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*Published in:*
Mathematical Methods of Operations Research

*Publication date:*
2005

# Drop Out Monotonic Rules for Sequencing Situations

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**Abstract**

This note introduces a new monotonicity property for sequencing situations. A sequencing rule is called drop out monotonic if no player will be worse off whenever one of the players decides to drop out of the queue before processing starts. This intuitively appealing property turns out to be very strong: we show that there is at most one rule satisfying both stability and drop out monotonicity. For the standard model of linear cost functions, the existence of this rule is established.

## 1 Introduction

For various classes of economic situations in which agents can cooperate, allocation rules have been developed, which handle the problem of allocating the rewards or cost savings from cooperation among the agents involved. Often, for such an economic situation, one can define a corresponding cooperative game. In this setting, *stable* rules are interesting, which assign allocations that are core elements of the corresponding games.

In this note, we consider stable rules, which behave well in case agents drop out, leaving a reduced economic situation. We say that a rule is *drop out monotonic* if applying the rule to the reduced situation yields an allocation, which, depending on the context, either makes all remaining players better off or all players worse off than in the original situation.

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If the cooperative games corresponding to reduced situations are subgames of the the original game, then a stable and drop out monotonic rule generates a population monotonic allocation scheme (pmas) for the original game (Sprumont (1990)). In the cases of linear production situations (Owen (1975)), airport situations (Littlechild and Owen (1973)) and holding situations (Tijs et al. (2000)), the game corresponding to a reduced situation after one player drops out is a subgame of the original game. So here, the existence of stable and monotonic rules boils down to the existence of a pmas. Such pmas-es do not always exist for linear production games. However, for airport situations, the Shapley value induces one of many stable and drop out monotonic rules. For holding games, the rule which gives all gains to the so-called holding house keeper is a pmas.

The property of drop out monotonicity introduced in this note is inspired by the so-called fairness condition introduced in Ambec and Sprumont (2002). They study the problem of water management from a game theoretical point of view: given a river of certain capacity flowing through a number of countries with certain demand for water, how should the water of the river be allocated?

The fairness condition states that whenever one of the countries ceases to demand water (drops out), all other countries should be better off. Contrary to the examples mentioned before, the reduced situation after a player drops out does not give rise to a subgame of the original game. Ambec and Sprumont show that there is a unique allocation rule which satisfies both stability (*ie*, generates a core element) and the fairness condition. This rule (the $\mu$ rule) is the marginal vector corresponding to the ordering of the countries along the river (from upstream to downstream).

This note studies the drop out monotonicity property in the context of sequencing situations, as introduced in Curiel et al. (1989), in which there is also a natural ordering of the players forming the initial queue. Indeed, in the most basic class of sequencing situations (with linear cost functions), a result similar to Ambec and Sprumont is established. Within a more general class of sequencing situations (with regular cost functions), it turns out that there is *at most* one stable and drop out monotonic rule, which must be the analogue of the $\mu$ rule.

## 2 Sequencing situations

In this section, we briefly review the model of sequencing situations as introduced in Curiel et al. (1989). A *sequencing situation* is a triple $(N, p, k)$, where $N = \{1, ..., n\}$ is a finite set of players waiting in front of a machine in order to have a job processed, $p \in \mathbb{R}^N_{++}$ is a vector of processing times, where $p_i > 0$ represents the time that the job of player $i \in N$ requires to be processed, and $k = (k_i)_{i \in N}$ is a vector of cost functions. For each player $i \in N$, $k_i(t)$ represents the costs of player $i$ if his job is completed in $t$ time units. Costs are assumed to be additive: the total costs of a coalition $S \subset N$ equal the sum of the individual costs of the members of $S$. Furthermore, the cost functions are regular, *ie*, for all $i \in N$, $k_i(t)$ is increasing in $t$ and $k_i(0) = 0$.

We pay special attention to the class of sequencing situations with linear cost functions: $k_i(t) = \alpha_i t$ for all $t \in \mathbb{R}_+$ with $\alpha_i \geq 0$. A *sequencing situation with linear cost functions* is denoted by $(N, p, \alpha)$ with $\alpha = (\alpha_i)_{i \in N}$.

In a sequencing situation, there is an initial ordering on the players in the queue, which without loss of generality we assume to put player 1 at the front and player $n$ at the back. By swapping *adjacent* places in the queue, players are allowed to save costs (*cf.* Curiel et al. (1989)). As only neighbours can switch, the cost savings of a coalition of players equals the sum of the savings made by its connected components. In this note, we consider the resulting *sequencing cost game* $(N, c)$, where for a coalition $S \subset N$, $c(S)$ is defined as the minimal total costs of the members of $S$ over all their admissible rearrangements in the queue.[1]

The core of a cost game $(N, c)$ is defined by

$$C(c) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = c(N), \forall_{S \subset N} : \sum_{i \in S} x_i \leq c(S)\}.$$

Core elements are stable in the sense that if such a vector is proposed as cost allocation for the grand coalition, no coalition will have an incentive to split off and cooperate on their own.

A *sequencing rule* is a function $f$ assigning to every sequencing situation $(N, p, k)$ a vector $f(N, p, k) \in \mathbb{R}^N_+$ such that $\sum_{i \in N} f_i(N, p, k) = c(N)$. A rule $f$ is called *stable* if $\mu(N, p, k) \in C(c)$ for every sequencing situation $(N, p, k)$. In this note, we investigate the $\mu$ rule, defined by

[1] Curiel et al. (1989) consider the related *cost savings* game. The definition of the cost game is analogous and omitted for the sake of brevity.

$$\mu_j(N,p,k) = c(P_j) - c(P_{j-1})$$

for all $j \in N$, where $P_j = \{1,\ldots,j\}$. In case the cost functions are linear, we can rewrite this as

$$\mu_j(N,p,\alpha) = c(\{j\}) - \sum_{i \in N: i<j} g_{ij}.,$$

where $g_{ij} = \max\{0, \alpha_j p_i - \alpha_i p_j\}$ equals the cost savings attainable by player $i$ and $j$ when $i$ is directly in front of $j$, regardless of the exact position in the order. According to the $\mu$ rule, the gain $g_{ij}$ goes fully to player $j$, who is behind $i$ in the queue.

Since every sequencing game is $\sigma$-component additive (*cf.* Curiel et al. (1995)), the $\mu$ rule is stable. So letting the players at the front of the queue pay the highest costs and attributing the gains to the players at the back of the queue results in a stable outcome.

# 3 Drop out monotonicity

Suppose that one player in the queue decides to wait no longer and drops out. One natural question in this situation is how the costs of the other players will be affected by this. It seems natural that none of the players should be worse off if one of them drops out of the queue. Formally, a rule $f$ is called *drop out monotonic* if for all sequencing situations $(N,p,k)$ and all $q \in N$ we have

$$f_j(N,p,k) \geq f_j((N,p,k)^{-q})$$

for all $j \in N \backslash \{q\}$, where $(N,p,k)^{-q} = (N\backslash\{q\}, (p_i)_{i \in N\backslash\{q\}}, (k_i)_{i \in N\backslash\{q\}})$ is the reduced situation without player $q$, in which the initial ordering on the remaining players is the same as in the original situation.

**Proposition 3.1** *$\mu$ is drop out monotonic on the class of sequencing situations with linear cost functions.*

**Proof:** Let $(N,p,\alpha)$ be a sequencing situation with linear cost functions, let $q \in N$ and let $j \in N\backslash\{q\}$. If $j < q$, then $\mu_j(N,p,\alpha) = c(\{j\}) - \sum_{i \in N: i<j} g_{ij} = \mu_j((N,p,\alpha)^{-q})$. If $j > q$, then $\mu_j((N,p,\alpha)^{-q}) = (\sum_{i=1}^{j} p_i - p_q)\alpha_j - \sum_{i \in N: i<j} g_{ij} + g_{qj} = ((\sum_{i=1}^{j} p_i)\alpha_j - \sum_{i \in N: i<j} g_{ij}) + (g_{qj} - p_q\alpha_j) = \mu_j(N,p,\alpha) - \min\{p_j\alpha_q, p_q\alpha_j\} \leq \mu_j(N,p,\alpha)$. $\square$

Proposition 3.1 shows that the $\mu$ rule is drop out monotonic in case the cost functions are linear. The question now arises whether this is the only rule satisfying this property. In the following theorem, we show that within the class of sequencing situations with regular cost functions (not necessarily linear), the $\mu$ rule is the only possible stable and drop out monotonic rule.

**Theorem 3.2** *Let $f$ be a sequencing rule. If $f$ is stable and drop out monotonic, then $f$ equals the $\mu$ rule.*

**Proof:** Let $(N, p, k)$ be a sequencing situation with regular cost functions and let $f$ be a stable and drop out monotonic rule. Denote the corresponding game by $(N, c)$ and denote $f_i^S = f_i(S, (p_j)_{j\in S}, (k_j)_{j\in S})$ and $\mu_i = \mu_i(N, p, k)$ for all $i \in N$ and $S \subset N, S \neq \emptyset$. We show that $f = \mu$ by an inductive argument.
First, from drop out monotonicity it follows that $f_1^N \geq f_1^{\{1\}}$. From stability we have $f_1^N \leq c(\{1\}) = f_1^{\{1\}}$. Hence, $f_1^N = f_1^{\{1\}} = c(\{1\}) = \mu_1$.
Next, let $j \in \{2, \ldots n\}$. Assume $f_i^N = f_i^{P_{j-1}} = \mu_i$ for all $i \in P_{j-1}$. From drop out monotonicity we have $f_i^N \geq f_i^{P_j}$ for all $i \in P_j$, so $\sum_{i\in P_j} f_i^N \geq \sum_{i\in P_j} f_i^{P_j} = c(P_j)$. By stability, $\sum_{i\in P_j} f_i^N \leq c(P_j)$. So $\sum_{i\in P_j} f_i^N = c(P_j)$ and, using the induction hypothesis, $f_j^N = c(P_j) - \sum_{i\in P_{j-1}} f_i^N = c(P_j) - c(P_{j-1}) = \mu_j$.
Hence, we may conclude that $f = f^N = \mu$. $\square$

It follows from Proposition 3.1 and Theorem 3.2 that drop out monotonicity and stability together characterise the $\mu$ rule on the class of sequencing situations with linear cost functions.

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