et al.* (2004). However, Bassett *et al.* (2004) do not perform a statistical comparison of the mean-variance and mean-expected shortfall efficient portfolio weights, while Bertsimas *et al.* (2004) use Monte-Carlo simulations instead of asymptotic theory.

As a benchmark we consider a sample of returns drawn from a three-variate normal distribution with population vector of means $[0.06, 0.08, 0.08]^T$ and covariance matrix:

$$\begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.04 \end{bmatrix}.$$

This simple example corresponds to a portfolio of assets with normally and independently distributed returns with an annual standard deviation of 20%. The independence of the returns makes the diversification motive very simple, so that it is easy to see which outcome in a portfolio optimization to anticipate. We expect the efficient mean-variance and mean-CRR portfolio weights to be equal in this case, because the considered risk measures are proportional under normality. We shall refer to this case with the abbreviation "NORM".

Next, we simulate returns from a three-variate Student distribution with the same vector of expected returns and covariance matrix as in the normal case. This example might be more realistic than the multivariate normal one since observable market returns usually have fat distributional tails. Nevertheless, from a theoretical point of view the standard deviation and the expected shortfall are equivalent in the case of a Student t-distribution from a portfolio optimization perspective. This is so because the Student t-distribution belongs to the class of elliptically symmetric distributions. We shall refer the simulation from the Student distribution with an abbreviation "t".

To illustrate the difference between a CRR measure and the standard deviation in a portfolio choice framework, we consider a sample of returns drawn from a three-variate asymmetric distribution, "ASYM", using returns on the following independent assets. Asset A has a lognormal distribution such that its log return is normally distributed with mean 0.06 and variance 0.04. Asset B consists of a long position in an equity and an at-the-money European call option written on this equity. We assume normally distributed equity log-returns and use the Black-Scholes formula to calculate the price of the option. We normalize the distribution of log-returns on asset B to have mean 0.08 and variance 0.04. Its distribution is significantly skewed to the left. Asset C consists of a long position in an equity and the money market account and a short position in the European call option on the equity. We normalize the distribution of the log-returns on asset C to have mean 0.08 and variance 0.04. This distribution is skewed to the right. Figure 3.1 shows kernel density estimates of the simulated log-return distributions for the assets A, B, and C.[13]

Summary statistical information on all considered assets is provided in Table 3.1. It can be seen that for the returns simulated from the three-variate normal distribution, NORM, the values of skewness and kurtosis are close to the theoretical

[13] We use the Gaussian kernel density with the bandwidth chosen according to the Silverman's rule of thumb, see Silverman (1986).

ones, i.e., 0 and 3, respectively. For the returns simulated from the three-variate Student t-distribution we observe significantly higher sample kurtosis than for the normal case. As the returns are generated from a t-distribution with 6 degrees of freedom, the sample kurtosis is close to 6, the theoretical result for a t-distribution with six degrees of freedom. Finally, for the case of asymmetric returns, we observe a substantial positive sample skewness for asset B and a negative sample skewness for asset C, while the kurtosis of all assets in the portfolio is close to 3, i.e., not very different from the normal case.

For the three simulated classes of returns we first perform a statistical comparison of the efficient portfolio weights resulting from the mean-variance and mean-expected shortfall portfolio optimization problems. We apply the asymptotic test for equality of the portfolio weights developed in Section 3.2 to all three cases of the simulated returns, NORM, t and ASYM. Since we want to make sure that a particular test result is not due to a specific portfolio expected return or shortfall probability threshold, we apply this test for different expected returns on the risk-efficient portfolio and different probability thresholds for the expected shortfall. The expected returns of the efficient portfolios are chosen to guarantee that the resulting portfolio belongs to the upper part of the efficient frontier. In particular, annual returns of 10%, 12%, 14% and 16% were chosen as portfolio target returns. Table 3.2 contains the corresponding p-values of the test.

The results indicate that there is no statistical difference in the mean-variance and expected shortfall efficient portfolio weights for the multivariate normal and t-distribution of the asset returns. In fact, this result aligns well with the theoretical predictions for elliptically symmetric distributions, see Bertsimas *et al.* (2004), and Embrechts *et al.* (1999). For the ASYM case, when the asset returns are simulated from a three-variate asymmetric distribution, we generally see a statistically significant difference between the variance and expected shortfall based portfolio weights. For the probability threshold of 2.5% the result holds in the whole range of the required portfolio expected returns at the 5% significance level. For required portfolio expected returns 14% and 16% and probability thresholds in the range of 2.5%-10% there is a difference between mean-variance and mean-expected shortfall portfolio weights significant at the 10% significance level. The test statistics become insignificant for the probability threshold of 12.5% and required portfolio expected returns of 14% and 16%. Usually, as can be noticed, the p-values of the test increase with the threshold probability and the required portfolio expected return. This means that the sensitivity of the expected shortfall to changes in the portfolio weights differs from the sensitivity of the standard deviation mostly in the tail area. The two risk measures become closer to each other as we increase the tail probability or portfolio expected return.

The expected shortfall gives the value of expected loss in the portfolio provided that the loss exceeds a certain quantile. For an investor such a measure of risk might not be the best reflection of riskiness of the position because for different quantiles the expected loss can behave differently with respect to portfolio weights. Therefore, a more general coherent risk measure can be a better choice. Here we consider the case of the point mass approximation (PMA) of a CRR measure described in section 3.2. In particular, we choose an equally weighted PMA CRR with probability thresholds of 2.5%, 5%, 7.5%, 10%, and 12.5%, which aggregates the expected shortfalls used for portfolio weight comparison before. Table 3.3 shows p-values for the comparison test between the mean-PMA CRR portfolio weights and the mean-variance portfolio weights. Similar to the results for expected shortfall reported in Table 3.2, the equality hypothesis is strongly rejected only for portfolios of asymmetric returns. The rejection holds for all required expected portfolio returns.

In addition, we investigate the economic effect of the differences between the mean-variance and the mean-shortfall portfolio weights. In Table 3.4 we report the decrease in the expected shortfall, which results from shifting from the standard deviation to the expected shortfall in a portfolio allocation decision. These numbers can be interpreted as a decrease in the expected loss in the portfolio for a given loss probability threshold. As can be seen from Table 3.4, the results support our statistical conclusions. The economic significance of the difference between the mean-variance and the mean-shortfall efficient portfolios is economically negligible for the returns simulated from the multivariate normal and the multivariate Student t-distributions. The effect from using the expected shortfall instead of the standard deviation is substantially less than a one-percent decrease in the expected conditional loss. In the case of the asymmetric returns the situation is different. We can observe a significant reduction in the expected loss for small probability thresholds and medium expected portfolio returns. In this example the effect decays as the probability threshold and the expected portfolio return increase. Overall, we observe more pronounced results in the tail of the portfolio return distribution.

In summary, the example in this section indicates that the portfolio allocations based on the mean-shortfall optimization can significantly differ from those based on the mean-variance approach. Furthermore, this difference depends on the choice of the risk level for the expected shortfall risk measure. This suggests that for portfolios of assets with asymmetric distributions of returns, such as equity and credit derivatives, an investor can benefit from using the expected shortfall risk measure when making an allocation decision. By doing so, he can better avoid the risk exposure from the extreme tail events while taking advantage on a positive skewness of the returns, i.e., extreme events from the positive side. Clearly, the standard deviation, which treats positive and negative returns symmetrically, cannot do the job of distinguishing the positively skewed returns form the negatively skewed ones.

3.3.2 Market returns

It is well known that returns observed in the market usually substantially deviate from the normal distribution. Generally, asset returns have fat tails and negative or positive skewness. These empirical facts potentially make the CRR measure an attractive alternative to the standard deviation. However, in reality, asset allocation decisions involve work with empirical data, including estimation procedures, so that there is always a level of uncertainty in the obtained result. As a consequence, the question of statistical and economic significance of the difference between CRR and variance based allocation decisions arises. In this section we compare the mean-variance and mean-CRR efficient portfolio weights for portfolios of market returns. We consider three cases: the daily exchange rates for the British pound, the Canadian dollar, the German mark, and the Japanese yen ("ER") with respect to the US dollar; the daily returns on the Fama-French size/book-to-market portfolios ("Fama-French"); the daily returns on S&P 500 index, US government bond JPM index, and Small Caps S&P 500 index ("Index"). The sample statistics for these portfolios are shown in Table 3.5.

It follows from the table that for most of these portfolios the deviation from the normal distribution is very substantial. In particular, all exchange rates in the ER portfolio have excess kurtosis, with the Japanese yen being the most fat tailed. It is also the case for the Japanese yen exchange rate that its empirical distribution is substantially positively skewed. The deviation from the normal distribution for the Fama-French and Index portfolios is even more pronounced. In particular, we observe large negative skewness for all returns in the Fama-French portfolio. For the indexes, we see that the S&P 500 and the Small Cap returns are negatively skewed. All reported returns have a large excess kurtosis with the S&P 500 being the most fat tailed. As the deviation of the reported returns from the normal distribution is so striking, we could expect substantially different weights for the variance and CRR based efficient portfolios as well.

Table 3.6 shows the outcomes of the equality test between the mean-variance and mean-expected shortfall efficient portfolios for different required portfolio expected returns and probability thresholds. These results ignore the estimation inaccuracy of the expected returns, see section 3.5.

Surprisingly, the results from Table 3.6 indicate that the variance and the shortfall-based efficient portfolio weights are not always significantly different. The weight-equality hypothesis cannot be rejected at standard significance levels for the portfolios of exchange rates. For the Fama-French efficient portfolios the equality hypothesis is strongly rejected for the low probability threshold of 2.5%, but cannot be rejected at the 5% significance level for higher thresholds. Significance levels of the test are especially high for the probability thresholds higher than 5%, where the equality hypothesis is generally accepted. For the Index portfolios the situation is reversed. The equality hypothesis is accepted at conventional significance levels for the low probability threshold of 2.5%, while for higher thresholds the equality hypothesis is usually rejected. These results indicate that mean-expected shortfall portfolio weights depend on the tail behavior of the return distribution function. If the sensitivities of the expected shortfall with respect to portfolio weights are proportional to those of the standard deviation, then the resulting portfolio weights are similar. Otherwise, they are different. One interesting point is that even though the market returns are usually fat tailed and negatively skewed, the portfolio weights produced by the expected shortfall and the standard deviation are not necessarily statistically different. As we have already seen in the example of the multivariate t-distribution, fat tails do not always mean a difference in allocation between the mean-variance and the mean-shortfall portfolios, because distributions of the returns can still be close to elliptically symmetric. Now, we discover that skewness per se might not matter as well. There are two overlapping factors which determine the test outcomes. First, the test results are driven by the covariance matrix of the portfolio weights, which depends on the sample variance of the returns. Thus, the test outcome is dependent on the relation between skewness and variance in the return distributions. Second, the difference between the mean-expected shortfall and mean-variance portfolios is due to the asymptotic tail behavior of the return distributions. Skewness and kurtosis are only partial measures of this behavior and cannot completely reflect the sensitivity of the risk measures with respect to the portfolio weights. Table 3.7 illustrates the change of the difference between the mean-expected shortfall and mean-variance portfolio weights with the probability threshold for the Fama-French and Index portfolios with a required annualized expected portfolio return of 10%. The results confirm the conclusions of the tests in Table 3.6. In particular, the difference between the three first mean-expected shortfall and mean-variance portfolio weights is relatively large and statistically significant for the probability threshold 2.5% in the Fama-French portfolio. These outcomes suggest that the rejection of the equality hypothesis in the Fama-French portfolio for the probability threshold of 2.5% in Table 3.6 was caused by differences between the mean-expected shortfall and mean-variance portfolio weights of Big/Med, Big/High, and Small/Low size/book-to-market factors. As we increase the probability threshold to 7.5%, the behavior of the expected shortfall risk measure becomes similar to the behavior of the standard deviation. As a result, the differences between the mean-expected shortfall and mean-variance portfolio weights become small and insignificant. The same effect is observed in Table 3.6. For the Index portfolio we observe a reverse situation: the increase of the probability threshold leads to significant difference between the mean-expected shortfall and mean-variance portfolio weights. As Table 3.7 suggests, the rejection of the null-hypothesis in Table 3.6 for higher probability thresholds is caused by the difference between the mean-expected shortfall and mean-variance portfolio weights of the Small Cap index. For the low probability threshold of 2.5% this difference is insignificant, and so is the test statistic in Table 3.6.

Additionally, as in the case of simulated returns, we perform a statistical comparison of the mean-PMA CRR and mean-variance portfolio weights. The equally weighted probability thresholds for the point mass approximation are chosen to be 2.5%, 5%, 7.5%, 10%, and 12.5%. Table 3.8 reports p-values of the test for different required expected portfolio returns. Even thought the results of this table align well with the results for the expected shortfall reported in Table 3.6, they indicate the statistical difference between the mean-variance and mean-CRR portfolios better. In particular, p-values of the Fama-French and Index portfolios are relatively small, which can be attributed to the contribution of the corresponding expected shortfalls with small significant test statistics.

Finally, Table 3.9 shows the economic size of the difference between the variance and the shortfall-based portfolio allocations.

For the Fama-French and Index portfolios the results support our statistical conclusions as we observe higher economic effect for those required portfolio expected returns and probability thresholds for which we also had smaller p-values of the equality test. The smaller economic effect is observed for the required portfolio expected returns and probability thresholds for which the equality hypothesis was not rejected.

Surprisingly, we observe high economic effect for portfolios of exchange rates (ER), where the decrease in expected loss with a given probability is up to 9%. At the same time the equality hypothesis is not rejected for these portfolios, see Table 3.6. The explanation for this phenomenon is high volatility of the exchange rates. The standard errors for the economic effects of the ER portfolios are relatively high, so that we can attribute the high p-values of the test statistics in Table 3.7 to the high volatility of the ER portfolio weights.

We conclude that for a typical portfolio of equities the expected shortfall and the standard deviation might produce statistically and economically different results. However, in certain cases the difference in portfolio weights is offset by the estimation error. When portfolios with asymmetric returns are considered, the portfolio weights for shortfall and standard deviation are significantly different, as in the ASYM case.

In this situation it might be beneficial to use Choquet risk measures which account for downside returns.[14]

3.4 Spanning tests

Comparison of the mean-variance and mean-CRR approach is not confined to the comparison of the portfolio weights. Additionally, one might ask the question whether the introduction of a new asset that shifts the mean-variance frontier has the same effect on the mean-CRR efficient frontier or conversely. Statistically, shifts in efficient frontiers can be characterized by spanning tests. In this section we are going to apply tests for mean-variance and mean-CRR spanning to several sets of assets, including the simulated returns from the previous section, the Fama-French value-book-to-market portfolios, and the S&P500 industry index returns. The results for the mean-variance and mean-CRR spanning tests are compared. In principle, as described in chapter 2, we can perform the spanning test for an arbitrary CRR measure. However, to make our analysis concise we focus on the mean-expected shortfall and mean-PMA CRR cases.

3.4.1 Simulated returns

In this subsection we apply the mean-variance and the mean-CRR spanning tests to the sets of returns simulated in the previous section. First, for the three sets of assets, NORM, t, and ASYM, we perform market efficiency tests with respect to the first asset, which we denote by R_1. The null hypothesis is that the asset R_1 is market efficient, so that the remaining assets, which we respectively denote by R_2 and R_3

[14] A natural extension of this study would be to investigate asymetric portfolios that include options or credit derivatives. However, due to non-stationarity problems, caused by the maturity of derivative contracts, the methodology would have to be significantly adjusted. We postpone this for a separate study.

are redundant. We perform three spanning tests. First, as a benchmark, the test for mean-variance spanning is performed. Then, the mean-expected shortfall efficiency for probability thresholds 2.5%, 7.5%, and 12.5% is tested. Finally, we implement the mean-PMA CRR spanning test, with equally weighted probability thresholds of 2.5%, 5%, 7.5%, 10%, and 12.5%. The risk-free interest rate is assumed to be 2.5%.

The test p-values are reported in Table 3.10. It can be clearly seen that the spanning hypothesis is strongly rejected for all risk measures, which means that the remaining assets R_2 and R_3 are not redundant. We do not report significance levels for asset R_1 as it should be, of course, redundant. The inclusion of the assets R_2 and R_3 in a mean-risk portfolio improves diversification from both the mean-variance and mean-CRR perspectives.

The difference between the mean-variance and mean-CRR spanning tests can be shown by testing the spanning hypothesis for a mean-variance market efficient portfolio. We form this portfolio from the three available assets R_1, R_2, and R_3. Table 3.11 reports p-values of the spanning tests with respect to the mean-variance portfolio of the available assets. The null hypothesis is that assets R_2 and R_3 are redundant.

As could be anticipated, the mean-variance hypothesis cannot be rejected at the conventional significance levels for all sets of assets. The mean-CRR spanning hypothesis cannot be rejected[15] for the returns simulated from the normal (NORM) and multivariate t-distributions. At the same time the spanning hypothesis is strongly rejected for the portfolio of asymmetric returns ASYM. This demonstrates that the difference between the outcomes of mean-variance and mean-CRR spanning tests should be expected for portfolios of non-standard instruments with asymmetric return distributions. Such instruments could include equity derivatives or pooled credit securities.

The spanning tests in Tables 3.10 and 3.11 can be interpreted from the point of

[15] The same results are obtained if the estimation error in mean returns is ignored.

view of an investor who considers the given 3 equities as an investment possibility set. The fact that the spanning hypothesis is accepted for an individual equity indicates the redundancy of this equity with respect to the market portfolio (or the set of other equities from which the "market portfolioïs formed). Rejection of the spanning test for the asset R_1 in Table 3.10 means that from the investor's perspective this asset cannot be viewed as a market portfolio, neither from the mean-variance nor from the mean-CRR perspective. The mentioned redundancy is related to a risk measure that is used by the investor for allocation purposes. Suppose that the mean-CRR investor forms a portfolio based on the mean-variance principle. In this case she invests her wealth in the combination of the risk-free asset and the mean-variance market portfolio. The results in Table 3.11 for asymmetrically distributed returns show that assets R_2 and R_3 are not redundant to such an allocation, i.e., the portfolio can still be improved from the mean-CRR perspective. On the other hand, an investor, who uses the mean-variance instead of the mean-CRR analysis gets almost the same diversification in the case of elliptically symmetric returns, NORM or t.

3.4.2 Market returns

Skewness and excess kurtosis of empirical distributions of asset returns is a frequent phenomenon observed in the market. In this subsection we apply spanning tests to the set of Fama-French portfolios based on the size and book-to-market factors as well as to the set of the S&P 500 sector indexes to check whether the mean-CRR spanning test produces significantly different conclusions from the mean-variance one. Sample statistics of the observed returns are reported in Table 3.12. The sample returns demonstrate substantial excess kurtosis and, in most of the cases, negative skewness.

Table 3.13 reports the results of the spanning tests. For the Fama-French set we perform the spanning tests with respect to the Fama-French market portfolio.

For the set of sector indexes the tests are performed with respect to the S&P 500 composite index.

The results indicate that for the portfolio of small companies with high and medium book-to-market ratio as well as for the portfolio of big companies with high book-to-market ratio the spanning hypothesis is strongly rejected in all tests. At the same time it can be seen that for the portfolio of small companies with low book-to-market ratio the p-value of the mean-variance spanning test is almost twice as high as the p-values of the mean-CRR tests, which could possibly indicate a difference between the two tests. Generally, the market portfolio is not optimal both from the mean-variance and mean-CRR perspectives. Its risk can be further diversified by inclusion of Small/High, Small/Medium, and Big/High Fama-French portfolios.

Testing the S&P 500 composite index for market efficiency with respect to the S&P 500 sector indexes shows that no test can reject the spanning hypothesis at the conventional significance levels. The mean-variance and mean-CRR spanning tests produce the same conclusions and similar p-values.

Since both mean-variance and mean-CRR spanning tests lead to the same conclusion in both the Fama-French and the S&P 500 examples, one could wonder whether these spanning tests can be distinguished at all for sets of common assets, such as stocks, stock indexes, etc. To check this we form the optimal mean-variance portfolios in both the Fama-French and S&P 500 sector index sets. For these portfolios we perform the mean-expected shortfall and mean-CRR PMA spanning tests. The results are reported in Table 3.14. The spanning hypothesis cannot be rejected by any of the tests at the conventional significance levels,[16] which means that the mean-CRR and mean-variance optimal portfolios are statistically similar. Thus, for portfolios of common equities mean-variance and mean-CRR spanning tests can be used interchangeably.

[16]Ignoring estimation errors in mean returns lead to the same conclusions.

3.5 Estimation inaccuracy in expected returns

The results on the portfolio weight equality tests discussed in section 3.3 are considered from the viewpoint of Markowitz (1952) who suggests that there are *a priori* believes about the future expected returns. Given these believes an investor compares two alternative approaches in portfolio allocation decision: mean-variance or mean-CRR. In this section we investigate the effect of estimation inaccuracy in expected returns when these are also estimated using sample averages. It is known that the portfolio weights in the mean-variance analysis are very sensitive to errors in expected returns, see, for example, Chopra and Ziemba (1993). The same is the case for the mean-CRR portfolios. The asymptotic variance of the equality tests would typically increase due to the estimation inaccuracy, so that the test statistics yield insignificant results in practical sample sizes. In this section we use the portfolio weight equality tests to illustrate this. Table 3.15 shows the p-values of the portfolio comparison tests for the ASYM, the Fama-French, and the Index portfolios when the estimation inaccuracy in expected returns is taken into account. Comparing these results to the results in Tables 3.2 and 3.6, we see the increase in significance levels of the tests due to the estimation inaccuracy in expected returns. As a result, the majority of test statistics becomes insignificant at the standard significance levels, confirming the findings of the sensitivity analysis by Chopra and Ziemba (1993).

3.6 Conclusion

In this chapter we empirically investigated the statistical implications of coherent risk measures, advocated in the literature, to the portfolio selection problem. We showed that efficient portfolio weights generated by mean-variance and mean-CRR optimizations can be statistically different for various portfolios of stocks if the estimation error in the mean returns can be ignored. Our results suggest that a CRR measure can better account for the downside risk in the case when one can include derivatives or other assets with asymmetric returns in the portfolio. In this case mean-variance and mean-CRR portfolio weights are likely to be statistically different. Economic differences between the mean-variance and the mean-CRR approaches align well with the statistical ones. The differences in expected loss between mean-variance and mean-expected shortfall portfolios are high for portfolios of asymmetric returns and relatively low for portfolios of common equities.

Secondly, we applied the mean-CRR spanning test to simulated returns, the Fama-French portfolios, and a number of sector indexes included in the S&P500. We showed that the difference between the mean-variance and the mean-CRR tests is especially pronounced for portfolios of asymmetric returns. For elliptically symmetric distributions of returns, as well as for portfolios of common equities, the mean-variance and mean-CRR tests lead to the same statistical conclusions. Both tests strongly reject the hypothesis that the market portfolio spans the set of Fama-French size-book-to-market portfolios. At the same time, both mean-variance and mean-CRR tests cannot reject market efficiency of the S&P 500 composite index. This means that the S&P500 composite index fulfills the role of market portfolio both for mean-variance and mean-CRR investors. Our results demonstrate that the mean-variance and the mean-CRR approaches are often statistically and economically similar for the equity asset classes considered.

Finally, we considered the sample mean estimation inaccuracy effect on the mean-variance and mean-CRR portfolio weight equality tests. In line with the existing literature on the sensitivity of the mean-variance analysis to the sampling error, the test statistics become insignificant.

3.A Limit distribution of a constrained extremum estimator

Our optimal portfolio problem can be expressed as a constrained extremum estimator problem

$$\min_{\theta \in \mathbb{R}^p} Ef(\theta) \ s.t. \ Eg(\theta) = 0, \tag{3.13}$$

The first order conditions of this problem are

$$\begin{aligned} &E\left[\nabla_\theta f\right] - \lambda E\left[\nabla_\theta g\right] = 0, \\ &Eg(\theta) = 0, \end{aligned}$$

where λ is the Lagrange multiplier of the equality constraint. Denote by $\psi_{\nabla f}(\theta)$ and $\psi_{\nabla g}(\theta)$ the influence functions of the gradient functionals $E[\nabla_\theta f]$ and $E[\nabla_\theta g]$ respectively. Let the influence function of the constraint functional $E[g(\theta)]$ be $\psi_g(\theta)$. Then, using the first order Taylor expansion of the FOC, we obtain:

$$\begin{bmatrix} H & -G \\ -G' & 0 \end{bmatrix} \sqrt{n} \begin{bmatrix} \widehat{\theta} - \theta \\ \widehat{\lambda} - \lambda \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} \sum_{i=1}^n \left(\lambda \psi_{\nabla gi} - \psi_{\nabla fi}\right) \\ \sum_{i=1}^n \psi_{gi} \end{bmatrix} + \begin{bmatrix} r_{1n} \\ r_{2n} \end{bmatrix},$$

where

$$\begin{aligned} H &= E\left[\nabla^2_\theta f\right] - \lambda E\left[\nabla^2_\theta g\right], && (3.14) \\ G &= E\left[\nabla_\theta g\right], && (3.15) \end{aligned}$$

and r_{1n} and r_{2n} are the residual terms converging in probability to zero (under appropriate conditions, see, for example, Van der Vaart (1998)). Solving this system of linear equations for $\sqrt{n}(\widehat{\theta} - \theta)$, we obtain the result for the asymptotic distribution of the constrained extremum estimator $\widehat{\theta}$ expressed in terms of the influence functions $\psi_f(\theta)$ and $\psi_g(\theta)$

$$\sqrt{n}\left(\widehat{\theta} - \theta\right) = H^{-1} \begin{bmatrix} bGG'H^{-1} - I_p \\ -bG \end{bmatrix}' \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{bmatrix} \psi_{\nabla fi} - \lambda \psi_{\nabla gi} \\ \psi_{gi} \end{bmatrix} + o_p(1), \tag{3.16}$$

where $b = (G'H^{-1}G)^{-1}$. Notice, that the Lagrange multiplier λ for a given optimal θ can be found from the first order condition, for example,

$$\lambda = (i'G)^{-1} i' E\left[\nabla_\theta f\right], \tag{3.17}$$

where i stands for a $p \times 1$ vector of ones.

Finally, for the case when the constraint and gradient functionals $E[g(\theta)]$, $E[\nabla_\theta f]$ and $E[\nabla_\theta g]$ do not involve a non-parametric estimation of population distribution functions, their influence functions can be found in a usual way, i.e., $\psi_g = g$, $\psi_{\nabla f} = \nabla_\theta f$ and $\psi_{\nabla g} = \nabla_\theta g$.

Suppose now that one wants to eliminate the estimation uncertainty from the constraint in (3.13). In this case the problem can be reformulated as

$$\min_{\theta \in \mathbb{R}^p} Ef(\theta) \ s.t. \ g(\theta) = 0.$$

It is straightforward to see that as a result all the constraint related terms in (3.16) disappear so that the limit distribution of the constrained extremum estimator is given by

$$\sqrt{n}\left(\widehat{\theta} - \theta\right) = H^{-1}\left(bGG'H^{-1} - I_p\right)\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi_{\nabla fi} + o_p(1).$$

3.B Limit distribution of portfolio weights

The Mean-CRR portfolio problem is obtained from the mean-risk problem (3.6) when a CRR measure (3.3) is used as an objective function

$$\min_{\theta \in \mathbb{R}^p} \int_0^1 s_\alpha(y + x'\theta)) d\phi(\alpha) \ s.t. \ E[y + x'\theta] = \nu.$$

This mean-CRR portfolio problem can be reformulated as an extremum estimation problem as discussed in Appendix 3.A, since a CRR measure can be expressed as an expectation. To simplify the exposition we use the notation $v = y + x'\theta$ for the portfolio return and F_v for its cumulative distribution function. Both v and F_v depend on the portfolio weights θ. First, we express the expected shortfall $s_\alpha(v)$ as an expectation

$$s_\alpha(v) = -\alpha^{-1} E\left[vI(F_v(v) \leq \alpha)\right].$$

Substituting this expression into equation (3.3) we obtain a CRR measure as an expectation

$$\begin{aligned}
\rho(v) &= -\int_0^1 \alpha^{-1} E\left[vI(F_v(v) \leq \alpha)\right] d\phi(\alpha) \\
&= -E\left[v \int_0^1 \alpha^{-1} I(F_v(v) \leq \alpha) d\phi(\alpha)\right] \\
&= -E\left[v \int_{F_v(v)}^1 \alpha^{-1} d\phi(\alpha)\right].
\end{aligned}$$

The mean-CRR optimal portfolio problem becomes

$$\min_\theta E\left[-v \int_{F_v(v)}^1 \alpha^{-1} d\phi(\alpha)\right] \;\; s.t. \; E\left[v\right] = \nu. \tag{3.18}$$

Problem (3.18) is a constrained extremum estimator problem, so the asymptotic results derived in Appendix 3.A apply. The asymptotic distribution of the mean-CRR portfolio weights can be expressed through the influence function $\xi(x, v)$ of the estimated portfolio weights[17]

$$\sqrt{n}\left(\widehat{\theta} - \theta\right) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi(x_i, v_i) + o_p(1) \rightarrow_d N\left(0, E[\xi\xi']\right),$$

where the index i identifies a particular observation in the sample. The influence function of the portfolio weights that ignores constraint estimation inaccuracy is

$$\xi(x_i, v_i) = H^{-1}\left(bGG'H^{-1} - I_p\right)\psi_{\nabla fi}. \tag{3.19}$$

[17]Notice, that we assumed the asset sample returns to be identically and independently distributed. Our results, however, can be straightforwardly extended to the case of stationary sample returns, see Newey and West (1987).

The influence function of the mean-CRR portfolio weights that takes into account the estimation inaccuracy in asset expected returns is

$$\xi(x_i, v_i) = H^{-1} \begin{bmatrix} bGG'H^{-1} - I_p \\ -bG \end{bmatrix}' \begin{bmatrix} \psi_{\nabla fi} - \lambda\psi_{\nabla gi} \\ \psi_{gi} \end{bmatrix}. \tag{3.20}$$

The vector G is the gradient of the constraint function with respect to portfolio weights $G = E[x]$, and λ is the Lagrange multiplier

$$\lambda = -e'\frac{\partial}{\partial\theta} E\left[v\int_{F_v(v)}^{1} \alpha^{-1} d\phi(\alpha)\right] E[e'x]^{-1},$$

matrix H is the Hessian of the objective function with respect to portfolio weights

$$H = -\frac{\partial^2}{\partial\theta\partial\theta'} E\left[v\int_{F_v(v)}^{1} \alpha^{-1} d\phi(\alpha)\right],$$

functions $\psi_{\nabla f}$ and $\psi_{\nabla g}$ are the influence functions of the the objective and constraint function gradient functionals correspondingly, function ψ_g is the influence function of the constraint functional, and scalar b is the notation

$$b = \left(G'H^{-1}G\right)^{-1}.$$

The exact expressions for the Lagrange multiplier λ, the Hessian H, and the influence function $\psi_{\nabla f}$ in case of a mean-CRR portfolio are as follows

$$\begin{aligned}
\lambda &= -E\left[e'x\int_{F_v(v)}^{1} \alpha^{-1} d\phi(\alpha)\right] E[e'x]^{-1}, \\
H &= E\left[\frac{\phi'(F_v(v))f(v)}{F(v)}\mathrm{Cov}(x|v)\right], \\
\psi_{\nabla f} &= \chi_{\nabla f} - E\left[\chi_{\nabla f}\right], \\
\chi_{\nabla f} &= -\int_{F_v(v)}^{1} \left(x - E\left[x|v = F_v^{-1}(\alpha)\right]\right)\alpha^{-1} d\phi(\alpha).
\end{aligned}$$

The derivation details can be found in chapter 2. Finally, the influence functions $\psi_{\nabla g}$ and ψ_g are

$$\begin{aligned}
\psi_{\nabla g} &= x - E[x], \\
\psi_g &= v - \nu.
\end{aligned}$$

3.B.1 Expected shortfall

In the case of expected shortfall the probability function $\phi(\alpha)$ is

$$\phi(\alpha) = I(\alpha \geq \tau),$$

so that the influence function of the mean-expected shortfall portfolio weights is given by (3.19) or (3.20) with

$$\begin{aligned}
\lambda &= -\tau^{-1} E\left[e'xI(F_v(v) \leq \tau)\right] E[e'x]^{-1}, \\
H &= \tau^{-1} f(F_v^{-1}(\tau)) \mathrm{Cov}(x|v = F_v^{-1}(\tau)), \\
\chi_{\nabla f} &= \tau^{-1} I(F_v(v) \leq \tau) \left(x - E[x|v = F_v^{-1}(\tau)]\right).
\end{aligned}$$

3.B.2 Point mass approximation (PMA) of a CRR measure

In the case of PMA CRR measure the probability function $\phi(\alpha)$ is a stepwise function

$$\phi(\alpha) = \sum_{k=1}^{m} \phi_k I(\alpha \geq \tau_k),$$

so that the influence function of the mean-expected shortfall portfolio weights is given by (3.19), if one wants to ignore the estimation error in the asset expected returns, or (3.20), if one wants to take into account the estimation constraint uncertainty, with

$$\begin{aligned}
\lambda &= -\sum_{k=1}^{m} \phi_k \tau_k^{-1} E\left[e'xI(F_v(v) \leq \tau_k)\right] E[e'x]^{-1}, \\
H &= \sum_{k=1}^{m} \phi_k \tau_k^{-1} f(F_v^{-1}(\tau_k)) \mathrm{Cov}(x|v = F_v^{-1}(\tau_k)), \\
\chi_{\nabla f} &= -\sum_{k=1}^{m} \phi_k \tau_k^{-1} I(F_v(v) \leq \tau_k) \left(x - E[x|v = F_v^{-1}(\tau_k)]\right).
\end{aligned}$$

3.B.3 Mean-variance portfolio weights

Using the same notations as in (3.6) we write the mean-variance portfolio problem

$$\min_{\theta} E\left[(y+x'\theta)^2\right] \;\; s.t. \; E\left[y+x'\theta\right]=\nu.$$

This problem can also be viewed as a constrained extremum estimator problem, so, again, the limit distribution results of the Appendix 3.A apply. The influence function of the mean-variance portfolio weights is given by expression (3.19) or (3.20) with the Lagrange multiplier λ given by

$$\lambda \;\; = \;\; e'\frac{\partial}{\partial\theta}E\left[(y+x'\theta)\right]E[e'x]^{-1}=E\left[(y+x'\theta)e'x\right]E[e'x]^{-1},$$

and the Hessian H of the objective function given by

$$H \;\; = \;\; \frac{\partial^2}{\partial\theta\partial\theta'}E\left[(y+x'\theta)\right]=E\left[xx'\right].$$

Finally, the influence functions of the gradient and constraint functionals are

$$\begin{aligned} \psi_{\nabla f} &= (y+x'\theta)x-E\left[(y+x'\theta)x\right], \\ \psi_{\nabla g} &= x-E[x], \\ \psi_{g} &= y+x'\theta-\nu. \end{aligned}$$

3.C Statistical comparison of portfolio weights

Let β be the vector of mean-variance portfolio weights, and θ be the vector of mean-CRR portfolio weights. Denote by $\eta(x,v)$ the influence function of the mean-variance portfolio weights, and by $\xi(x,v)$ the influence function of the mean-CRR portfolio weighs. The exact expressions for these influence functions are provided in Appendix 3.B. The joint asymptotic distribution of the mean-variance and the mean-CRR weights is

$$\sqrt{n}\left(\widehat{\gamma}-\gamma\right) \equiv \sqrt{n}\begin{pmatrix}\widehat{\beta}-\beta\\ \widehat{\theta}-\theta\end{pmatrix} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix}\eta(x_i,v_i)\\ \xi(x_i,v_i)\end{pmatrix} + o_p(1) \rightarrow_d N\left(0,\Omega\right),$$

where

$$\Omega = E\begin{bmatrix}\eta\eta' & \eta\xi'\\ \xi\eta' & \xi\xi'\end{bmatrix}.$$

The hypothesis $H_0 : \beta = \theta$ vs. $H_1 : \beta \neq \theta$ can be tested in a standard way. Introduce the restriction matrix $R = [I_p, -I_p]$, then

$$\widehat{\gamma}'R'\left(R\widehat{\Omega}R'\right)^{-1}R\widehat{\gamma} \rightarrow_d \chi^2_p$$

3.D Tables and figures

Portfolios	Assets	N Obs	Avg. Return	Skewness	Kurtosis	Covariance		
NORM	Asset 1	3000	0.06	-0.04	3.09	0.04	0.00	0.00
	Asset 2		0.08	0.00	3.00	*	0.04	0.00
	Asset 3		0.08	-0.01	2.99	*	*	0.04
t	Asset 1	3000	0.06	0.22	5.40	0.04	0.00	0.00
	Asset 2		0.08	-0.12	6.45	*	0.04	0.00
	Asset 3		0.08	-0.13	5.10	*	*	0.04
ASYM	Asset A	3000	0.06	-0.04	3.01	0.04	0.00	0.00
	Asset B		0.08	0.66	3.16	*	0.04	0.00
	Asset C		0.08	-0.79	4.03	*	*	0.04

Tabel 3.1: Sample statistics of simulated asset returns. NORM - returns from the three-variate normal distribution, t - returns from the three-variate t-distribution, ASYM - returns from the three-variate asymmetric distribution.

		Expected Portfolio Return			
Portfolios	Probability Threshold	10%	12%	14%	16%
NORM	2.5%	76.0%	86.8%	81.7%	87.1%
	5%	63.6%	63.5%	56.1%	74.7%
	7.5%	39.7%	50.2%	63.0%	64.3%
	10%	54.4%	42.4%	47.0%	41.1%
	12.5%	60.6%	68.1%	68.1%	56.0%
t	2.5%	94.4%	87.6%	89.0%	90.4%
	5%	80.2%	95.1%	96.1%	96.8%
	7.5%	67.7%	82.5%	95.7%	95.2%
	10%	99.3%	92.4%	96.4%	94.2%
	12.5%	98.5%	92.6%	88.0%	99.4%
ASYM	2.5%	0.0%	0.0%	0.0%	0.1%
	5%	0.0%	1.3%	3.5%	7.8%
	7.5%	0.0%	0.9%	5.3%	7.1%
	10%	0.0%	0.9%	6.3%	9.9%
	12.5%	0.0%	3.9%	13.3%	22.5%

Tabel 3.2: p-values of the test for equality of the mean-variance and the mean-shortfall portfolio weights in portfolios of simulated returns.

	Expected Portfolio Return			
Portfolios	10%	12%	14%	16%
NORM	86.1%	68.3%	60.8%	63.6%
t	85.5%	85.0%	92.9%	93.8%
ASYM	0.0%	0.1%	0.9%	2.3%

Tabel 3.3: p-values of the test for equality of the mean-variance and the mean-PMA CRR portfolio weights in portfolios of simulated returns. The probability thresholds for the PMA CRR measure are 2.5%, 5%, 7.5%, 10%, and 12.5%.

		Expected Portfolio Return			
Portfolios	Probability Threshold	10%	12%	14%	16%
NORM	2.5%	0.03%	0.01%	0.01%	0.03%
		0.08	*0.10*	*0.19*	*0.16*
	5%	0.05%	0.10%	0.09%	0.08%
		0.09	*0.19*	*0.35*	*0.28*
	7.5%	0.04%	0.10%	0.21%	0.32%
		0.12	*0.21*	*0.24*	*0.31*
	10%	0.03%	0.11%	0.13%	0.14%
		0.06	*0.16*	*0.24*	*0.35*
	12.5%	0.01%	0.02%	0.05%	0.11%
		0.04	*0.07*	*0.10*	*0.18*
t	2.5%	0.01%	0.29%	0.32%	0.30%
		0.14	*0.53*	*0.83*	*0.99*
	5%	0.04%	0.02%	0.02%	0.01%
		0.11	*0.06*	*0.07*	*0.06*
	7.5%	0.01%	0.01%	0.00%	0.01%
		0.12	*0.13*	*0.05*	*0.09*
	10%	0.00%	0.00%	0.00%	0.00%
		0.00	*0.04*	*0.03*	*0.05*
	12.5%	0.00%	0.00%	0.01%	0.00%
		0.00	*0.02*	*0.06*	*0.00*
ASYM	2.5%	3.88%	4.62%	5.48%	6.46%
		1.10	*1.50*	*2.04*	*2.45*
	5%	2.14%	2.15%	2.34%	2.59%
		0.63	*0.84*	*0.99*	*1.10*
	7.5%	1.52%	1.13%	1.16%	1.32%
		0.42	*0.58*	*0.74*	*0.93*
	10%	1.16%	0.90%	0.77%	0.72%
		0.35	*0.45*	*0.49*	*0.60*
	12.5%	0.88%	0.67%	0.53%	0.48%
		0.33	*0.33*	*0.37*	*0.39*

Tabel 3.4: Economic size of the difference between the mean-shortfall and mean-variance simulated efficient portfolios. The effect is measured as a decrease in the expected shortfall when switching from the standard deviation to the expected shortfall risk measure in portfolio optimization. The standard errors are given in italics.

Portfolios	Assets	N Obs	Avg. Return	Skewness	Kurtosis	Covariance					
ER	BP	3913	0.34%	-0.21	6.61	0.009	0.001	0.007	0.004		
	CAN		-0.70%	0.00	5.39	*	0.003	0.001	0.000		
	DM		1.03%	0.03	4.72	*	*	0.012	0.005		
	JAP		1.15%	0.77	10.70	*	*	*	0.013		
Fama- French	Small/Low	10448	9.6%	-0.67	11.66	0.028	0.019	0.018	0.022	0.017	0.017
	Small/Med		14.9%	-0.86	13.78	*	0.015	0.014	0.015	0.013	0.013
	Small/High		16.9%	-0.88	14.75	*	*	0.014	0.014	0.012	0.013
	Big/Low		10.7%	-0.47	17.25	*	*	*	0.026	0.019	0.018
	Big/Med		11.8%	-1.10	31.06	*	*	*	*	0.018	0.016
	Big/High		13.6%	-0.89	24.21	*	*	*	*	*	0.018
Index	S&P 500	4797	8.6%	-2.08	46.41	0.030	0.022	0.001			
	Small Caps		7.6%	-0.94	16.61	*	0.025	0.000			
	Gov. Bonds		2.6%	-0.04	7.48	*	*	0.002			

Tabel 3.5: Sample statistics for market returns. ER - exchange rates, Fama-French - returns on the Fama-French portfolios, Index - returns on market indexes.

		Expected Portfolio Return			
Portfolios	Probability Threshold	10%	12%	14%	16%
ER	2.5%	39.8%	29.7%	28.6%	25.2%
	5%	50.1%	55.9%	56.3%	50.9%
	7.5%	39.2%	41.4%	45.8%	48.3%
	10%	16.2%	14.8%	15.3%	13.9%
	12.5%	28.7%	22.7%	28.0%	24.4%
Fama-French	2.5%	0.0%	0.0%	0.0%	0.0%
	5%	7.1%	6.3%	10.3%	5.1%
	7.5%	60.1%	87.3%	64.2%	26.2%
	10%	53.0%	51.3%	69.6%	61.1%
	12.5%	46.1%	88.2%	80.1%	52.6%
Index	2.5%	67.5%	71.2%	80.4%	63.2%
	5%	0.6%	2.0%	2.0%	1.4%
	7.5%	6.7%	5.3%	4.0%	3.5%
	10%	4.7%	5.2%	5.9%	5.5%
	12.5%	5.0%	5.2%	4.1%	4.5%

Tabel 3.6: p-values of the test for equality of the mean-variance and mean-shortfall portfolio weights in portfolios of market returns.

Comparison of Portfolio Weights					
		M-ShF	M-V	Diff	Std. Err.
FF, 2.5%	Big/Med	1.55	1.36	0.20	0.055
	Big/High	1.13	1.28	-0.16	0.050
	Small/Low	0.45	0.27	0.19	0.074
	Small/Med	-0.19	-0.17	-0.02	0.064
	Small/High	-0.22	-0.18	-0.03	0.045
FF, 7.5%	Big/Med	1.38	1.36	0.02	0.052
	Big/High	1.24	1.28	-0.04	0.040
	Small/Low	0.32	0.27	0.05	0.035
	Small/Med	-0.14	-0.17	0.03	0.037
	Small/High	-0.17	-0.18	0.02	0.028
Index, 2.5%	Small Caps	0.47	0.52	-0.05	0.114
	Gov. Bonds	-0.23	-0.24	0.01	0.015
Index, 5%	Small Caps	0.29	0.52	-0.24	0.086
	Gov. Bonds	-0.21	-0.24	0.03	0.011

Tabel 3.7: Effect of the probability threshold on the difference between mean-expected shortfall and mean-variance portfolio weights. Portfolio weights are reported for the required expected portfolio return of 10%. Portfolio names and probability thresholds are given in the left column.

	Expected Portfolio Return			
Portfolios	10%	12%	14%	16%
ER	25.6%	21.8%	22.3%	28.6%
Fama-French	2.1%	5.1%	13.2%	4.3%
Index	4.2%	4.9%	4.5%	4.5%

Tabel 3.8: p-values of the test for equality of the mean-variance and mean-shortfall portfolio weights in portfolios of market returns. The probability thresholds for the PMA CRR measure are 2.5%, 5%, 7.5%, 10%, and 12.5%.

		Expected Portfolio Return			
Portfolios	Probability Threshold	10%	12%	14%	16%
ER	2.5%	5.83%	7.16%	8.23%	9.36%
		11.54	*14.33*	*16.69*	*19.76*
	5%	2.60%	3.53%	4.51%	5.44%
		4.39	*5.10*	*6.02*	*7.08*
	7.5%	2.63%	3.12%	3.83%	4.58%
		4.46	*5.21*	*5.89*	*6.60*
	10%	4.19%	5.07%	5.89%	6.54%
		3.88	*4.83*	*5.80*	*6.71*
	12.5%	2.40%	2.79%	3.21%	3.71%
		2.48	*3.42*	*3.81*	*4.39*
Fama-French	2.5%	2.74%	0.96%	0.97%	0.88%
		1.41	*0.87*	*0.80*	*1.05*
	5%	0.67%	0.26%	0.28%	0.62%
		0.55	*0.35*	*0.35*	*0.54*
	7.5%	0.30%	0.22%	0.31%	0.25%
		0.32	*0.28*	*0.31*	*0.26*
	10%	0.16%	0.17%	0.14%	0.23%
		0.20	*0.20*	*0.17*	*0.28*
	12.5%	0.16%	0.07%	0.11%	0.21%
		0.15	*0.11*	*0.17*	*0.25*
Index	2.5%	0.40%	0.17%	0.06%	0.22%
		1.13	*1.15*	*0.90*	*2.01*
	5%	3.78%	4.74%	5.47%	5.99%
		3.23	*3.19*	*3.81*	*4.85*
	7.5%	2.35%	2.95%	3.91%	4.48%
		1.94	*2.62*	*3.04*	*3.48*
	10%	1.91%	2.30%	2.75%	3.12%
		1.44	*1.71*	*1.97*	*2.33*
	12.5%	1.46%	1.90%	2.35%	2.79%
		1.17	*1.43*	*1.78*	*2.03*

Tabel 3.9: Economic size of the difference between the mean-shortfall and mean-variance market efficient portfolios. The effect is measured as a decrease in the expected shortfall when switching from the standard deviation to the expected shortfall risk measure in portfolio optimization. The standard errors are given in italics.

	M-V	Mean-Expected Shortfall 2.5%	7.5%	12.5%	M-PMA
	Simulated NORM Returns vs. R1				
R_2	0.00%	0.00%	0.00%	0.00%	0.00%
R_3	0.00%	0.00%	0.00%	0.00%	0.00%
	Simulated t Returns vs. R1				
R_2	0.00%	0.00%	0.00%	0.00%	0.00%
R_3	0.00%	0.00%	0.00%	0.00%	0.00%
	Simulated ASYM Returns vs. R1				
R_2	0.00%	0.00%	0.00%	0.00%	0.00%
R_3	0.00%	0.00%	0.00%	0.00%	0.00%

Tabel 3.10: p-values of the spanning tests for simulated returns with respect to the asset R_1. The reported results are for the mean-variance (M-V), mean-expected shortfall, and mean-PMA CRR (M-PMA) spanning. The PMA probability thresholds are 2.5%, 5%, 7.5%, 10%, and 12.5% with equal weights of 20%.

	M-V	Mean-Expected Shortfall 2.5%	7.5%	12.5%	M-PMA
	Simulated NORM Returns vs. MV				
R_2	100.00%	63.02%	65.61%	56.34%	68.99%
R_3	100.00%	47.48%	99.94%	59.33%	76.13%
	Simulated t Returns vs. MV				
R_2	100.00%	76.39%	94.53%	97.04%	86.91%
R_3	100.00%	74.38%	75.91%	58.06%	64.14%
	Simulated ASYM Returns vs. MV				
R_2	100.00%	0.00%	0.00%	0.00%	0.00%
R_3	100.00%	0.00%	0.00%	0.00%	0.00%

Tabel 3.11: p-values of the spanning tests for simulated returns with respect to the optimal mean-variance portfolio. The reported results are for the mean-variance (M-V), mean-expected shortfall, and mean-PMA CRR (M-PMA) spanning. The PMA probability thresholds are 2.5%, 5%, 7.5%, 10%, and 12.5% with equal weights of 20%.

Portfolios	Assets	N Obs	Avg. Return	Skewness	Kurtosis	Volatility
Fama- French	Mkt	10448	11.13%	-0.75	21.17	14.2%
	Small/Low		9.59%	-0.67	11.66	16.8%
	Small/Med		14.91%	-0.86	13.78	12.1%
	Small/High		16.89%	-0.88	14.75	11.7%
	Big/Low		10.74%	-0.47	17.25	16.1%
	Big/Med		11.77%	-1.10	31.06	13.5%
	Big/High		13.63%	-0.89	24.21	13.6%
S&P 500 Ind.	SP	2609	9.5%	-0.01	6.13	18.0%
	COD		10.6%	0.00	7.84	20.8%
	CST		8.8%	-0.09	9.09	16.4%
	ENE		12.0%	0.09	5.27	21.8%
	FIN		14.0%	0.20	5.82	23.4%
	HCR		13.5%	-0.06	6.42	20.7%
	IND		10.6%	-0.10	7.02	19.7%
	INT		14.1%	0.37	6.55	34.4%
	MAT		6.4%	0.22	6.16	21.3%
	TEL		4.1%	0.05	6.27	24.4%
	UTL		4.8%	-0.22	10.03	18.3%

Tabel 3.12: Annualized sample statistics of the market returns used for spanning tests. Fama-French are the returns on the Fama-French size/book-to-market portfolios with MKT being the market portfolio. S&P500 Ind. are returns on the S&P 500 industrial indexes. GICS sectors: consumer discretionary (COD), consumer staples (CST), energy (ENE), financials (FIN), health care (HCR), industrials (IND), information technology (INT), materials (MAT), telecommunications services (TEL), and utilities (UTL). SP is the S&P 500 composite index.

	M-V	**Mean-Expected Shortfall**			**M-PMA**
		2.5%	**7.5%**	**12.5%**	
	Fama-French Size/Book-to-Mkt. Portfolios vs. MKT				
Small/Low	20.42%	10.61%	10.36%	10.69%	10.03%
Small/Medium	0.00%	0.00%	0.00%	0.00%	0.00%
Small/High	0.00%	0.00%	0.00%	0.00%	0.00%
Big/Low	4.91%	13.32%	8.79%	7.50%	10.16%
Big/Medium	8.45%	6.94%	6.44%	6.59%	6.59%
Big/High	0.07%	0.06%	0.06%	0.07%	0.06%
	S&P 500 Sector Indexes vs. S&P 500 Composite				
COD	74.67%	78.19%	74.84%	75.24%	76.02%
CST	68.49%	72.71%	66.66%	64.80%	67.54%
ENE	43.00%	45.04%	45.79%	44.56%	44.73%
FIN	30.68%	31.10%	27.89%	27.37%	28.70%
HCR	27.04%	26.87%	28.53%	28.93%	28.18%
IND	66.05%	80.24%	68.21%	65.47%	71.24%
INT	78.81%	69.60%	75.60%	79.44%	74.74%
MAT	70.15%	62.75%	68.82%	71.04%	67.87%
TEL	36.49%	33.41%	36.05%	35.66%	35.28%
UTL	70.17%	59.93%	64.10%	67.58%	63.91%

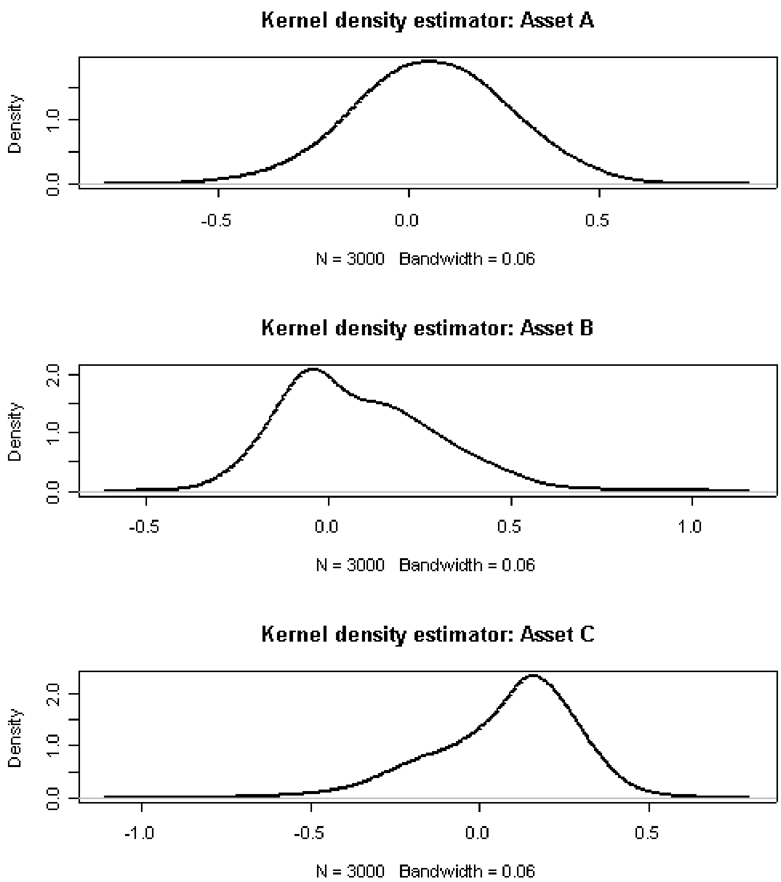
Tabel 3.13: p-values of the spanning tests for the Fama-French size/book-to-market portfolios with respect to the market portfolio and S&P 500 sector indexes with respect to the S&P 500 composite index. The reported results are for the mean-variance (M-V), mean-expected shortfall, and mean-PMA CRR (M-PMA) spanning. The PMA probability thresholds are 2.5%, 5%, 7.5%, 10%, and 12.5% with equal weights of 20%. GICS sectors: consumer discretionary (COD), consumer staples (CST), energy (ENE), financials (FIN), health care (HCR), industrials (IND), information technology (INT), materials (MAT), telecommunications services (TEL), and utilities (UTL).

	M-V	**Mean-Expected Shortfall**			**M-PMA**
		2.5%	**7.5%**	**12.5%**	
	Fama-French Size/Book-to-Mkt. Portfolios vs. MV				
Small/Low	99.35%	41.53%	66.74%	73.00%	57.98%
Small/Medium	99.40%	42.92%	68.67%	74.80%	59.54%
Small/High	99.35%	34.99%	66.35%	74.39%	55.39%
Big/Low	99.62%	89.45%	95.54%	93.35%	94.40%
Big/Medium	99.68%	61.39%	94.41%	96.74%	83.82%
Big/High	99.82%	67.45%	93.98%	99.66%	86.65%
	S&P 500 Sector Indexes vs. MV				
COD	99.99%	93.59%	95.06%	91.27%	93.97%
CST	100.00%	92.04%	88.98%	94.99%	92.67%
ENE	99.98%	90.40%	96.18%	96.86%	95.36%
FIN	100.00%	79.76%	85.73%	89.68%	85.87%
HCR	99.98%	97.23%	96.33%	95.22%	98.60%
IND	100.00%	87.71%	93.34%	93.05%	92.48%
INT	100.00%	97.49%	99.65%	97.21%	98.00%
MAT	99.98%	86.69%	81.94%	91.49%	86.92%
TEL	99.99%	81.44%	90.92%	90.35%	88.20%
UTL	99.98%	94.35%	91.95%	99.66%	94.02%

Tabel 3.14: p-values of the spanning tests for the Fama-French size-book-to-market portfolios and S&P 500 sector indexes with respect to the optimal mean-variance portfolio. The reported results are for the mean-variance (M-V), mean-expected shortfall, and mean-PMA CRR (M-PMA) spanning. The PMA probability thresholds are 2.5%, 5%, 7.5%, 10%, and 12.5% with equal weights of 20%. GICS sectors: consumer discretionary (COD), consumer staples (CST), energy (ENE), financials (FIN), health care (HCR), industrials (IND), information technology (INT), materials (MAT), telecommunications services (TEL), and utilities (UTL).

		Expected Portfolio Return			
Portfolios	Probability Threshold	10%	12%	14%	16%
ASYM	2.5%	2.4%	8.6%	8.9%	13.2%
	5%	10.5%	41.1%	50.9%	58.5%
	7.5%	7.7%	38.4%	56.4%	60.4%
	10%	9.2%	43.6%	60.9%	66.5%
	12.5%	16.1%	59.8%	71.2%	77.1%
Fama-French	2.5%	21.3%	93.1%	95.9%	91.4%
	5%	84.5%	99.7%	99.7%	98.8%
	7.5%	94.4%	99.8%	99.9%	99.9%
	10%	94.6%	99.7%	99.9%	99.8%
	12.5%	95.8%	99.9%	99.9%	99.9%
Index	2.5%	94.7%	95.1%	96.5%	93.8%
	5%	78.5%	79.7%	81.2%	82.5%
	7.5%	87.3%	87.3%	85.6%	85.2%
	10%	81.5%	80.8%	81.0%	81.3%
	12.5%	73.0%	74.1%	73.1%	72.6%

Tabel 3.15: p-values of the test for equality of the mean-variance and mean-shortfall portfolio weights for ASYM, Fama-French and, Index portfolios with inaccuracy in the mean returns taken into account.



Figuur 3.1: Kernel density of the returns simulated from ASYM distribution.

Hoofdstuk 4

Option Pricing and Dynamics of the Implied Prices of Volatility Risk: An Empirical Analysis

4.1 Introduction

It is generally agreed upon that empirically relevant probabilistic models for asset prices usually describe markets that are incomplete in terms of the underlying assets. For example, stochastic volatility, used in many financial models, usually leads to market incompleteness[1]. This means that derivatives on the underlying asset cannot be priced by no-arbitrage arguments alone, or, in other words, that the risk-neutral probability measure is not unique[2]. In a stochastic volatility model, the risk-neutral probability measure can be uniquely specified by the volatility risk premium. This is a term which determines Girsanov's transformation with respect to the innovations

[1]There are some exceptions. For example, models studied by Kallsen and Taqqu (1998) or Hobson and Rogers (1999). A model which allows stochastic volatility is also complete if we assume the existence of a tradable portfolio perfectly correlated with stochastic volatility.

[2]When determined in terms of the underlying asset.

in stochastic volatility that cannot be hedged by innovations in the asset price. In an equilibrium, including in the market both the underlying and the derivatives, the volatility risk premium can be fully endogenised. Alternatively, without a full equilibrium, but imposing absence of arbitrage opportunities, we can use derivative prices observed in the market to get additional information about this volatility risk premium, which then reflects the attitude of the market towards risk and allows one to characterize the pricing mechanism. In this paper, we define the volatility risk premium as a product of the price of volatility risk and the instantaneous volatility itself. This specification is also used, for example, by Heston (1993), Chernov (2003), and Jiang and Knight (2002). The price of volatility risk can then be estimated on a daily basis from observed option prices. We estimate it by minimizing an appropriate distance between observed and theoretical European call option prices. The resulting process of implied prices of volatility risk shows substantial variability. This chapter analyzes its statistical properties and propose several specifications to model its dynamics. We demonstrate that taking into account the dynamics of the implied prices of volatility risk significantly improves out-of-sample prediction of option prices with respect to common approaches used in the literature.

The problem of finding the empirically relevant volatility risk premium in a stochastic volatility model, or, equivalently, of finding the pricing kernel or the risk-neutral probability measure, is extensively studied in the literature. Hull and White (1987) assume idiosyncratic volatility risk, or, in other words, a zero volatility risk premium to overcome incompleteness. Melino and Turnbull (1990) study the pricing of currency exchange options under stochastic volatility. They discovered that stochastic volatility models allowing for a non-zero volatility risk premium describe option prices better than the models with idiosyncratic volatility risk.

In general, an appropriate volatility risk premium is an empirical question. There are two approaches to the empirical analysis of market incompleteness. One is semi- or nonparametric in the sense that no or very few restrictions are put on the risk-neutral probability measure. This approach is followed, for example, by Rosenberg and Engle (2002) who estimate empirical pricing kernels from the option and S&P500 index data nonparametrically. Further, they study the dynamics of the risk-aversion implied by the estimated pricing kernels. Ait-Sahalia and Lo (1998) use a nonparametric kernel density estimator to obtain the relation between derivatives' prices and accompanying characteristics, like the prevailing price of the underlying. Finally, Poteshman (1998) considers a nonparametric specification of a continuous time stochastic volatility model for the daily S&P500 index. Option data are used to get nonparametric estimates of the volatility drift, the volatility diffusion coefficient, and the volatility risk premium as a function of the current level of the volatility. An advantage of the semi- or nonparametric approach is that the data are fitted very well. A disadvantage is that the estimates are less efficient and the resulting model may have insufficient prediction power. Moreover, such an approach is less suited when studying dynamic properties of volatility risk premiums due to the curse of dimensionality.

An alternative can be provided by a parametric approach. In this case, the risk-neutral probability measure is specified parametrically, so that the resulting estimates can be interpreted as parameter values in the model that minimize a certain "distance" between model and actual prices. An example of this approach is Chernov (2003). He uses a multi-factor stochastic volatility model to describe the dynamics of traded assets. He also directly specifies the volatility risk premium and recovers the pricing kernel implied by the model. Duan (1995) imposes the so-called Locally Risk Neutral Valuation Relationship (LRNVR) in a discrete time GARCH model to obtain the relevant risk-neutral probability measure. Kallsen and Taqqu (1998) show that the LRNVR is essentially equal to assuming a piece-wise constant likelihood ratio of the risk-neutral probability measure with respect to the physical one. Bakshi *et al.* (1997), and Heston (1993) directly specify the relevant risk-neutral probability measure.

Our approach is parametric in the sense that, following Cox *et al.* (1985) and Heston (1993), we specify the functional form for the volatility risk premium parametrically. In addition, we introduce extra flexibility to our parametric specification by using so called ïmplied"prices of volatility risk. The methodology for extracting implied prices of volatility risk is similar to the case of Black-Scholes implied volatilities. However, while the Black-Scholes implied volatility is option specific, our implied price of volatility risk is a market wide parameter. Hence, we estimate the empirically relevant prices of volatility risk on a daily basis and then study the statistical properties of the resulting time series. We use S&P500 index data from Jan 1, 1992 to Dec 31, 1998 and corresponding European call option data from Jan 1, 1992 to Aug 8, 1997. We find that the implied prices of volatility risk are non-constant, exhibit significant autocorrelation, and that appropriate modelling leads to significantly better prediction of future volatility risk premiums.

With respect to the existing literature, our approach is closest to Melino and Turnbull (1990). They estimate a continuous time stochastic volatility model for the Canada-US exchange market. They consider the pricing of foreign currency options imposing a non-zero but constant volatility risk premium. Their conclusion is that theoretical option prices are sensitive to the actual value of the risk premium and that an imposed non-zero risk premium does produce more accurate predictions of option prices. Melino and Turnbull (1990) try only several fixed values of the volatility risk premium. Instead, we propose to estimate the implied prices of volatility risk on a daily basis. We show that this produces even more accurate predictions of option prices. Guo (1998) also finds evidence of time varying risk premiums for the foreign exchange market. However, Guo (1998) considers the implied risk premiums only over annual and semiannual periods. This does not allow a thorough investigation of the short-run dynamic properties of the implied risk premiums. Jiang and van der Sluis (1999) consider not only stochastic volatility, but also stochastic interest rates. That paper analyzes pricing errors for options using previous day's implied price of volatility risk as a predictor of today's. The main conclusion of the paper is that allowing for stochastic interest rates hardly improves the results and that volatility risk is clearly not idiosyncratic.

The remainder of this chapter is organized as follows. Section 4.2 describes the estimation methodology and the relevant theoretical background. Section 4.3 discusses the data. In Section 4.4 we present the estimation results, analyze the dynamics of implied prices of volatility risk, and compare the out-of-sample performance of the selected dynamic model for the implied prices of volatility risk with other specifications. Section 5.5 concludes.

4.2 Methodology

In this section we first formulate the stochastic volatility model under the physical probability measure. To describe the dynamics of the S&P500 we use the Heston (1993) model, which belongs to the class of affine diffusion processes. The affine structure of the stochastic differential equations allows one, in principle, to get analytical solutions for transition probabilities. To price options one subsequently needs to obtain the model dynamics under the risk-neutral probability measure. In the stochastic volatility model this transformation depends on a specification of the volatility risk premium. The volatility risk premium fully describes the risk-neutral probability measure. Following Heston (1993), we specify it in such a way that, under the risk-neutral probability measure, the model still belongs to the class of affine-diffusion processes. This allows us to get closed-form solutions for the risk-neutral transition probabilities. However, in contrast with Heston (1993) we allow the parameter describing the volatility risk premium to be time-varying and study the benefits of such an approach for pricing derivatives.

The parameters of the stochastic volatility model under the risk-neutral probability measure can be divided into parameters identifiable from the dynamics of the underlying index value (these parameters also enter the stochastic volatility model under the physical probability distribution) and parameters which can only be identified from derivative prices (risk-preference parameters as far as they concern non-hedgeable innovations in the stochastic volatility). To be more precise, our model contains four parameters, identifiable from the observed index value process and one parameter, which reflects the market's incompleteness. Our methodology of estimating the parameters is twofold. First, we estimate the parameters of the stochastic volatility model from the observed time-series of S&P500 prices and filter instantaneous volatilities, using an E-GARCH specification. Subsequently, we estimate the implied volatility risk premium from observed option prices by minimizing an appropriate distance between theoretical and market option prices. The implied volatility risk premium is the only variable estimated from derivative prices.

4.2.1 Estimation of the stochastic volatility Model

In our work we use the specification of Jiang and Knight (2002) for the underlying index value process: **Lin**

$$\begin{aligned} d\ln S_t &= \mu dt + V_t^{1/2} dW_t^{(1)}, && (4.1)\\ dV_t &= \beta\left(\alpha - V_t\right) dt + \sigma V_t^{1/2} dW_t^{(2)}, && (4.2)\\ d\left\langle W^{(1)}, W^{(2)}\right\rangle_t &= \rho dt, \end{aligned}$$

where S_t is the index value at time t, $W_t^{(1)}$ and $W_t^{(2)}$ are two dependent Brownian motions with instantaneous correlation ρ, and $V_t^{1/2}$ is the stochastic instantaneous volatility at time t. The log-price and volatility processes are described by parameters $\mu, \alpha, \beta, \sigma$ and ρ. The coefficient μ determines the expected drift of the log-value of the underlying asset. The volatility process is mean-reverting. The parameter α, called the long-run volatility, represents the mean-reversion volatility level to which the volatility reverts. The parameter β, called the volatility mean reversion parameter, determines the speed or intensity of the volatility attraction to the mean-reverting level α. The parameter σ is a dispersion parameter of the volatility process, which determines how volatile the volatility process itself is. The parameter ρ is the correlation coefficient between the price and volatility processes. The magnitude of this parameter is responsible for the degree of skewness of the index return distribution. It also captures the often observed leverage effect. We assume that the instantaneous interest rate is constant over time, so that the dynamics of the money market account is given by

$$d \ln B_t = r dt. \tag{4.3}$$

Specifications similar to (4.1) – (4.2) are often used in empirical work when modelling the S&P500 index dynamics. For recent examples see also Chacko and Viceira (2003), Chernov and Ghysels (2000), and Chernov (2003). In principle, the specification of a dynamic model for an underlying asset should be able to capture empirical features of observable daily returns, such as skewness, excess kurtosis, and autocorrelation of squared returns. The stochastic volatility model (4.1) – (4.2) is able to do this to a great extent.[3] Chernov *et al.* (2003) recommend to use affine diffusion models for option pricing purposes, since this allows an analytical treatment of the pricing problem. We follow their advise, considering a single stochastic volatility factor model proposed by Heston (1993).

To estimate the model we use the General Method of Moments (GMM). We apply it to a system of unconditional moment restrictions on the index log-returns taken from Jiang and Knight (2002). Denote the index log-returns $R_t = \ln S_{t+1} - \ln S_t$. Then the moment conditions that we use are:

[3] Also extreme events can be taken into account by modelling jumps as in Pan (2002) or using CEV models as in Jones (2003).We, however, consider a relatively calm post crash period 1992-1998.

$$\begin{aligned}
E\left[R_t\right] &= \mu, \\
Var\left[R_t\right] &= \alpha, \\
E\left[\left(R_t-\mu\right)^3\right] &= \frac{3}{\beta^2}\left(e^{-\beta}+\beta-1\right)\alpha\rho\sigma, \\
E\left[\left(R_t-\mu\right)^4\right] &= 3\alpha^2+\phi_0. \\
Cov\left[\left(R_t-\mu\right)^2,\left(R_{t+\tau}-\mu\right)^2\right] &= \frac{1}{2\beta^3}e^{-(\tau+1)\beta}\left(e^{\beta}-1\right)\phi_1, \quad \tau \geq 1,
\end{aligned}$$

with

$$\begin{aligned}
\phi_0 &= \frac{3}{\beta^3}\left(e^{-\beta}+\beta-1+4\left(\left(2+\beta\right)e^{-\beta}+\beta-2\right)\rho^2\right)\alpha\sigma^2>0, \\
\phi_1 &= \left(e^{\beta}-1+4\rho^2\left(e^{\beta}-\beta-1\right)\right)\alpha\sigma^2>0.
\end{aligned}$$

The analytical expressions for these moments are derived using the joint characteristic function of returns, which is available in closed form for this affine diffusion process, see Jiang and Knight (2002). These particular moments were chosen due to their empirical relevance. The first two allow one to identify the expected index log-return μ and long-run volatility α. The third and the fourth one account for the excess skewness and kurtosis of the empirical log-return distribution. The final moment matches the empirical autocovariance pattern in squared returns. Note that these moments can be used to develop general intuition on how well the SV model can describe the empirical findings. For instance, it immediately follows from the expression of the third central moment that the sign of the correlation coefficient ρ determines the direction of the skewness of the log-return distribution, while the magnitudes of both σ and ρ affect the size of the effect. The fourth moment shows that the excess kurtosis of log-returns equals $\varphi_0/\alpha^2 > 0$. The autocovariance of squared log-returns is always positive. Jiang and Knight (2002) use these moments to estimate the stochastic volatility model from a shorter time-series of the S&P500 index values.

4.2.2 Filtering of the instantaneous volatilities.

One of the problems with SV models is that they contain the latent volatility, which needs to be filtered for the purposes of option pricing. There are several approaches to solve this problem in the literature. Gallant and Tauchen (1998) suggest a reprojection method for filtering conditional volatilities from the continuous stochastic volatility specification. This method requires Monte-Carlo simulation of the dynamics of the stochastic volatility process, using an Euler discretization scheme. Another, more direct method, is to filter the volatilities from a discrete time GARCH or E-GARCH specification. Nelson (1992) shows that this filter is consistent, which means that, as the discretization step goes to zero, the filtered volatility process converges to the true one under mild assumptions. An advantage of this method is that it allows one to obtain consistent estimates of the volatility even in the case when the model for the asset price dynamics is misspecified. In other words, the GARCH or E-GARCH filter gives general consistency and robustness. This happens because of the continuity of the volatility process, see Nelson (1992) for the details. A disadvantage of the method is that the estimated volatilities might be somewhat less efficient than in case of the reprojection method. However, since the reprojection method involves the choice and estimation of an auxiliary model as well as simulations, the efficiency of the reprojection method is not straightforward in an empirical application.

In this paper we filter instantaneous volatilities from the discrete E-GARCH specification of the stochastic volatility model. E-GARCH is chosen because, similar to the continuous-time stochastic volatility model as specified by (4.1) and (4.2), the E-GARCH allows for skewness, excess kurtosis, and a leverage effect.[4]

[4] In principle, GARCH specification also provides a consistent filter for instantaneous volatilities. When comparing E-GARCH and GARCH, we find that the empirical results are similar.

4.2.3 Stochastic volatility option pricing

To price derivatives in the model (4.1) – (4.2) one needs to derive the asset price dynamics under the risk-neutral probability distribution. As is well known, the derivative price can then be expressed as an expectation of the normalized future payoff

$$\frac{C_t}{N_t} = E_t^Q \left[\frac{C_T}{N_T} \right]$$

where C_T is the future payoff of the derivative, C_t is the derivative price, N_t is the numeraire at time t, and Q indicates taking expectations under the risk-neutral probability measure. For the stochastic volatility model (4.1)-(4.2), the change of the probability measure can be characterized by the Radon-Nikodym derivative, following from Girsanov's theorem:

$$\left(\frac{dQ}{dP} \right)_T = \exp \left(-\frac{1}{2} \int_0^T \left(\eta_t^2 + \lambda_t^2 \right) dt - \int_0^T \eta_t dW_t^{(1)} - \int_0^T \lambda_t dW_t^{(2)} \right)$$

where η_t and λ_t are the drift transformations corresponding to $W^{(1)}$ and $W^{(2)}$ respectively. We assume that these transformations are functions of the state variables S, V, and time, i.e., $\eta_t = \eta \left(S_t, V_t, t \right)$ and $\lambda_t = \lambda \left(S_t, V_t, t \right)$. They enter Girsanov's equation and determine the return and volatility risk premiums. In vector differential form the Girsanov transformation of the Brownian motions $W^{(1)}$ and $W^{(2)}$ is the following:

$$\begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \end{bmatrix} = \begin{bmatrix} \eta \left(S_t, V_t, t \right) \\ \lambda \left(S_t, V_t, t \right) \end{bmatrix} dt + \begin{bmatrix} d\widetilde{W}_t^{(1)} \\ d\widetilde{W}_t^{(2)} \end{bmatrix}, \tag{4.4}$$

here $\eta \left(S_t, V_t, t \right)$ is the index value risk premium and $\lambda \left(S_t, V_t, t \right)$ is the volatility risk premium. The index value risk premium $\eta \left(S_t, V_t, t \right) = \left[\mu - r + d \right] / \sqrt{V_t}$, with d standing for the continuous dividend pay-out rate, is fixed by the no-arbitrage argument due to the fact that the index is a tradable asset. On the contrary, the volatility risk premium $\lambda(S_t, V_t, t)$ cannot be fixed by no-arbitrage arguments alone, which reflects the market incompleteness. Each possible choice of the volatility risk premium, satisfying appropriate integrability conditions, excludes arbitrage opportunities.

Following Heston (1993) and Cox *et al.* (1985) we make the assumption that the volatility risk-premium is proportional to the instantaneous volatility

$$\lambda(t, S_t, V_t) = \lambda \frac{1}{\sigma} V_t^{1/2}.$$

We call the coefficient of proportionality λ *the price of volatility risk.* It is the central object of our study, characterizing the derivative pricing mechanism in the incomplete market. Using this assumption and applying Girsanov's transformation to (4.1) – (4.2), we find

$$d \ln S_t = \left(r - d - \frac{1}{2} V_t \right) dt + V_t^{1/2} d\widetilde{W}_t^{(1)}, \tag{4.5}$$

$$dV_t = (\beta + \lambda) \left[\frac{\alpha\beta}{\beta + \lambda} - V_t \right] dt + \sigma V_t^{1/2} d\widetilde{W}_t^{(2)}, \tag{4.6}$$

$$d \left\langle \widetilde{W}^{(1)}, \widetilde{W}^{(2)} \right\rangle_t = \rho dt, \tag{4.7}$$

where $\widetilde{W}^{(1)}$ and $\widetilde{W}^{(2)}$ are risk-neutral Brownian motions.

Heston (1993) shows that the price for a European call option with the exercise price K and time to maturity τ can, in general, be written in the following form

$$C_t = e^{-d\tau} S_t Q_S \{S_T \geq K\}_t - e^{-r\tau} K Q_B \{S_T \geq K\}_t, \tag{4.8}$$

where $Q_S \{S_T \geq K\}$ is the risk-neutral conditional probability that the option expires in the money, with the index value S_t as a numeraire, and $Q_B \{S_T \geq K\}$ is the risk-neutral conditional probability that the option expires in the money, with the money market account B_t as a numeraire.

Generally, closed-form solutions for these conditional risk-neutral probabilities are not available. Duffie *et al.* (2000) demonstrate that, in an affine jump-diffusion model, the solution for the conditional characteristic function of S_t is available in analytical form. The conditional characteristic function for our stochastic volatility model are derived in Heston (1993). The probabilities Q_S and Q_B are then the inverse Fourier transforms of the corresponding conditional characteristic functions:

$$Q_S\left(S_T \geq K\right)_t = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\omega \ln K}\phi_S\left(s, V, \omega\right)}{i\omega}\right] d\omega \tag{4.9a}$$

$$Q_B\left(S_T \geq K\right)_t = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\omega \ln K}\phi_B\left(s, V, \omega\right)}{i\omega}\right] d\omega \tag{4.9b}$$

where $\phi_S\left(s, V, \omega\right)$ is the conditional characteristic function of the logarithm of the underlying value under the risk-neutral probability measure with S_t as the numeraire, and $\phi_B\left(s, V, \omega\right)$ is the conditional characteristic function of the logarithm of the underlying value under the risk-neutral probability measure with the money market account B_t as the numeraire.

4.2.4 Estimation of the implied price of volatility risk.

The price of volatility risk λ enters only the stochastic volatility model under the risk-neutral probability measure, so information on the underlying value dynamics cannot be used to estimate λ. On the contrary, option prices depend on the price of volatility risk through the risk-neutral conditional probabilities. Thus, observed option prices can be used to estimate the price of volatility risk. We estimate the prices of volatility risk on a daily basis by minimizing an appropriately chosen distance between observed option prices and the theoretical option prices for the stochastic volatility model, given in equation (4.8). Note, however, that the model itself assumes a constant price of volatility risk. In this respect, our prices of volatility risk estimated on a daily frequency are analogous to the implied volatilities of Black and Scholes. We call our estimates for this reason implied prices of volatility risk. Following Bakshi *et al.* (1997), we choose the sum of the relative squared errors as a distance measure between observed and theoretical option prices. There are two reasons for this choice. First, one could think of practical considerations. An investor, investing in different option contracts, wants to minimize the percentage of his wealth at risk. The relative error of the stochastic volatility model indicates exactly the percentage of wealth that can be lost due to mispricing. Second, we would like to have a measure which does not overweight expensive options with respect to cheap ones. The relative squared error becomes then a simple and natural choice. It is also consistent with the measure of out-of-sample pricing performance. Summarizing, we estimate the implied price of volatility risk as follows:

$$\widehat{\lambda}_t = \operatorname{Arg}\min_{\lambda} \sum_{i=1}^{J_t} \left[\frac{C_{i,t}\left(\lambda, \widehat{\theta}, \widehat{V}_t\right) - C_{i,t}}{C_{i,t}} \right]^2 \tag{4.10}$$

where $C_{i,t}$ is the observed price of the ith option at day t, $C_{i,t}\left(\lambda_t, \widehat{\theta}, \widehat{V}_t\right)$ is the theoretical price of the ith option from (4.8), and J_t is the number of option contracts observed at day t. Other parameters $\theta = (\alpha, \beta, \sigma, \rho)$, necessary to obtain the theoretical option prices, are estimated from the dynamics of the underlying value, and the instantaneous volatilities V_t of the process are filtered using the discrete E-GARCH specification, as described in Sections 2.1 and 2.2.

4.3 Data

In our analysis we use daily values of the S&P500 index from January 1, 1992 to December 31, 1998 and European call options written on the S&P500 index from January 1992 to August 1997. The annualized summary statistics for the daily S&P500 returns are reported in Table 4.1. The average return on the index is 15.4% with annualized volatility of 13%. The distribution of the index return is negatively skewed. The distribution has substantial excess kurtosis. The autocorrelations of the index returns are low and statistically insignificant, while the autocorrelations of squared returns are positive, significant, and diminishing with the order.

We also use European call option data on the S&P500 index for the period from January 2, 1992 to August 21, 1997. To eliminate possible data errors and liquidity problems we apply several screening criteria to our option data set. The criteria we use are based on Bakshi *et al.* (1997). We exclude from our sample options with moneyness[5] less than 0.9 and greater than 1.1 as well as contracts with maturity less than 5 trading days. Options with mid quotes less than \$3/8 and implied Black-Scholes volatilities greater than 90% are also eliminated from our sample as being illiquid contracts. Finally, we exclude from the sample call options which violate the no-arbitrage restriction $C_t \geq Max\left[0, S_t e^{-d\tau} - Ke^{-r\tau}\right]$. Table 4.2 contains the sample characteristics for the selected call options subdivided in different categories according to moneyness and time to maturity. For each category we report the average bid-ask midpoint price, the average bid-ask spread,[6] the average Black-Scholes implied volatility and the number of contracts.

The implied volatilities are calculated using the end-of-the-day option mid price, S&P500 index level, the risk-free interest rate, time to maturity, and dividend yield. Since the companies in the S&P500 index pay dividends we have to take them into account. Following Chernov (2003) we assume a continuous annual dividend yield of 2%, which is consistent with historical data. The risk-free interest rates are obtained from US interbank interest rates for 1, 3, 6 and 12 months. The risk-free rate for a particular option is calculated by linear interpolation of the US interbank interest rates that straddle the option's maturity.

[5]We define moneyness as the Index-to-Strike ratio: $I = S/K$.

[6]Defined as ask price minus bid price.

4.4 Estimation results

4.4.1 Estimation of the SV model

The GMM estimation of the parameters of the stochastic volatility model is based on the marginal and joint moment conditions as outlined in Section 2. In principle, there is an infinite number of moments available for GMM estimation. Our choice of moments is mainly guided by the Monte Carlo evidence on GMM estimation of a stochastic volatility model as, for instance, in Andersen and Sorensen (1996). While choosing the number of moments we have to take into account the usual trade-off between efficiency of the parameter estimates and precision of the optimal weighting matrix. The absolute moments of the index return, as opposed to a discrete model, cannot be derived in closed form for the continuous-time model, so we cannot use them. Andersen and Sorensen (1996) show that inclusion of absolute moments brings only minor gains to estimation performance. Further, in choosing the exact moments, we take into account that the SV model allows for skewness and excess kurtosis. This makes the first four unconditional moments important. Finally, autocorrelation of squared returns is determined by the volatility process and the leverage effect, hinting that the joint moments of squared returns are important for identification of the parameters of the volatility process. To capture the changes of the autocorrelation in lag order we use the first five lags of these moments. Thus, the first four central moments of the index returns and the first five orders of the autocorrelation of the squared returns are used in the GMM estimation. To estimate the optimal weighting matrix we use Newey and West (1987) with a fixed lag number of 13. The initial parameter values are set equal to the method of moment estimates, obtained by matching the first four central moments and the first order autocorrelation of the squared returns to the data. Table 4.3 reports the parameter estimates and asymptotic standard errors for the SV model.

The estimated expected return μ on S&P500 is 15.4%, which is consistent with the sample properties. The estimated mean-reverting volatility level α is equal to 0.0173, which corresponds to 13.1% standard deviation of the index return. Thus, the difference between the sample standard deviation and our estimate is not substantial. The mean-reverting coefficient β is estimated to be equal to 16.18. The estimate is significant at the 10% level. The half-life of a volatility shock, according to the estimated mean-reverting coefficient, is equal to $\ln 2/\beta \approx 11$ trading days.

The estimate of the volatility parameter σ of the volatility process is significantly different from zero and equals to 1.3. The parameter ρ is insignificantly different from zero. We use the estimated parameters of the stochastic volatility model for option pricing.

Our estimation results are comparable to those of Chacko and Viceira (2003) and Jiang and Knight (2002). While our estimate of the mean-reversion parameter is very close to the one in Chacko and Viceira (2003), the estimate of volatility of the volatility process is lower. As was noted in several papers, the mean-reverting coefficient and volatility of the volatility process are interrelated parameters. There is also evidence that the estimates of those two parameters tend to change significantly depending on the estimation method and the sample.

4.4.2 Filtering instantaneous volatilities.

We filter instantaneous volatilities using the E-GARCH(1,1) specification. It gives consistent estimates of the true volatility process, see Nelson (1992). The discrete time E-GARCH model as well as the continuous time stochastic volatility model allows for leverage effect and excess kurtosis. We obtain the instantaneous volatility estimates, which are subsequently used in option pricing. Figure 4.1 shows the estimated annualized instantaneous volatilities. Volatility spikes in Figure 4.1 reflect turbulent times on the stock market. It is possible to see that the variability of the volatility process is substantial. The mean-reverting pattern of the volatility process is apparent as well. Option prices, in case of the stochastic volatility model, will depend positively on the instantaneous volatilities. However, this dependence will be weaker for contracts with long time to maturity, especially if the mean-reverting parameter β (or the price of volatility risk λ) is high.

4.4.3 Estimation of implied prices of volatility risk

As outlined before, European call option prices in a stochastic volatility model depend on the parameters of the model under the physical probability measure as well as the price of volatility risk λ. We use the estimated parameters of the stochastic volatility model given in Table 1 to analyze the sensitivity of the European call option prices, relative to the value of the underlying, with respect to the price of volatility risk. Figure 4.2 shows call option prices as functions of the price of volatility risk λ. The maturity of the option contracts is 1 year. The relative European call option prices are shown for moneyness equal to 0.94, 1, and 1.06. The Black and Scholes European call option price is also shown for the sake of comparison. The volatility parameter in the Black and Scholes model is chosen to be equal to the mean-reverting level α of the volatility process in the stochastic volatility model. For the stochastic volatility model, the current level of the instantaneous volatility is chosen as $V_t = \alpha$.

As can be seen, the price of a European call option in the stochastic volatility model decreases as the price of volatility risk increases. This is true for all maturities and all values of moneyness of the option contract and due to the fact that a higher price of volatility risk makes the long-run volatility less uncertain (the process reverses faster). The relative call option price is a nonlinear convex function of the price of volatility risk. This means that negative changes in the price of volatility risk lead to higher absolute changes in option price than the positive ones. Overall the stochastic volatility model is a more flexible model than the Black-Scholes. This explains the well known finding that the stochastic volatility model with a non-zero price of volatility risk is capable to explain the systematic biases of the observed option prices from the Black and Scholes European call option price.

As explained in subsection 4.2.4, we estimate the implied prices of volatility risk on a daily basis by minimizing the relative distance between observed and model-predicted option prices. Figure 4.3 shows the dynamics of the estimated prices of volatility risk. Estimates of the implied prices of volatility risk show non-trivial dynamics with a high persistence.

The estimated implied prices of volatility risk have the same measurement units as the mean-reverting coefficient β in the stochastic volatility model (4.6). Thus, we can roughly interpret the implied price of volatility risk as the extent to which mean-reversion speed changes under the risk-neutral probability distribution. Negative values of the implied price of volatility risk imply that the European call option prices are higher in comparison to the idiosyncratic volatility risk. From the upper panel of Figure 4.3 we can see that the estimated implied prices of volatility risk are generally negative, which means that a model with idiosyncratic volatility would underprice options. The average estimated implied price of volatility risk is -6.08 with a standard deviation of 0.13. The negative price of volatility risk is in line with the empirical literature. For example, Bakshi and Kapadia (2003) provide a strong evidence of the negative volatility risk premium by studying statistical properties of the delta-hedged gains. Also Pan (2002) obtains a negative volatility risk premium for stochastic volatility specification.

The high persistence of the estimated implied prices of volatility risk suggests that the dynamics of the implied prices of volatility risk can be modeled as an AR process. However, we estimate a dynamic model for daily changes in the implied price of volatility risk (see the lower panel of Figure 4.3) to avoid problems with unit roots. Table 4.4 shows sample characteristics of daily changes in the implied prices of volatility risk.

It can be seen that the autocorrelation coefficients are negative and decrease with the autocorrelation order.

We search for an appropriate time series model to describe the dynamics of daily changes in the implied prices of volatility risk in the ARMA class of linear models. As a choice criterion the Schwarz information criterion (SIC) is used. Applying SIC, we pick the ARMA(2,1) to model the dynamics of the implied prices of volatility risk. The corresponding parameter estimates are given in Table 4.5. All the coefficients are statistically significant at the 5% significance level.

The estimates in Table 4.5 show that daily changes in the implied price of volatility risk are persistent. One of the roots of the ARMA(2,1) is equal to 0.8. It also follows from the model, that the implied price of volatility risk is predictable from its past. The model explains about 6% of variation in daily changes of the implied price of volatility risk.

We also tried to include lags of other financial variables such as index daily return, index value spreads, instantaneous volatilities, and the price adjusted trade volumes for prediction of the implied prices of volatility risk. It turned out that these variables help little in explaining the dynamics of the implied price of volatility risk.

4.4.4 Out-of-sample pricing performance

We analyze the performance of our model on a 1, 5, and 20 day horizon. We use the estimated ARMA (2,1) model to form 1, 5 and 20 day ahead predictions for the implied price of volatility risk. In order to evaluate the importance of the dynamic properties of the implied prices of volatility risk, we investigate the out-of-sample performance of the chosen dynamic model with respect to predicting future option prices conditionally on the index value price and instantaneous volatility. This strategy gives insight into the errors of a hedging strategy based on predicting implied prices of volatility risk. We compare our dynamic specification of implied prices of volatility risk with several other approaches used in the literature.

Prediction of option prices, conditional on the index value and the instantaneous volatility, in our model boils down to the prediction of the implied price of volatility risk. Note that very similar techniques are often applied in practice with respect to implied Black-Scholes volatilities. There, one tries to predict future values of the implied volatility using past values and characteristics of the options like maturity and moneynessȮur approach is more structural, since we model part of the Black-Scholes pricing errors by allowing for non-idiosyncratic volatility risk.

We compare our dynamic model with several alternative specifications, which were proposed in the literature. The first specification assumes idiosyncratic volatility risk, i.e., $\lambda = 0$. Such a specification was proposed and investigated from a theoretical point of view by Hull and White (1987). Empirical results usually do not support this specification. Here, it is included for the sake of comparison. The second specification assumes a constant price of volatility risk. In the framework of a non-affine stochastic volatility process, a comparable specification was considered by Melino and Turnbull (1990). As third specification we assume that today's implied price of volatility risk is simply equal to the one in the previous trading day. This specification is also used in Jiang and van der Sluis (1999) whose focus is, however, not on the dynamic properties of the price of volatility risk, but the effect of random interest rates. Finally, we also include the Black-Scholes model as a benchmark. We compare these four specifications to our dynamic ARMA(2,1) model. Overall, we consider the following five specifications:

- BS: Black and Scholes specification (constant volatility)
- $\widehat{\lambda}_t = 0$: idiosyncratic volatility risk
- $\widehat{\lambda}_t = \lambda$: constant price of volatility risk
- $\widehat{\lambda}_{t+1|t} = \widehat{\lambda}_t$: unpredictable changes in the implied price of volatility risk
- ARMA(2,1): dynamic specification

We base the evaluation of the different pricing models on the basis of the average (absolute) relative distance between predicted option prices and observed option prices. The predicted option prices are calculated according to the stochastic volatility option pricing equation. We use the parameter vector $(\alpha, \beta, \sigma, \rho)$ estimated over the whole sample, the actual S_{t+s} and $\widehat{V}_{t+s}$, and forecasts for the implied price of volatility risk, $\widehat{\lambda}_{t+s|t}$, obtained from the dynamic model. The predicted option prices are confronted with the observed price of the corresponding option contract and the relative pricing error is calculated. Table 4.6 shows the results of the comparison among the different specification based on the average absolute relative pricing error. Standard errors are added to indicate the variability of the average pricing errors in our sample.[7] It follows that the out-of-sample pricing performance substantially improves if we allow the price of volatility risk to change over time. First, note that modelling stochastic volatility improves the pricing performance in comparison to the Black-Scholes model: relative pricing error decreases almost by 6 percentage points. A further 2 percentage points improvement can be achieved by allowing a constant price of volatility risk. Finally, as we let the price of volatility risk to change in time, the pricing error decreases by virtually 5% (for 1 day forecasts). Notice, however, that the difference between the random walk $\left(\widehat{\lambda}_{t+1|t} = \widehat{\lambda}_t\right)$ and the ARMA(2,1) is marginal.

The results in Table 4.6 clearly show that modelling the dynamics of the implied price of volatility risk improves the out-of-sample pricing performance of the stochastic volatility model, aggregated over all options traded on a single day. Table 4.7 shows the out-of-sample pricing performance for different groups of options constructed with respect to moneyness and maturity. The average absolute relative errors are shown for a one day forecasting horizon. It turns out that the stochastic volatility model with a dynamic implied price of volatility risk prices expensive op-

[7] These standard errors cannot be used for the statistical inference on the model pricing error since the model sampling error is not taken into account.

tions better than cheap ones. In particular, options with long maturities are priced better than contracts with short time to maturity. For deep in-the-money options with short time to maturity the performance of the model with constant price of volatility risk is comparable with the performance of dynamic models. In general, the dynamic models for the implied price of volatility risk substantially outperform their alternatives.

4.5 Conclusion

In this paper we present an empirical application of the Heston (1993) stochastic volatility model. The modelled market is incomplete in terms of the underlying assets, so that pricing by arbitrage of derivatives is impossible: the volatility risk premium is not fixed. By using option prices observed in the market, we are able to estimate the empirically relevant prices of volatility risk. Our finding, that the estimated implied price of volatility risk changes over time, is in line with the existing literature. We model explicitly the dynamics and investigate statistical properties of the implied prices of volatility risk. We show that modelling dynamics of the implied prices of volatility risk improves out-of-sample option pricing performance with respect to the specifications studied before.

4.A Tables and figures

S&P500 returns: Sample Characteristics							
	N	Mean	Std.Dev.	Skewness	Kurtosis	Maximum	Minimum
$R(t)$	1769	0.154	0.131	-0.607	12.116	12.572	-17.924
	Autocorrelations						
	N	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(8)$	$\rho(10)$	$\rho(15)$
$R(t)$	1769	-0.002	-0.025	-0.028	-0.009	0.056	-0.016
t-stat		0.09	-0.72	-0.97	-0.05	0.48	-0.27
$R^2(t)$	1769	0.265	0.14	0.079	0.086	0.087	0.076
t-stat		5.9	5.7	5.5	4.2	4.1	2.9

Tabel 4.1: Annualized summary statistics of S&P 500 index returns for the period 1992-1998.

Moneyness		Maturity (days) ≤ 60	$60-180$	>180
<0.94	Mid price	1.24	2.28	10.18
	Bid-Ask spread	0.24	0.3	0.57
	Implied volatility	14%	12%	12%
	N contracts	646	3064	2369
$0.94-0.97$	Mid price	2.12	5.95	18.51
	Bid-Ask spread	0.25	0.4	0.75
	Implied volatility	12%	12%	13%
	N contracts	3961	5761	2070
$0.97-1.00$	Mid price	5.62	13.02	28.18
	Bid-Ask spread	0.35	0.58	0.84
	Implied volatility	13%	13%	14%
	N contracts	7291	6834	2688
$1.00-1.03$	Mid price	15.23	23.46	39.74
	Bid-Ask spread	0.63	0.76	0.95
	Implied volatility	15%	14%	15%
	N contracts	7366	6800	2857
$1.03-1.06$	Mid price	28.42	35.24	50
	Bid-Ask spread	0.86	0.88	0.96
	Implied volatility	17%	16%	16%
	N contracts	6670	6273	2839
>1.06	Mid price	44.26	49.42	63.46
	Bid-Ask spread	0.96	0.94	1.01
	Implied volatility	21%	17%	17%
	N contracts	5724	5479	2111

Tabel 4.2: Characteristics of European call option data: The average midpoint price, the average bid-ask spread, the average implied volatility and the total number of observations for different moneyness and the time-to-maturity.

S&P annualized		
	Estimates	Std. Err.
μ	0.154	0.045
α	0.0173	0.0022
β	16.18	9.47
σ	1.3	0.40
ρ	-0.67	0.43

Tabel 4.3: Annualized estimated parameters of the SV model. The left column contains standard errors of the estimates.

$\Delta\lambda(t)$: **Sample Characteristics**							
	N	Mean	Std.Dev.	Skewness	Kurtosis	Maximum	Minimum
$\Delta\lambda(t)$	1408	-0.0034	1.15	-0.0401	6.548	7.676	-6.255
	Autocorrelations						
	N	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(4)$	$\rho(5)$	$\rho(6)$
$\Delta\lambda(t)$	1408	-0.203	-0.071	-0.01	-0.006	-0.023	0.016

Tabel 4.4: Sample Characteristics of the Implied Prices of Volatility Risk.

	ARMA(2,1)	
Param.	Estimates	t-statistic
AR(1)	0.66	11.38
AR(2)	0.11	2.71
MA(1)	-0.90	-21.74

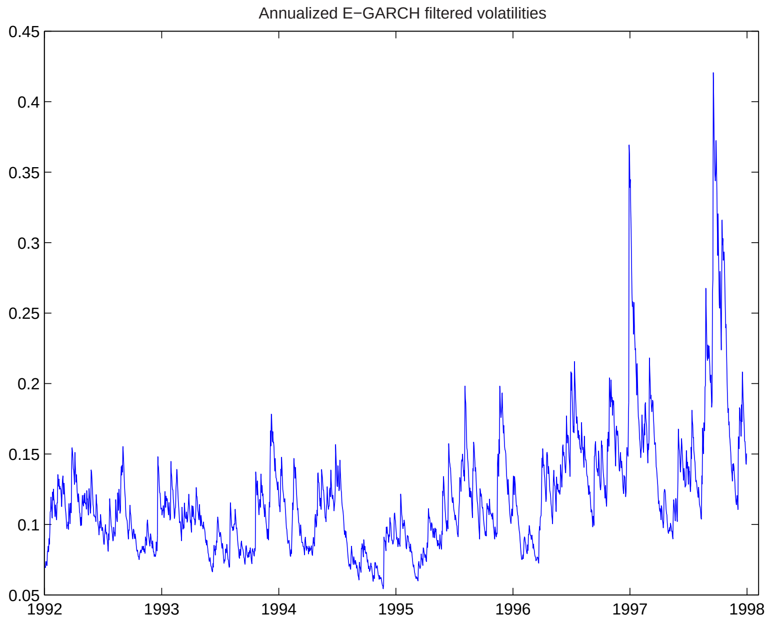
Tabel 4.5: Modelling the dynamics of the Implied Price of Volatility Risk, parameter estimates.

Average Absolute Relative Prediction Errors					
	BS	$\lambda = 0$	$\lambda(t) = \lambda$	$\lambda(t+1) = \lambda(t)$	ARMA(2,1)
1 day	22.7%	17.00%	15.00%	10.19%	10.16%
	0.15%	0.07%	0.07%	0.05%	0.05%
5 days	*	*	*	10.83%	10.76%
	*	*	*	0.05%	0.05%
20 days	*	*	*	12.03%	11.8%
	*	*	*	0.06%	0.06%

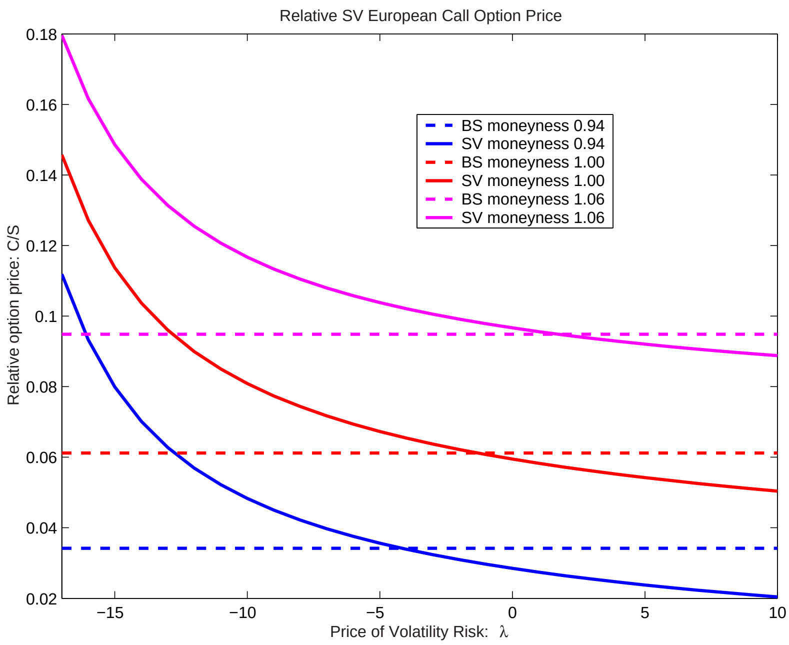
Tabel 4.6: Out-of-sample predictive performance for the stochastic volatility model with various specifications of the implied price of volatility risk λ. The numbers are the average absolute relative errors and their standard errors.

BS	$\lambda=0$	$\lambda(t)=\lambda$	RW	ARMA	BS	$\lambda=0$	$\lambda(t)=\lambda$	RW	ARMA	BS	$\lambda=0$	$\lambda(t)=\lambda$	RW	ARMA
Moneyness=<0.94 Maturity$<=60$					Moneyness=<0.94 Maturity $60-180$					Moneyness=<0.94 Maturity>180				
46.3%	80.0%	61.7%	40.0%	40.7%	86.7%	45.8%	38.9%	14.8%	14.7%	37.2%	22.7%	30.1%	21.5%	21.1%
Moneyness $0.94-0.97$ Maturity$<=60$					Moneyness= $0.94-0.97$ Maturity $60-180$					Moneyness= $0.94-0.97$ Maturity>180				
77.1%	58.5%	45.7%	39.7%	39.8%	58.4%	31.8%	26.7%	11.7%	11.7%	19.6%	16.0%	19.6%	14.8%	14.4%
Moneyness $0.97-1.00$ Maturity$<=60$					Moneyness= $0.97-1.00$ Maturity $60-180$					Moneyness= $0.97-1.00$ Maturity>180				
38.5%	34.3%	27.8%	25.3%	25.4%	20.7%	17.6%	14.5%	6.6%	6.6%	11.4%	11.6%	13.0%	11.3%	11.0%
Moneyness $1.00-1.03$ Maturity$<=60$					Moneyness= $1.00-1.03$ Maturity $60-180$					Moneyness= $1.00-1.03$ Maturity>180				
10.3%	10.0%	8.4%	6.6%	6.7%	9.4%	9.2%	8.3%	4.1%	4.1%	8.2%	8.1%	9.7%	8.1%	7.9%
Moneyness $1.03-1.06$ Maturity$<=60$					Moneyness= $1.03-1.06$ Maturity $60-180$					Moneyness= $1.03-1.06$ Maturity>180				
5.5%	4.1%	3.6%	2.9%	2.9%	6.3%	5.2%	5.3%	2.9%	2.9%	6.6%	5.9%	7.6%	6.2%	6.1%
Moneyness >1.06 Maturity$<=60$					Moneyness>1.06 Maturity $60-180$					Moneyness>1.06 Maturity>180				
3.5%	2.3%	2.1%	1.8%	1.8%	4.9%	3.3%	3.7%	2.2%	2.2%	5.7%	4.6%	6.2%	4.7%	4.5%

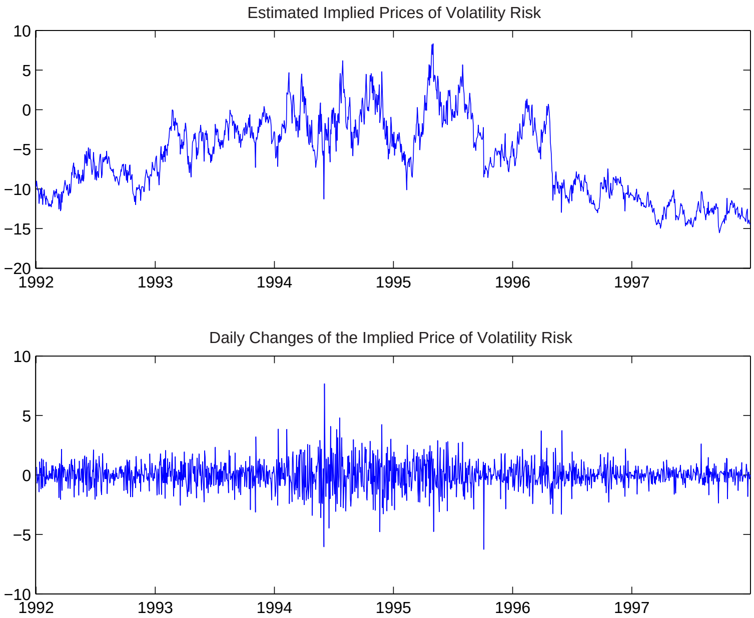
Tabel 4.7: Out-of-sample predictive performance for various specifications of the implied price of volatility risk and per option category. Forecasting horizon 1 trading day. The numbers are the average absolute relative errors. RW corresponds to $\widehat{\lambda}_{t+1|t}=\widehat{\lambda}_t$ and ARMA to the ARMA(2,1) model.



Figuur 4.1: Annualized E-GARCH filtered volatilities.



Figuur 4.2: Relative European call option prices in the stochastic volatility model as functions of the implied price of volatility risk.



Figuur 4.3: Estimated implied prices of volatility risk.

Hoofdstuk 5

Assessing credit with equity: A constant elasticity of variance model with jump to default

5.1 Introduction

Theoretical credit risk models developed in the financial literature can be divided into two categories: structural models and reduced-form models. Introduced by Merton (1974), structural models describe default as an event triggered by underlying processes of state variables related to the capital structure of the debt issuer.[1] As opposed to the structural models, which link default explicitly to the first time asset falls below a certain level, a more recent literature adopted the reduced-form approach, assuming that the default arriving intensity exists and expressing it directly as a function of latent state variables or predictors of default, see, for example, Jarrow and Turnbull (1995), Artzner and Delbaen (1995), and Duffie and Singleton (1999). This approach allows straightforward application of statistical methods for

[1] Examples of this approach are, among others, Cathcart and El-Jahel (1998), Leland and Toft (1996), and Longstaff and Schwartz (1995).

estimating the incidence of default. It has recently been shown by Duffie and Lando (2001) and Giesecke (2003) that structural models augmented with incomplete information can be consistent with a reduced form representation.

Investors have been showing an increasing appetite for models that simultaneously handle credit and equity instruments, which is important in managing a portfolio including these two instruments. Indeed, cross-asset trading of credit risk has been gaining momentum among credit hedge funds and banks. The rise of capital structure arbitrage[2] is a good example, see, for instance, Yu (2004). Due to a weak and indirect linkage to the firm's capital structure, reduced-form models of credit risk might not be of great help. Among the papers that actually model this linkage through a default hazard rate factor model, see, for example, Bakshi *et al.* (2004). Structural models are driven by the value evolution in the firm's assets. The assets-value evolution is often assumed to be diffusive[3] so that the default can be seen predictably coming by observing changes in the capital structure of the firm (see the seminal papers of Merton (1974) and Black and Cox (1976)). While appealing, structural models reveal certain disadvantages when it comes to applications. The underlying (the sum of the firm's liabilities and equity) is illiquid and often non-tradable. Obtaining accurate asset volatility forecasts and reliable capital structure leverage data is difficult. In addition, predictability of the default event implies the counterfactual prediction of zero credit spreads for short maturities.[4] Finally, arbitrary use of the structural default barrier is often a temptation hard to resist, while endogenous barriers are impractical because of the unrealistic capital-

[2]Capital structure arbitrage is a term used in the financial industry for positions in credit instruments hedged with equity or equity derivatives.

[3]There are few exceptions which incorporate jumps. Examples are Huang and Huang (2002) and Zhou (2001). Duffie and Lando (2001) and Giesecke (2003) take into account incomplete accounting information.

[4]See, for example, the empirical studies by Sarig and Warga (1989) and Beneish and Press (1995).

structure assumptions under which they are derived, see, as an example, Hui *et al.* (2003).

We propose a parsimonious credit risk model that does look at the firm's balance sheet, but avoids the application mishaps of structural models. We take as underlying the most liquid and observable corporate security: equity. This modelling choice brings in hedging viability and the possibility of reliable model calibration since infrequent and often noisy leverage information from book values can be avoided. We parsimoniously represent default as equity value hitting the zero barrier either diffusively or with a jump. The presence of an equity-value drop to zero has its credit-risk foundation in the incompleteness of accounting information (see Duffie and Lando (2001)) and rules out default predictability. The model is especially appealing for pricing credit securities of distressed companies. The equity price, being an imperfect hedge against default events, becomes a very informative credit indicator as the company approaches bankruptcy.

We assume that the continuous-path part of the equity value is a Constant-Elasticity-of-Variance (CEV) diffusion,[5] which enables absorption at zero, and that the jump to default is driven by a Poisson process. Such distributional assumptions allow us to obtain closed forms for Corporate Bond (CB) prices and Credit Default Swap (CDS) rates, from which hedge ratios can be easily calculated. These assumptions and a careful specification of the state-price density also empower analytic credit-risk management. We provide closed form solutions for the objective default probabilities in the presence of systematic jump-to-default risk. Albanese and Chen (2004) and Campi and Sbuelz (2004) also use a CEV-equity model to price credit instruments, but they disregard the default predictability issue. In deriving closed-

[5]The CEV process has been first introduced to finance by Cox (1975). The CEV-based asset-pricing literature includes the works of Albanese *et al.* (2001), Beckers (1980), Boyle and Tian (1999), Cox and Ross (1976), Davydov and Linetsky (2001), Emanuel and MacBeth (1982), Forde (2005), Goldenberg (1991), Leung and Kwok (2005), Lo *et al.* (2000), Lo *et al.* (2001), Lo *et al.* (2004), Sbuelz (2004), and Schroder (1989).

form values, we build upon a CEV result in Campi and Sbuelz (2004). Brigo and Tarenghi (2004), Naik *et al.* (2003) and Trinh (2004) introduce a hybrid debt-equity model that considers equity as a primitive, but that, like structural models, necessitates an exogenous default barrier, which is then left to potentially *ad-hoc* uses. Equity value is usually assumed to be a geometric Brownian motion, except in Brigo and Tarenghi (2004)[6]. Das and Sundaram (2003) have proposed an equity-based model that accounts for default risk, interest risk, and equity risk using a lattice framework. As such, they do not seek hedger-friendly analytic solutions. Numerical credit risk pricing based on equity has also been suggested by the convertible bond literature (see, for example, Andersen and Anreasen (2000), Andersen and Buffum (2003), and Tsiveriotis and Fernandes (1998); McConnel and Schwartz (1986) ignore the possibility of bankruptcy). Linetsky (2005) builds upon the convertible bond literature to assess zero-coupon CB prices within a geometric-Brownian motion model with jump-like bankruptcy, where the hazard rate of bankruptcy is a negative power of the share price. The dependence of the hazard rate on the share price severely complicates the analysis[7].

The remainder of the chapter is organized as follows. Section 5.2 describes the underlying equity value process. Section 5.3 provides analytic results for corporate bonds and credit default swaps. Section 5.4 specifies a pricing kernel that permits analytic objective default probabilities. Section 5.5 concludes. An Appendix gathers proofs and technical details.

[6]Brigo and Tarenghi (2004) and Hui *et al.* (2003) employ a flexible time-varying default barrier. Hui *et al.* (2003) do not take equity as the underlying.

[7]The valuation formulae in Linetsky (2005) are spectral expansions that embed single integrals with respect to the spectral parameter and calculations imply the use of numerical-integration routines.

5.2 The equity value

We fix the risk-neutral probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and the information filtration $(\mathbb{F}) = \{\mathcal{F}_t : t \geq 0\}$, satisfying the usual conditions.[8] Suppose further that the equity price S of a debt issuer is a Markov process with respect to $(\mathbb{F})$ in $\mathbb{R}_+$, solving the stochastic differential equation

$$dS_t = (r + \lambda - q) S_{t-} dt + \sigma S_{t-}^{\rho} dz_t - S_{t-} dN_t, \tag{5.1}$$

where z is an $(\mathcal{F}_t)$-standard Brownian Motion in $\mathbb{R}$ and N is a pure jump process with exponentially distributed arrival time τ. The parameters of the stochastic differential equation (5.1) are the risk-free interest rate r, the dividend payout rate q, the constant scale factor for the diffusive volatility $\sigma > 0$, and the elasticity parameter[9] of the diffusive volatility ρ. We denote the left-continuous version of the process S_t by $S_{t-} = \lim_{\varepsilon \searrow 0} S_{t-\varepsilon}$, which is the left time limit. The process N is defined as

$$N_t = \begin{cases} 0 & \text{if } 0 \leq t < \tau \\ 1 & \text{if } t \geq \tau \end{cases},$$

where the arrival time τ has the exponential probability density function

$$f_\tau(t) = \lambda e^{-\lambda t},$$

with intensity parameter $\lambda > 0$. The process N can be interpreted as a first-jump-stopped Poisson process with respect to filtration $(\mathcal{F}_t)$. Notice, that the Brownian Motion z and the Poisson jump process N defined on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and with respect to the same filtration $(\mathcal{F}_t)$ are independent by construction, see, for example, Shreve (2004), Corollary 11.5.3. Moreover, any random variable depending only on the path of z will be independent on any random variable depending only on the path of N.

[8] $\mathcal{F}_0$ contains all the null sets of $\mathbb{F}$ and $\{\mathcal{F}_t\}$ is right continuous.

[9] Note, that the elasticity of volatility is $\rho - 1$.

The equity process S is the default-state process. We assume that the default happens as soon as S_t becomes zero for the first time. According to the SDE (5.1) the default can happen either when the process is diffusively absorbed at zero, or when a jump happens. To rigorously define these defaults we introduce a pure diffusive counterpart S^c of the process S, satisfying the stochastic differential equation

$$dS_t^c = (r + \lambda - q)\, S_t^c dt + \sigma (S_t^c)^\rho dz_t. \tag{5.2}$$

The paths of the processes S_t and S_t^c coincide before the jump time τ, i.e.,

$$S_t = \begin{cases} S_t^c & \text{if } 0 \leq t < \tau \\ 0 & \text{if } t \geq \tau \end{cases}.$$

We define the stopping time $\xi = \inf\{t : S_t^c = 0\}$ as the time of diffusive absorbtion at zero[10]. We call the stopping time ξ the *time of diffusive default*, and the stopping time τ the *time of jump default*. The default time η is defined as the minimum between τ and ξ

$$\eta \equiv \tau \wedge \xi = \inf\{t : S_t = 0\}.$$

Notice, that since τ and ξ are independent the default survival probability is the product of the diffusive default survival probability and the jump default survival probability

$$\mathbb{P}^{\mathbb{Q}}(\eta > t) = \mathbb{P}^{\mathbb{Q}}(\tau > t)\, \mathbb{P}^{\mathbb{Q}}(\xi > t)\,.$$

[10] According to the boundary classification, an inverse relationship between volatility and share price ($\rho < 0$) is necessary to have absorbtion at zero. Such an assumption is unlikely to be counterfactual.

5.3 Analytic results for corporate bonds and credit default swaps

Let $V^{\mathbb{Q}}(S,T,y)$ be the T-truncated Laplace transform of the default time η probability density function under the risk-neutral probability measure $\mathbb{Q}$

$$V^{\mathbb{Q}}(S,T,y) \equiv E_0^{\mathbb{Q}}\left[\exp(-y\eta)1_{\{\eta\leq T\}}\right],$$

where y is the parameter of the Laplace transformation. This quantity is the building block for the analytic pricing of corporate bonds (CB) and credit default swaps (CDS). There is a simple interpretation of the Laplace transform of the default time probability density function. The value $V^{\mathbb{Q}}(S,T,y)$ represents the present value of 1 unit of currency at default discounted at rate y, if default occurs within the time interval $[0,T]$. It is straightforward that the Laplace transform $V^{\mathbb{Q}}(S,T,r)$ is the fair present value of the contract with time to maturity T that pays 1 unit of recovery at default, and $V(S,T,0)$ is the risk-neutral probability of default within the time interval $[0,T]$.

The next proposition is a neat and useful result stemming from the independence between the standard Brownian motion $\{z\}$ and the Poisson jump process $\{N\}$. It gives an analytic characterization of the T-truncated Laplace transform $V^{\mathbb{Q}}(S,T,y)$. The proposition states that the Laplace transform is the linear convex combination of the adjusted risk-neutral probability of default within time T (with weight $\frac{\lambda}{y+\lambda}$) and of the $(y+\lambda)$-discounted value of 1 unit of currency at the diffusive default within T (with weight $\frac{y}{y+\lambda}$). The latter is the T-truncated Laplace transform of the diffusive default time probability density function with Laplace parameter $y+\lambda$,

$$E_0^{\mathbb{Q}}\left[\exp\left(-\left(y+\lambda\right)\xi\right)1_{\{\xi\leq T\}}\right],$$

and its closed form[11] has recently been derived by Campi and Sbuelz (2004). The closed form is provided in Appendix 5.A.

[11] Davydov and Linetsky (2001), see pp. 953 and 956, point out that the T-truncated Laplace

Proposition 5.1 *Under the above assumptions, the T-truncated Laplace transform of η's $\mathbb{Q}$-p.d.f. with Laplace parameter y is*

$$
\begin{aligned}
V^{\mathbb{Q}}(S,T,y) &= \frac{\lambda}{y+\lambda}\left[1-\exp\left(-\left(y+\lambda\right)T\right)\left(1-E_0^{\mathbb{Q}}\left[1_{\{\xi\leq T\}}\right]\right)\right] \\
&\quad + \frac{y}{y+\lambda}E_0^{\mathbb{Q}}\left[\exp\left(-\left(y+\lambda\right)\xi\right)1_{\{\xi\leq T\}}\right].
\end{aligned}
$$

Proof. See Appendix 5.A. ■

Proposition 1 empowers analytic pricing of corporate bonds (CB) and credit default swaps (CDS). Consider a reference entity's CB that has face value F and pays an (annualized) coupon C at dates $T_1 < T_2 < \ldots < T_k = T$ up to its maturity T. The fair CB price is

$$
\begin{aligned}
P_{CB}(S,T,r) &= \sum_{j=1}^{k}\exp\left(-rT_j\right)\left[1-V^{\mathbb{Q}}(S,T_j,0)\right]C \\
&\quad + \exp(-rT)\left[1-V^{\mathbb{Q}}(S,T,0)\right]F \\
&\quad + V^{\mathbb{Q}}(S,T,r)\,RF,
\end{aligned}
$$

where R is the recovery rate at default, which is a fixed input parameter in applications. CB's defaultable part is assessed under the assumption of Recovery of Face Value at Default (RFV), which bears the value $V^{\mathbb{Q}}(S,T,r)\,RF$. Under RFV, CB holders receive the same fractional recovery R of the face value F at default for CBs issued by the reference entity regardless of maturity. Guha and Sbuelz (1991) show that the RFV recovery form is consistent with a typical bond indenture language (for example, the claim acceleration clause), defaulted bond price data, and stylized

transform of ξ's $\mathbb{Q}$-p.d.f. with Laplace parameter $y+\lambda$ can be obtained by numerically inverting the closed-form non-truncated Laplace transform

$$
\frac{1}{a}E_0^{\mathbb{Q}}\left[\exp\left(-\left(y+\lambda+a\right)\xi\right)\right],
$$

where the inversion parameter is $a > 0$.

facts that are relevant for interest rate hedging (for example, the low duration of high-yield bonds).

Consider a CDS related to the CB just described. It offers a protection payment of $(1-R)\,F$ in exchange for an (annualized) fee f_{CDS} paid at the dates $T_1^* < T_2^* < \ldots < T_m^* = T^*$ up to the contract's maturity[12]. The fair CDS rate is

$$f_{CDS}\left(S, T^*, r\right) \;=\; \frac{V^{\mathbb{Q}}\left(S, T^*, r\right)(1-R)}{\sum_{j=1}^{m} \exp\left(-rT_j^*\right)\left[1 - V^{\mathbb{Q}}\left(S, T_j^*, 0\right)\right]}.$$

The holder of a CB can achieve total recouping of the face value F at default by being long a CDS with similar maturity and payment dates. Being short $\frac{\partial}{\partial S} P_{CB}\left(S, T, r\right)$ shares the CB holder can hedge against the pre-default price shocks driven by diffusive news. Recent evidence shows that such equity-based hedges perform reasonably well for high-yield CBs (see Naik *et al.* (2003)). Given analytic CB prices, an easy and effective measure of the Delta-hedge ratio is

$$\frac{\partial}{\partial S} P_{CB}\left(S, T, r\right) \;\simeq\; \frac{P_{CB}\left(S+\varepsilon, T, r\right) - P_{CB}\left(S-\varepsilon, T, r\right)}{2\varepsilon},$$

where ε is a small positive number. Finally, parallel shifts of the (flat) term structure of the interest rates can be hedged by selling a portfolio of default-free bonds that has interest-rate sensitivity equal to $\frac{\partial}{\partial r} P_{CB}\left(S, T, r\right)$. Such a hedge ratio can be easily calculated in our model. More details on model-based CB hedging are in Appendix 5.C.

5.4 The objective default probability

Our equity-based model of credit risk (5.1) is specified under the risk-neutral probability measure $\mathbb{Q}$ since the focus of the analysis so far was on pricing credit derivatives. It is, however, sometimes of interest to be able to determine the objective default probabilities as well as to analyze dynamics of the underlying equity under

[12]Notice, that the CDS rate payment dates need not coincide with the coupon payment dates of the reference CB. In fact, they are usually different.

the physical probability measure $\mathbb{P}$ empirically. The specification (5.1) implies a link between the risk-neutral measure $\mathbb{Q}$ and the physical probability measure $\mathbb{P}$ through the specification of the Radon-Nikodým derivative or the pricing kernel.

Suppose that we have an objective probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a Brownian motion $z_t^{\mathbb{P}}$ and a Poisson process $N_t^{\mathbb{P}}$ with intensity $\lambda_{\mathbb{P}}$ are defined with respect to the same filtration $\mathcal{F}_t, t \geq 0$. The specification (5.1) of the equity price process under the risk-neutral probability measure $\mathbb{Q}$ imposes restrictions on the dynamics of the price process under the physical probability measure $\mathbb{P}$. In particular, dynamics of the price process under the physical probability measure should satisfy the SDE

$$dS_t = \left[\sigma S_{t-}^{\rho} \Theta(t) + (r - q + \lambda) S_{t-}\right] dt + \sigma S_{t-}^{\rho} dz_t^{\mathbb{P}} - S_{t-} dN_t^{\mathbb{P}},$$

which is implied by the Radon-Nikodým derivative of the form

$$\begin{aligned} \Pi(t) &= \Pi_1(t)\Pi_2(t), \\ \Pi_1(t) &= \exp\left\{-\int_0^t \Theta(u) dz_u^{\mathbb{P}} - \frac{1}{2}\int_0^t \Theta^2(u) du\right\}, \\ \Pi_2(t) &= \left(\frac{\lambda}{\lambda_{\mathbb{P}}}\right)^{N_t^{\mathbb{P}}} \exp\left\{(\lambda_{\mathbb{P}} - \lambda)t\right\}, \end{aligned}$$

where $\Theta(t)$ is an arbitrary bounded adapted càglàd (left continuous with right limits) process.[13] In order to guarantee that the dynamics of the equity price process under the physical probability measure is described by a CEV-jump process with a constant drift, analogous to (5.1), one needs to assume the specific functional form of the adapted process $\Theta(t)$, which is

$$\Theta(t) = \theta S_{t-}^{1-\rho}.$$

The dynamics of the equity price process $\{S\}$ under the objective measure follows straightforwardly:

$$dS_t = (\mu_{\mathbb{P}} + \lambda_{\mathbb{P}}) S_{t-} dt + \sigma S_{t-}^{\rho} dz_t^{\mathbb{P}} - S_{t-} dN_t^{\mathbb{P}}. \tag{5.3}$$

[13] See, for example, Theorem 11.6.9 in Shreve (2004).

The relation between the SDE parameters under $\mathbb{P}$ and $\mathbb{Q}$ are the following

$$\mu_{\mathbb{P}} = r - q + \theta\sigma + (\lambda - \lambda_{\mathbb{P}}),$$

where we call $\theta\sigma$ the premium for the diffusive risk, and $\delta \equiv \lambda - \lambda_{\mathbb{P}}$ the premium for the jump-like default risk. Such a terse specification of $\{S\}$'s $\mathbb{P}$-dynamics makes a neat account of systematic jump-like default risk. The sign of the jump-like risk premium λ is an empirical question. Nevertheless, given the fact that investors would like to be compensated for the unanticipated default risk, it is reasonable to assume that the jump-to-default intensity under $\mathbb{Q}$ is always greater than its level under $\mathbb{P}$, i.e., $\delta \geq 0$. If the systematic nature of the jump-like default risk is washed away, so that risk-neutral and objective jump-to-default intensities tend to coincide ($\delta \searrow 0$), then the jump-risk is not priced.

As far as the diffusive risk is concerned, if its premium faints, it is either because such a risk is not priced ($\theta \searrow 0$) or because the risk is dimming ($\sigma \searrow 0$).

Since the objective drift is constant ($E_t^{\mathbb{P}}[dS_t] = \mu_{\mathbb{P}} S_{t-}$), arguments similar to those behind Proposition 5.1 lead to an analytic expression for the quantity $V^{\mathbb{P}}(S, T, y)$:

$$\begin{aligned} V^{\mathbb{P}}(S,T,y) &= \frac{\lambda_{\mathbb{P}}}{y+\lambda_{\mathbb{P}}}\left[1 - \exp\left(-\left(y+\lambda_{\mathbb{P}}\right)T\right)\left(1 - E_0^{\mathbb{P}}\left[1_{\{\xi \leq T\}}\right]\right)\right] \\ &\quad + \frac{y}{y+\lambda_{\mathbb{P}}} E_0^{\mathbb{P}}\left[\exp\left(-\left(y+\lambda_{\mathbb{P}}\right)\xi\right) 1_{\{\xi \leq T\}}\right], \end{aligned}$$

where the T-truncated Laplace transform of ξ's $\mathbb{P}$-p.d.f. with Laplace parameter $y + \lambda_{\mathbb{P}}$ is analytic (see Campi and Sbuelz (2004)). Its closed form is provided in Appendix 5.B.

In summary, we achieve analytic objective default probabilities by augmenting the original parameter set $\{r, q, \sigma, \rho, \lambda\}$ with two preference-based parameters, θ for the diffusive risk, and δ for the jump-like default risk.

5.5 Conclusion

We present an equity-based credit risk model that, by taking as primitive the most liquid and observable part of a firm's capital structure, overcomes many of the problems suffered by structural models in pricing and hedging applications. Our parsimonious model avoids any assumption on the firm's liabilities. It empowers the analytical pricing of CBs and CDSs and it can match non-zero short-maturity spreads. Cross-asset hedging is viable and easy to implement. A careful specification of the diffusion part of the equity price process under the physical probability measure enables analytic credit-risk management in the presence of systematic jump-to-default risk.

5.A Laplace transform

From the independence between ξ and τ we have that

$$E_0^{\mathbb{Q}}\left[1_{\{\tau\wedge\xi>s\}}\right] = E_0^{\mathbb{Q}}\left[1_{\{\tau>s\}}1_{\{\xi>s\}}\right] = E_0^{\mathbb{Q}}\left[1_{\{\tau>s\}}\right]E_0^{\mathbb{Q}}\left[1_{\{\xi>s\}}\right],$$

Hence, the time-s-evaluated $\mathbb{Q}$-p.d.f. of the stopping time $\eta = \tau\wedge\xi$ is

$$\begin{aligned}
f_\eta(s) &= -\frac{d}{ds}E_0^{\mathbb{Q}}\left[1_{\{\tau>s\}}1_{\{\xi^c>s\}}\right] \\
&= -\frac{d}{ds}\left(E_0^{\mathbb{Q}}\left[1_{\{\tau>s\}}\right]E_0^{\mathbb{Q}}\left[1_{\{\xi>s\}}\right]\right) \\
&= f_\tau(s)\,E_0^{\mathbb{Q}}\left[1_{\{\xi>s\}}\right] + f_\xi(s)\,E_0^{\mathbb{Q}}\left[1_{\{\tau>s\}}\right] \\
&= \lambda\exp(-\lambda s)\,E_0^{\mathbb{Q}}\left[1_{\{\xi>s\}}\right] + f_\xi(s)\exp(-\lambda s).
\end{aligned}$$

The T-truncated Laplace transform of η's $\mathbb{Q}$ -p.d.f. with Laplace parameter y is

$$\begin{aligned}
V^{\mathbb{Q}}(S,T,y) &= E_0^{\mathbb{Q}}\left[\exp(-y\eta)\,1_{\{\tau\wedge\xi\leq T\}}\right] \\
&= \int_0^T \exp(-ys)\,f_{\tau\wedge\xi}(s)\,ds \\
&= \lambda Y_1 + Y_2,
\end{aligned}$$

$$\begin{aligned}
Y_1 &= \int_0^T \exp(-(y+\lambda)s)\,E_0^{\mathbb{Q}}\left[1_{\{\xi>s\}}\right]ds, \\
Y_2 &= \int_0^T \exp(-(y+\lambda)s)\,f_\xi(s)\,ds.
\end{aligned}$$

Y_2 is the T-truncated Laplace transform of ξ's $\mathbb{Q}$-p.d.f. with Laplace parameter $y+\lambda$,

$$Y_2 = E_0^{\mathbb{Q}}\left[\exp(-(y+\lambda)\xi)\,1_{\{\xi\leq T\}}\right].$$

Its closed form has been derived by Campi and Sbuelz (2004) and it can be found below after this proof. An integration by parts gives

$$\begin{aligned} Y_1 &= \frac{-1}{y+\lambda}\exp\left(-\left(y+\lambda\right)s\right)E_0^{\mathbb{Q}}\left[1_{\{\xi>s\}}\right]\Big|_0^T \\ &\quad -\int_0^T \frac{-1}{y+\lambda}\exp\left(-\left(y+\lambda\right)s\right)\left(-f_\xi\left(s\right)\right)ds \\ &= \frac{1}{y+\lambda}\left[1-\exp\left(-\left(y+\lambda\right)T\right)E_0^{\mathbb{Q}}\left[1_{\{\xi>T\}}\right]\right]-\frac{1}{y+\lambda}Y_2. \end{aligned}$$

This completes the proof.

5.B The discounted value of cash at ξ within $[0,T]$

The T-truncated Laplace transform of ξ's $\mathbb{Q}$-p.d.f. with Laplace parameter w ($w \geq 0$) has been shown by Campi and Sbuelz (2004) to be

$$E_0^{\mathbb{Q}}\left[\exp\left(-w\cdot\xi\right)1_{\{\xi\leq T\}}\right] = \lim_{\epsilon\searrow 0}\sum_{n=0}^{\infty} a_n\left(A,B\right)\left(\frac{x}{2}\right)^n\frac{\Gamma(\nu-n,\frac{x}{2K},\frac{x}{2\epsilon})}{\Gamma(\nu)},$$

$$\begin{aligned} \Gamma(\nu) &= \int_0^{+\infty} u^{\nu-1}e^{-u}du && \text{(Gamma Function)}, \\ \Gamma(\nu-n,\frac{x}{2K},\frac{x}{2\epsilon}) &= \int_{\frac{x}{2K}}^{\frac{x}{2\epsilon}} u^{-n}u^{\nu-1}e^{-u}du && \text{(Generalized Incomplete Gamma Function)}, \\ a_n\left(A,B\right) &= \left(-1\right)^n C\left(B,n\right)A^n\ , \\ C\left(B,n\right) &= \frac{\prod_{k=1}^{n}\left(B-\left(k-1\right)\right)}{n!}1_{\{n\geq 1\}}+1_{\{n=0\}}\ , \end{aligned}$$

$$\begin{aligned} x &= S^{2(1-\rho)}, \\ \nu &= \frac{1}{2(1-\rho)}, \\ A &= \frac{2\left(r-q+\lambda\right)}{\sigma^2(1-\rho)}, \\ K &= \frac{\sigma^2(1-\rho)}{2\left(r-q+\lambda\right)}\left(1-e^{-2T(r-q+\lambda)(1-\rho)}\right), \\ B &= \frac{w}{2\left(r-q+\lambda\right)\left(1-\rho\right)}. \end{aligned}$$

The Generalized Incomplete Gamma Function, the Incomplete Gamma Function, and the Gamma function are built-in routines in many computing softwares like MATLAB and Mathematica, which makes the above expressions fully viable.

5.C Model-based CB hedging

Full dynamic hedging of a long position in a CB implies being short δ units of stocks and γ units of CDSs a given the CDS rate f. Let Z be the recovery rate, and X be the notional of the CDS. Introduce the notation

$$D_{CDS} = V^{\mathbb{Q}}\left(S,T^{*},r\right)\left(1-Z\right)X - \sum_{j=1}^{k}\exp\left(-rT_{j}^{*}\right)\left[1-V^{\mathbb{Q}}\left(S,T_{j}^{*},0\right)\right]f$$

Then δ and γ are adapted processes that satisfy the following system of risk-nullifying equations:

$$\begin{aligned}\frac{\partial}{\partial S}P_{CB}-\delta-\gamma\frac{\partial}{\partial S}D_{CDS} &= 0,\\ R\cdot F-P_{CB}\left(S,T,r\right)+\delta-\gamma\left(\left(1-Z\right)X-D_{CDS}\right) &= 0.\end{aligned}$$

Our model also states that, in the case of a jump to default ($\eta=\tau$), pure Delta hedging recoups a fraction

$$\frac{\frac{\partial}{\partial S}P_{CB}\left(S_{\tau-},T-\tau,r\right)S_{\tau-}}{P_{CB}\left(S_{\tau-},T-\tau,r\right)-R\cdot F}$$

of the CB loss suffered at default.

5.D The objective probability of default at ξ within T

The replacement of the risk-neutral intensity-added drift $r-q+\lambda$ with the objective intensity-added drift $\mu_{\mathbb{P}}+\lambda_{\mathbb{P}}$ implies that the T-truncated Laplace transform of ξ 's $\mathbb{P}$-p.d.f. with Laplace parameter w ($w \geq 0$) has this analytic expression:

$$E_0^{\mathbb{P}}\left[\exp\left(-w\cdot\xi\right)1_{\{\xi\leq T\}}\right] = \lim_{\epsilon\searrow 0}\sum_{n=0}^{\infty} a_n\left(A_{\mathbb{P}},B_{\mathbb{P}}\right)\left(\frac{x}{2}\right)^n \frac{\Gamma(\nu-n,\frac{x}{2K_{\mathbb{P}}},\frac{x}{2\epsilon})}{\Gamma(\nu)},$$

$$A_{\mathbb{P}} = \tfrac{2(\mu_{\mathbb{P}}+\lambda_{\mathbb{P}})}{\sigma^2(1-\rho)}, \qquad K_{\mathbb{P}} = \tfrac{\sigma^2(1-\rho)}{2(\mu_{\mathbb{P}}+\lambda_{\mathbb{P}})}\left(1-e^{-2T(\mu_{\mathbb{P}}+\lambda_{\mathbb{P}})(1-\rho)}\right),$$

$$B_{\mathbb{P}} = \tfrac{w}{2(\mu_{\mathbb{P}}+\lambda_{\mathbb{P}})(1-\rho)}.$$

The analytic expression of the objective probability of diffusive default within time T is retrieved by taking $w=0$.

Hoofdstuk 6

Samenvatting

De hoekstenen van de moderne financieringstheorie zijn portefeuillekeuzetheorie en het prijzen via arbitrage. De moderne portefeuilletheorie, geïntroduceerd door Markowitz (1952), beoogt te verklaren hoe individuele of institutionele beleggers hun vermogen (zouden moeten) alloceren over risicovolle financiële activa. De theorie van het prijzen via arbitrage, in eerste instantie gebruikt voor het pijzen van opties door Black en Scholes (1973) en Merton (1973), en verder ontwikkeld door Harrison en Kreps (1979), Harrison en Pliska (1981), en gegeneraliseerd door Delbaen en Schachermayer (1994, 2005), betreft het prijzen van financiële activa via afwezigheid-van-arbitrage argumenten. Dit proefschrift bevat vier essays op het terrein van de portefeuillekeuze en het prijzen van activa via arbitrage.

Een portefeuillekeuze wordt gewoonlijk beschouwd als een afweging tussen rendement en risico van de portefeuille. Investeerders, die portefeuilles met hogere rendementen prefereren, pogen in het algemeen te volatiele activa te vermijden. Vanuit het perspectief van kapitaalreguleringvereisten hebben institutionele investeerders vaak ook belang bij het beperken van hun risicoblootstelling. Een traditionele benadering in de moderne portefeuilletheorie is ontwikkeld door Markowitz (1952), die heeft voorgesteld om de variantie van de portefeuille als risicomaatstaf te gebruiken en het verwachte rendement als beloningsmaatstaf. Gedurende talloze jaren was deze benadering de industriestandaard, vooral ook vanwege de eenvoud van de berekeningswijze. Echter, vanuit het perspectief van risicobeheer is de variantie geen bevredigende risicomaatstaf. Ten eerste worden vanwege de symmetrie in geval van de variantie winsten en verliezen als even ongewenst beschouwd. Dit nadeel werd in het bijzonder duidelijk door de ontwikkeling van activaderivaten, zoals opties, en kredietgestructureerde produkten, zoals *portefeuillefaillesementsrenteruilovereenkomsten* (portfolio default swaps) en onderpandige schuldverplichtingen. Ten tweede is de variantie ongeschikt om risico's op extreme gebeurtenissen met een kleine kans, zoals faillissementen, te beschrijven. Ten slotte, en gezien vanuit een theoretisch perspectief, is de verwachte rendement-variantie benadering niet in overeenstemming met tweede orde stochastische dominantie en dus ook niet met de verwachtenutsbenadering voor portefeuilleselectie.

Er zijn alternatieve modellen bedacht waarin de afweging tussen beloning en risico blijft gehandhaafd, maar met alternatieve risicomaatstaven voor de variantie om de modellen geschikter te maken voor praktische toepassingen. Gelijktijdig is er een axiomatische benadering ontwikkeld voor de theorie van risicometing door Artzner et al. (1999), die het concept van een coherente risicomaatstaf hebben geïntroduceerd, zodanig dat de risiscomaatstaf voldoet aan eigenschappen wenselijk vanuit het oogpunt van regulering. Bijzondere aandacht, zowel vanuit theoretisch als vanuit praktisch oogpunt, is geschonken aan *verwacht-tekort* (expected shortfall), een coherente risicomaatstaf consistent met tweede orde stochastische dominantie. Basset et al. (2004) en Portnoy en Koenker (1997) hebben laten zien dat een verwacht-rendement-verwacht-tekort- portefeuilleselectieprobleem, gebruikmakend van steekproefgegevens, kan worden geherformuleerd als een lineair-programmeringsprobleem dat op een efficiënte wijze kan worden opgelost met behulp van bestaande simplex en inwenig-punt-algoritmes. Zoals aangetoond door Kusuoka (2001) kan verwacht-tekort worden gegeneraliseerd tot de klasse van coherente reguliere risicomaatstaven (CRR-maatstaven) die de wenselijke eigenschappen van verwacht-tekort behouden.

In de hoofdstukken 2 en 3 worden de statistische en economische eigenschappen van verwacht-rendement-CRR-portefeuilles bestudeerd.

In Hoofdstuk 2 wordt een statistische spanningstoets ontwikkeld voor verwacht-rendement-CRR-efficiënte-grenzen, toegepast in hoofdstuk 3. Spanningstoetsen voor de combinatie verwacht-rendement-variantie, geïntroduceerd door Huberman en Kandel (1987), maken gebruik van regressie-analyse om te toetsen of de verwacht-rendement-variantie-efficiënte-grens gegenereerd door een bepaalde verzameling activa statistisch overeenkomt met de verwacht-rendement-variantie-efficiënte-grens gegenereerd door een deelverzameling van de activa. Sindsdien zijn verschillende modificaties voorgesteld van deze verwacht-rendement-variantie-spanningstoets. Een aardig overzicht hiervoor is De Roon en Nijman (2001). Zodra een investeerder ertoe besluit over te gaan van de conventionele verwacht-rendement-variantie benadering op de verwacht-rendement-CRR-portefeuilleselectie, ontstaat de noodzaak voor vergelijkbare toetsen in de nieuwe situatie. Analoog aan de verwacht-rendement-variantie-efficiënte-grens in de verwacht-rendement-variantie benadering kan men verwacht-rendement-CRR-efficiënte-grenzen construeren. De spanningstoets voor verwacht-rendement-CRR-efficiënte-grenzen is een belangrijke statistisch middel om de eventuele redundantie te beoordelen van bepaalde deelverzamelingen van activa vanuit het gezichtspunt van verwacht-rendement-CRR-efficiënte-grenzen. Zoals hoofdstuk 2 laat zien, kan deze toets, geheel in de geest van Huberman en Kandel (1987), worden uitgevoerd via een eenvoudige semiparametrische instrumentele-variabelen-regressie, waar de instrumentele variabelen een directe link hebben met de stochastische verdisconteringsvoet. De toets is gebaseerd op een relatie ontwikkeld door Tasche (1999), die geldt voor alle activa die voorkomen in de verwacht-rendement-CRR-efficiënte-portefeuille. Toepassingen van de verwacht rendement-CRR-spanningstoets voor verschillende coherente reguliere risicomaten, inclusief het welbekende verwacht-tekort, worden geïllustreerd.

In hoofdstuk 3 worden de verwacht-rendement-variantie-efficiënte-portefeuille en de verwacht-rendement-CRR-efficiënte-portefeuille vergeleken, zowel statistisch als economisch. CRR-maten worden steeds populairder in empirische toepassingen. Echter, Bertsimas et al. (2004) laten zien dat de variantie en een CRR-maat dezelfde optimale portefeuilles zullen opleveren voor activarendementen met elliptisch-symmetrische verdelingen. Alhoewel theoretische voordelen van een CRR-maat ten opzichte van de variantie in talloze studies zijn aangetoond, blijft de vraag bestaan naar de praktische significantie van het verschil tussen de beide benaderingen. Dit is in het bijzonder het geval voor typische financiële activa, zoals aandelen, valuta en marktindices, waarvan de rendementsverdelingen vaak bij benadering elliptisch symmetrisch blijkt te zijn. De vergelijking in hoofdstuk 3 vereist de afleiding van de asymptotische verdeling van optimale portefeuillegewichten verkregen uit een steekproefgebaseerde verwacht-rendement-risico-optimalisatie. De resultaten doen vermoeden dat zelfs voor typische activa de uitkomsten van verwacht-rendement-risico en verwacht-rendement-CRR optimalisaties statistisch en economisch verschillend kunnen zijn. De toetsen, ontwikkeld in dit hoofdstuk, laten tevens zien hoe schattingsonzekerheid, veroorzaakt door steekproeffouten in verwachte rendementen, hetgeen, zoals gerapporteerd door Chopra en Ziemba (1993) problematisch kan zijn in de context van portefeuillekeuze, als het ware kan worden uit- en aangezet. Ten slotte worden diverse verwacht-rendement-CRR-spanningstoetsen, ontwikkeld in hoofdstuk 2, toegepast op verschillende marktindices. De uitkomsten van de verwacht-rendement-variantie- en de verwacht-rendement-CRR-spanningstoetsen blijken voor conventionele activaklassen typisch dezelfde uitkomsten op te leveren. Echter, in geval van activa met asymmetrische rendementen wordt de verwacht-rendement-CRR-efficiëntie van verwacht-rendement-variantie-efficiënte portefeuilles verworpen. Dit suggereert superioriteit van de CRR-maat in geval van portefeuilles bestaande uit niet-standaard produkten, zoals combinaties van kredietinstrumenten en -derivaten. In geval van conventionele activa, zoals aandelen en valuta, leveren de verwacht-rendement-variantie en verwacht-rendement-CRR-benaderingen vergelijkbare uitkomsten.

De hoofdstukken 4 en 5 van het proefschrift bestuderen toepassingen van activaprijstheorieën voor het prijzen van opties en het modelleren van kredietrisico. De activaprijstheorie betreft gewoonlijk het prijzen zonder arbitrage van derivaten geschreven op enkele onderliggende basisactiva, waarvan de dynamiek statistisch wordt gemodelleerd. Een mooi voorbeeld van deze benadering wordt gegeven door Black en Scholes (1973) en Merton (1973), die de prijzen afleiden van Europese opties, geschreven op een onderliggend activum, dat een geometrisch Wienerproces volgt. Met groeiende georganiseerde en onderhandse markten voor afgeleide instrumenten is de activaprijstheorie een belangrijk hulpmiddel geworden voor het prijzen van afgeleide instrumenten. Optieprijsmodellen worden wijdverbreid gebruikt in de industrie, soms met geavanceerde veronderstellingen betreffende de onderliggende activa. Gemotiveerd door het empirisch bewijs aangaande de scheefheid van de geïmpliceerde volatiliteit biedt Heston (1993) een gesloten vorm oplossing in geval van een stochastisch volatiliteitsoptieprijsmodel. In dit model wordt in de optieprijzen ook rekening gehouden met de additionele volatiliteitsrisicofactor, die het model realistischer maakt door de rendementsverdeling aan te passen aan vaak waargenomen bovenmatige kurtosis en negatieve scheefheid. Duffie et al. (2000) generaliseren Hestons stochastische volatiliteitsmodel tot de klasse van affiene-sprong diffusies. Net zoals de activaprijstheorie kan worden toegepast op het prijzen van aandeelderivaten kan de activaprijstheorie ook toegepast worden op kredietinstrumenten. Merton (1974) heeft de zonder-arbitrage-prijsprincipes toegepast op het prijzen van bedrijfsschulden, door gebruik te maken van de hefboomratio als onderliggend proces waarvan de dynamiek statistisch wordt gemodelleerd. Talloze variaties op Mertons idee zijn toegepast in kredietrisicomodellen die worden gebruikt door financiële instellingen. Mertons model dient ook als basis voor structurele vorm benaderingen voor het modelleren van kredietrisico in de academische literatuur.

Het belangrijkste oogmerk van hoofdstuk 4 is de empirische kant van het prijzen van opties onder de veronderstelling van Hestons stochastische volatiliteit. Samenklontering en stochastische dynamiek van rendementsvolatiliteit is een empirisch feit dat allicht opgenomen dient te worden in realistische, statistische modellen voor activaprijsgedrag. Diverse ARCH en GARCH modellen ontsproten aan Engle (1982) en Bolerslev (1986) zijn geopperd om rekening te houden met de geobserveerde heteroskedasticiteit in activarendementen in discrete-tijdmodellen. Nelson (1991) introduceerde het E-GARCH-model dat ook het hefboomeffect in rendementsverdelingen kan modelleren.

Naast het modelleren van realistische dynamiek van activarendementen heeft de empirische literatuur aangaande optieprijzen laten zien dat het Black-Scholes model toegepast op waargenomen optieprijzen resulteert in een verschijnsel bekend als de geïmpliceerde volatiliteitsglimlach of -scheefheid, die inconsistent is met het model. Dit verschijnsel wordt vooral toegeschreven aan zowel het hefboomeffect in activarendementen als aan de dikke staarten van de empirische rendementsverdeling, welke worden genegeerd in het Black-Scholes model. Stochastische volatiliteitsoptieprijsmodellen corrigeren de inconsistentie tussen de optieprijs en de onderliggende aandeeldynamiek gedeeltelijk. Echter, het is bekend dat in geval van stochastische volatiliteitsmodellen de financiële markten in het algemeen incompleet zijn in termen van het onderliggende activum, aangezien de stochastische volatiliteit niet kan worden afgedekt. Dit betekent dat de volatiliteitsrisicopremie niet identificeerbaar is op basis van uitsluitend de onderliggende activumdynamiek. Verhandelde optiecontracten kunnen worden gebruikt om de ontbrekende informatie over het prijsmechanisme te achterhalen. In het bijzonder kunnen, analoog aan de geïmpliceerde volatiliteiten in het Black-Scholes-model, de geïmpliceerde prijzen van het volatiliteitsrisico worden geschat via optieprijzen. De prijs van het volatiliteitsrisico kan worden geïnterpreteerd als de markthouding jegens risico. Hoofdstuk 4 analyseert de dynamiek van de geïmpliceerde prijzen van het volatiliteitsrisico vanuit dit perspectief. Het onderzoekt de dynamiek van de geïmpliceerde prijzen van het volatiliteitsrisico en laat zien dat het modelleren van de dynamiek hiervan een significante bijdrage levert aan het verbeteren van de optieprijsprestatie buiten de steekproef.

Hoofdstuk 5 stelt een alternatieve wijze voor om kredietrisico van bedrijven in nood te modelleren. Bestaande structurele-vorm-kredietrisicomodellen vereisen het gebruik van infrequente en vaak verstoorde informatie over de kapitaalstructuur van een bedrijf. De resulterende prijsprestaties van deze modellen, met name voor bedrijven in nood, is niet naar tevredenheid, zie Eom et al. (2004). De aandeelprijs van een bedrijf in nood daarentegen kan een informatieve indicatie zijn van de kredietrisico volgens de markt. Als imperfecte afdekking tegen faillissement wordt de aandeelprijs informatiever als het bedrijf dichter aankomt tegen faillissement. Vanuit een econometrisch oogpunt is het modelleren van een faillissement via de aandeelprijs ook aantrekkelijk vanwege de betere kwaliteit en frequentere beschikbaarheid van data. In tegenstelling tot de srructurele- en herleide-vorm-modellen voor kredietrisico, stelt het model in hoofdstuk 5 voor om het aandeel te gebruiken als liquide en waarneembare primitieve om analytisch bedrijfsobligaties en kredietfaillissementsrenteruilovereenkomsten te modelleren. Op deze wijze worden restringerende veronderstellingen aangaande de bedrijfskapitaalstructuur vermeden. Faillissement wordt eenvoudigweg weergegeven als de aandeelprijs die de nulgrens passeert of op continue wijze of via een sprong, hetgeen een kredietwijdte ongelijk nul impliceert voor korte looptijden. Eenvoudige kruisactivumafdekking wordt mogelijk. Via een bondig geformuleerde Radon-Nikodym afgeleide maken we ook analytisch kredietrisicomanagement mogelijk in geval van systematisch sprong-naar-faillissementsrisico.

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