Tilburg University

Skill and strategy in games

Dreef, M.R.M.

Publication date:
2005

Skill and Strategy in Games

Skill and Strategy in Games

Proefschrift

ter verkrijging van de graad van doctor aan de Universiteit van Tilburg, op gezag van de rector magnificus, prof.dr. F.A. van der Duyn Schouten, in het openbaar te verdedigen ten overstaan van een door het college voor promoties aangewezen commissie in de aula van de Universiteit op vrijdag 30 september 2005 om 14.15 uur door

Marcellinus Ronaldus Maria Dreef

geboren op 31 augustus 1978 te Breda.

Promotores: Prof. dr. P.E.M. Borm
Prof. dr. B.B. van der Genugten

Beproeft alle dingen;
behoudt het goede.

Bijbel, 1 Tessalonicensen 5:21.

Preface

When looking back at my years as a PhD student, I have no difficulty convincing myself that it was a good decision to accept this scientific challenge in the first place. Partly this feeling is based on the useful lessons and experiences that came with the research. The main reason for the satisfaction with my choice, however, is formed by the people that supported me and kept me company along the way. At this place, I want to thank them.

First of all, I want to express my gratitude towards my two supervisors, Peter Borm and Ben van der Genugten. Their enthusiasm has inspired me, even at times when I was not too enthusiastic about the results of my work myself. I am grateful that they have given me the *chance* to improve my *skill* in doing research under their guidance.

Equally inspiring was my cooperation with Stef Tijs, who is co-author of chapter 7. He introduced his open question concerning the value of coin games to me and I am glad we finally succeeded in answering it.

The fourth person who had a significant influence on parts of this thesis, is Mark Voorneveld. He has supplied me with critical comments upon early versions of the articles on which chapters 2, 4 and 5 are based.

These four people have also formed my thesis committee, together with Joseph Kadane, Fioravante Patrone, Maarten Mastboom and Cyriel Fijnaut. I want to thank them all for their interest in my work.

During the largest part of my stay at the Department of Econometrics and Operations Research, I have shared an office with Ruud Hendrickx. I want to thank him for critically checking my manuscript for typos and other small mistakes, for the discussions on everything and more, for the game practice sessions and for his company.

Together with Ruud, Hendri Adriaens is responsible for some of the TEXnical tricks required to make the layout of this thesis as it is. Thanks for being avail-

able as a "help desk" at the times I needed it.

Many other colleagues at the department who may not have had direct influence on my thesis were important in making daily life at Tilburg University as pleasant as it was for me.

Finally, I am grateful that Bart-Jeroen Haselbekke and Joline van Sorge have agreed to act as my *paranimfen*.

The last and most important paragraph of this section is saved for the people who are closest to me. I want to thank my parents, who have been there for me all my life and have supported me in everything I did. And last, but definitely not least, I want to tell Carmelina how happy I am that she is always there to show me the things in life that are more important than work.

Marcel Dreef
July 2005

Contents

Chapter 1

Introduction

1.1 Motivation

Games, played in the casino or at home, form an interesting topic of discussion, for players as well as for spectators. Almost everyone has an opinion about how to play a game like roulette or blackjack. Moreover, everyone tends to have an opinion on how much skill is involved in playing these games. However, in general the amount of skill that is ascribed to a game, varies widely among the discussants. Take the game of poker as an example. Fanatic poker players will be convinced that poker is a game of skill, thereby justifying the amount of time they spend on the game. For other people, like chess devotees, merely the fact that dealing of cards is involved, is sufficient to qualify poker as a game of chance.

This tendency of disagreement on the skill level of games becomes a problem when the exploitation of the games is concerned. Gaming acts, in The Netherlands and in other European countries, but also in many states in the USA, distinguish between games of skill and games of chance. Generally speaking, games in which random factors are the main determinants of the outcome are games of chance, while games in which the behaviour of the players predominantly influences the game result are games of skill. According to the Dutch Gaming Act, a licence is required for a casino to exploit a game of chance, whereas anyone is allowed to offer a game of skill. Therefore, from a juridical perspective, it is important that one can objectively determine for a game whether the players have sufficient influence on the game result to classify it as a game of skill. The determination of the relative skill level of a game, by comparing the players' influence on game results to the influence of the random factors, is the first primary topic of this thesis.

The other main subject discussed in this dissertation, is the computation of optimal strategies in two-person games with zero-sum payoffs. For a one-person game like blackjack, the optimal strategy may be too complex for human players to memorize and execute it perfectly. However, the computation of it, using probability theory to deal with the uncertainty generated by the unknown cards, is relatively simple. In games with two players, like poker, optimal play is still well-defined as long as the payoffs of the players sum to zero. However, the computations are difficult, since the quality of a player's strategy depends on the strategy used by the opponent. We investigate the computation of optimal play in two-person variants of poker in this thesis. We also discuss optimal strategies for a class of take-and-guess games that used to be rather popular in bars to determine who has to pay for the beer.

1.2 Outline

The dissertation consists of two parts. Part I, consisting of chapters 2 to 4, mainly deals with relative skill and the role of random factors in games. Part II, consisting of chapters 5 to 7, is devoted to the computation of optimal strategies in two interesting classes of games, poker and take-and-guess games.

Part I starts with chapter 2, which presents and motivates a quantitative measure for the relative skill level of a game. Although the concept of skill in games is mentioned in the literature much earlier, e.g., by Borel (1953), only in the last two decades it started to gain more attention. Larkey et al. (1997) provided an interesting discussion on the interpretation and the relevance of skill in analyzing and solving games. The reasons they had for writing their article, however, did not include the motivation that is behind the study of skill involved in games, as one finds it in this thesis: the current source of inspiration is the law. The Dutch Gaming Act has been the starting point for the way in which we tackle the difficult problem of measuring relative skill in games with random factors. In casino games, random generators like the dealing of cards or the spinning of a wheel in roulette imply objective probabilities for the occurrence of the uncertain events in these games. In principle, it is possible to repeat these games under the same conditions over and over again. This makes it possible to speak about a player's average game result in the long run or, equivalently, his expected game result. Expected game results for beginners and advanced (or optimal) players are compared to determine whether strategic choices by the players influence the outcome of the game. A comparison with the influence of the chance elements is made by investigating what game results advanced players could attain if they were informed about outcomes of the chance moves before making their strategic decisions. This framework for measuring relative skill in games was initiated by the work of Borm and Van der Genugten (1998) and extended in Dreef, Borm and Van der Genugten (2004b).

Chapter 3 studies a type of sports-related competitions that has become popular in recent years: management games. A participant in such a game acts as the manager of a fictive sports team. Examples of sports for which management games are organized are soccer, tennis, cycling and Formula One racing. Given a set of restrictions, the participant selects players and possibly additional elements for his team. His team earns points for certain events that occur in the sports competition to which the management game is related. The primary goal of the game is to maximize the number of points earned during the competition. Basically, a team scores well in the management game if the team members do well in the real competition. The large number of people participating in management games on the Internet has turned this type of entertainment into a profitable business. However, since this business concerns "exploiting games with monetary prizes", the Gaming Act may require a licence to organize a management game, depending on the participants' influence on their results.

In contrast to casino games, in a management game there is no objectively defined randomization process influencing the scores of the participants. However, these scores are subject to a different type of uncertainty: the uncertainty about the results of the real sports events can be interpreted as the random factor in the management game. With this type of random factors, it is not possible to compute the expected game result of a player by taking into account predetermined probabilities. Therefore, we have to adapt the methods from chapter 2, before we can apply them to a management game in order to determine whether it should be classified as a game of chance or as a game of skill. We use a statistical model to estimate the random factor in the scores of the participants. The notion of the expected game result of a certain type of player is replaced by the notion of the average result of a group of players of this type.

In chapter 4 we return to the class of games in which random generators or chance moves generate uncertainty using objective probabilities. Think again of the cards in poker or the throw of dice in backgammon. We focus on how much the uncertainty created by these chance moves restricts players in their control over the outcome of a game. When determining a strategy for the game, a player has to take into account all possible outcomes of the chance moves. The formulation of a good strategy would be easier if he would know the outcome of the chance moves in advance. The information about this outcome is valuable to the player. But how valuable is it? How much is he willing to pay for this information if he could buy it? Of course, this depends on the amount by which he can increase (or decrease) his expected payoff using this extra knowledge. Loosely formulated, the difference between what a player can do with and without the information, is called the value of information. In contrast to other definitions of the value of information in the literature (see, e.g., Borm (1988) and Kamien et al. (1990)) the model in chapter 4 takes into account that this value may depend on the opposition the player faces. For example, it might be very useful to have the information if the opponent does not have it, while it is less valuable to know the outcome of the chance move if the opponent knows it too. The computations of the value of information in chapter 4 use a pre-game that was introduced by Sakaguchi (1993). In such a pre-game, both players get the opportunity to buy information about the outcome of the chance moves before the start of the game.

Part II of this thesis, consisting of chapters 5 to 7, is devoted to the computation of optimal strategies in two-person zero-sum games. For one-person games, computing optimal strategies is in general not difficult. However, when two players are involved, the complexity of the computations increases, since the quality of a strategy depends on the strategy the opponent uses. Another complicating factor for determining optimal play in games is formed by the uncertainty players face as a consequence of random factors. The nature of the topics studied in Part II influences the style of writing: this part is more technical than Part I.

Chapter 5 presents the analysis of a simple poker model. The analysis of poker is interesting for a wider audience than just for poker players. The game provides an excellent domain for investigating problems of decision making under uncertainty. It raises interesting questions about the role of information in the game and brings challenges to research in artificial intelligence. And, of course, it is a class of games that is interesting for application of the skill analysis described in chapter 2. Since poker does not involve playing out cards, as opposed to a game like bridge, all strategic aspects in the game concern the bidding by the players. Unfortunately, even though the strategic structure of the game is relatively simple, real poker games are difficult to analyze. From a deck of cards, millions of different poker hands can be drawn, so that the dimension of the representation of the game quickly becomes too large to analyze, even for modern high-speed computers.

To handle this problem of the large numbers of hands, we can order them and represent them by numbers between zero and one on the real line. The highest possible poker hand, a royal flush, then corresponds to one, while the lowest hand corresponds to zero. To make the analysis of the game simpler, one can model the card distribution as a continuous distribution on the interval $[0, 1]$, thereby implicitly increasing the number of possible hands from "very large" to infinity. This approach is followed in this chapter, which studies a two-player poker game with alternate bidding that was introduced by Von Neumann and Morgenstern (1944, chapter 19). In the original model, the hands of the players are drawn from a continuous uniform distribution on $[0, 1]$. We extend the model by allowing for other than uniform hand distributions. We analytically compute the value of the game as well as optimal strategies for both players. Next, we translate the strategic results in the continuous game to the situation where the game is played with a deck of 52 cards, from which real five-card poker hands are drawn. Finally, we determine the relative skill level of the game.

Chapter 6 discusses the computation of optimal strategies in poker models with a betting structure that is more difficult than the model from chapter 5. Whereas the model of Von Neumann and Morgenstern (1944) is sufficiently simple to find the equilibria of the game directly using a mathematical analysis, such a direct approach to find equilibria in the game with continuous card distributions is not always possible in games with more complex betting structures. We present a way to find an equilibrium in such a game by sophisticatedly using information from an equilibrium in a related discrete game. The chapter is concluded by a presentation of the equilibrium of the largest continuous poker model that we are able to analyze completely. The model includes a raising possibility for both players. For this game we also determine the relative skill level.

Chapter 7 studies a less famous but mathematically equally interesting class of games, formed by the so-called take-and-guess games. This class can be divided into two subclasses. In both subclasses, each of the two players has to take a number of objects out of a given private set of objects. After that, they have to guess the total amount of objects taken by both of them. In the first class, the morra games, both players have to announce their guesses simultaneously. In the other class, the so-called coin games, the players announce their guesses sequentially.

Take-and-guess games differ from poker in the fact that no external chance moves are involved. The uncertainty for a player is solely generated by his opponent. Especially for the coin games, this does not guarantee that the computation of optimal strategies is easy. We give an overview of the values for morra and coin games and we describe optimal strategies for both players for all possible numbers of coins explicitly.

Although it is still difficult to determine the relative skill level of a given game in practice, Part I of this thesis can help in reaching agreement about it. This may resolve discussions between players and spectators, but, more importantly, the methods for measuring relative skill presented in this part of the dissertation can serve as a tool for a judge who has to decide whether the exploitation of a game is allowed or not. Part II may be more of direct interest for those who like playing games, in particular for players of poker and take-and-guess games. They could use the global information about optimal strategies to increase their playing level.

1.3 Publication background

Most of the contents of this thesis has already been published in articles or in research reports. The exposition of the skill analysis in chapter 2 is to a large extent based on Dreef, Borm and Van der Genugten (2004b) and the overview article by Dreef, Borm and Van der Genugten (2004a). The skill analysis for management games that is described in chapter 3 was originally carried out for the research report Van der Genugten, Borm and Dreef (2004).

Chapter 4, describing the role of chance moves and information in two-person games, is based on Dreef and Borm (2005).

Of the two chapters on poker, chapter 5 and 6, only the first is based on an earlier publication. The contents of chapter 5 have originally been published as Dreef, Borm and Van der Genugten (2003), except for part of the skill computations. These computations have served as an illustration in Dreef, Borm and Van der Genugten (2004a).

The results of chapter 7 have appeared as the discussion paper Dreef and Tijs (2004).

Part I

Skill and information in games

Chapter 2

Measuring skill in games

2.1 Introduction

How should one define skill in games? The definition of skill that one finds in a typical dictionary, is "the special ability to do something well, especially as gained by learning and practice". To be able to use such a broad definition within the context of games, it should be refined. Larkey, Kadane, Austin and Zamir (1997) defined skill as "the extent to which a player, properly motivated, can perform the mandated cognitive and/or physical behaviours for success in a specific game". Whereas this definition concerns the player, we are interested in defining the skill level of the game for the whole population of players. To make the definition of skill applicable to games instead of individual players within the same game, we modify it such that it expresses how *useful* the player's abilities can be for him in the game. In a game with a high skill level, skillful players can have a significant advantage over the less competent participants, whereas this advantage should be relatively small in games with a low skill level. To give a simple example, a perfect memory may not help you in roulette, but in poker it does. As the articles of Larkey et al. (1997) and Reep et al. (1971) indicate, the notion of skill can be defined for a large class of games, including various ball games as well as card games and mind sports. The current chapter concentrates on games for which the outcome can be expressed in terms of money and in which players can be identified by their strategies. Moreover, the games can, at least in theory, be repeated under the same conditions. For these games it is possible to give an objective quantification of uncertainty in terms of probability. This is an important property of casino games. Although this property applies to a much wider class of games than what is generally understood by this term, we refer to the class of games under consideration as *casino games*.

For casino games, we define skill as the relative extent to which the outcome of a game is influenced by the players, compared to the extent to which the outcome depends on the influence of the random factors involved. For random factors one can think of the spinning of a roulette wheel or the dealing of cards. The larger the influence of the players on the game outcome, the higher the skill level. Games without random elements, in which only the players have influence on the outcome, are called pure games of skill; games in which only the random factors affect the outcome, are pure games of chance. A game like chess belongs to the first category, while roulette is intuitively classified as a pure game of chance.[1] Although the classification is easy for these two games, there is a large number of games in which the two sources of influence are both present and for which the skill level lies somewhere in the area between the pure games of skill and the pure games of chance.

From a juridical perspective, it is important that one can determine for these games in the grey area whether the players are sufficiently influential in a game to classify it as a game of skill or not. According to the Dutch Gaming Act, a licence is required to exploit a game of chance, whereas anyone is allowed to offer a game of skill. Similar laws apply in other European countries, as well as in many states in the USA. It is not difficult to imagine that the organizer of a game and the legislator have different opinions about the role of chance in a game. Qualitatively judging the role of chance is rather subjective and the exploitation of games of chance is a lucrative business, since these games are appealing to a large audience. Caillois (1979, p. 115) argues:

> "[Games of chance] promise the lucky player a more modest fortune than he expects, but the very thought of it is sufficient to dazzle him. Anyone can win. This illusory expectation encourages the lowly to be more tolerant of a mediocre status that they have no practical means of improving. Extraordinary luck—a miracle—would be needed. It is the function of *alea* to always hold out

[1] Even pure games of chance are not always classified as such by the participants. The way the game is presented may lead to misperceived skillful influence over non-controllable events, as Wohl and Enzle (2002) show.

> hope of such a miracle. That is why games of chance continue to prosper. The state itself even profits from this. Despite the protest of moralists, it establishes official lotteries, thus benefiting from a source of revenue that for once is accepted enthusiastically by the public."

The observation that the state itself profits from the appeal of games of chance is also true for the Netherlands. In practice, the government only grants the required licence to Holland Casino, a state-owned company. The government has both the control and the profits of this market. In fact, obtaining the profits of the legal gambling activities was one of the main goals of the revision of the Dutch Gaming Act in 1964.

Borm and Van der Genugten (1998, 2001) presented the basics of a method that can be used to determine whether a game can be classified as a game of skill or not. This method is based on the Dutch Gaming Act. In the current chapter, which is based on Dreef, Borm and Van der Genugten (2004a, 2004b), we discuss the relevant aspects of this method as well as some related practical issues and we present a slightly modified definition. The general framework is described in section 2.2. The sections 2.3 to 2.8 are devoted to the description of the details of the analysis.

Whereas the skill measure is meant to determine the skill involved in the game as a whole, it is in general interesting to study the skill level of individual players as well. In sports player skill levels can be recognized, for example, in the form of handicaps assigned to golf players. Within the class of games we focus on, one can think of the ELO ratings of chess players that determine their position on the world ranking. Section 2.9 contains some discussion on this topic.

In section 2.10 we illustrate the computations involved with the skill analysis by two examples. The concluding section, section 2.11, sketches some possibilities to investigate the skill level of games using empirical data. This last subject will get more attention in chapter 3, in which we discuss a case study.

2.2 A relative measure of skill

The method that Borm and Van der Genugten (1998, 2001) developed is based on the following important passage in the Dutch Gaming Act[2], which gives a qualitative characterization of the class of games for which a licence is needed:

> [...] it is not allowed to: exploit games with monetary prizes if the participants in general do not have a predominant influence on the winning possibilities, unless in compliance to this act, a licence is granted [...].

All games that satisfy this definition, are called *games of chance*. By definition, all games to which this definition does not apply because the players' influence on the outcomes is sufficiently large, are referred to as *games of skill*. Borm and Van der Genugten (1998) give the following three qualitative requirements which summarize the basic ideas underlying the Dutch legislation concerning the exploitation of games with chance elements.

(R1) The legislation applies exclusively to situations which involve the exploitation of games with monetary prizes.

(R2) The skill of a player should be measured as his average game result in the long run, i.e., in terms of expected result. For a game to be qualified as a game of skill, it is necessary that these expected results vary among players.

(R3) The fact that there is a difference between players with respect to their expected payoffs does not immediately imply that the underlying game is a game of skill. For a game of skill it is sufficient that the chance elements involved do not prohibit these differences to be substantial.

Using the requirements (R1)-(R3), we are ready to give the general framework of the relative skill measure for one-person games. To take into account requirement (R1), we restrict attention in our analysis to games in which the "game result" of a player can be expressed in terms of money.

[2]For a brief description of the legal framework in the USA we refer to Ware and Kadane (2002). They apply the skill analysis to electronic draw poker and conclude that "the theory of Borm and Van der Genugten seems promising as a way to describe the relative importance of skill and chance in games such as electronic draw poker".

The difference in player results that is required by (R2), can be measured by what is called the *learning effect* in the game. According to (R3), it is not the absolute size of the learning effect that determines the skill level of a game, but the relative size of this effect in relation to the restrictive possibilities within the game set by the chance elements. Therefore, one should also quantify this restrictive influence of the random factors. One can do this by investigating the possibilities of the players in the absence of these random moves. This restriction by the chance elements is captured in the *random effect* of the game. Using these two effects, Borm and Van der Genugten (1998) defined

$$\text{skill level} = \frac{\text{learning effect}}{\text{learning effect} + \text{random effect}}. \tag{2.1}$$

Formal definitions of the learning effect and the random effect will follow later, but let us already note that these concepts will be defined such that the corresponding numbers will be nonnegative. This implies that

$$\text{pure games of chance} \quad 0 \leq \quad \text{skill level} \quad \leq 1 \quad \text{pure games of skill.}$$

Games in which the random effect dominates the learning effect will have a low skill level. For games in which the learning effect dominates, the skill level will be high.

The following sections will make clear how the concepts described above are formally defined in order to obtain an objective way to fix the skill level of a specific casino game and to compare games on this aspect. The choice of the appropriate bound between games of chance and games of skill is the task of a judge. We come back to this issue at the end of section 2.10.

2.3 Player types

The jurisprudence regarding the Dutch Gaming Act indicates how one should interpret the framework that we presented in the previous section in practice. Both the learning effect and the random effect should be measured by comparing different types of players. For more details concerning the Gaming Act and the corresponding jurisprudence we refer to Van der Genugten and Borm (1994) and Van der Genugten, Das and Borm (2001, chapter 9). In this section we describe and briefly discuss the three player types that are used in the analysis. We call them beginners, optimal players and fictive players.

2.3.1 Beginners

A beginner is a player who has only just familiarized himself with the rules of the game. He plays a relatively simple, naive strategy. In game theoretic terms, a beginner can be thought of as a specific type of boundedly rational player.

It is not always easy to determine the behaviour of beginners in a specific game. In general, we distinguish three ways to do this. First of all, in games with a structure like roulette, we think it is reasonable to assume that a beginner chooses randomly between all pure strategies that are not obviously stupid. The category of games for which this method is suitable, however, is not the most interesting category with respect to the analysis of skill. In many games this approach does not make sense. In a poker game, for example, even a beginner can figure out a more sophisticated strategy than randomly selecting any of the available actions for each of the $2,589,960$ poker hands that he can receive.

Secondly, the behaviour of beginners can be determined by means of observation. This method has two disadvantages. The collection of data could be a costly affair and is only possible for games that are already exploited.

The third way to gain insight applies to games that are not (yet) exploited in practice: have the rules and structure of the game studied by a gambling expert. This person can use his expert knowledge to formulate an idea for the beginner's strategy that satisfies some general ideas of how people act in games they are not really familiar with. In section 2.4 we devote some more attention to this approach.

2.3.2 Optimal players

Optimal players have completely mastered the rules of the game and exploit their knowledge maximally in their strategies. Optimal players can be seen as the formal representatives of the more natural category of advanced players. Advanced players are observed in practice in any skillful casino game that has been around for a longer period. They play a smart strategy which yields them game results close to the theoretical maximum.

The payoffs of the optimal player can be computed analytically or approximated by means of simulation. In a one-person game the optimal player just solves the underlying maximization problem. In a two-person zero-sum game optimal play is defined by minimax strategies. In more-person games, it is not immediately clear what constitutes optimal play. We return to the topic of optimal play in more-person games when we speak about opposition in section 2.7. Game theorists refer to this type of players as rational players.

2.3.3 Fictive players

Fictive players know in advance the realization of the random elements in the game. However, they cannot influence the randomization process. We distinguish between two kinds of random elements. In the first place, a fictive player is informed about the outcome of the *external* factors or chance moves. External chance moves are, for example, the dealing of the cards and the spinning of the roulette wheel. The other sort of chance move a player can face, occurs in more-person games and is caused by his opponents. Players may generate uncertainty for their opponents by playing mixed strategies. We call these random elements *internal* chance moves. Besides having information regarding the external chance moves, a fictive player can be informed about these internal chance moves of the other players and he can anticipate their actions. The concept of a fictive player is introduced to obtain a natural upper bound for the maximal realization of game results.

2.3.4 The use of the three player types in the skill measure

We will see now how the player types that were defined in the previous three subsections fit in the framework that was set up in section 2.2. In formula (2.1), we have seen that the two basic quantities of the relative skill measure are the learning effect and the random effect. The learning effect is defined as the difference between the result of the optimal player and the beginner. This effect measures how much a player can gain in the game by figuring out how to play a good strategy. The random effect is defined as the difference in game result between the fictive player and the optimal player. This difference reflects to what extent the optimal players' maximum expected gains are restricted by the uncertainty that is created by the random factors in the game.

Regarding the random effect, one has to be careful. Two different definitions are used. Borm and Van der Genugten (1998, 2001) use the game results of the fictive player that is only informed about the external factors or chance moves. Later in this chapter, we describe an alternative approach, in which we compare the result of the optimal player to the result of the fictive player to whom also the realization of the internal chance moves is revealed.

A player type that was not mentioned above, but which certainly is of theoretical interest when one studies a casino game, is the average player. When compared to the results of the player types we just introduced, the results of the average casino visitor in a specific game could be helpful when determining the skill level of this game. Borm and Van der Genugten (1998) indeed use the average player in the development of the measure, but they also explain why this type does not make it into the final model: it is often hard, if not impossible, to reach agreement about the strategic behaviour of the average player.

2.4 Beginners

In this section, the beginner is the central player type. In contrast to the optimal and the fictive player, the expected game result of the beginner cannot always objectively be determined. For a specific game, the results of beginners may even vary with the context in which the game is offered: the way new players play a game, depends on the general popularity of the game at a certain place at a given time, for example via the information about "smart" beginners' strategies they obtain from other players.

As mentioned in section 2.3.1, in general there are three ways to formulate a beginner's strategy: assuming a random selection of actions, observing naive players in practice or asking the help of a gambling expert. When choosing the third possibility, this person can use his expert knowledge to formulate a strategy that satisfies some general ideas about how people act in games they are not really familiar with. An example of the combined application of the second and the third method can be found in section 5.5.1: for a simple poker game we determine a reasonable beginner's strategy by projecting a general tendency among poker players on the strategy space of this particular game.

Especially for the last method, where the beginner's strategy is determined by way of the judgment of a gambling expert, it would be helpful if there would exist some rules of thumb for formulating such a strategy. Kadane (1986) makes an attempt to list these rules, when he tries to determine the skill level of electronic draw poker. From the discussion that follows his short list, it is clear how difficult it is to formulate a set of rules which the strategy of a naive player always satisfies.

In principle, it is not necessary for the analysis of skill to define the strategy of a beginner very precisely; in the end it is his expected game result that is important. However, in most situations, the best way to determine this number is via specification of his strategy in the game.

Larkey et al. (1997) investigate the skill of twelve different types of players in a simplified version of stud poker. Players are defined as algorithms that determine their strategic choices in the game. By carefully varying certain characteristics over the algorithms, skill differences among players are created. To determine the skill level of the game, one would like to know which of the twelve player types is most representative of a beginner in this stud poker game. The game results of that player can then be used in the formula for the skill level. So, we want to know which characteristics of the strategies (or: which steps in the algorithms) can or cannot be ascribed to inexperienced players.

In sections 2.4.1 and 2.4.2, we list a number of possible deviations from rational play, discussed in behavioural economics and psychology literature, that may give insight in the way beginners act in casino games.

2.4.1 Behavioural aspects

During recent years, the attention for the integration of psychology into economics has greatly increased. This has lead to a large stream of literature on behavioural (or psychological) economics. The goal of researchers in this field is to investigate departures from the standard assumptions about human behaviour that are made by economists. To have a concrete frame of reference, we first formulate the classical model of individual choice under uncertainty. Economic agents are assumed to maximize the expected value of a utility function of the form

$$\max_{x \in X} \sum_{s \in S} \pi(s) U(x|s), \tag{2.2}$$

where X is the agent's set of possible choices, S is the state space, $\pi(s)$ are the agent's subjective beliefs updated using Bayes' rule, and U is a utility function that represents the agent's preferences over all available choices. In the remainder of this section, we discuss some of the psychological phenomena that give rise to alternative models of individual decision making. We use the same division into three categories as Rabin (2002) did in his Alfred Marshall Lecture: assumptions about preferences (section 2.4.1.1), heuristics and biases in judgment (section 2.4.1.2) and lack of "stable utility maximization" (section 2.4.1.3).

2.4.1.1 Assumptions about preferences

The first category of departures from the standard theory consists of attempts to make $U(x|s)$ more realistic. Important lessons in this direction can be learnt from prospect theory, the theory that was introduced by Kahneman and Tversky (1979). Prospect theory uses two functions to characterize choices: the *value function*, which replaces the utility function in standard expected utility theory, and the *decision weight function*, which transforms probabilities into decision weights. One of the key properties of the decision weight function seems to be important for the behaviour of beginners in games: small probabilities are overweighed. As an example, consider video poker players who draw new cards too often, hoping for the royal flush that they will probably receive only once in their lifetimes. They overestimate the probability of receiving such a good hand.

The value function in prospect theory has three important characteristics:

1. changes in wealth are important, not final asset positions;
2. the function is S-shaped; it is concave for gains and convex for losses;
3. the part regarding losses is steeper than the gains part: this reflects loss aversion.

The first point is taken into account in the standard game theoretic analysis of casino games, since game rules are nearly always presented in terms of bets and gains. As a consequence, this definition forms the logical basis for the analysis of the beginners in the game. The other two characteristics may give rise to some discussion about a beginner's strategy, because they may give clues about which strategies are avoided and which strategies will be more attractive in specific games. Epstein (1977) also discusses the characterics of utility functions in the context of gambling.

It is also interesting to note that preferences may change over time. They need not even be constant during an evening in the casino. Participants may consider the history of play relevant for their strategic choices, even in games in which plays are independent of each other. Their decisions may be influenced by losses and gains that were made during previous plays. Thaler and Johnson (1990) present an interesting investigation of the effects of both prior gains and prior losses on preferences. Under some circumstances a prior gain can increase a person's willingness to accept certain gambles. This phenomenon is called the *house money effect*. This change in preferences is explained by the tendency of gamblers to perceive a loss as a reduction of previously made gains in this situation. In the case of prior losses, gambles which offer the possibility of breaking even should be treated differently from those who do not. The first case is discussed by Kahneman and Tversky (1979, p. 287). They conclude that "a person who has not made peace with his losses is likely to accept gambles that would be unacceptable to him otherwise". Thaler and Johnson (1990) conjecture that it is important in the examples presented by Kahneman and Tversky that the second gamble always offers a possibility to return to the point of departure. If such a possibility is not present, prior losses may often lead to increased risk aversion. The above findings are all phrased in terms of preferences over gambles (prospects), but it is not difficult to apply the results to (preferences over) strategies of a player.

Another class of modifications of the utility function is formed by the alternative social preferences. The idea underlying these modifications is that self-interest is not the only motivation that individuals use when making choices. People can also be interested in another person's well-being. This altruism can be based on the context in which the decision maker is active (see, e.g., Bester and Güth (1998) for some examples), but it can also be based on considerations of fairness or reciprocity, as Rabin (1993) argues. At first sight, things like fairness and altruism seem to be an unlikely explanation for deviation from rational play in casino games. After all, most participants will have increasing personal welfare as a goal (or at least as a subgoal, besides the utility they may receive from gambling). Most casino games are, possibly apart from some entrance fee, zero-sum. Being altruistic in a zero-sum game is equivalent to being masochistic. A reason why reciprocity may play a role, however, is the following. In more-person casino games, such as poker, it is difficult for a professional player to make a profit sitting at a table with other professionals. When beginners are joining the game, there are possibilities for the professionals to gain by taking advantage of their weaknesses. This way of acting by the advanced players is completely rational: "the best way for one to play a game depends on how others *actually* play, not on how some theory dictates that rational people *should* play" (Goeree and Holt (2001, p. 1419)). If a beginner somehow notices that some of his opponents are playing "against him", he may see this as a motivation to try to keep them from making profit, instead of focusing on trying to make profit himself. Although it is not immediately clear how this effect could be incorporated in a beginner's strategy, such reciprocity considerations could play a role.

2.4.1.2 Heuristics and biases in judgment

Whereas the first category of departures from the standard expected utility model has to do with taste, the category that we discuss in this section is about mistakes made by the decision maker. These errors include overconfidence and a biased judgment about various game elements, but also the inability to randomize correctly.

The first phenomenon of interest is overconfidence. This reflects the tendency of players to overestimate their own abilities, their prospect for success or the probability of positive outcomes. In behavioural economics a large stream of research has been devoted to this subject; see, e.g., Camerer and Lovallo (1999) and Hvide (2002). Overestimating one's own abilities, relative to the others, is sometimes referred to as the "better than average effect": more than half of the people think they will perform better than the average person. A too positive idea about one's own skill can also lead to unrealistic optimism about the chances of attaining good outcomes; see, e.g., Weinstein (1980). The combination of these types of errors forms a good explanation of the inexperienced poker players who bet (bluff) too often and with relatively bad hands.

Another thing that forms a problem for inexperienced players, is randomization. They believe in the "law of small numbers", as Tversky and Kahneman (1971) phrase it. That is, they wrongfully assume that the pattern of a large population will be replicated in all of its subsets. This is reflected, for example, in roulette: people expect a black number to come up after a series of red numbers. But it is also applicable in games in which equilibrium play requires mixed strategies. As an example one can think of bluffing elements in poker: with a low hand you often fold, but sometimes you bet to mislead your opponent. Series of decisions that are based on randomization, which should be independent, will often show a negative correlation if the randomization is done by beginners. In this way, beginners become preys for the professionals, because their "random sequences" are predictable. Not only beginners have problems with this aspect of game play, this is a tendency among people in general. Palacios-Huerta (2003) claims that an exception is formed by professional soccer players taking penalties: in his study, he finds that "professionals play minimax".

A final type of mistakes made by beginners is simply having an incorrect or incomplete image of the game they are playing. They make *mental models* of the game that need not coincide with the standard game representation, e.g., by a tree or a normal form. People tend to focus on specific strategies for various reasons and often they ignore the payoffs of the opponent.[3] The mental model that someone forms of a game will also depend on the way (and order) in which the rules are explained to him. For examples and a more elaborate discussion, we refer to Warglien, Devetag and Legrenzi (1999).

[3] In general, even if they take the opponent's payoffs into account, people only do a few steps of iterated reasoning, so that this information is only partly used in strategic decisions; see, e.g., Camerer (2003, chapter 5).

2.4.1.3 Lack of "stable utility maximization"

The last category of modifications of the standard assumptions is based on psychological findings that suggest that there do not exist well-defined utilities $U(x|s)$ such that behaviour is best described by assuming that people maximize a function of the form that is given in formula (2.2). For an overview of utility theory, including a discussion of the preference relations underlying utility functions, we refer to Luce and Raiffa (1957, chapter 2) or to Fishburn (1970).

An example of a phenomenon that may be relevant for analyzing beginners, is the tendency of people to "rationalize the past" as Eyster (2002) calls it. A past choice that is suboptimal given a current action may not be suboptimal given another current action. If so, then a person can rationalize the past choice by changing his current action; often someone can choose a current action consistent with his past choice having been optimal. In casino games, this phenomenon can be observed when a poker player keeps raising just because he raised the first time, even though his estimates of the winning probabilities might have drastically lowered as a result of the actions of his opponents.

A second issue that keeps inexperienced players from maximizing a formula like (2.2), is the fact that they find it difficult to think through disjunctions: according to the sure-thing principle (STP), if a person would prefer a to b knowing that X occured, and if he would also prefer a to b knowing that X did not occur, then he definitely prefers a to b. Shafir (1994) reviews a number of experimental studies of decision under uncertainty that exhibit violations of STP in simple disjunctive situations. The author argues that a necessary condition for such violations is people's failure to see through the underlying disjunctions. In a game theoretic context, this implies that players may not always be capable of looking ahead in game trees. The more complex the game is, the more this will be a problem for a beginner who tries to formulate a good strategy.

2.4.2 Psychological aspects

Although it is not possible to draw a solid line between psychology and behavioural economics, we devote a separate section to some "purely psychological" aspects that might affect the perceived behaviour of beginners.

The problem that gave rise to this section on naive game play is the analysis of skill. We want to find an objective measure of the skill involved in a game. Since it is already difficult for experts to distinguish between games of skill and games of chance, it is not surprising that many casino visitors cannot make this distinction. Often they overestimate their own influence on the game result: they accredit a too high skill level to games of pure chance like roulette. On this subject Cohen (1960, p. 85) writes the following.

> "Success in many types of gambling seems to the player to depend, and indeed does in fact depend, on a certain degree of skill and on an element of chance. Success, that is to say, seems to him to be determined by two kinds of factor, one kind within, and the other outside, his control. At one extreme he believes that success is almost entirely due to his individual skill, the element of chance being, so he thinks, negligible. At the other extreme, he believes that success depends almost wholly on 'chance' factors outside his control. Of course his beliefs do not necessarily tally with the 'objective' state of affairs. Nor does he necessarily act in accordance with any truly 'objective' evaluation."

Psychologists refer to this belief as the *illusion of control*. Gamblers throw dice hard to produce high numbers. People want to pay more for specific numbers in a lottery than for numbers that are randomly assigned (Langer, 1975). For the case of roulette, Oldman (1974) discusses the illusion of control in detail. Wohl and Enzle (2002) extend the illusion of control model by Langer (1975) by including perceptions of personal luck as a potential source of misperceived skillful influence over uncontrollable events. Participants in their experiments acted as if luck could be transmitted from themselves to a wheel of fortune and thereby positively affect their perceived chance of winning.

To conclude this section, we want to remark that the phenomenon of illusory control is closely related to the judgment errors due to overconfidence that we mentioned in section 2.4.1.2. Clearly, the distinction between the mathematical character of a game and the way the game is perceived by the players also relates to the mental models discussed in section 2.4.1.3.

2.4.3 General remarks

In this section we make a few general remarks on the incorporation of findings from behavioural economics and psychology in the analysis of beginners in casino games. In the first place, one should try not to stick with global models and general results when analyzing a game. Local, game-specific considerations are often more useful. Moreover, the environment in which the game is played, may influence the strategy of inexperienced players. Think, for example, of casinos organizing sessions for new players to become familiar with the rules of the games. In such sessions strategic advise may be given to the audience or the complex rules of a game can be presented in a simplified way. People who start playing the game with this information in mind may play a strategy that is completely different from the strategies used by players who did not attend such an introductory session.

A second issue that deserves some attention is the distinction between experiments that are run by psychologists and experiments that are carried out by economists. Psychologists do not use repetition; they are interested in initial behaviour. Economists ask their subjects to perform a task repeatedly, because they want to learn something about equilibrium behaviour. People who do not get the opportunity to learn may be seen as “real beginners” who play a game for the very first time. On the other hand, from the behaviour of subjects in economic experiments we may draw conclusions about the decisions of people during their whole first evening in the casino.

The last possibility to learn about a beginner’s strategic behaviour that we want to mention, is the direct observation of the way a player processes information. A nice example of an investigation of decision rules, which also measures how subjects attend to payoff information, is the paper by Costa-Gomes, Crawford and Broseta (2001). The authors “get behind the subjects’ eyes” and see payoffs during the same amount of time and in the same order as the subjects do. For a discussion of this research we refer to Camerer (2003, section 5.6).

2.5 Measuring the game result

In the preceding sections we introduced the learning effect, the random effect and the three player types whose game results are used to determine these effects. However, we did not yet define exactly a player's game result. As Borm and Van der Genugten (1998) already suggested, the relevant numbers that should be taken into account are the payoff to the player and the stakes (bets) that are needed to obtain this payoff. Two sensible definitions of a player's game result that one can come up with, using these numbers, are *(net) gains* and *returns*:

$$\text{gains} = \text{payoffs} - \text{stakes}, \qquad \text{returns} = \frac{\text{gains}}{\text{stakes}}.$$

One should be careful when making a selection. Implicitly, the choice of measurement implies an assumption about the goals of the players in a game. In general, a player's strategy will depend on his focus: the strategy that maximizes the expected net gain is not necessarily the same as the strategy optimizing the expected returns. In practice, mostly players seem to aim for the highest possible gain.

There are games, however, in which expected gain does not form an appropriate strategy evaluation. A practical example is the game of roulette. Intuitively, roulette is a pure game of chance. A player cannot influence his expected results by varying his strategy; i.e., if results are measured in terms of expected returns. Of course, by betting twice as much, one can double the expected gain, but the expected returns are not affected. If we define the strategy of a beginner, we have to make assumptions about the bet size he uses. For roulette we know that the optimal player will bet the minimum, since the expectation of his gain is negative.[4] If we assume that a beginner plays a strategy that assigns a positive probability to a bet larger than the minimum, his expected gain will be smaller than for the optimal player and, as a result, roulette will have a positive learning effect. This positive learning effect will not occur if we use expected returns to evaluate the player's strategies.[5]

[4]We don't consider "not playing" as a strategic option; we analyze the behaviour of a rational player, *given* that he participates in the game.

[5]To be complete, we note that there is a difference in expected returns between simple strategies (e.g., red or black, even or odd) and non-simple strategies (e.g., single numbers). However, the learning effect will always be small compared to the random effect that is caused by the fictive player who always bets maximally on the winning number.

The use of expected returns has some disadvantages. In the first place, the linearity of the game results is lost. This makes computations more difficult. Besides that, in more-person games we have the complication that zero-sum games are turned into games of which the payoffs are not zero-sum. There is an alternative that seems to use the best of two worlds: one could determine the strategies in the linear, zero-sum environment, focusing on maximimum expected gain, and consequently compute the corresponding expected returns and use these in the relative skill measure. This possibility has a theoretical drawback. The expected gain of a beginner will be smaller than or equal to the expected gain of an optimal player and an optimal player will never have an expected gain that is strictly higher than the expected gain of the fictive player. However, this logical ordering is not necessarily preserved when when we look at the expected returns that correspond to the strategies of the three player types.

Another option is to model the bet size as a *pre-game decision* of how many unit games to play at the same time, where the *unit game* is the game with fixed, normalized bet size. We can use this way of modelling if the following conditions are satisfied:

(C1) the size of the bet that is chosen does not affect the course of the game;

(C2) at the moment the bet size is chosen, no information about the outcome of the chance move is available yet;

(C3) the structure of the payoff function is such that the expected gain of a player is linear in the bets of the player.

Within the class of one-person games we find games that satisfy the three conditions above. For example, in roulette, deciding to bet 10 euro on black, is comparable to deciding to play 10 games of "unit roulette" simultaneously, in which you bet 1 euro (the fixed, normalized bet size) on black. A similar decomposition is possible for instance for trajectory games like golden-ten, but also for blackjack played with an automatic card-shuffling machine.

We think that what one really wants to know if one asks for the skill involved in a game, is the skill level of the unit game. When playing multiple instances of a game simultaneously, one has the same relative influence on the expected result as in one instance of the game. In defining our three player types, we can therefore restrict ourselves to defining the strategies they use in the unit game. Measuring expected gain is then equivalent to measuring expected returns and the ordering problem will not occur anymore.

In general, in more-person games condition (C1) is no longer satisfied. E.g., in a two-person game where the players do not move simultaneously and where the second player is informed about the amount bet by the first player, different bets of the first player lead to different information sets of the second player. This type of bet of the first player is an example of a *strategic bet*, whereas the bets that satisfy the conditions above are called *non-strategic bets*. In a game that contains strategic bets a reduction to the analysis of a unit game is not possible. This is not a problem, since for more-person games there is no need for an alternative definition of game results; expected gains can serve this purpose very well. The only assumption we have to make in the skill analysis of more-person games, is that all participants have sufficiently large resources. In this way, buying out an opponent by means of extraordinarily large (bluffing) bets is not possible and, as a consequence, the analysis only takes into account the "real" strategic features of the game.

2.6 One-person games

We are now ready to give the formal definition of the relative skill measure for one-person games. The definitions apply to games for *one player*, possibly with chance moves. Here, the number of chance moves must be finite and for each chance move the number of possible outcomes must be finite. Moreover, the rules of the game determine a finite, non-empty set X of pure strategies of the player. For this game we also consider the *fictive* situation in which the player knows the outcomes of the chance moves before he has to make a decision. This knowledge extends the set of pure strategies to a finite set $\bar{X} \supseteq X$.

The quality of a strategy $\bar{x} \in \bar{X}$ is determined by its corresponding *expected gain*, as (R2) suggests. The expectation of the player's gain with respect to the external chance moves is given by the function U that assigns expected gain $U(\bar{x})$ to each strategy $\bar{x} \in \bar{X}$. Let $\Delta(A)$ denote the set of all probability distributions on a finite set A:

$$\Delta(A) = \left\{ p : A \to [0,1] \mid \sum_{a \in A} p(a) = 1 \right\}.$$

This defines the sets of mixed strategies $\Delta(X)$ and $\Delta(\bar{X})$. Using expectations with respect to these internal chance elements, the extension $U(\bar{\sigma})$ for mixed strategies $\bar{\sigma} \in \Delta(\bar{X})$ is immediate.

The definition of (potential) *relative skill* is based on the expected gains of three *types* of players: the *beginner*, the *optimal player* and the *fictive player*. A beginner is associated with a given strategy $\sigma^0 \in \Delta(X)$ with corresponding expected gain

$$U^0 := U(\sigma^0).$$

The optimal player uses a strategy with maximal expected gain, i.e.,

$$U^m := \max_{x \in X} U(x) = \max_{\sigma \in \Delta(X)} U(\sigma).$$

Clearly, the fact that the optimal player maximizes over his set of pure strategies, instead of its mixed extension, does not affect his maximum expected gain. The fictive player has the extra information on the outcome of the chance moves and can do at least as good as the optimal player, but possibly better. He uses a strategy in $\bar{X}$ with maximal expected gain. We write

$$U^f := \max_{\bar{x} \in \bar{X}} U(\bar{x}) = \max_{\bar{\sigma} \in \Delta(\bar{X})} U(\bar{\sigma}).$$

These definitions lead to an ordering of the expected gains of the three player types: $U^0 \leq U^m \leq U^f$.

We call the difference between the expected gain of the optimal player and the beginner, $U^m - U^0$, the *learning effect* (*LE*) in the game, while we refer to the difference between the expected gain for fictive and optimal players, $U^f - U^m$, as the *random effect* (*RE*) in the game. Clearly, the learning effect relates to the variation of expected results among players that is mentioned in (R2) in section 2.2. The random effect is used to determine whether this variation is substantial in relation to the restrictive influence of the chance elements, as (R3) requires.

The definition of the relative skill level RS of the game is based on the ratio of the learning effect and the random effect:

$$RS = \frac{LE}{LE + RE} = \frac{U^m - U^0}{U^f - U^0}.$$

Obviously, both the learning effect and the random effect are nonnegative. As a consequence, $0 \leq RS \leq 1$. For RS to be equal to its lower bound, the learning effect must be zero. Therefore $RS = 0$ indicates a *pure game of chance*. On the other hand, we have $RS = 1$ if there is no random effect in the game. Therefore, this extreme case corresponds to a *pure game of skill*.

For the sake of completeness we define $RS = 1$ if $LE = RE = 0$. This boundary case is only of theoretical importance, because in practice this will not occur. In a game with $LE = RE = 0$, the chance elements do not have a restrictive influence on the maximal expected gain a player can attain, but the game is so easy that even a beginner can figure out how to play optimally (e.g., tic-tac-toe).

2.7 Definition of opposition

The framework for the skill analysis that was introduced in section 2.2 is not only applicable to one-person games, but also to games with more players. Although in one-person games the game results for the three player types are unambiguously determined by the strategies chosen by the players, in more-person games the payoff of a player clearly depends on the way the opposition acts.

In the analysis of skill two approaches are used to model the opposition of the beginners, the optimal players and the fictive players. Borm and Van der Genugten (1998, 2001) compute what would be (jointly) optimal for the opponent(s) against an optimal player. Next, the three player types are evaluated against this resulting optimal (joint) strategy of the opposition. In section 2.8, we use a different approach: we assume that the opponents play in such a way that they offer maximal opposition to the player type under consideration.

Whereas this direct opposition is clear in two-person zero-sum games, in games with more participants it is not. In a game with three or more players the mutual competition is of a more indirect and complex nature. Although money is still only reallocated in an n-person zero-sum game, two particular participants cannot be viewed as direct adversaries in the sense that they should (or could) act such that they oppose each other as strongly as possible, regardless of what the other players do. As a result, it is not directly clear how to determine the expected gain of a given player type facing maximal opposition. The solution that is chosen for this problem in the skill analysis, is the following. In an n-person game the $n-1$ opponents of a specific player are assumed to act as one. In terms of cooperative game theory these $n-1$ players form a coalition. By defining the payoff of the coalition as the sum of the individual member payoffs, we obtain a two-person zero-sum game again, in which optimal play is well-defined. Using this pessimistic assumption, we can find the optimal opposition for any player in the more-person game in the familiar way.

2.8 More-person games

In this section, we present the generalization of the definition of relative skill for one-person games to n-person games. We consider a finite game with player set $N := \{1, \dots, n\}$, again possibly with chance moves. In the analysis, we refer to the players in N as *player roles*, thereby indicating that these are the roles or positions that players can take in the game. The finite, non-empty set X_i contains the pure strategies of player i. The set of strategy profiles of the players is then $X := \prod_{i \in N} X_i$. For each player i, the fictive situation that he knows the outcomes of the chance moves leads to the extended set $\bar{X}_i \supseteq X_i$ of strategies. This leads to the extension $\bar{X} := \prod_{i=1}^{n} \bar{X}_i$. For player i, $\Delta(X_i)$ and $\Delta(\bar{X}_i)$ denote his sets of mixed strategies as a normal and as a fictive player respectively. Each player makes his strategic choices independently of his opponents. Therefore, the product sets $\prod_{i=1}^{n} \Delta(X_i)$ and $\prod_{i=1}^{n} \Delta(\bar{X}_i)$ contain all possible strategy profiles.

For each $i \in N$ the function U_i assigns to each strategy profile $\bar{x} \in \bar{X}$ the expected gain $U_i(\bar{x})$ of player i. The vector $U(\bar{x}) = (U_1(\bar{x}), \dots, U_n(\bar{x}))$ specifies the gains of all players. Using expectations, the extension $U(\bar{\sigma})$ for mixed strategy profiles in the set $\prod_{i=1}^{n} \Delta(\bar{X}_i)$ is straightforward.

As before, we base our definition of *relative skill* on the expected gains of three types of players: the beginner, the optimal player and the fictive player.

However, these types must now be defined for each *role* a player can take. After all, in most games player roles are not symmetric. The difficulty, compared to the one-player case, is that a player's gain may now depend on the strategic choices of his opponents. Borm and Van der Genugten (2001) dealt with this difficulty in the following way. For each player $i \in N$ the strategy choices of the other players are considered fixed. The uniform reference for the three player types in the role of player i is a minimax strategy of the coalition of all opponents of player i in the related two-person zero-sum game.

A drawback of this method is that the coalition of opponents of player i in general has multiple minimax strategies. The value of the skill measure will therefore depend on the minimax strategy selected. Although it does not influence the expected gain of player i as an optimal player, it does influence these numbers for this player as a beginner and as a fictive player. Borm and Van der Genugten (2001) solved this problem by replacing the minimax strategy by an approximation obtained by fictitious play with prescribed accuracy and starting with the strategy profile consisting of beginners' strategies. However, from a numerical point of view this is not a simple solution, so it can still be judged as a drawback of the concept.

As announced in section 2.7, we drop the earlier idea of a fixed and uniform reference of the opponents against each type of player in a specific player role. Instead we let the opponents react optimally, depending on the type of player. Playing optimally must be interpreted as giving maximal opposition. This assumption on the behaviour of the coalition of opponents is only reasonable for zero-sum games. After all, for a zero-sum game, the coalition's aggregate gain is higher as the gain of player i is lower (and vice versa), while this relation does not hold for nonzero-sum games. This is not really a restriction, since any practical casino game you can think of can, maybe apart from some entrance fee, be modelled as a zero-sum game. If a bank (or dealer) is involved, this person should be considered as an extra player with only one strategy.

Another consequence of the the definition of skill proposed by Borm and Van der Genugten (2001) that deserves attention is the fact that each game without external chance elements is a game of skill by definition. This is a result of the fact that the effects of the use of mixed strategies, the so-called internal chance elements, are not taken into account. The following example serves as an illustration.

Example 2.8.1 (Matrix game) Consider the following zero-sum game for two players. Both players have a coin. They simultaneously put their own coin on a table and cover it with one hand. The players can choose which side of the coin will be up, H(eads) or T(ails). If both players decide the same, then player 1 receives one euro from his opponent. Otherwise, player 1 has to pay one euro to player 2. The players are allowed to use randomization in the selection of their strategies. The matrix below summarizes the expected gains of player 1, the row player.

$$\begin{array}{cc} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}$$

This two-person zero-sum game has no external chance moves. Therefore, the random effect is equal to zero. According to the definition from section 2.2, the consequence is a skill level of one, whatever the learning effect may be. However, in practice in this game players will always randomize between the strategies available to them; this is pure gambling. Anyone observing this game will intuitively associate this randomization with a game of pure chance. $\triangleleft$

The message of this example is that merely the fact that optimal play needs randomization should influence the measure of relative skill. The following alternative definition of relative skill for more-person games also incorporates this idea.

We now provide the formal definitions. Let $X_{-i} = \prod_{j \neq i} X_j$ denote the pure (coalition) strategies of the opponents of player i. Then for player i as *beginner* with strategy $\sigma_i^0 \in \Delta(X_i)$ the gain with optimal play by opponents is

$$U_i^0 := \min_{x_{-i} \in X_{-i}} U_i(\sigma_i^0, x_{-i}).$$

The expected gain for player i as an *optimal player* is given by his expected gain in a Nash equilibrium of the related two-person zero-sum game against the coalition of the other players:

$$U_i^m := \max_{\sigma_i \in \Delta(X_i)} \min_{x_{-i} \in X_{-i}} U_i(\sigma_i, x_{-i}) = \min_{\sigma_{-i} \in \Delta(X_{-i})} \max_{x_i \in X_i} U_i(x_i, \sigma_{-i}).$$

Note that the equality follows from the minimax theorem of Von Neumann (1928) and that U_i^m is exactly the *value* of the two-person zero-sum game. For player i as a *fictive player* we assume that he does not only know the outcome of the chance moves, but also the outcome of the randomization process of his opponents. This is the key change with the aim of a better incorporation of the randomization of the players in the definition of relative skill. A fictive player can anticipate future actions of his opponents. So in optimal play against a fictive player randomization has no effect at all. Therefore, the opponents will choose a pure strategy from X_{-i}, minimizing the maximum gain of a fictive player i. Player i will choose a strategy from $\bar{X}_i$ that maximizes his expected gain, given the strategy of his opponents. This leads to the expected gain U_i^f of the fictive player:

$$U_i^f := \min_{x_{-i} \in X_{-i}} \max_{\bar{x}_i \in \bar{X}_i} U_i(\bar{x}_i, x_{-i}).$$

It is not difficult to see that, for a specific player i, just as in the one-person case, we have for the ordering of expected gains of the different player types that $U_i^0 \leq U_i^m \leq U_i^f$. To find the expected gain in the game for each player type, we take the average over all n possible player roles. For the beginners, this leads to $U^0 = \frac{1}{n}\sum_{i=1}^n U_i^0$. Similarly, we have $U^m = \frac{1}{n}\sum_{i=1}^n U_i^m$ and $U^f = \frac{1}{n}\sum_{i=1}^n U_i^f$ for the expected gains for optimal and fictive players, respectively.

The learning effect is again the difference between the expected gains for beginners and optimal players: $LE = U^m - U^0$. The contribution of player i to this learning effect is $\frac{1}{n}(U_i^m - U_i^0)$. Analogously, the random effect of the game is $RE = U^f - U^m$, with $\frac{1}{n}(U_i^f - U_i^m)$ as contribution of player i. Now we are ready to give the extension of the measure of skill for more-person games. Analogous to the measure for one-person games we define

$$RS = \frac{LE}{LE + RE} = \frac{U^m - U^0}{U^f - U^0} = \frac{\frac{1}{n}\sum_{i=1}^n (U_i^m - U_i^0)}{\frac{1}{n}\sum_{i=1}^n (U_i^f - U_i^0)}.$$

To conclude this section, let us illustrate the formulas with the matrix game from example 2.8.1.

Example 2.8.2 (Matrix game (continued)) In this example we show how to calculate the skill measure for the matrix game we defined in example 2.8.1, starting from a certain characterization of beginner's play. Recall that the payoff matrix A of the game is as follows.

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

where the first row and column correspond to H. Both players can choose from the same set of pure strategies: $X_1 = X_2 = \{H, T\}$. Beginners will probably choose between H and T randomly, so $\sigma_1^0 = \sigma_2^0 = \frac{1}{2}H + \frac{1}{2}T$. To compute the expected gain of a beginner in the role of player 1, we check what his expected gain is if player 2 plays optimally against σ_1^0. Player 2 can choose any strategy to obtain a payoff of 0. Therefore, $U_1^0 = 0$ and, because of symmetry, $U_2^0 = 0$.

To compute the expected gains of the optimal players, U_1^m and U_2^m, we compute the Nash equilibrium of the matrix game. It is not difficult to see that this equilibrium is unique and that for each player the equilibrium strategy is equal to the beginner's strategy. The value of the game, $v(A)$, is zero, so the optimal players have expected gains $U_1^m = U_2^m = v(A) = 0$.

In the fictive situation that player 1 can observe the outcome of the possible randomization of his opponent, he can always put his coin with the same side up. So, his gain will be 1, independent of the strategy choice of player 2. The same reasoning holds for player 2 as a fictive player, so we have $U_1^f = U_2^f = 1$.

Using these numbers, we can now compute the learning effect and the random effect for our matrix game:

$$\begin{aligned} LE &= \tfrac{1}{2}\textstyle\sum_{i=1}^2 (U_i^m - U_i^0) = \tfrac{1}{2}((0-0)+(0-0)) = 0, \\ RE &= \tfrac{1}{2}\textstyle\sum_{i=1}^2 (U_i^f - U_i^m) = \tfrac{1}{2}((1-0)+(1-0)) = 1. \end{aligned}$$

The last step is to combine these effects to find the value of the skill measure:

$$RS = \frac{LE}{LE + RE} = \frac{0}{0+1} = 0.$$

Thus, following the new definition, we conclude that this is a pure game of chance. We stress, however, that changes in the assumptions on the behaviour of the beginners may influence the value of the skill measure. Although the skill analysis in principle is applicable to any matrix game, one should be careful with conclusions based on the skill level. The beginner's strategy may depend on "the story behind the matrix".

Note that the expected gains of the optimal players do not influence the value of RS in a two-person zero-sum game, since these values cancel out in the formula. However, if we want to see how much the players individually contribute to the learning effect and the random effect, we need to know the values of U_1^m and U_2^m. ◁

Example 2.8.2 was just a small illustration of the computations the skill analysis requires. We present the analysis of two more interesting games in section 2.10.

2.9 Player skill versus game skill

Before we turn to two particular examples in section 2.10, we would like to spend a few words on the relation between player skill and game skill. As the first paragraph of the introduction to this chapter already indicates, there is a distinction between the two concepts. In a *game with a high skill level*, the *skillful players* can have a significant advantage over the less competent participants, whereas this advantage is relatively small in games with a low skill level. The previously mentioned article of Larkey et al. (1997) focuses on skill differences between players. Their ideas are presented by means of a large example, in which twelve different player types play a simplified version of stud poker against each other. Each player is described by means of a complete, algorithmically described strategy. Skill differences are created by carefully varying certain characteristics over the twelve strategies. These players play a complete tournament and in the end the table with game results is used to draw conclusions about skill differences between players and about different types of skill that can be useful in the poker game. Their results make clear that a player's performance strongly depends on the opposition he faces. The authors can, given the opposition, distinguish between more and less skillful player types. However, it is not directly clear how one could use their results to say something about the skill level of the poker variant itself.

How does this work in our skill analysis? In section 2.2 we defined the notions of learning effect and random effect that are used to compute the relative influence of a player on the game result. If we consider a one-person game, we can just fill in these numbers in formula (2.1) to find the skill level of the game. For a one-person game, the relative influence that the player has determines the skill level of the game. For more-person games some extra work is required. For each player (or player role) in the game we can compute the learning effect and the random effect.[6] Next, there are two ways to use these numbers to draw conclusions about the skill level of the game. In the first place, we can compute the overall learning effect and random effect by taking the average over the, say, n players and use the results in formula (2.1) to compute the skill level. This is the approach followed by Borm and Van der Genugten (1998, 2001), as well as in our approach of section 2.8. An alternative would be to compute the relative skill level for each player role separately and take the average over these n numbers to find the skill of the game as a whole. Both methods seem to make sense, but in general they do not yield the same results.

We use a subscript to indicate whether we speak about a player or the game itself and rewrite the formula for the method used in section 2.8:

$$RS_{\text{game}} = \frac{LE_{\text{game}}}{LE_{\text{game}} + RE_{\text{game}}} = \frac{\frac{1}{n}\sum_{i=1}^{n} LE_{\text{player }i}}{\frac{1}{n}\sum_{i=1}^{n}(LE_{\text{player }i} + RE_{\text{player }i})}. \tag{2.3}$$

The alternative method boils down to

$$RS_{\text{game}} = \frac{1}{n}\sum_{i=1}^{n} RS_{\text{player }i} = \frac{1}{n}\sum_{i=1}^{n} \frac{LE_{\text{player }i}}{LE_{\text{player }i} + RE_{\text{player }i}}. \tag{2.4}$$

An example in which the difference between two methods is easily illustrated, is presented by blackjack. In principle, blackjack is a one-person game. Although the dealer draws cards too, he cannot make any strategic decisions. For this one-person game, we can compute the learning effect LE_{BJ}, the random effect RE_{BJ} and the resulting skill level RS_{BJ}. Next, we modify the game such that you can play it with two players. In each play one of the participants takes the role of the bank. Having the role of the bank, a player has no choices; he has to play a fixed, predescribed strategy. Therefore, a beginner and a fictive player will have the same expected game result as an optimal player. As a result, the learning effect, and thus the relative skill of this player role are zero. For the other player we already have the numbers LE_{BJ}, RE_{BJ} and RS_{BJ}. If we use formula (2.3) to determine the skill level of the new game, we find that it is equal to the skill level of "standard" blackjack, whereas the

[6]One should not confuse player roles with player types; e.g., in a five-person poker game we have to analyze for five *player roles* each of the three *player types* (beginner, optimal player and fictive player) that are needed for the skill analysis.

skill level turns out to be halved according to formula (2.4). Which of the two alternatives is preferred, depends on the context in which it is used. When one want to make a consistent comparison with other games, the availability of information regarding skill for these games may determine the selection of the formula. Moreover, practical restrictions can require the use of the first alternative. After all, when empirical data is collected, it is much harder to collect a player's results for each player role in a game separately than to collect aggregated results. If only these aggregated numbers are available, then it is not possible to use formula (2.4).

2.10 Examples

In this section we illustrate the computations of the skill measure presented in section 2.8 for two simple, but realistic more-person games. Section 2.10.1 discusses a coin game. Subsequently, in section 2.10.2 we analyze the skill involved in a simple poker game among three players in which an external chance element, the dealing of cards, plays a role.

2.10.1 A coin game

In this section we consider a generalization of the n-coin game that was explored by Schwartz (1959). The n-coin game is a two-person zero-sum game in which both players have n coins available to play with. In the game we discuss, this number of coins is not necessarily the same for both players: player 1 has m coins and player 2 has n coins. We call this generalization the (m, n)-coin game. Both players also know the number of coins available to the opponent. We come back in detail to (m, n)-coin games in chapter 7.

First, each player takes a number of his coins, possibly zero, in his hand. A player cannot see how many coins the opponent has taken in his hand. Now, first player 1 guesses the total number of coins taken by both players and then player 2 does the same, but he is not allowed to guess the same total as player 1. Subsequently, both players show their hands, so that the actual total can be determined. The game is won by the player who guessed the total number of coins correctly. The winner receives one euro from the opponent. If neither of the two guesses the right number, then nothing is paid.

We calculate the relative skill RS of the $(1,2)$-coin game. A pure strategy of player 1 has the form $(i;j)$ where i is the number of coins he takes and j represents the sum he guesses. Player 2 plays a strategy of the form $(k;l_0,l_1,l_2,l_3)$. In this notation, k denotes the number of coins player 2 takes, while l_j tells us what number player 2 will guess if player 1 guessed the total number of coins to be j.

To find the value of this game, we want to know its normal form. Since the normal form is rather large, we restrict attention to the *quasi-reduced normal form*. Two pure strategies x_i and x_j of a player are called *realization equivalent* if they lead to the same terminal node for every given specification of the strategies for the other player, i.e., if x_i and x_j differ only at irrelevant information sets. The quasi-reduced normal form considers for player i a subset Q_i of his collection of pure strategies X_i, such that no two elements of Q_i are realization equivalent. We can construct this quasi-reduced normal form for the (1,2)-coin game and delete the (weakly) dominated strategies of player 2 from it. These are the strategies for which there is another pure strategy for player 2 which gives him a payoff against any pure strategy of player 1 that is at least as high, and a strictly higher payoff against at least one pure strategy of player 1. These dominated strategies in the coin game typically are strategies with which player 2 cannot win the game. An example is a strategy in which player 2 takes 0 coins and guesses 2 or 3, while he knows that player 1 can have at most one coin in his hand. In this way we see that the game is equivalent to the reduced normal form game that is displayed in Table 2.10.1. Player 1 is the row player, while the columns of the table correspond to pure strategies of player 2.

It is easily verified that, by playing $\frac{1}{5}(0;0)+\frac{2}{5}(0;2)+\frac{2}{5}(1;2)$, player 1 can guarantee a value of $-\frac{1}{5}$. Analogously, player 2 can guarantee this value by playing $\frac{2}{5}(0;1,0,0,1)+\frac{1}{5}(1;1,2,1,2)+\frac{2}{5}(2;2,3,3,2)$. Therefore, the value of this game is $-\frac{1}{5}$. Then we know the expected gains for the optimal players, both in the role of player 1 and player 2: $U_1^m=-U_2^m=-\frac{1}{5}$.

The next step is to determine the strategies for the beginners. For player 1 two pure strategies can be considered "unreasonable" at first sight. After all, choosing $(0;3)$ or $(1;0)$ ensures him that he will not win the game, because the sum guessed is not reachable with his own choice of coins. Furthermore, randomization is a logical thing to do. Therefore, we assume that a naive player 1 chooses each of the remaining six pure strategies with equal probability. A best reply of player 2 is the pure strategy $(1;1,2,1,2)$. This results in the expected gain of player 1 as a beginner: $U_1^0 = -\frac{1}{3}$.

For player 2 there are a number of strategies that can immediately be seen to be irrelevant. It makes no sense for player 2 to guess a sum that is smaller than the number of coins he has in hand. Even someone who plays the game for the first time will see this. Consequently, all strategies with $l_j < k$ for any j are left out of consideration. A beginner in the role of player 2 will play a fair randomization over the remaining 109 pure strategies. A best response of player 1 against this strategy σ_2^0 is to play (0;0). With this response, player 1 wins the game with probability $\frac{81}{109}$ and loses with probability $\frac{10}{109}$ against σ_2^0. In this way player 2 has to pay an expected value of $\frac{71}{109}$. So $U_2^0 = -\frac{71}{109}$. Taking a fair average over the player roles yields us an expected beginner's gain of $U^0 = -\frac{161}{327}$.

Recall that fictive players are assumed to face optimal opposition and recall that they are informed about all internal and external random moves. Computation of the expected payoffs of these players is based on the normal form of the game. This implies that a fictive player can anticipate the opponent's moves at later stages in the extensive form game. Note that this is consistent with the model of Borm and Van der Genugten (2001), in which fictive players

		Player 2											
		$(0;1,0,0,0)$	$(0;1,0,0,1)$	$(0;1,0,1,0)$	$(0;1,0,1,1)$	$(1;1,2,1,1)$	$(1;1,2,1,2)$	$(1;2,2,1,1)$	$(1;2,2,1,2)$	$(2;2,2,3,2)$	$(2;2,3,3,2)$	$(2;3,2,3,2)$	$(2;3,3,3,2)$
	$(0;0)$	1	1	1	1	-1	-1	0	0	-1	-1	0	0
	$(0;1)$	-1	-1	-1	1	1	1	1	1	-1	0	-1	0
	$(0;2)$	-1	-1	0	0	-1	-1	-1	-1	1	1	1	1
Player 1	$(0;3)$	-1	0	-1	0	-1	0	-1	0	-1	-1	-1	-1
	$(1;0)$	-1	-1	-1	-1	0	0	-1	-1	0	0	-1	-1
	$(1;1)$	1	1	1	1	-1	-1	-1	-1	0	-1	0	-1
	$(1;2)$	0	0	-1	-1	1	1	1	1	-1	-1	-1	-1
	$(1;3)$	0	-1	0	-1	0	-1	0	-1	1	1	1	1

Table 2.10.1: The reduced normal form of the $(1,2)$-coin game.

know the fixed equilibrium strategy of their opponent. In the new approach the fictive player also knows the result of any possible randomization of the other player. Therefore, randomization becomes useless in giving maximal opposition against a fictive player.

For both player roles in the $(1,2)$-coin game, a fictive player can respond to any pure strategy of the opponent by taking a number of coins so that the other player's guess is incorrect. Moreover, he can guess the correct total himself. After all, a fictive player knows how many coins the opponent has taken in his hand. With this information, player 1 can always win the game by guessing the sum of this number and the (arbitrarily chosen) number of coins in his own hand. There is nothing player 2 can do to prevent him from winning. Similarly, player 2 as a fictive player will succeed in winning too. Hence, the expected gains for the fictive players are $U_1^f = U_2^f = 1$. Consequently, the average expected gain for a fictive player is $U^f = 1$.

We can now compute the learning effect and the random effect for the $(1,2)$-coin game following the definitions of section 2.8:

$$\begin{aligned} LE &= \tfrac{1}{2}\textstyle\sum_{i=1}^{2}(U_i^m - U_i^0) = \tfrac{1}{2}((-\tfrac{1}{5}+\tfrac{1}{3}) + (\tfrac{1}{5}+\tfrac{71}{109})) = \tfrac{161}{327}, \\ RE &= \tfrac{1}{2}\textstyle\sum_{i=1}^{2}(U_i^f - U_i^m) = \tfrac{1}{2}((1+\tfrac{1}{5}) + (1-\tfrac{1}{5})) = 1. \end{aligned}$$

The last step is to combine these effects to find the value of the skill measure:

$$RS = \frac{LE}{LE+RE} = \frac{161}{488} \approx 0.33.$$

The resulting skill value of 0.33 strongly depends on our definition of how beginners act. If the explanation of the (1,2)-coin game to a new player includes an advise on strategy selections, then a starting player may do significantly better than "our" beginner and, consequently, the skill level we find for this game would be lower.

When we compute the skill level for each player separately, we find

$$\begin{aligned} LE_1 &= \tfrac{2}{15}, & RE_1 &= \tfrac{6}{5}, & RS_1 &= \tfrac{1}{10}, \\ LE_2 &= \tfrac{464}{545}, & RE_2 &= \tfrac{4}{5}, & RS_2 &= \tfrac{116}{225}. \end{aligned}$$

It is clear now that there is more skill involved in this game for the second player role, who has to make use in a smart way of the information that is contained in player 1's guess. Averaging the two player skill levels according to the alternative formula (2.4) yields an overall skill level of $\frac{277}{900} \approx 0.3078$ for the game.

The next section gives an illustration of the way the skill measure should be computed for games with more than two players in which an external chance element is involved.

2.10.2 Drawpoker

In this section we consider the simplified version of standard drawpoker that is discussed by Binmore (1992, Chapter 12). The first simplification is that there is no second dealing round in which players can change a number of cards. The second simplification is that we do not use a standard deck with the usual types of hands of five cards. Instead of that, all cards have distinct values and every player gets only one card, dealt with or without replacement. In this way we get a poker game which still contains many strategic features of standard drawpoker. First we give a formal description of the game and thereafter we compute the relative skill of the three-person version of this game that Binmore (1992) also analyzed. For general references on specific aspects of poker, we refer to Epstein (1977) and Scarne (1990). We study poker games in more detail in chapter 5 and chapter 6.

We describe the rules of the most general form of our version of drawpoker precisely. The game is played with n players, numbered from 1 to n, where $n \geq 2$. At the beginning of the game, each player places the initial bet (or ante) a into the stakes. Then each player gets one card. This card is randomly chosen out of a deck of c cards with card values $0, 1, \ldots, c-1$. According to the set-up of the game, cards are dealt with ($r = 1$) or without ($r = 0$) replacement. If they are dealt with replacement, we should have $c \geq 2$ to make the game interesting; in the case of dealing without replacement, we require $c \geq n + 1$.

Then the betting starts, beginning with player 1, followed by player 2 and so on. Player 1 can *pass* or open the game with *bet*. If he bets, then he chooses an amount b_i from a given set of s betting possibilities $b_1 < \ldots < b_s$ and adds this to the pot. As long as the game is not opened with a bet, the next players have the same choice of moves. If all players pass, then the game is a draw and the ante is returned to the players.

As soon as the game is open, the player whose turn it is can choose between three actions. He may *call*, *raise* or *fold*. His successors have the same choice. A call means placing an amount into the pot equal to that of the last bet made. A raise means that the last bet is raised with some extra amount chosen from the set of betting possibilities and this new bet is placed into the pot. A fold means that the player drops out of the game and loses his contribution to the stakes.

The total number of raises, including the first bet, must not exceed a certain maximum m. A player cannot raise or call again on his own bet or raise. So the betting round ends if at a stage where the player whose turn it would be next is the last player who did not call or fold. If at such a stage this player is the only player still in the game, then he wins the pot. Otherwise, a showdown follows. In a showdown all players that are still in the game show their cards. The player with the highest card wins the pot. If more than one player has the highest card, which is only possible if cards were dealt with replacement, then the pot is equally divided between them.

To compute the skill level of our drawpoker game, we have to specify the strategy of a beginner in all possible player roles. It is not easy to judge how a beginner would play this game in all variants. Perhaps a very simple way is to imagine that he has two card values in mind: a raise card C_R and a fold card C_F. He uses these values to play as follows.

- If his card value is less than or equal to C_F, then he folds whenever this choice is available to him, otherwise he passes.
- If his card is greater than or equal to C_R, then he raises the maximum, b_s, whenever this choice is available to him; after a pass, he bets b_s.
- If his card is between C_F and C_R, then he does not pass, but bets the minimum, b_1. If the game is already open, then he calls.

Of course the values of C_F and C_R will depend on the game parameters. It seems reasonable to assume that a beginner bets or raises if, roughly spoken, his card value is among the highest 10% in the deck. With respect to the determination of the fold card, we think that the beginner will in general play with much opportunism and will not pass or fold unless his card is among the lowest 50% in the deck. If the number of players becomes larger, this percentage may increase, but it will probably not exceed 70%. This choice is reflected in the following formula for C_F. Clearly, this is too difficult to compute for a beginner, but the resulting value for the fold card satisfies the preceding description of the behaviour of our beginners:

$$\begin{aligned} C_F &= \max\left\{0, (c-2) - \left\lfloor (c-1)\left(\tfrac{1}{2} - \tfrac{1}{10}\ln(n-1)\right)\right\rfloor\right\}, \\ C_R &= (c-1) - \left\lfloor \tfrac{1}{10}(c-1)\right\rfloor. \end{aligned}$$

Here, $\lfloor x \rfloor$ denotes the integer part of x. This notion is used to make sure that the boundaries are given by (integer) card numbers.

So far we kept our notation and definitions with respect to the game as general as possible. In the remainder of this section we restrict our attention to the variant for which Binmore (1992) computed the Nash equilibrium. This is the 3-person drawpoker game with 2 cards, L (low) and H (high), that are dealt *with* replacement ($r = 1$). The ante is $a = 2$ and the only possible betting amount is $b_1 = 8$. Only one bet is allowed; when the game is open, players are only allowed to fold or call. This parameter choice enables us to do a large part of the skill analysis manually. Binmore already showed that the pure strategy space for all three players can be reduced enormously.

For this game the beginners' strategies we proposed above boil down to passing or folding with an L and raising or calling with an H.

For our analysis we are interested in the expected gains of the three players when the coalition formed by the two opponents gives maximal opposition. Let us consider the two-person zero-sum game that corresponds to the situation in which player 1 plays against the coalition of the players 2 and 3. After elimination of dominated strategies player 1 has two pure strategies left: $X_1 = \{x_1, x_2\}$. Strategy x_2 is the beginner's strategy, i.e., passing or folding with an L and raising or calling with an H. The other pure strategy, x_1, differs from x_2 in one position; it prescribes passing with a high card. This phenomenon in which a player with a good hand tries to mislead his opponents is called *sandbagging.* The coalition of players 2 and 3 also has only two undominated pure strategies: $X_{23} = \{y_1, y_2\}$. Translation of these strategies in terms of strategies for the two individual players in this coalition shows that player 2 always plays the same strategy as the beginner. Player 3 does that too in strategy y_1, but in y_2 he bets with a low card when neither of his predecessors has opened the game. This strategic aspect of poker is more familiar than sandbagging; it is called *bluffing*. The matrix $A_{1,23}$ below displays the net expected gains for player 1 in this reduced two-person zero-sum game.

$$A_{1,23} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{pmatrix} -\frac{1}{2} & \frac{3}{4} \\ 0 & -\frac{1}{4} \end{pmatrix} \end{array}$$

In the unique Nash equilibrium of this game player 1 plays $\frac{1}{6}x_1 + \frac{5}{6}x_2$, while his opponents use the mixed strategy $\frac{2}{3}y_1 + \frac{1}{3}y_2$. The resulting game value is $v(A_{1,23}) = -\frac{1}{12}$. Therefore, the expected gain for optimal player 1 is $U_1^m = -\frac{1}{12}$.

When player 1 plays the strategy of a beginner, then a possible best answer of the coalition is to let player 2 play the beginner's strategy and let player 3 play the bluffing strategy, i.e., to bet with an L when the other players have not opened the game. Since player 1 already reveals his card in the opening move, the other players make sure he does not gain anything unnecessary with an L and that the coalition does not lose any betting amounts if player 1 has an H. In this way player 1 makes an expected loss of $\frac{1}{4}$ and therefore his expected gain is $U_1^0 = -\frac{1}{4}$.

Now, let us see what the possibilities of the fictive player are. This player knows the cards of the other players as well as the result of their possible randomization. Furthermore, he can anticipate future actions of opponents, since our analysis is based on the normal form of the game. His opponents are aware of all this and give maximal opposition. Therefore, for players 2 and 3 bluffing and sandbagging become useless if they face a fictive player. They both pass or fold with an L and they bet or call with an H. So, if player 1 knows that neither player 2 nor player 3 has a higher card than he has, the best he can do with an L is to bet (bluff). The opponents will fold. Of course, if he has a high card, he will always bet. This is how player 1 as a fictive player can make an expected gain of $U_1^f = \frac{1}{2}$.

For the other players similar reasoning and computations lead to the values that are displayed in Table 2.10.2. From the numbers in this table we can compute the learning effect and the random effect for 3-person drawpoker:

$$\begin{aligned} LE &= \tfrac{1}{3}\textstyle\sum_{i=1}^{3}(U_i^m - U_i^0) = \tfrac{1}{3}((-\tfrac{1}{12}+\tfrac{1}{4}) + (-\tfrac{1}{12}+\tfrac{1}{4})) + (\tfrac{1}{10}-0)) = \tfrac{13}{90}, \\ RE &= \tfrac{1}{3}\textstyle\sum_{i=1}^{3}(U_i^f - U_i^m) = \tfrac{1}{3}((\tfrac{1}{2}+\tfrac{1}{12}) + (\tfrac{1}{2}+\tfrac{1}{12}) + (\tfrac{1}{2}-\tfrac{1}{10})) = \tfrac{47}{90}. \end{aligned}$$

	player 1	player 2	player 3
beginner	$-\frac{1}{4}$	$-\frac{1}{4}$	0
optimal	$-\frac{1}{12}$	$-\frac{1}{12}$	$\frac{1}{10}$
fictive	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Table 2.10.2: The expected gains in 3-person drawpoker.

The last step is to combine these effects to find the value of the skill measure:

$$RS = \frac{LE}{LE + RE} = \frac{13}{60} \approx 0.22.$$

When we compute the skill level for each player separately, we find

$$\begin{array}{llllll} LE_1 & = \frac{1}{6}, & RE_1 & = \frac{7}{12}, & RS_1 & = \frac{2}{9}, \\ LE_2 & = \frac{1}{6}, & RE_2 & = \frac{7}{12}, & RS_2 & = \frac{2}{9}, \\ LE_3 & = \frac{1}{10}, & RE_3 & = \frac{2}{5}, & RS_3 & = \frac{1}{5}. \end{array}$$

The result of averaging these numbers according to formula (2.4) yields an average skill level of $\frac{29}{135} \approx 0.21$, which is close to the value of RS we just found. Skill does not vary very much over the three player roles in this game.

Just as in the coin game example of section 2.10.1, we would like to stress that this result depends on our definition of beginners' behaviour. If this game is played in a casino where the brochure with the game description ends with some words on bluffing and sandbagging, then a beginner may use a strategy that differs heavily from the one we assumed him to use. This may also affect the skill level we find for the drawpoker game.

2.11 Using empirical data

In the foregoing we have described, discussed and illustrated general aspects concerning a skill analysis of casino games. In the last section of the chapter we want to indicate briefly what could be the role of empirical data in determining the skill level of a game.

In the first place one could think about collecting player results in a casino and using the resulting numbers as input for the skill measure that was given in formula (2.1). However, one should be careful. For a one-person game, we can imagine that it is possible to collect information about the game results of beginners and advanced players, or otherwise about the average player that was mentioned in section 2.3.4. The expected results for the fictive player should still be computed, since this is a theoretical player type.

For more-person games, the collection of useful data is more difficult. Of course, it is still possible to observe and collect the game results of beginners and advanced players. However, one should now know exactly against what kind of opposition the results in this data set are obtained. In an ideal situation one should obtain detailed information about the decisions made by all players. After all, the results for the fictive players still have to be computed and for these computations information about the opposition is needed.

If information is not available in so much detail, it may still be possible to come up with a specific solution for the particular situation in which the skill analysis is applied. In chapter 3, an example of such a context-specific approach is given: the framework sketched in this chapter is used, in combination with an analysis of variance, to determine the skill level of a "management game".

As a final remark we wish to mention the possibility of designing experiments to collect data for a specific game. Yu and Cowan (1995) give an example of a statistical model using duplicate tournaments to deduce information about the luck-skill balance in the game. They argue, however, that it is difficult to separately estimate effects of luck and effects of the actions chosen by a player.

Chapter 3

Case study: management games

3.1 Introduction

In recent years the popularity of a certain type of sports-related competitions has increased: management games. A participant in such a game acts as the manager of a fictive sports team. Examples of sports for which management games are organized, are soccer, tennis, cycling and Formula One racing. Given a set of restrictions, the participant selects players and possibly additional elements for his team. His team earns points for certain events that occur in the sports competition to which the management game is related. The goal of the game is to maximize the number of points earned during the competition. Basically, a team scores well in the management game if the team members do well in the real competition.

Management games are made attractive by promising interesting prizes for the best managers. Moreover, participation fees are kept low. This combination naturally leads to a large number of participants. With many paying participants the exploitation of a management game turns into a profitable business. However, since this business concerns "exploiting games with monetary prizes", it is interesting to know whether or not a licence is required for the organization of a management game, according to the passage of the Dutch Gaming Act that was cited at the beginning of section 2.2. Recall that such a licence is required if the game is classified as a game of chance. If it is a game of skill, exploitation is allowed to anyone. The investigation of the participants' influence on their winning possibilities or, phrased differently, the investigation of the skill involved in a management game is the subject of the current chapter.

The development of the Internet is probably the main reason for the increased popularity of management games. While such competitions were organized by newspapers and sports magazines before, the Internet has made the organization of these games much easier and less costly. It is not difficult anymore to deal with large groups of participants, since the participants themselves can be made responsible for all their management actions. Using smart programming work and clear instructions, the interaction of the game organizers with the game can be restricted to adding new results from the real sports competition to a central database from which the team scores are derived.

The example which is studied in detail in this chapter, is a management game called Grand Prix Manager 2003 (GPM 2003). In this game a team consists of two drivers, a chassis, an engine and a set of tires that were active in the FIA Formula One World Championship 2003. Drivers or car parts score positively for a team in GPM 2003 by reaching good qualifying positions, achieving good race results, setting fastest lap times, and so on. Crashing or not finishing for other reasons leads to negative scores in the management game.

The remainder of this chapter is organized as follows. Section 3.2 discusses the details of the skill analysis for management games in general, while section 3.3 contains the complete analysis of the example game, GPM 2003. The computations of this example were originally carried out for the research report of Van der Genugten et al. (2004).

3.2 The analysis of skill

In management games the participants can win prizes and they have to pay a fee to take part in the competition. Typically, in these games both the skill of the participants and random factors influence the game results. Therefore, a logical question is: do management games classify as games of chance according to the Gaming Act? Skill can be applied by carefully studying the competition to which the management game is related in order to make a good estimate of the expected scores of all potential team elements. Data from the past, from related competitions can be used for this purpose. However, no matter how well one is prepared, one never knows whether all predictions will come true. The uncertainty about the results of the real sports events can be interpreted as the random factor in the management game. In contrast to casino games, in a management game there is no objectively defined randomization process from which one can derive information about the probabilities. Therefore, we have to adapt the methods from chapter 2 slightly, before we can apply them to a management game in order to determine if it should be classified as a game of chance or as a game of skill.

We list the characteristics of a management game that are important for our analysis:

- the game consists of several comparable rounds of play;
- in each round the participants earn points and the scores of participants do not (or hardly) influence the scores of other participants;
- the bets (or participation fees) are the same for all participants;
- the distribution of prizes is based on the comparison of the scores of all participants: a participant's gains (prizes) in the management game depend on his own score, but also on the scores of the other participants;
- the game has at least a few thousand participants.

For games with these characteristics it is possible to determine the relative skill level, based on realized scores of players, the prize schedule and other game specific information about possibilities for team formation. This is done by estimating probabilities from the available game data. Clearly, this analysis can only be carried out *after* the game has ended. However, it is definitely reasonable to use the results of such an analysis to draw conclusions about management games that will be organized in the future, if these games show a large extent of similarity with the game for which the analysis was carried out.

Since the bets are the same for all participants, we do not have to take them into account in our analysis. This influences neither the learning effect nor the random effect, so the relative skill level is also not affected. Therefore, in the remainder of this chapter, *game results* can be interpreted both as gains and as (the monetary value of) prizes.

The analysis is based on estimates of the game results of the three player types *beginner*, *advanced player* and *fictive player*. In the analysis of this practical situation, the advanced player takes the role of the optimal player, as discussed in section 2.3.2.

Since the game has a large number of participants, it is reasonable to assume that there will be both beginners and advanced players among them. Beginners will reach low scores, while the advanced players will attain high scores that bring them close to the top of the ranking. We assume that the population of advanced players will be smaller than the population of beginners. Therefore, we fix the score of the type beginner at the average of the lowest five percent of the scores in the final ranking of the management game. The score of the type advanced player is defined as the average score of the top ten of the same ranking. This is a modification of the analysis of chapter 2: instead of taking expectations with respect to the probabilities determined by a given randomization process, we take the averages of groups of comparable participants.

The scores of the players are influenced by the uncertain events in the sports competition. Therefore, we make a statistical model of the scores of all participants in each round of play. In this model the player influence on the score is a systematic component, while the fluctuations around this component are attributed to the uncertain events. Our estimation of the model is based on the realized scores of the participants. The scoring possibilities in the management game are directly determined by the events that occur in the underlying sports competition. They can vary over rounds. In the example of GPM 2003, a round of play corresponds to a Grand Prix race. In a race with many crashing cars there are less (positive) scoring possibilities than in races in which almost all cars reach the finish. This difference will have a systematic effect on the scores for the corresponding round of play in the management game. So, in our statistical model we have to take into account this variation.

Therefore, we use a two-way analysis of variance (ANOVA) with *player influence* and *round influence* as explanatory factors to estimate the distribution of the scores of all participants. The fit of this model turns out to be good. Using the given prize schedule of the game, these estimated distributions also lead to the expected game results of the player types beginner and advanced player. For the fictive player we do not use the statistical model. His game result can be derived from the game data. The fictive player knows in advance the result of all uncertain events: he knows which team elements score well in each round. He can use this information to set up a "perfect team", leading to the maximum game result that is achievable.

In section 3.3 we present the details of this analysis of skill for the case of GPM 2003.

3.3 Grand Prix Manager

In this section we give a complete overview of the analysis of relative skill for the management game Grand Prix Manager 2003 (GPM 2003). This is a game that was offered by the Dutch company Sportdreams BV on the web site http://www.f1manager.com. The game is related to the FIA Formula One World Championship 2003.

In section 3.3.1, we give a description of the basic features and rules of the game. An overview of the information that was available to us about results of the management game and of the FIA Championship is given in section 3.3.2. In section 3.3.3, we use this information to determine the skill level of GPM 2003.

3.3.1 The rules of the game

GPM 2003 is a management game based on the FIA Formula One World Championship 2003. Each participant uses a given budget to compose a team, consisting of two drivers, an engine, a chassis and a set of tires. He manages this team during the championship and earns points for the achievements of his team elements in the real Formula One competition.

The goal of the game With the team he manages, each participant earns points during each of the sixteen Grand Prix races of the championship. The goal of the game is to maximize this number of points in order to reach a top position in the general ranking. Prizes are available for the top positions in that ranking.

Composition of a team Each participant gets a budget of € 100 million to buy the required elements for his team. All drivers, engines, chassis and tires are given a value in euros. These values are determined by Sportdreams and based on the results of the FIA Formula One World Championship 2002. These values are not changed during the season. Table 3.1 gives an overview of all values.

Drivers

Name	Price (€)	Team
Coulthard	33 mln	McLaren
Raïkkonen (2)	25 mln	McLaren
M. Schumacher	60 mln	Ferrari
Barrichello (2)	43 mln	Ferrari
Fisichella	18 mln	Jordan
Firman (2)	20 mln	Jordan
Webber	20 mln	Jaguar
Wilson (2)	18 mln	Jaguar
Montoya	34 mln	Williams
R. Schumacher (2)	42 mln	Williams
Trulli	19 mln	Mild Seven Renault
Alonso (2)	25 mln	Mild Seven Renault
Sato (Villeneuve)	19 mln	BAR
Button (2)	18 mln	BAR
Heidfeld	24 mln	Sauber
Frentzen (2)	20 mln	Sauber
Panis	19 mln	Toyota
Da Matta (2)	18 mln	Toyota
Verstappen	17 mln	Minardi
Kiesa (2)	12 mln	Minardi

Tires

Name	Price (€)
Bridgestone	5 mln
Michelin	3 mln

Chassis

Name	Price (€)
McLaren	31 mln
Ferrari	52 mln
Jordan	18 mln
Jaguar	15 mln
Williams	41 mln
Mild Seven Renault	23 mln
BAR	17 mln
Sauber	22 mln
Toyota	16 mln
Minardi	11 mln

Engines

Name	Price (€)
Mercedes	31 mln
Ferrari	56 mln
Jordan Ford	17 mln
Jaguar Cosworth	15 mln
BMW	41 mln
Renault	21 mln
BAR Honda	16 mln
Petronas	23 mln
Toyota	15 mln
Minardi Cosworth	11 mln

Table 3.1: Values of all drivers and car parts in GPM 2003.

Some parts of this overview need extra explanation. A (2) after the name of a driver indicates that he is the second driver of the Formula One team. Second drivers score differently from first drivers in GPM 2003. The chassis is just named after the team. The ten different engines are listed in the same order as the teams that use them. The only extra information we need to know exactly how all possible team elements are related, is the usage of tires by the different teams. This information is given in Table 3.2.

Team values and transfers All values given in Table 3.1 are constant during the competition. However, the team budget changes after each Grand Prix, depending on the score the team has achieved. For each point the team earns, the budget is increased by € 0.01 million. All team elements can be sold and changed after each Grand Prix. As the competition progresses and

Team	Tires
McLaren	Michelin
Ferrari	Bridgestone
Jordan	Bridgestone
Jaguar	Michelin
Williams	Michelin
Mild Seven Renault	Michelin
BAR	Bridgestone
Sauber	Bridgestone
Toyota	Michelin
Minardi	Bridgestone

Table 3.2: Tires used by the ten different race teams.

a team does well, its manager is able to select more expensive drivers or car parts for the team. Each transfer entails transaction costs: 10% of the value of the sold item is subtracted from the team budget.

Score system The game is based on a rather complex score system. We give an overview of the details, starting with the points that the drivers can earn.

The drivers get points for their position at the start of the race, which is determined during the qualifying session. They earn points for their final race position, even if they don't reach the finish. Furthermore, points can be won by setting the fastest lap during the race, finishing in the Grand Prix and for the difference between the qualifying position ($\mathrm{Pos}_{\mathrm{qual}}$) and the final race position ($\mathrm{Pos}_{\mathrm{race}}$). The latter difference can also lead to a negative score. Negative scores are also obtained, when a car has to leave the race. This can either be the result of an error of the driver or an accident with another car. Even if the accident is caused by another driver, this leads to a negative score. A driver is punished with a negative score for each stop-and-go penalty that is imposed on him during the race. Second drivers, marked with a (2) in Table 3.1, can earn extra points by finishing before their team-mate. Table 3.3 gives an overview of the score system for the drivers.

The engines score points for the manager in a similar way. Points can be earned with the qualifying position and the position in the final ranking of the Grand Prix. Apart from that, three things affect the engine score: driving the fastest lap, finishing and an engine fault leading to elimination of the car from the race. Since each engine is used in two cars during each Grand Prix, it is possible that an engine can get a positive score for finishing and a negative

Position	Qualifying	Race
1	50	100
2	45	90
3	40	80
4	36	72
5	32	64
6	28	56
7	24	48
8	20	40
9	17	34
10	14	28
11	10	20
12	9	18
13	8	16
14	7	14
15	6	12
16	5	10
17	4	8
18	3	6
19	2	4
20	1	2

Event	Score
Fastest lap (first driver)	5
Fastest lap (second driver)	10
Finishing (first driver)	5
Finishing (second driver)	10
Second driver before team-mate	10
Race versus qualifying	$\text{Pos}_{\text{qual}} - \text{Pos}_{\text{race}}$
Each stop-and-go penalty	-5
Not finished: driver fault	-10
Not finished: accident	-5

Table 3.3: GPM 2003 scoring system for drivers.

score for not finishing as a result of an engine fault at the same time. Table 3.4 shows the details of the engine scores.

Position	Qualifying	Race
1	25	50
2	23	45
3	20	40
4	18	36
5	16	32
6	14	28
7	12	24
8	10	20
9	8	17
10	7	14
11	5	10
12	5	9
13	4	8
14	4	7
15	3	6
16	3	5
17	2	4
18	2	3
19	1	2
20	1	1

Event	Score
Fastest lap	15
Finishing	5
Not finished: engine fault	-10

Table 3.4: GPM 2003 scoring system for engines.

The third type of team element that can win or lose points for the manager's team, is the chassis. Scores are awarded to the chassis in a way that is very close to the way the engine scores are determined. The difference is that the chassis doesn't get a score for the fastest race lap. Just like an engine, a chassis is awarded two different scores in each race, since it is used in two cars.

Table 3.5 gives an overview of the chassis scores.

Position	Qualifying	Race
1	25	50
2	23	45
3	20	40
4	18	36
5	16	32
6	14	28
7	12	24
8	10	20
9	8	17
10	7	14
11	5	10
12	5	9
13	4	8
14	4	7
15	3	6
16	3	5
17	2	4
18	2	3
19	1	2
20	1	1

Event	Score
Finishing	5
Not finished: chassis fault	−10

Table 3.5: GPM 2003 scoring system for the chassis.

The final team element affecting the score, is the set of tires. Tires score for the pole position in the qualification, for the fastest race lap and for winning a Grand Prix. Each of these events leads to a score of five points, as can be seen in Table 3.6. So the maximum number of points won with a set of tires in a Grand Prix is fifteen.

Event	Score
Pole position	5
Fastest lap	5
Winning the Grand Prix	5

Table 3.6: GPM 2003 scoring system for the tires.

Quiz Besides the points that can be earned with the team, a participant in GPM 2003 can also earn 10 points in each round of play by giving the correct answer to a question related to Formula One racing. No extra team budget is awarded for a correct answer, so the quiz score only affects the participant's position in the ranking.

Prizes For the managers of the teams that end at the top positions in the final ranking, prizes are available. Apart from that, prizes can also be won in each individual round of play. We could not retrieve the exact prize schedule for GPM 2003. However, since prize schedules for similar games are not altered much from year to year, the scheme for the next edition of the game, GPM 2004, can very well be used as a approximation. This is the approach that we followed. Table 3.7 gives an overview of the prizes that can be won in GPM 2004. We have estimated the monetary value of each prize, to be able to compare them. These estimates are based on price information from the Internet, mostly from sites to which the Sportdreams web site directly refers.

Prizes for the final ranking

Position	Value (€)	Prize (description)
1	800.00	3-day trip to Grand Prix in Barcelona
2	200.00	cap and photo, signed by Michael Schumacher
3- 7	100.00	Formula One fan-package (cap, shirt and flag)
8-12	41.00	lithograph of a driver (of own choice)
13-22	39.95	DVD with overview of the Formula One season 2003

Prizes for each round of play

Position	Value (€)	Prize (description)
1	64.50	Formula One car (1:18 scale model)
2	40.00	voucher for car maintenance
3	29.00	one year subscription to magazine
4-5	7.95	book about Formula One

Table 3.7: Prizes in GPM 2004 and their estimated values.

3.3.2 Available data

For GPM 2003, we had access to the scores of all participants (managers) for each separate round of play (corresponding to a Grand Prix). During the Formula One season of 2003, sixteen races were organized.

For a number of participants, we have not taken into account the scores. The most important reason was that some participants had scored zero points in two or more rounds of play. Since the number of negative scores in each round of play is very small, it is not likely that these zeroes are scored with teams that satisfy all conditions. Is is more likely that the participants have not actively participated in the management game during these rounds of play. An extra argument that supports this conjecture, is that the zeroes are mainly obtained in the first rounds of the game. Probably, these participants have subscribed only after the start of GPM 2003. Therefore, we do not consider their results relevant for our analysis of the skill level of the management game.

After this elimination of scores according to the criterion we have just described, we still have the results of 10,566 participants. Table 3.8 shows a small part of the table with the scores.

Position	GP 1	GP 2	GP 3	⋯	GP 14	GP 15	GP 16	Total
1	428	502	478		450	349	428	6,796
2	380	443	488		480	399	455	6,776
3	384	582	486		470	399	488	6,772
4	444	454	530		480	422	329	6,763
5	369	577	535		480	399	251	6,718
6	428	492	478		435	422	508	6,711
7	367	492	468		435	399	513	6,703
8	428	502	478		530	399	395	6,695
9	428	502	478		470	434	394	6,642
10	401	443	488		480	399	325	6,637
⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋮
10,557	0	225	260		159	239	125	2,485
10,558	0	235	184		198	266	104	2,344
10,559	0	105	270		133	172	186	2,335
10,560	118	102	237		125	240	149	2,250
10,561	128	96	189		63	253	171	2,190
10,562	107	91	95		136	193	136	2,156
10,563	0	90	85		92	173	203	2,070
10,564	111	95	61		119	184	122	1,862
10,565	116	89	186		97	217	91	1,797
10,566	0	95	51		129	184	112	1,731

Table 3.8: A sample of the scores of the participants of GPM 2003.

Apart from this information, we got the complete race results for all drivers, for all sixteen races of the FIA Formula One World Championship 2003. Results for the other team elements (engines, etc.) follow from Table 3.1 and Table 3.2. Together with the information about the score system of GPM 2003, which is described in section 3.3.1, we can reconstruct the points for all elements for each separate Grand Prix.

3.3.3 Analysis of skill

This section contains the details of the skill analysis for GPM 2003. Section 3.3.3.1 describes the statistical model we formulated for the participants' scores. Sections 3.3.3.2-3.3.3.4 contain the derivation of the expected game results for the three player types involved in the analysis of skill: beginners, advanced players and fictive players.

3.3.3.1 A statistical analysis of the scores

The scores of the players are influenced by the uncertain events in the sports competition. Therefore, we make a statistical model of the scores of all participants in each round of play. We denote the score of participant i in round j as y_{ij}. Assume that y_{ij} can be explained by an overal mean (ν), the influence of the player (α_i) and the influence of the round of play (β_j) and an error term (ε_{ij}):

$$y_{ij} = \mu_{ij} + \varepsilon_{ij} \qquad \text{for } i = 1, \ldots, a \text{ and } j = 1, \ldots, b,$$

with

$$\mu_{ij} = \nu + \alpha_i + \beta_j.$$

Here $a = 10,566$ is the number of players and $b = 16$ is the number of rounds. For identification we assume that $\sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{b} \beta_j = 0$. Furthermore, we assume that the errors ε_{ij} are i.i.d. with normal distribution $N(0, \sigma^2)$. The LS-estimate of the model is

$$\begin{aligned} y_{ij} &= z_{ij} + e_{ij}, \\ z_{ij} &= \bar{y} + (\bar{y}_{i\cdot} - \bar{y}) + (\bar{y}_{\cdot j} - \bar{y}) \end{aligned}$$

with

$$\bar{y} = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij}, \quad \bar{y}_{i\cdot} = \frac{1}{b} \sum_{j=1}^{b} y_{ij} \quad \text{and} \quad \bar{y}_{\cdot j} = \frac{1}{a} \sum_{i=1}^{a} y_{ij}.$$

Table 3.9 gives the two-way ANOVA-table of the round scores. For each source of variation, the sum of squares (SS), the degrees of freedom (d.f.), the mean square (MS), the F-ratio and the 0.95-quantile of the corresponding F-distribution are given. From this table, we see that both factors, the player (A)

source	SS	d.f.	MS	F-ratio	$F_{0.05}$
player (A)	362,587,336.1	10,565	34,319.7	5.9	1.02
round (B)	476,040,019.6	15	31,736,001.3	5,483.4	1.67
model (M)	838,627,355.8	10,580	79,265.3	13.7	1.02
error (E)	917,194,902.1	158,475	5,787.6		
corr. total	1,755,822,257.9	169,055			

$$R^2 = 0.48$$

Table 3.9: Two-way analysis of variance of the round scores.

and the round of play (B), have a significant influence on a player's round score. The model effect is significant as well. The mean squared error, $s^2 = \text{MSE}$, estimates σ^2. Therefore, the estimate of σ is $s = \sqrt{\text{MSE}} = 76.08$. To check some model assumptions, we have plotted a histogram of the standardized residual values e_{ij}/s. Figure 3.1 shows the result. Clearly, the assumption of the normal distribution is reasonable. There seem to be no outliers either: just over four percent of the observations lies outside the interval $[-2s, 2s]$.

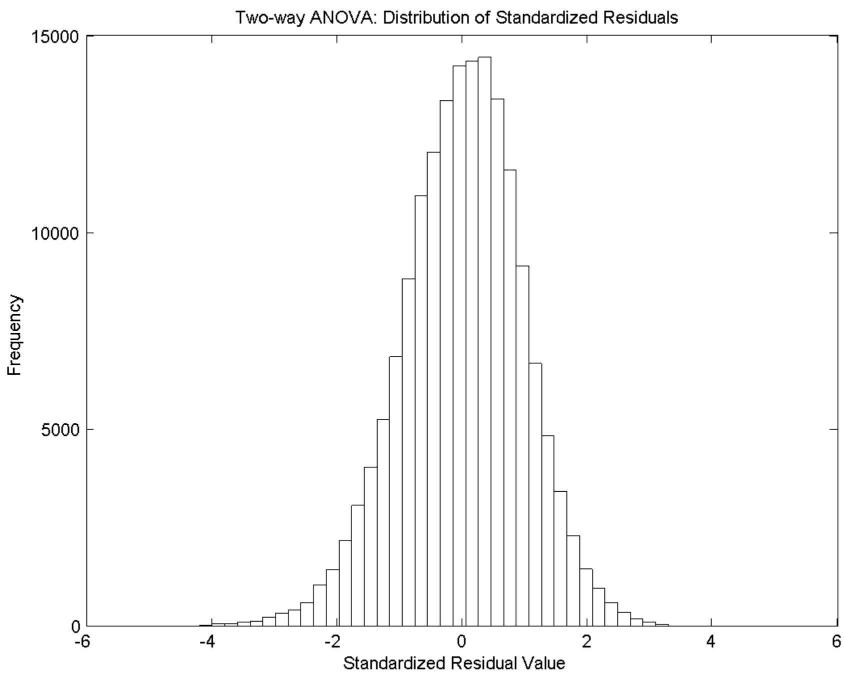


Figure 3.1: Histogram of the standardized residuals in the two-way ANOVA for GPM 2003.

The round effects β_j have no influence on the positions of the players, since they have the same effect on all player scores within one round of play. Furthermore, $\sum_{j=1}^{b} \beta_j = 0$. Therefore we eliminate the round effects. So the round scores of player i are now assumed to be distributed according to $N(\nu + \alpha_i, \sigma^2)$. This distribution is estimated by $N(\bar{y}_{i\cdot}, s^2)$, since $\nu + \alpha_i$ is estimated by $\bar{y} + (\bar{y}_{i\cdot} - \bar{y}) = \bar{y}_{i\cdot}$. The total score of player i has distribution $N(b(\nu + \alpha_i), b\sigma^2)$, estimated by $N(b\bar{y}_{i\cdot}, s^2_{\text{tot}})$ with $s^2_{\text{tot}} = bs^2$. The cumulative distribution of the expected total scores of the participants is displayed in Figure 3.2.

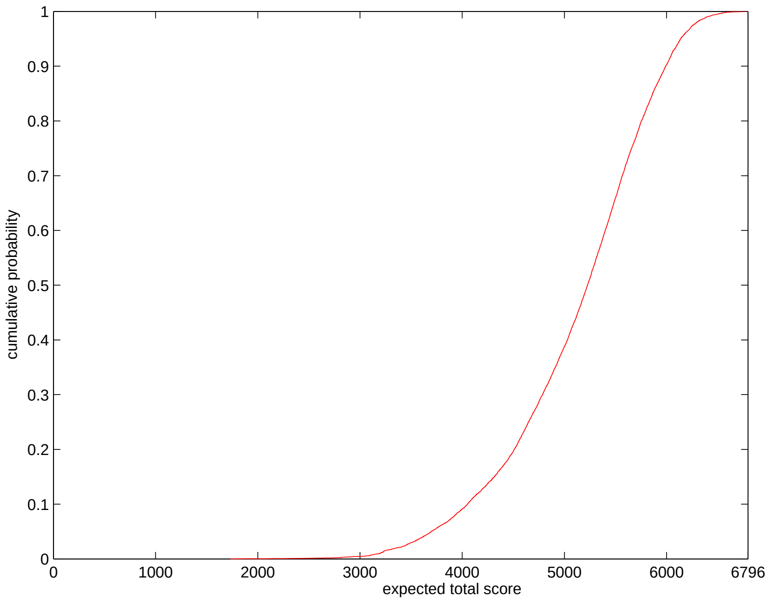


Figure 3.2: Cumulative distribution of the expected total scores for the participants in GPM 2003 (the standard deviation for the expected total scores is $s_{\text{tot}} = 304.31$).

In theory, the simultaneous distribution of the prizes follows from the simultaneous distribution of the total scores. Using the information from Table 3.7, this distribution of the prizes gives us the expected gains of all participants. Since we cannot compute the expected gains analytically, we do it by simulation. Using the estimated distributions of the round scores of the players, we have simulated over 3.5 million repetitions of GPM 2003 with the given population of participants. This number of repetitions is large enough for obtaining the desired accuracy. Figure 3.3 shows clearly that the expected gains are very small for most participants.

In the following three subsections we give the expected game results as well

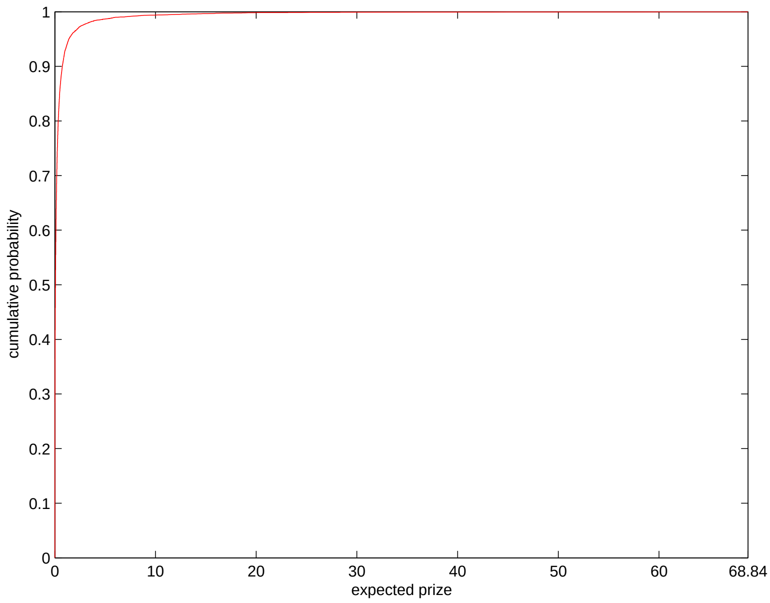


Figure 3.3: Cumulative distribution of the expected prizes (€) for the participants in GPM 2003.

as the expected prizes for the three player types that are considered in the skill analysis.

3.3.3.2 Beginners

For the skill analysis, as introduced in chapter 2, we need to know the expected game result of beginners in GPM 2003. As mentioned in section 3.2, we consider the participants that end up in the lowest five percent of the final ranking as beginners. For this group, the average total score is equal to $3,367$ points. The expected prize of a beginner in GPM 2003 is equal to € 0.00.

3.3.3.3 Advanced players

For the advanced players in GPM 2003, we determine the expected prize and the expected total score in the same way as we did for the beginners in section 3.3.3.2. We use the the top ten of the final ranking. The expected total score is therefore directly derivable from Table 3.8 and is equal to $6,721$ points. For the expected prize of the advanced players, we use the simulation results to find an amount of € 47.70.

3.3.3.4 Fictive players

For determination of the results of the fictive player, we cannot use the statistical approximation of section 3.3.3.1. The fictive player is informed about the realization of all elements of uncertainty before determining his strategy. For GPM 2003 this means that a fictive player knows in advance the race results of all Grand Prix races and the resulting scores for all drivers, engines, chassis and tires in the management game.

Since this information is collected during the season, we can only determine the optimal strategy of the fictive player in GPM 2003 *after* the end of the Formula One season. In theory, we could use non-linear programming to solve the optimization problem that determines this strategy. However, the dimensions of this problem are too large for the memory of our computer. Therefore, we used an approximation: we found a well-scoring collection of sixteen race teams (one for each race), that satisfies all rules of the management game. The team compositions for all rounds of play are given in Table 3.10. The total score for each Grand Prix includes a maximum quiz score of ten points.

GP	Driver 1	Driver 2	Engine	Chassis	Tires	Price	Score
1	Raïkkonen (2)	Frentzen (2)	Mercedes	BAR	Michelin	96 mln	446
2	Raïkkonen (2)	Alonso (2)	Renault	MS Renault	Michelin	97 mln	582
3	Raïkkonen (2)	Alonso (2)	Renault	MS Renault	Michelin	97 mln	486
4	Raïkkonen (2)	Alonso (2)	Renault	MS Renault	Michelin	97 mln	374
5	Montoya	Alonso (2)	Renault	MS Renault	Bridgestone	108 mln	463
6	Raïkkonen (2)	Button (2)	Mercedes	BAR	Bridgestone	96 mln	488
7	Raïkkonen (2)	Trulli	Mercedes	MS Renault	Michelin	101 mln	512
8	Montoya	Alonso (2)	Renault	Jaguar	Michelin	98 mln	441
9	Montoya	Alonso (2)	Renault	Jaguar	Michelin	98 mln	398
10	Raïkkonen (2)	Kiesa (2)	BMW	Jaguar	Michelin	96 mln	433
11	Raïkkonen (2)	Trulli	Mercedes	Toyota	Bridgestone	96 mln	432
12	Montoya	Trulli	Renault	Toyota	Michelin	93 mln	512
13	Webber	Alonso (2)	Renault	MS Renault	Michelin	92 mln	536
14	Raïkkonen (2)	Montoya	BAR Honda	Minardi	Michelin	89 mln	346
15	Raïkkonen (2)	Frentzen (2)	Toyota	Sauber	Michelin	85 mln	446
16	Raïkkonen (2)	Button (2)	BAR Honda	BAR	Bridgestone	81 mln	453
						Total	7,348

Table 3.10: A lower bound for the score of the fictive player in GPM 2003: a well-scoring strategy.

So we have found a strategy that gives a lower bound for the score of the fictive player. We can also determine an upper bound by computing an optimal strategy in a game in which the rules are slightly relaxed. When we assume that no transfer costs are involved with selling drivers, engines, chassis or tires, a player can select the teams that are shown in Table 3.11.

GP	Driver 1	Driver 2	Engine	Chassis	Tires	Price	Score
1	Raïkkonen (2)	Frentzen (2)	Mercedes	BAR	Michelin	96 mln	446
2	Raïkkonen (2)	Alonso (2)	Renault	MS Renault	Michelin	97 mln	582
3	Raïkkonen (2)	Fisichella	Mercedes	McLaren	Bridgestone	110 mln	583
4	Raïkkonen (2)	Alonso (2)	Mercedes	McLaren	Michelin	115 mln	468
5	Barrichello (2)	Alonso (2)	Renault	MS Renault	Bridgestone	117 mln	502
6	Raïkkonen (2)	Button (2)	Mercedes	McLaren	Bridgestone	110 mln	530
7	Raïkkonen (2)	Montoya	Mercedes	McLaren	Michelin	124 mln	586
8	Raïkkonen (2)	Alonso (2)	BMW	Williams	Michelin	135 mln	549
9	Montoya	Button (2)	BMW	Williams	Michelin	137 mln	533
10	Raïkkonen (2)	Montoya	BMW	Williams	Michelin	144 mln	619
11	Raïkkonen (2)	Barrichello (2)	Ferrari	MS Renault	Bridgestone	152 mln	566
12	Coulthard	Montoya	BMW	MS Renault	Michelin	134 mln	547
13	Raïkkonen (2)	Alonso (2)	BMW	Williams	Michelin	135 mln	608
14	Raïkkonen (2)	Montoya	Ferrari	Williams	Bridgestone	161 mln	579
15	Raïkkonen (2)	M. Schumacher	Ferrari	Sauber	Bridgestone	168 mln	519
16	Raïkkonen (2)	Barrichello (2)	Mercedes	McLaren	Bridgestone	135 mln	566
						Total	8,783

Table 3.11: An upper bound for the score of the fictive player in GPM 2003: a very well scoring strategy (which would be possible in a game without transfer costs).

For the strategies given in Table 3.10 and 3.11, we can also determine to which prizes they would have led in GPM 2003, as it was played in practice. In this way, we compute a lower and an upper bound for the prize of the fictive player. The lower bound is € 993.50, while the upper bound we find in this way, is equal to € $1,832.00$.

After manually exploring improvement possibilities for the strategy that gives the lower bound, we think this bound is rather tight. Since the team prices in the upper bound strategy are much larger than the real player budget allows, the upper bound probably gives a large overestimation of the possibilities of the fictive player. These considerations have led to our estimate of the score and the prize of the fictive player: we have taken a weighted average of the bounds we computed, in which the weight of the lower bound is four times the weight of the upper bound. This gives a score of $7,635$ points for the fictive player, and an expected prize of € $1,161.20$.

3.3.4 Relative skill of GPM 2003

Using the definitions from section 3.2 and the numerical results from sections 3.3.3.2-3.3.3.4, we can determine the relative skill of GPM 2003. Table 3.12 gives an overview of all numbers that are relevant for the skill analysis.

		Prizes	Scores
Expected result beginner	€	0.00	3,367
Expected result advanced player	€	47.70	6,721
Expected result fictive player	€	1,161.20	7,635
Learning effect (LE)	€	47.70	3,354
Random effect (RE)	€	1,113.50	914
Relative skill (RS)		0.041	0.786

Table 3.12: Relative skill in GPM 2003.

We distinguish the analysis which is based on the prizes in the game from the analysis based on the scores of the participants.

We notice a serious difference between the RS based on prizes and RS based on game scores. If the prize a participant wins would simply be his score, then the RS of GPM 2003 would be approximately 0.79. However, in the management game, as it is offered by Sportdreams BV, the scores are only an intermediate result, on which the distribution of prizes is based. Taking into account the prize schedule, the RS of GPM 2003 turns out to be approximately 0.04.

Following the jurisprudence concerning the Dutch Gaming Act, which implicitly puts the bound between games of chance and games of skill somewhere between 0.05 and 0.15, GPM 2003 should be classified as a game of chance. Consider once again the hypothetical situation in which the total game scores would be the amount of money won by the participants. In this case, even the lower bound of RS is much larger than 0.15. This tells us that in this case the management game should be classified as a game of skill. However, we stress again that the underlying assumption is invalid: prizes are not equal to scores in the real game.

These two (apparently contradicting) conclusions are both given to illustrate the big difference between an analysis based on the total game scores and an analysis based on the real prize distribution. The prizes of the participants are a function of their game scores. The skill analysis based on prizes therefore clearly depends on the specific form of this function. A qualitative explanation for the difference in GPM 2003 is as follows. The scores of beginners differ much from the scores of advanced players. However, there is a relatively large group of good players outside of the top ten that reach total scores that are only slightly lower than the scores of the advanced players. To

		Prizes	Scores
Expected result fictive player (lower bound)	€	993.50	$7,348$
Expected result fictive player (upper bound)	€	$1,832.00$	$8,783$
Random effect (lower bound)	€	945.80	627
Random effect (upper bound)	€	$1,784.30$	$2,062$
Relative skill (lower bound)		0.026	0.619
Relative skill (upper bound)		0.048	0.843

Table 3.13: Bounds for relative skill in GPM 2003, based on the bounds for the results of the fictive player.

be more precise, recall from Figure 3.2 that the estimated standard deviation s_{tot} of the total scores is approximately 304. More than 50 players in GPM 2003 have a total score that is within a distance of s_{tot} of the score of the winner. And even more than 400 players have ended within a distance of $2s_{\text{tot}}$ in the final ranking. The scores of these players are within the margins that are generated by the uncertain factors in the game: if they would have been lucky, they could have won the game. Since the number of prizes is small, this random factor plays a serious role in the distribution of the prizes. One can think of the distribution of the prizes as a lottery between good players with more or less the same score. For beginners this random factor hardly plays a role. They never reach prize winning positions in the final ranking, not even if they are really lucky.

Finally, we use the analysis of section 3.3.3.4 to do some sensitivity analysis. Table 3.13 lists the bounds we found for the score and prize of the fictive player. Furthermore, it shows the lower bound and the upper bound for the random effect (RE) that follow from the bounds for the expected results for the fictive player. The lower bound for the relative skill (RS) level then follows from the upper bound for RE and the upper bound for RS is computed using the lower bound for RE. It turns out that our (subjective) choice for the weighted average that has led to the results of the fictive player does not influence the conclusions. Both the upper and the lower bound for the skill level concerning prizes lead to the conclusion that GPM 2003 is a game of chance.

Chapter 4

Chance moves and information in two-person games

4.1 Introduction

Picture yourself sitting at a table, playing poker against one opponent. You play for money and your objective is to make as much money as possible in this game. Of course, your opportunities to make money depend on a few things: the dealing of the cards, the strategy of your opponent and your own strategy. The first two factors are outside your control; you can only influence the third aspect, your own strategy. This strategy tells you, for each possible poker hand that you can be dealt and for each possible action taken by the opponent, what actions you will take. The prescribed action depends on your hand, but it cannot depend on the hand of your opponent, because you simply don't know his cards. But what if there were a possibility to learn your opponent's hand, for example by paying someone to hold up a mirror behind his back? To what extent would this increase your possibilities? Can you use this information to improve your expected profits in the game? And if that is the case, with what amount does your expected profit change? Or, in other words, how much are you willing to pay this person who holds up the mirror?

A one-sided cheating option such as this mirror leads to interesting questions already, but things become even more interesting when your opponent can be active behind your back too. What if he also knows your cards? Does this change the value that "your" man with the mirror has for you?

In this chapter, which is based on Dreef and Borm (2005), we provide an answer to the questions above. We do this for the class of two-person zero-sum games in which a chance move at the start of the game determines which game exactly is to be played. We discuss what will be the consequences when the outcome of this chance move is revealed to one or both of the players before the players have to choose their actions. Loosely formulated, the difference between what a player can do with and without the information is called the *value of information*. We take into account that this value may depend on the kind of opponent this player faces. For example, it might be very useful to have the information if the opponent does not, while it is less valuable to know the outcome of the chance move if the opponent knows it too. Besides that, one may wonder if it is always valuable to know the random outcome if the opponent has the irrational objective of trying to lose as much as possible. We formalize the various kinds of opponents later in this chapter.

The value of information has been a notion of interest for a long time. It has been studied both in a non-strategic and a strategic setting. For the strategic setting, game theoretic analysis has been applied to many classes of games. Ponssard (1975) called the class of games that we study, games with an initial chance move, *games with "almost" perfect information*. These games were also subject of study in the papers of Ponssard and Sorin (1980, 1982). Value of information in two-person zero-sum games has been studied by Ponssard and Zamir (1973) in the context of sequential games. Ponssard (1976) considers the constant-sum case, while general bimatrix games are the object of study in the articles of Levine and Ponssard (1977), Borm (1988) and Kamien, Tauman and Zamir (1990).

In these papers, most of the definitions concerning the value of information in a strategic conflict are based on the difference between two numbers. However, we think that more numbers may be important if one wants to quantify the worth of information in a game to one or both of the players. In computing the value of information, we use the idea of an information buying pre-game that Sakaguchi (1993) introduced. In such a pre-game, both players get the opportunity to buy information about the outcome of the initial chance move before the start of the game. The value of information is then determined by setting the "price of information" in this pre-game at a reasonable level.

The values of information that we compute will be used to determine how restrictive the chance move in the game is for the players. In fact, these values will be used to quantify the extent to which the players have influence on the game result by defining the (derived) notion of *relative control.* This notion will also be referred to as *relative influence.*

An interesting aspect of our way of analyzing information in two-person zero-sum games, is that it makes use of various game-theoretic concepts within a larger framework. Apart from the zero-sum games themselves, coordination games and amalgamations of games play an important role. Coordination games form a nice subclass of the (exact) potential games. For an extensive overview of potential games we refer to Voorneveld (1999). Amalgamations of games were introduced by Borm, García-Jurado, Potters and Tijs (1996).

The analysis of relative player influence is closely related to the analysis of the skill level of a game, which is described in detail in chapter 2. The main goal of the skill analysis is rather similar to the goal of the current chapter: with both methods one can draw conclusions about the role of the chance moves in a game. Central in the skill analysis are three types of players who can play the game: beginners, optimal players and fictive players. The second and third category will also appear in our analysis of relative player influence. The category of the beginners, which generally is the most difficult to describe, will not play a role here.

The chapter is organized as follows. In the next section, we give some preliminaries and introduce the most important basic notions that are used in the text. Then, the sections 4.3 and 4.4 describe the way in which the value of information and the role of the chance moves are studied. Section 4.5 illustrates the analysis with an example that is based on a simple poker game. To conclude, section 4.6 contains a few remarks about our model.

4.2 Notation and definitions

In this section we introduce the notation that we use throughout this chapter. A *(strategic) two-person game* is a tuple $G = \langle X_1, X_2, u_1, u_2 \rangle$, where

- X_i denotes the finite, nonempty set of pure strategies of player i,
- each player i has a payoff function $u_i : X_1 \times X_2 \to \mathbb{R}$ specifying for each strategy profile $x = (x_1, x_2) \in X_1 \times X_2$ player i's payoff $u_i(x) \in \mathbb{R}$.

The set of pure strategy profiles will be denoted by $X = X_1 \times X_2$. A two-person game is called *zero-sum* if $u_1(x) = -u_2(x)$ for each strategy profile $x \in X$. The set of probability distributions over a finite set S is denoted $\Delta(S)$:

$$\Delta(S) = \{p : S \to [0,1] \mid \sum_{s \in S} p(s) = 1\}.$$

The mixed extension of the finite game $G = \langle X_1, X_2, u_1, u_2 \rangle$ allows each player i to choose a mixed strategy from $\Delta(X_i)$; a mixed strategy for player 1 (2) is denoted by p (q). Payoffs are extended to mixed strategies as follows:

$$u_i(p, q) = \sum_{x \in X} p(x_1) q(x_2) u_i(x),$$

i.e., the payoff to a mixed strategy profile is simply the expected payoff. A pure strategy $x_i \in X_i$ can be identified with the mixed strategy that assigns probability one to x_i.

A mixed-strategy profile $(p, q) \in \Delta(X_1) \times \Delta(X_2)$ is a *(mixed-strategy) Nash equilibrium* of the game G if

$$\text{for all } x_1 \in X_1 \quad : \quad u_1(p, q) \geq u_1(x_1, q) \quad \text{and} \tag{4.1}$$

$$\text{for all } x_2 \in X_2 \quad : \quad u_2(p, q) \geq u_2(p, x_2). \tag{4.2}$$

Let A and B be two matrices of equal size. With slight abuse of notation, we define a *bimatrix game* $\langle A, B \rangle$ as a strategic two-person game with payoff functions $u_1(p, q) = p^\top A q$ and $u_2(p, q) = p^\top B q$. Here, p is a column vector of which the ith element gives the probability with which player 1 plays his ith pure strategy; q is defined analogously. The column vector corresponding to the ith pure strategy of a player is written as e_i. A *matrix game* is a bimatrix game with $B = -A$, written as $\langle A \rangle$. We write matrix games and bimatrix games without the brackets if they are used as the argument of a function.

A *bimatrix game with almost perfect information* is a bimatrix game in which the payoff matrices A and B are formed in a special way. A chance move determines which of k possible bimatrix games will be played. The game $\langle A_i, B_i \rangle$ is played with probability μ_i ($1 \leq i \leq k$). The μ_i are such that each element is selected with positive probability ($\mu_i > 0$) and the sum of the probabilities equals one ($\sum_{i=1}^k \mu_i = 1$). The payoff matrix A is formed by taking the weighted sum of the k underlying payoff matrices: $A = \sum_{i=1}^k \mu_i A_i$.

Similarly, $B = \sum_{i=1}^{k} \mu_i B_i$. All matrices A_i and B_i must have the same size. In the naming of this type of games we follow Ponssard (1975). A *matrix game with almost perfect information* is a matrix game $\langle A \rangle$ that is based on the matrix games $\langle A_i \rangle$ in the sense that is described above.

4.3 The strategic possibilities of the players

In the remainder of this chapter, the basic object of study is a matrix game with almost perfect information $\langle A \rangle$, based on the matrix games $\langle A_1 \rangle, \ldots, \langle A_k \rangle$. $\langle A_i \rangle$ will be played with probability μ_i $(1 \leq i \leq k)$.

4.3.1 Player types and related games

For the investigation of the possibilities of the players and the role of the chance move in this game, we distinguish four types of players. On the one side players can be either egoistic or altruistic, whereas on the other side players can be clairvoyant or not. *Egoistic* players want to maximize their own payoffs, while the aim of an *altruistic* player is to minimize his own payoff. The naming stems from the fact that the latter type of player helps his opponent when playing a zero-sum game. A similar distinction of behavioural patterns in non-cooperative games is given by Szép and Forgó (1985); for zero-sum games our altruistic players coincide with both their *masochist* and *philantropic* types. Also in evolutionary settings the distinction between altruistic and selfish attitudes is made. A discussion on the context dependence of these types of preferences can be found in Bester and Güth (1998). The *clairvoyance* relates to the outcome of the chance move: clairvoyant players know beforehand which matrix game will be played. However, they cannot influence the randomization procedure. Table 4.1 summarizes the terms we use when we refer to the resulting four possible player types as well as the corresponding abbreviations.

	Not clairvoyant	Clairvoyant
Altruistic	worst player (W)	fictive worst player (FW)
Egoistic	optimal player (O)	fictive optimal player (FO)

Table 4.1: Four types of players.

For a given matrix game of the type that we discussed at the beginning of this section, we want to know to what extent the players are in control. More precisely, we determine the range of payoffs that can be reached by the players, given the rules of the game. For each player we want to know how well he can do, but we are also interested in how badly he can do. Moreover, we want to know if the uncertainty that is caused by the chance move really restricts the possibilities of the players. To investigate these questions, we let each of the four player types take both player roles in the matrix game and we let all combinations of player types play the game. If we assume that players always know what type of opponent they are facing, this idea gives rise to the 16 games that are given in Table 4.2.

Player 1 type \ Player 2 type	FW	W	O	FO
FW	$\langle -A_i \rangle$ $(1 \leq i \leq k)$	$\langle -A^1 \rangle$	$\langle -A^1, -A^1 \rangle$	$\langle -A_i, -A_i \rangle$ $(1 \leq i \leq k)$
W	$\langle -A^2 \rangle$	$\langle -A \rangle$	$\langle -A, -A \rangle$	$\langle -A^2, -A^2 \rangle$
O	$\langle A^2, A^2 \rangle$	$\langle A, A \rangle$	$\langle A \rangle$	$\langle A^2 \rangle$
FO	$\langle A_i, A_i \rangle$ $(1 \leq i \leq k)$	$\langle A^1, A^1 \rangle$	$\langle A^1 \rangle$	$\langle A_i \rangle$ $(1 \leq i \leq k)$

Table 4.2: All combinations of player types and the resulting games.

Let us explain the contents of table 4.2 in more detail. The basic situation is the case where two optimal players face each other. These players both try to maximize their payoffs in the matrix game $\langle A \rangle$. If player 2 acts as a worst player and thus tries to obtain the lowest possible payoff in $\langle A \rangle$, the resulting strategic situation can be modelled by the coordination game $\langle A, A \rangle$. In this game the payoff for player 1 is the same as in the original game, whereas player 2 now acts as if he maximizes his payoff, pretending that his payoff matrix is A instead of $-A$. However, after we have found an equilibrium in this game, we have to reverse the sign of player 2's payoff again. This reasoning explains the four cells in the middle, the situations where two non-clairvoyant players meet.

The notations A^1 and A^2 require some explanation too. We give the interpretation for A^1; the story for A^2 goes analogously. The payoff matrix A^1 is used in the cells where player 1 is fictive and player 2 is not. Such a situation can be modelled as an *amalgamation of games*, following the definition of Borm, García-Jurado, Potters and Tijs (1996). In an amalgamation of games the player set is partitioned into two parties and the game is an aggregation of the conflicts between two players, one in each party. The payoff of a player is just the sum of the payoffs in the separate conflicts. In the situation we study here, we have k "instances" of player 1 (all members of the first party) playing against one opponent (the only member of the second party). Taking into account the fact that instance i of player 1 is called to play with probability μ_i, the matrix $\mu_i A_i$ is the payoff matrix that is used in the game between instance i of player 1 and his opponent ($1 \leq i \leq k$). Borm et al. (1996, p. 574, Proposition 1) showed for the $(k+1)$-person game that models this situation that each equilibrium corresponds to an equilibrium in the related two-person *correlation game*. Their result follows from the results by Kuhn (1953) on behavioural strategies in games with perfect recall. In the correlation game the k instances of player 1 are considered as one player and are allowed to pick correlated strategies. Together they choose a strategy from $\Delta(\prod_{i=1}^{k} X_1^i)$, where X_1^i is the set of pure strategies of player 1 in the matrix game $\langle A_i \rangle$. The payoff matrix of this correlation game, with player 1 as a fictive player, is represented as A^1. Together with the reasoning about the reversed sign as before, this explains the contents of the eight cells where one player is clairvoyant and his opponent is not.

The remaining four cells, corresponding to the situations with two clairvoyant players, speak more or less for themselves. Both players know which of the k matrix games they play, so they can optimize their strategic behaviour for each of these games separately.

The following example will illustrate the types of games described above.

Example 4.3.1 Let $\langle A \rangle$ be a matrix game with almost complete information, based on the matrix games $\langle A_1 \rangle$ and $\langle A_2 \rangle$ that are played with equal probability ($\mu_1 = \mu_2 = \frac{1}{2}$). The payoff matrices are given in Figure 4.1. Figure 4.2 gives the payoff matrices as well as the available strategies both for the situation where player 1 is a fictive optimal (FO) player and the situation where he is

	f_1	f_2
e_1	5	0
e_2	0	2

$\langle A_1 \rangle$

	f_1	f_2
e_1	2	5
e_2	4	0

$\langle A_2 \rangle$

Figure 4.1: The matrix games underlying $\langle A \rangle$.

an optimal (O) player. With the strategy $e_a e_b$ we denote the choice of player 1 to play e_a if payoff matrix A_1 is used and play e_b if A_2 is used.

	f_1	f_2
e_1	$\frac{7}{2}$	$\frac{5}{2}$
e_2	2	1

$\langle A \rangle$

	f_1	f_2
$e_1 e_1$	$\frac{7}{2}$	$\frac{5}{2}$
$e_1 e_2$	$\frac{9}{2}$	0
$e_2 e_1$	1	$\frac{7}{2}$
$e_2 e_2$	2	1

$\langle A^1 \rangle$

Figure 4.2: The resulting games with player 2 as an optimal player and player 1 as an optimal player ($\langle A \rangle$) and as fictive optimal player ($\langle A^1 \rangle$).

$\triangleleft$

4.3.2 Expected payoffs

We want to compare the equilibrium payoffs of the games that are played in each of the 16 cells of Table 4.2. Since the central game in this analysis is a zero-sum game, we can restrict our attention to player 1's payoff. Player 2's payoff is the same number with opposite sign. Note that it is possible to interchange the roles of the players. With player 2 as the row player the basic game would be $\langle -A^\top \rangle$ and we could construct Table 4.2 in a similar way as we did for $\langle A \rangle$.

In half of the cases equilibrium payoffs are unambiguous. If the players are both egoistic, the games we have to solve are matrix games and therefore they have a uniquely defined value. The same holds if both players are altruistic. In the other eight cells, with one player being altruistic while his opponent is egoistic, we have to make a selection out of the possibly many Nash equilibria of bimatrix games. In fact, all bimatrix games we have to solve are coordination games. This type of games forms a nice subclass of the (exact) potential games. For an extensive overview of potential games we refer to Voorneveld (1999) or to Monderer and Shapley (1996). An obvious Nash equilibrium refinement choice for these games is the potential maximizer. For coordination games, the logical choice for a potential function is u_1. We denote the payoff for the players for any x that maximizes u_1 in the game G as $u_{PM}(G)$.

We give three reasons justifying the choice of the potential maximizer. First of all, we are exploring the range of possibilities the players have in the game. Potential-maximizing strategies certainly form an extremity of the strategic options of the players. It gives a theoretic bound of the game. Secondly, the potential maximizer forms an attractive focal point for the players, being a pure-strategy equilibrium with high payoffs. Finally, Reijnierse, Voorneveld and Borm (2003) showed that the potential maximizer is in the set of informationally robust equilibria of the game. This equilibrium refinement concept is closely related to our idea of leaking of information to one (or both) of the players. Table 4.3 presents the expressions for the expected payoffs for player 1 in each of the 16 situations.

		Player 2 type			
		FW	W	O	FO
Player 1 type	FW	$-\sum_{i=1}^{k}\mu_i v(-A_i)$	$-v(-A^1)$	$-u_{PM}(-A^1,-A^1)$	$-\sum_{i=1}^{k}\mu_i u_{PM}(-A_i,-A_i)$
	W	$-v(-A^2)$	$-v(-A)$	$-u_{PM}(-A,-A)$	$-u_{PM}(-A^2,-A^2)$
	O	$u_{PM}(A^2,A^2)$	$u_{PM}(A,A)$	$v(A)$	$v(A^2)$
	FO	$\sum_{i=1}^{k}\mu_i u_{PM}(A_i,A_i)$	$u_{PM}(A^1,A^1)$	$v(A^1)$	$\sum_{i=1}^{k} v(A_i)$

Table 4.3: The expected payoffs of player 1 for all combinations of player types.

Example 4.3.2 We computed the numbers that are given in Table 4.3 for the game we introduced in Example 4.3.1. The results are in Table 4.4. ◁

The following lemma states two simple observations regarding the interchange of player roles in a matrix game $\langle A\rangle$ and a coordination game $\langle A, A\rangle$.

Player 1 type \ Player 2 type	FW	W	O	FO
FW	$2\frac{1}{7}$	$1\frac{5}{7}$	0	0
W	$2\frac{3}{7}$	2	1	0
O	5	$3\frac{1}{2}$	$2\frac{1}{2}$	$1\frac{6}{7}$
FO	5	$4\frac{1}{2}$	$2\frac{11}{14}$	$2\frac{1}{7}$

Table 4.4: The expected payoffs of player 1 for each combination of player types.

Lemma 4.3.3

(C1) $v(-A^\top) = -v(A)$*;*

(C2) $u_{PM}(A, A) = u_{PM}(A^\top, A^\top)$*.*

The first part of the lemma states that equilibrium payoffs in a matrix game do not depend on who is the row player and who is the column player. The second part formalizes the fact that interchanging the player roles in a coordination game does not influence the potential maximizing payoffs. We use the observations in Lemma 4.3.3 to explain some symmetry arguments in the proof of Theorem 4.3.5.

Lemma 4.3.4 *Let* $\langle A, B\rangle$ *be a bimatrix game and let* $\langle A\rangle$ *be the (corresponding) matrix game. Then*

$$(\hat{p}, \hat{q}) \in NE(A, B) \Rightarrow \hat{p}^\top A\hat{q} \geq v(A).$$

Proof. Let $(\hat{p}, \hat{q}) \in NE(A, B)$. Then $\hat{p}^\top A\hat{q} = \max_p p^\top A\hat{q} \geq \min_q \max_p p^\top Aq = v(A)$. □

Let us introduce some notation for the payoffs that are given in Table 4.3. We write the collection of player types as $\mathcal{T} = \{FW, W, O, FO\}$ and we use the notation u_{τ_1,τ_2} for the expected payoff of player 1 in the game between player 1 of type τ_1 and player 2 of type τ_2 for all $\tau_1, \tau_2 \in \mathcal{T}$. Using these definitions, we can formulate Theorem 4.3.5. This theorem states that the payoffs in Table 4.3 are higher for egoistic players than for altruistic players against any opponent type and that clairvoyancy helps a player, independent of the type of opponent he faces.

Theorem 4.3.5

$$u_{\tau_1,FW} \geq u_{\tau_1,W} \geq u_{\tau_1,O} \geq u_{\tau_1,FO} \quad \textit{for all } \tau_1 \in \mathcal{T}, \tag{4.3}$$

$$u_{FW,\tau_2} \leq u_{W,\tau_2} \leq u_{O,\tau_2} \leq u_{FO,\tau_2} \quad \textit{for all } \tau_2 \in \mathcal{T}. \tag{4.4}$$

Proof. Since the potential maximizing strategies together form a Nash equilibrium, it follows directly from Lemma 4.3.4 that $u_{\tau_1,W} \geq u_{\tau_1,O}$ for all $\tau_1 \in \mathcal{T}$. By writing down the expression for $u_{FW,FW}$ we see that

$$\begin{aligned}
u_{FW,FW} &= -\sum_{i=1}^{k} \mu_i v(-A_i) = -\sum_{i=1}^{k} \mu_i \min_{q_i \in \Delta(X_2)} \max_{p_i \in \Delta(X_1)} p_i^\top (-A_i) q_i \\
&\geq -\min_{q \in \Delta(X_2)} \sum_{i=1}^{k} \max_{p_i \in \Delta(X_1)} p_i^\top (-\mu_i A_i) q \\
&= -\min_{q \in \Delta(X_2)} \max_{p \in \prod_{i=1}^{k} \Delta(X_1)} p^\top (-A^1) q \\
&\overset{(*)}{=} -\min_{q \in \Delta(X_2)} \max_{p \in \Delta(\prod_{i=1}^{k} X_1)} p^\top (-A^1) q = -v(-A^1) = u_{FW,W},
\end{aligned}$$

where $(*)$ was shown by Borm et al. (1996, p. 574, Proposition 1). Analogously, one can show that $u_{W,FW} \geq u_{W,W}$, $u_{O,O} \geq u_{O,FO}$ and $u_{FO,O} \geq u_{FO,FO}$. In a similar way, one can show that the inequality between $u_{O,FW}$ and $u_{O,W}$:

$$\begin{aligned}
u_{O,FW} &= u_{PM}(A^2, A^2) = \max_{(p,q) \in \Delta(X_1) \times \Delta(\prod_{i=1}^{k} X_2)} p^\top A^2 q \\
&= \max_{(p,q_1,\ldots,q_k) \in \Delta(X_1) \times \prod_{i=1}^{k} \Delta(X_2)} \sum_{i=1}^{k} p^\top (\mu_i A_i) q_i \\
&\geq \max_{(p,q) \in \Delta(X_1) \times \Delta(X_2)} \sum_{i=1}^{k} p^\top (\mu_i A_i) q \\
&= \max_{(p,q) \in \Delta(X_1) \times \Delta(X_2)} p^\top A q = u_{PM}(A, A) = u_{O,W},
\end{aligned}$$

where the third equality again is an application of the result of Borm et al. (1996). The validity of the remaining three inequalities in (4.3) can be shown analogously.

Finally, the inequalities in (4.4) can be derived from (4.3) by writing down a table like Table 4.3 with player 2 as the row player and applying Lemma 4.3.3 to the inequalities that (4.3) gives for the payoffs in this table. □

Theorem 4.3.5 states that the payoffs for player 1, as defined in Table 4.3, are (weakly) decreasing in value if we go through the table from bottom-left to top-right. Therefore, we know that differences like, for example, $u_{FO,O} - u_{O,O}$ and $u_{FW,W} - u_{FW,O}$ are nonnegative.

4.4 The role of information

In section 4.4.2 we use the definitions and results of section 4.3.2 to present a well-defined way to quantify the restrictive role of the chance moves in matrix games with almost perfect information. But first, in section 4.4.1, we will see which different types of value of information that are distinguished in the literature can be derived from Table 4.3.

4.4.1 The value of information

In this section we see how the numbers from Table 4.3 are related to the various definitions one finds in the literature on the value of information in a strategic context. For an overview of various types of information and a discussion on relations between them, we refer to Borm (1988).

The games between two egoistic players, one of the two being clairvoyant, are games with *private information* for the clairvoyant player. The difference $u_{FO,O} - u_{O,O}$ is often referred to as the *value of private information* for player 1. According to Theorem 4.3.5, this value is nonnegative. This is a confirmation of a result of Ponssard (1976). Two egoistic players, both being clairvoyant, play a game with *public information*. Therefore, the difference $u_{FO,FO} - u_{O,O}$ is called the *value of public information* for player 1. It is not possible to say anything about the sign of the value of public information; all we can say is that the value of public information for player 1 will be the opposite of the value of public information for player 2.

With each matrix game with almost perfect information, we want to associate eight values of information, one for each player in each possible combination of attitudes (egoistic or altruistic). We do this using the payoffs in Table 4.3. As the two definitions in the previous paragraph illustrate, most definitions concerning the value of information in a strategic conflict are based on the difference between two numbers. However, we think that at least four numbers are important if one wants to quantify the worth of information in a game to one or both of the players. If both players are egoistic, then the four numbers in the lower right part of Table 4.3 should be taken into account: $u_{O,O}$, $u_{FO,O}$, $u_{O,FO}$ and $u_{FO,FO}$. To do this, we follow the approach of Sakaguchi (1993), who defined an information buying pre-game. In this game both players have to decide whether to buy information or not. Player i has to pay an amount c_i to his opponent if he wants to be informed about the outcome of the chance move. These prices are set by an external person, someone like the "maven" from Kamien, Tauman and Zamir (1990). Figure 4.3 shows the payoffs of the information buying pre-game. The prices are set in such a way

		Player 2: Don't buy	Player 2: Buy
Player 1	Don't buy	$u_{O,O}$	$u_{O,FO} + c_2$
	Buy	$u_{FO,O} - c_1$	$u_{FO,FO} - c_1 + c_2$

Figure 4.3: The information buying pre-game when both players are egoistic.

that for both players the pre-game is interesting in the sense that information is neither too cheap nor too expensive. They both really have to think about buying or not buying it. More formally, c_1 and c_2 are set such that neither player has a strongly dominant strategy in the pre-game.

Buy is strongly dominant for player 1 if $c_1 < \underline{c}_1 = \min\{u_{FO,O} - u_{O,O}, u_{FO,FO} - u_{O,FO}\}$. In other words, it should be profitable to buy the information against an optimal player, but also against a fictive optimal player. Similarly, *Buy* is strongly dominant for player 2 if $c_2 < \underline{c}_2 = \min\{u_{O,O} - u_{O,FO}, u_{FO,O} - u_{FO,FO}\}$. Analogously, *Don't buy* is a strongly dominant strategy for player 1 if $c_1 > \overline{c}_1 = \max\{u_{FO,O} - u_{O,O}, u_{FO,FO} - u_{O,FO}\}$. Buying is neither profitable against an optimal player nor against a fictive optimal player. For player 2 *Don't buy* is a strongly dominant strategy if $c_2 > \overline{c}_2 = \max\{u_{O,O} - u_{O,FO}, u_{FO,O} - u_{FO,FO}\}$.

Only between the boundaries $\underline{c}_i$ and $\overline{c}_i$ it is possible for player i to be made indifferent between buying and not buying by his opponent. Therefore, the price of information for player i, c_i, should definitely lie between these boundaries. We follow the approach of Sakaguchi (1993) and set the prices at a level that ensures that buying the information with probability $\frac{1}{2}$ is an equilibrium strategy for both players. The prices for which the pre-game has this nice characteristic, are $c_i = \frac{1}{2}(\underline{c}_i + \overline{c}_i)$ for $i \in \{1, 2\}$. To make clear that these prices are the values of information for the players in the game when they both act egoistically, we write them as $c_1^{O,O}$ and $c_2^{O,O}$.

For the games in which altruistic players are involved, we can define the prices of information in a similar way. Consider the situation where player 1 is egoistic and player 2 is altruistic. The four payoffs of interest can then be found in the bottom-left part of Table 4.3: $u_{O,FW}$, $u_{O,W}$, $u_{FO,FW}$ and $u_{FO,W}$. Figure 4.4 shows the payoffs of the information buying pre-game for this situation. It is clear that the "fair" price of information for player 1 can

		Player 2: Don't buy	Player 2: Buy
Player 1	Don't buy	$u_{O,W}$	$u_{O,FW} + c_2$
	Buy	$u_{FO,W} - c_1$	$u_{FO,FW} - c_1 + c_2$

Figure 4.4: The information buying pre-game when player 1 is egoistic and player 2 is altruistic.

be set in the way that we described for the game between two optimal players. Does this method also work for player 2, who has altruistic motives? Yes, it does. We know from Theorem 4.3.5 that $u_{O,W} \leq u_{O,FW}$ and $u_{FO,W} \leq u_{FO,FW}$. The altruistic player 2 is better off if more money in the game is transferred to his opponent. Since the game is zero-sum, we already implicitly assumed that an altruistic player was willing to pay one unit for each unit the opponent gets extra. Therefore, consistent reasoning leads to the conclusion that player 2 is willing to pay at least $\underline{c}_2 = \min\{u_{O,FW} - u_{O,W}, u_{FO,FW} - u_{FO,W}\}$. Using similar reasoning, we can also define the upperbound $\overline{c}_2$. In order to give the game the property that buying the information with probability $\frac{1}{2}$ is an equilibrium strategy for both players, we have to define $c_i = \frac{1}{2}(\underline{c}_i + \overline{c}_i)$ for $i \in \{1, 2\}$ here too.

For the other two combinations of attitudes, with only player one or both players being altruistic, we can do similar computations. In this way we can associate with each matrix game *eight* values of information, four for each player. We give the numbers for our example.

Example 4.4.1 For the game that was discussed in examples 4.3.1 and 4.3.2, the value of information for player 1, in the situation where he is egoistic and his opponent is altruistic, should lie between $\underline{c}_1^{O,W} = \min\{5-5, 4\frac{1}{2} - 3\frac{1}{2}\} = 0$ and $\overline{c}_1^{O,W} = \max\{0,1\} = 1$. Taking the average, we get $c_1^{O,W} = \frac{1}{2}$. This number is given in the list below, together with the other seven relevant values of information.

$$\begin{array}{rclrcl} c_1^{O,O} & = & \frac{2}{7} & \qquad c_2^{O,O} & = & \frac{9}{14} \\ c_1^{O,W} & = & \frac{1}{2} & \qquad c_2^{O,W} & = & 1 \\ c_1^{W,O} & = & \frac{1}{2} & \qquad c_2^{W,O} & = & \frac{1}{2} \\ c_1^{W,W} & = & \frac{2}{7} & \qquad c_2^{W,W} & = & \frac{3}{7} \end{array}$$

$\triangleleft$

We use these values of information in the next section to quantify the relative influence of the players on the game result, compared to the influence of the initial chance move.

4.4.2 Player control and influence of the chance move

In this section we use the payoffs from Table 4.3, together with the corresponding values of information, to quantify the restrictive role of the chance moves. By Theorem 4.3.5 we know that the highest possible equilibrium payoff for player 1 occurs in the situation with two clairvoyant players, with player 1 being egoistic and player 2 being altruistic. In terms of Theorem 4.3.5, this payoff is written as $u_{FO,FW}$. Similarly, the minimal payoff for player 1 is $u_{FW,FO}$. These numbers represent the maximum and minimum payoff for player 1, given that the information on the chance moves can be used by the players. The difference between these numbers, $u_{FO,FW} - u_{FW,FO}$, indicates the size of the *fictive range* of the game's payoffs.

We want to compare numbers within this fictive range with the payoffs that can be attained by non-clairvoyant players, as displayed in the four central cells in Table 4.3. The payoff information in these four cells can be summarized in a logical way by considering four differences: for the two players, we compute the difference between the maximum and minimum payoff, both against a worst opponent and against an optimal opponent. These differences give the payoff variation that can be caused by the players themselves. We are interested in the relative size of these numbers, compared to the restrictions caused by the chance move. The restriction can be quantified by two numbers: the value of information in the case a player tries to minimize his payoff and the value that this information has to him when he tries to maximize his payoff.

We define γ_i^O (γ_i^W) to be the *relative control* level of player i against an optimal (worst) opponent. Formally,

$$\begin{aligned}
\gamma_1^O &= \frac{(u_{O,O} - u_{W,O})}{c_1^{O,O} + (u_{O,O} - u_{W,O}) + c_1^{W,O}}, \\
\gamma_1^W &= \frac{(u_{O,W} - u_{W,W})}{c_1^{O,W} + (u_{O,W} - u_{W,W}) + c_1^{W,W}}, \\
\gamma_2^O &= \frac{(u_{O,W} - u_{O,O})}{c_2^{O,W} + (u_{O,W} - u_{O,O}) + c_2^{O,O}}, \\
\gamma_2^W &= \frac{(u_{W,W} - u_{W,O})}{c_2^{W,W} + (u_{W,W} - u_{W,O}) + c_2^{W,O}}.
\end{aligned}$$

From these definitions and the result of Theorem 4.3.5, it is clear that $0 \leq \gamma_i^O \leq 1$ and $0 \leq \gamma_i^W \leq 1$ for $i \in \{1, 2\}$. If $\gamma_i^O = 1$, then the chance move is not restrictive at all for player i against an opponent playing optimally.

It is interesting to note that the opponents against whom the relative controls γ_2^O and γ_1^W are computed have the same objective. These opponents both try to maximize the expected payoff of player 1, but they have to operate from different role perspectives. In the first case, the opponent has to play from the "row position", whereas the opponent against whom γ_1^W is computed uses the "column position". If $\gamma_1^W < \gamma_2^O$, then player 2's control against a rational opponent, who is maximizing his own payoff, is higher if he can operate from the row position than if he can act as the column player. Similar comparisons can be made between γ_1^O and γ_2^W to say something about the relative control of player 1 against a payoff-maximizing opponent.

4.5 An example: minipoker

In this section we illustrate the analysis with a more lively example than the ones we used so far to explain our definitions. We study a two-person game called *minipoker*. This is a game of cards played by two players, player 1 and player 2, and with three cards, namely Q(ueen), K(ing) and A(ce). As usual, A is higher than K and Q is the lowest card of these three. Before play starts, both players donate 1 unit to the stakes. After (re)shuffling the deck of cards each player is dealt one card. Each player knows his own card, but not the card of his opponent. Thus, the card which remains in the deck is not shown to either of the players. Player 1 starts the play and has to decide between P(assing) and B(etting). If he decides to pass, a showdown follows immediately. In the showdown both cards are compared and the player with the highest card gets the stakes. If player 1 decides to bet, he has to add one extra unit to the stakes. Subsequently, player 2 has to decide between F(olding) and C(alling). If he decides to fold, player 1 gets the stakes. If player 2 decides to call, he also has to add one extra unit to the stakes and a showdown follows.

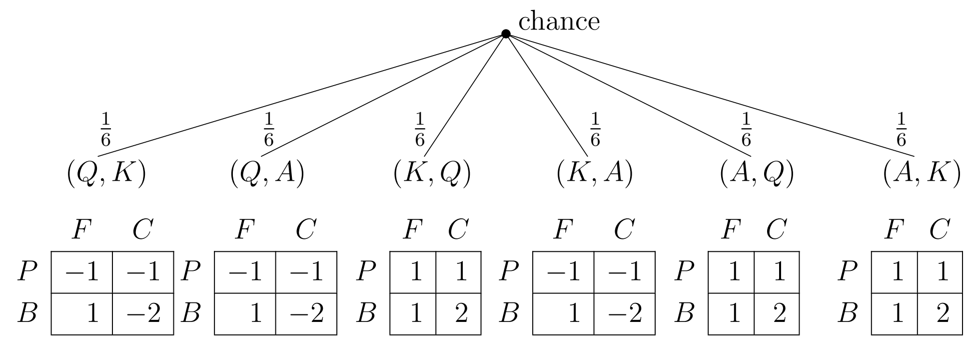


Figure 4.5: Minipoker as a matrix game with "almost" perfect information.

We can model this game as a matrix game with almost perfect information in the way that is shown in Figure 4.5: after the initial chance move, the players play one of the six 2×2 matrix games (all with equal probability). However, if we model minipoker this way, there is a difference between the normal players in this game and the normal players we have studied so far. So far, the normal players were not able to make any distinction between the k matrix games they could possibly face. In minipoker, both players can exclude outcomes of the chance move by looking at their own card. In fact, we can say that the outcome space of the chance move is the set

$$\{(Q, K), (Q, A), (K, Q), (K, A), (A, Q), (A, K)\}$$

and that player 1 faces can distinguish between elements of the partition

$$\{\{(Q, K), (Q, A)\}, \{(K, Q), (K, A)\}, \{(A, Q), (A, K)\}\}.$$

Similarly, player 2 faces the partition

$$\{\{(Q, K), (A, K)\}, \{(Q, A), (K, A)\}, \{(K, Q), (A, Q)\}\}.$$

The fact that the non-fictive players, together with this formulation of the game as a matrix game with almost perfect information, do not completely fit into the framework of this chapter is not a problem. It is not difficult to see that the proof of Theorem 4.3.5 only uses the fact that the information partition of the fictive players is a refinement of the partition of the normal players.

The payoffs in all 16 games that are relevant for determining the relative influence of the players on their game result are given in Figure 4.7. As an illustration, we give part of the matrix that corresponds to the game between a fictive optimal player 1 and an optimal player 2, $\langle A^1 \rangle$, in Figure 4.6. Player 1 can distinguish all six possible deals, so a pure strategy for him prescribes an action (bet or pass) for each of these situations. As a result, player 1 as a fictive player has 2^6 pure strategies. Only part of this set is listed in the figure. Player 2 only knows his own card. Therefore, a pure strategy for him dictates a decision for each of the three possible cards he can receive. Player 2 has 2^3 pure strategies. The subscripts in the strategy labels in Figure 4.6 indicate the cases to which the decisions correspond. For example, player 2's strategy $F_Q F_K C_A$ prescribes: fold with a queen or a king, but call with an ace, if player 1 bets. The value of this game is $\frac{1}{9}$.
We learn from Figure 4.7 that the value of the game is $\frac{1}{18}$. And, as we can see, the numbers in this table satisfy the order that we need to make all definitions regarding information and control sensible. Using the numbers in Figure 4.7, we can determine the values of information and relative control levels for both players in minipoker. We find that three of the eight prices of information are

	$F_QF_KF_A$	$F_QF_KC_A$	$F_QC_KF_A$	$F_QC_KC_A$	$C_QF_KF_A$	$C_QF_KC_A$	$C_QC_KF_A$	$C_QC_KC_A$
$P_{Q,K}P_{Q,A}P_{K,Q}P_{K,A}P_{A,Q}P_{A,K}$	0	0	0	0	0	0	0	0
$P_{Q,K}P_{Q,A}P_{K,Q}P_{K,A}P_{A,Q}B_{A,K}$	0	0	$\frac{1}{6}$	$\frac{1}{6}$	0	0	$\frac{1}{6}$	$\frac{1}{6}$
$P_{Q,K}P_{Q,A}P_{K,Q}P_{K,A}B_{A,Q}P_{A,K}$	0	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$P_{Q,K}P_{Q,A}P_{K,Q}P_{K,A}B_{A,Q}B_{A,K}$	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_{Q,K}B_{Q,A}B_{K,Q}B_{K,A}P_{A,Q}P_{A,K}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{3}$
$B_{Q,K}B_{Q,A}B_{K,Q}B_{K,A}P_{A,Q}B_{A,K}$	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{1}{6}$
$B_{Q,K}B_{Q,A}B_{K,Q}B_{K,A}B_{A,Q}P_{A,K}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{6}$	$-\frac{1}{2}$	$-\frac{1}{6}$
$B_{Q,K}B_{Q,A}B_{K,Q}B_{K,A}B_{A,Q}B_{A,K}$	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	1	$-\frac{1}{3}$	0

$\langle A^1 \rangle$

Figure 4.6: Minipoker between a fictive optimal player 1 and an optimal player 2.

player 1 type \ player 2 type	FW	W	O	FO
FW	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$
W	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$
O	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{1}{18}$	0
FO	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{1}{9}$	0

Figure 4.7: Expected payoffs for player 1 in minipoker, all combinations of player types.

unequal to zero: $c_1^{O,O} = \frac{1}{36}$, $c_2^{O,O} = \frac{1}{12}$ and $c_2^{O,W} = \frac{1}{6}$. Using these numbers, we find $\gamma_1^O = \frac{20}{21}$, $\gamma_1^W = 1$, $\gamma_2^O = \frac{46}{55}$ and $\gamma_2^W = 1$. For both players, it turns out that playing against a player who minimizes his own payoff is better from a control point of view.

What's the explanation for this extremely high level of relative control against a worst opponent? Well, let us consider the case of player 1 facing an opponent who tries to minimize his own payoff. If neither player is fictive and player 1 acts as an optimal player, then in the unique equilibrium player 1 always bets, while player 2 calls with a queen and folds with a king or with an ace. The expected payoff for player 1 is then $\frac{4}{3}$. If player 1 acts as a worst player against his worst opponent, the best thing he can do with any card is passing (or he can bet with a queen, but this will not be useful since player 2 will not risk winning by calling with a king or an ace). His expected payoff will then be 0. Since $c_1^{O,W} = c_1^{W,W} = 0$, revealing player 2's card to player 1 does not give player 1 additional strategic possibilities. The fact that $u_{O,W} > u_{W,W}$ is therefore sufficient to obtain a relative control level equal to 1.

From the numbers γ_i^O and γ_i^W $(i = 1, 2)$, we want to draw two conclusions. In the first place, since both γ_1^O and γ_2^O are smaller than 1, we know that the dealing of the cards really leads to restrictions in the possibilities of rational players. In the second place, since $\gamma_1^O < \gamma_2^W$ and $\gamma_2^O < \gamma_1^W$, we can conclude that both players in minipoker would prefer the position of the opponent from a control point of view. In the "column position" of player 2 one has more control about the expected payoff of player 1 when one is trying to maximize this payoff. Similarly, the "row position" of player 1 gives more control over the maximization of player 2's payoff.

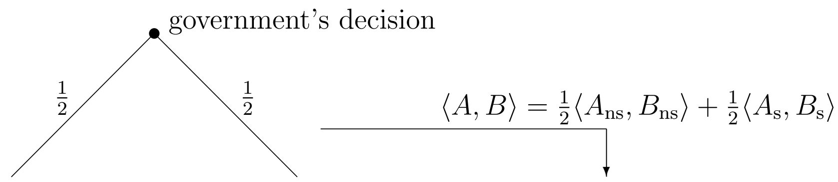
4.6 Concluding remarks

In this chapter we have presented a way to analyze the role of chance moves for a specific class games. We have given a method that enables us to determine the value of information about those chance moves for the players. Using this valuation of the information, we quantified the restrictive role of the chance moves with respect to the influence of the players on their own payoffs. To conclude, we wish to make a few remarks about our analysis.

4.6.1 Extension to other classes of games

The starting point in this chapter was a matrix game with almost perfect information. We used the zero-sum property in the proof of Theorem 4.3.5. Of course, it is interesting to check whether our analysis can be carried out for a broader class of bimatrix games. Example 4.6.1 supports the conjecture that this is possible. Before we give the example, we have to think for a moment about the definition of the altruistic players. Although we prefer this positive terminology over the term masochistic in the zero-sum context, we want to stress here that the original idea was that these players try to minimize their own payoffs. So in a bimatrix game, we interpret the worst player as a masochistic player.

Example 4.6.1 Consider the following duopoly situation. Two firms produce some good and they can choose to use *grey* or *green* energy for the production process. The second type is better for the environment, but it is also more expensive. Therefore, the selling price of the product should be higher. This leads to a decrease in demand. However, the government is discussing the possibility of giving subsidy to consumers who buy products that have been produced in an ecologically sound way. This subsidy is expected to stimulate the consumers so much that the demand for the product will increase, even if the price is higher. The probability that the government will decide to give this subsidy, is estimated by the firms to be 50%. We model this situation as a bimatrix game with almost perfect information. This game is shown in Figure 4.8.



(no subsidy)

	grey	green
grey	$4, 4$	$4, 3$
green	$3, 4$	$3, 3$

$\langle A_{\text{ns}}, B_{\text{ns}} \rangle$

(subsidy)

	grey	green
grey	$4, 4$	$3, 7$
green	$7, 3$	$5, 5$

$\langle A_{\text{s}}, B_{\text{s}} \rangle$

(compound game)

	grey	green
grey	$4, 4$	$3\frac{1}{2}, 5$
green	$5, 3\frac{1}{2}$	$4, 4$

$\langle A, B \rangle$

Figure 4.8: Duopoly with possible consumer subsidy for ecologically sound products.

Figure 4.9 contains the payoffs corresponding to the 16 games from which we wish to draw conclusions about relative control. The bimatrix game corresponding to the situation where firm 1 knows in advance whether the government will give the subsidy, is given in this figure too, since it helps in quickly checking the 16 payoffs. The presence of a strongly dominant strategy in each bimatrix game that we have to consider simplifies the computations. Using the information in Figure 4.9, we find that the price of information

	grey	green
(grey, grey)	$4,4$	$3\frac{1}{2},5$
(grey, green)	$5\frac{1}{2},3\frac{1}{2}$	$4\frac{1}{2},4$
(green, grey)	$3\frac{1}{2},4$	$3,5$
(green, green)	$5,3\frac{1}{2}$	$4,4$

$\langle A^1, B^1\rangle$

firm 1 type \ firm 2 type	FW	W	O	FO
FW	$3\frac{1}{2},3\frac{1}{2}$	$3\frac{1}{2},4$	$3,5$	$3,5\frac{1}{2}$
W	$4,3\frac{1}{2}$	$4,4$	$3\frac{1}{2},5$	$3\frac{1}{2},5\frac{1}{2}$
O	$5,3$	$5,3\frac{1}{2}$	$4,4$	$4,4\frac{1}{2}$
FO	$5\frac{1}{2},3$	$5\frac{1}{2},3\frac{1}{2}$	$4\frac{1}{2},4$	$4\frac{1}{2},4\frac{1}{2}$

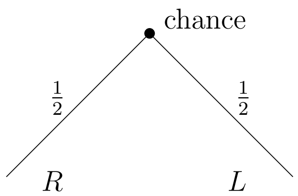
Figure 4.9: Information about the duopoly needed for analyzing the relative influence of the firms on their profits.

is $\frac{1}{2}$ for each combination of attitudes. These prices are computed in the same way as for the zero-sum case: we construct the information buying pre-game and we set the prices such that the situation in which both players buy information with probability $\frac{1}{2}$ is an equilibrium. These numbers lead to $\gamma_1^O = (4-3\frac{1}{2})/(\frac{1}{2}+(4-3\frac{1}{2})+\frac{1}{2}) = \frac{1}{3}$. Similarly, we find $\gamma_1^W = \frac{1}{2}$ and, by the symmetry of the game, $\gamma_2^O = \frac{1}{3}$ and $\gamma_2^W = \frac{1}{2}$. For both firms, relative control is smaller against an optimal opponent than against a worst opponent. Phrased differently, for both firms the uncertainty about the subsidy being given or not has more influence on the firm's profits when its competitor acts as a profit maximizer.

In non-zero sum games, an egoistic player and his masochistic opponent do not necessarily have the same objective. Minimizing one's own payoff is not the same as maximizing the payoff of the opponent anymore. So, a comparison between the relative control numbers γ_2^O and γ_1^W does not make sense. ◁

In the example above we have a nicely structured bimatrix game with almost perfect information for which we can carry out our analysis. The following example shows that our analysis in general does not work for bimatrix games, not even if the game is such that each of the underlying bimatrix games, as well as the compound bimatrix game itself, has an equilibrium in strongly dominating strategies.

Example 4.6.2 This example is based on an example given by Bassan, Scarsini and Zamir (1997). It is a bimatrix game with almost perfect information in which each of the underlying bimatrix games has a unique Nash equilibrium, consisting of strongly dominant strategies. The numbers that we need for de-



$\langle A, B\rangle = \frac{1}{2}\langle A_1, B_1\rangle + \frac{1}{2}\langle A_2, B_2\rangle$

	L	R
T	$0, 0$	$6, -3$
B	$-3, 6$	$5, 5$

$\langle A_1, B_1\rangle$

	L	R
T	$-20, -20$	$-7, -16$
B	$-16, -7$	$-5, -5$

$\langle A_2, B_2\rangle$

	L	R
T	$-10, -10$	$-\frac{1}{2}, -9\frac{1}{2}$
B	$-9\frac{1}{2}, -\frac{1}{2}$	$0, 0$

$\langle A, B\rangle$

Figure 4.10: The bimatrix game with almost perfect information from Bassan, Scarsini and Zamir (1997).

termining relative influence of the players in this game, are given in Figure 4.11. We see that player 1's payoffs in Figure 4.11 are not decreasing from bottom-

	L	R
(T, T)	$-10, -10$	$-\frac{1}{2}, -9\frac{1}{2}$
(T, B)	$-8, -3\frac{1}{2}$	$\frac{1}{2}, -4$
(B, T)	$-11\frac{1}{2}, -7$	$-1, -5\frac{1}{2}$
(B, B)	$-9\frac{1}{2}, -\frac{1}{2}$	$0, 0$

$\langle A^1, B^1\rangle$

		player 2 type			
		FW	W	O	FO
player 1 type	FW	$-7\frac{1}{2}, -7\frac{1}{2}$	$-11\frac{1}{2}, -7$	$-1, -5\frac{1}{2}$	$-5, -5$
	W	$-7, -11\frac{1}{2}$	$-10, -10$	$-\frac{1}{2}, -9\frac{1}{2}$	$-4, \frac{1}{2}$
	O	$-5\frac{1}{2}, -1$	$-9\frac{1}{2}, -\frac{1}{2}$	$0, 0$	$-3\frac{1}{2}, -8$
	FO	$-5, -5$	$\frac{1}{2}, -4$	$-8, -3\frac{1}{2}$	$-2\frac{1}{2}, -2\frac{1}{2}$

Figure 4.11: Information needed for analyzing the relative player control in the game of Bassan, Scarsini and Zamir (1997).

left to top-right, which is a necessary condition to enable computation of the value of information in a way that is analogous to the approach presented in this chapter. ◁

4.6.2 Relation with the analysis of skill

As stated in the introduction of this chapter, the analysis of relative player control is related to the skill analysis of a game. The goal of that analysis, of which we give a detailed description in chapter 2, is similar: with both methods one can draw conclusions about the role of the chance moves in a game. The methods even share two "building blocks": the expected payoffs of the optimal player and the fictive optimal player. The way the skill analysis defines an optimal player is exactly the same as the definition we used, but for the fictive optimal player two different definitions have been used in the skill analysis. In the analysis of Borm and Van der Genugten (2001), the fictive optimal player is assumed to have the same kind of information about the chance moves of the game as our clairvoyant, egoistic player. In the alternative approach, that we have described in section 2.8, the clairvoyancy of the player is assumed to be even stronger: the fictive optimal player also knows in advance the outcome of possible randomization by his opponent. This type of uncertainty, caused by the players themselves, is sometimes called an *internal* chance move. To indicate the contrast, the chance moves of the game itself are called *external* chance moves. As stated, our fictive optimal player, whose clairvoyance only helps him as far as the external chance moves are concerned, is like the fictive player that was used in the analysis of Borm and Van der Genugten (2001).

Part II

Strategy and equilibrium structure

Chapter 5

Von Neumann's poker model

5.1 Introduction

The analysis of poker is interesting for a wider audience than just for poker players. The game provides an excellent domain for investigating problems of decision making under uncertainty. It raises interesting questions about the role of information in the game and brings challenges to research in artificial intelligence. And, of course, it is a class of games that is interesting for application of the skill analysis described in chapter 2. Since poker does not involve playing out cards, as opposed to a game like bridge, all strategic aspects in the game concern the bidding by the players. Unfortunately, even though the strategic structure of the game is relatively simple, real poker games are difficult to analyze. From a deck of cards, millions of different poker hands can be drawn, so that the dimension of the representation of the game quickly becomes too large to analyze, even for modern high-speed computers.

To handle this problem of the large numbers of hands, we can order them and represent them by numbers between zero and one on the real line. The highest possible poker hand, a royal flush, then corresponds to one, while the lowest hand corresponds to zero. To make the analysis of the game simpler, one can model the card distribution as a continuous distribution on the interval $[0, 1]$, thereby implicitly increasing the number of possible hands from "very large" to infinity. This approach is followed in this chapter, which studies a two-player poker game with alternate bidding that was introduced by Von Neumann and Morgenstern (1944, chapter 19). This game is played as follows. First, both players pay an ante and receive a hand. Next, player 1 chooses between *betting* a fixed bet size, and *passing*. When player 1 has decided to bet, then player 2 can choose between *folding* and *calling*. In the first case, he gives up the ante. In the second case, he has to add the same fixed bet size as player 1 to the pot and then the same thing happens as when player 1 has passed: a showdown follows. In the showdown, the player with the better hand wins the pot. A specific variant of this game is also studied in the book of Binmore (1992).

In the model of Von Neumann and Morgenstern (1944) the hands of the players are drawn from a continuous uniform distribution on $[0, 1]$. In this chapter, we extend the model by allowing for other than uniform hand distributions. We compute the value of the game as well as optimal strategies for both players. Next, we translate our general strategic results to the situation where the game is played with regular playing cards. We need this information to approximate the skill level of this game using the methods described in chapter 2.

The chapter is organized as follows. First, we give an exact description of the specific poker game under consideration in section 5.2. In section 5.3 we compute the optimal strategies for both players and discuss equilibrium play in some more detail. Subsequently, we approximate optimal play for the case where this poker game is played with a regular deck of 52 cards. This is the subject of section 5.4. Finally, we measure the skill involved in this variant of poker and present the results in section 5.5.

5.2 Game description

We give a formal description of the rules of our poker game, which we call *minipoker* throughout this text. To begin the game, both players add an *ante* of size a to the stakes. Then the cards are dealt. Instead of considering the $\binom{52}{5}\binom{47}{5} = 3,986,646,103,440$ possible hand combinations that can be dealt in a general poker game, the hands are assumed to be real numbers, independently drawn from the interval $[0, 1]$. So, player 1's hand is the value u of a continuous random variable U and player 2's hand is the value v of a continuous random variable V. U and V are independently, identically distributed on $[0, 1]$ according to the cumulative distribution function $F : [0, 1] \to \mathbb{R}_+$.

The function $f : [0, 1] \to \mathbb{R}_{++}$ denotes the probability density function for this distribution and is assumed to be *positive* and *continuous* on its domain.

After seeing his hand, player 1 can choose between *passing* and *betting.* If he passes, a *showdown* follows immediately. In the showdown, the players compare their hands and the player with the highest hand wins the pot. Betting means adding an amount b to the stakes. After a bet by player 1, player 2 can decide to *fold* or to *call.* If he folds, then he loses his ante of a to player 1. To call, player 2 must put an extra amount b in the pot. In that case, a showdown follows and the player with the better hand takes the pot.

The difference with the case of Von Neumann and Morgenstern (1944) is that they only consider hands u and v that are drawn independently from uniform distributions on $[0, 1]$. Furthermore, they use a terminology for the strategic options of the players which is different from the usual one.

Figure 5.1 displays the essence of our poker model in extensive form. Player 2 receives a hand v. In the picture, two possible hands for player 1 are shown together with v: u_1 and u_2 (with $u_1 < v < u_2$). The decision of player 1 is displayed as if taken before v is dealt to player 2.

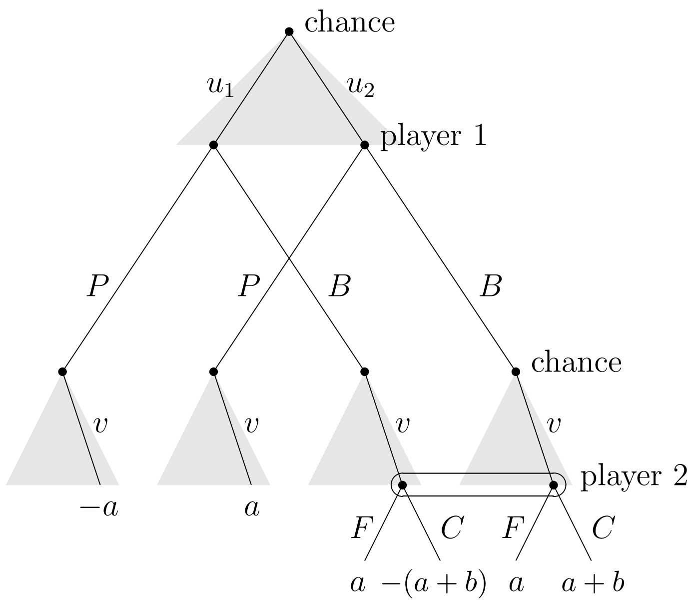


Figure 5.1: The extensive form of two-person minipoker ($u_1 < v < u_2$).

5.3 Optimal strategies

In this section we search for the Nash equilibria of minipoker. We restrict attention to behavioural strategies that are measurable functions of the player's hands. The structure of the analysis is similar to the way Binmore (1992, chapter 12) explains a specific variant of this game.

Pure strategies for Von Neumann's poker model are functions $g : [0,1] \to \{P, R\}$ and $h : [0,1] \to \{F, C\}$. A mixed strategy is therefore something rather complex. Since the game has the property of perfect recall, according to Aumann (1964) we can work with behavioural strategies as well, without restricting the players in their possibilities.

A behavioural strategy for player 1 is a Lebesgue measurable function $p : [0,1] \to [0,1]$, where $p(u)$ is the probability with which he bets if the value of his hand is u. Similarly, a behavioural strategy for player 2 is a function $q : [0,1] \to [0,1]$, where $q(v)$ is the probability with which he plans to call if he is dealt a hand with value v.

Suppose that the players use the behavioural strategies p and q. Then, given dealt cards (u,v), we can compute the expected gain $z(u,v)$ of player 1. This value depends on who has the better hand.
If $u > v$,

$$\begin{aligned} z(u,v) &= \overbrace{ap(u)(1-q(v))}^{B,F} + \overbrace{(a+b)p(u)q(v)}^{B,C} + \overbrace{a(1-p(u))}^{P} \\ &= a + bp(u)q(v). \end{aligned}$$

If $u < v$,

$$\begin{aligned} z(u,v) &= \overbrace{ap(u)(1-q(v))}^{B,F} - \overbrace{(a+b)p(u)q(v)}^{B,C} - \overbrace{a(1-p(u))}^{P} \\ &= 2ap(u) - (2a+b)p(u)q(v) - a. \end{aligned}$$

Even though player 1 does not know what hand player 2 is holding, he can now compute the expectation with respect to v of his own payoff for a given hand u.

$$\begin{aligned} E_1(u) &= \int_{v<u} z(u,v)f(v)dv + \int_{v>u} z(u,v)f(v)dv \\ &= \int_0^u (a+bp(u)q(v))f(v)dv \\ &\quad + \int_u^1 (2ap(u)-(2a+b)p(u)q(v)-a)f(v)dv \\ &= p(u)S_1(u)+T_1(u), \end{aligned}$$

with

$$\begin{aligned} S_1(u) &= 2a(1-F(u)) + b\int_0^u q(v)f(v)dv - (2a+b)\int_u^1 q(v)f(v)dv, \\ T_1(u) &= 2aF(u)-a. \end{aligned}$$

Analogously, for player 2, we get

$$\begin{aligned} E_2(v) &= -\int_{u<v} z(u,v)f(u)du - \int_{u>v} z(u,v)f(u)du \\ &= -\int_0^v (2ap(u)-(2a+b)p(u)q(v)-a)f(u)du \\ &\quad - \int_v^1 (a+bp(u)q(v))f(u)du \\ &= q(v)S_2(v)+T_2(v), \end{aligned}$$

with

$$\begin{aligned} S_2(v) &= (2a+b)\int_0^v p(u)f(u)du - b\int_v^1 p(u)f(u)du, \\ T_2(v) &= 2aF(v) - 2a\int_0^v p(u)f(u)du - 1. \end{aligned}$$

When we look for a Nash equilibrium $(\widetilde{p},\widetilde{q})$, all that matters are the signs of the functions $\widetilde{S}_1$ and $\widetilde{S}_2$, obtained by writing $q(v)=\widetilde{q}(v)$ and $p(u)=\widetilde{p}(u)$. How can we see this? Suppose that player 2 uses strategy $\widetilde{q}$. Then player 1 will get a payoff of $p(u)\widetilde{S}_1(u)+\widetilde{T}_1(u)$ if he raises with probability $p(u)$ when dealt u. The second component of this payoff is independent of $p(u)$, so to see what decision is optimal, we only need to know the sign of $\widetilde{S}_1(u)$. If $\widetilde{S}_1(u)>0$, the choice $p(u)=1$ is optimal. If $\widetilde{S}_1(u)<0$, the choice $p(u)=0$ is optimal. Only if $\widetilde{S}_1(u)=0$, other choices of $p(u)$ are optimal too. Applying similar considerations to player 2, we obtain the following necessary conditions for equilibrium strategies $\widetilde{p}$ and $\widetilde{q}$:

$$\begin{array}{rclcrcl} \widetilde{S}_1(u) > 0 & \Rightarrow & \widetilde{p}(u) = 1; & \quad & \widetilde{S}_2(v) > 0 & \Rightarrow & \widetilde{q}(v) = 1; \\ \widetilde{S}_1(u) < 0 & \Rightarrow & \widetilde{p}(u) = 0; & & \widetilde{S}_2(v) < 0 & \Rightarrow & \widetilde{q}(v) = 0; \\ 0 < \widetilde{p}(u) < 1 & \Rightarrow & \widetilde{S}_1(u) = 0; & & 0 < \widetilde{q}(v) < 1 & \Rightarrow & \widetilde{S}_2(v) = 0. \end{array}$$

In the graphs of Figure 5.2, for F we have chosen the uniform distribution on the interval $[0, 1]$, while the ratio $\frac{b}{a}$ of the bet size and the ante is equal to 1. Figure 5.2(a) shows what the graph of $\widetilde{S}_2(v)$ looks like. To check this, take a

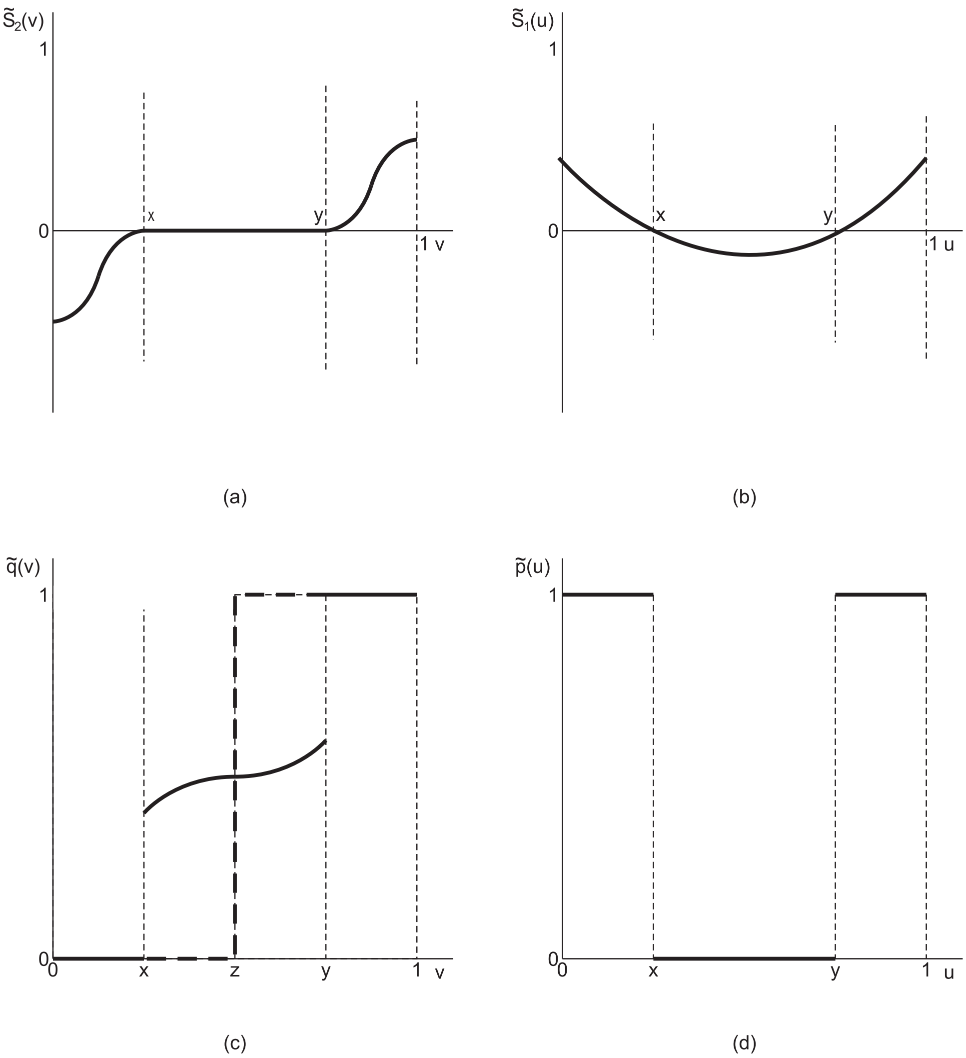


Figure 5.2: Finding the equilibrium strategies $\widetilde{p}$ and $\widetilde{q}$.

look at the expression for $\widetilde{S}_2(v)$. Since both a and b are positive numbers, $f(u)$

is assumed to be positive for all $u \in [0,1]$ and $p(u)$ can only take nonnegative values, it follows that the function $\widetilde{S}_2$ is weakly increasing in v. Substituting $v = 0$ in the formula for $\widetilde{S}_2(v)$ yields $\widetilde{S}_2(0) \leq 0$, while substitution of $v = 1$ tells us $\widetilde{S}_2(1) \geq 0$. Since $\widetilde{S}_2$ is continuous, there exist numbers x and y such that x is the smallest number in $[0,1]$ for which $\widetilde{S}_2(x) = 0$ and y is the largest number in $[0,1]$ for which $\widetilde{S}_2(y) = 0$. Note that, unless $x = y$, the function $\widetilde{S}_2$ cannot be strictly increasing. The information about $\widetilde{S}_2$, summarized in Figure 5.2(a), tells us much about the function $\widetilde{q}$. What we know about $\widetilde{q}$ is summarized in Figure 5.2(c).

The expression for $\widetilde{S}_2(v)$ is informative about the function $\widetilde{p}$ too. Since $\widetilde{S}_2(v)$ is constant for v on the interval $[x,y]$, we must have $\widetilde{p}(v) = 0$ on the interval (x,y). However, $\widetilde{p}(v)$ cannot be zero on a larger open interval I, because this would imply that $\widetilde{S}_2(v)$ would then be constant on I. This constant would need to be zero, because $\widetilde{S}_2(v) = 0$ on the interval $[x,y]$. However, this contradicts the fact that $[x,y]$ is the largest interval on which $\widetilde{S}_2(v) = 0$.

What we have learned about $\widetilde{p}$ tells us something about $\widetilde{S}_1$. It cannot be that $\widetilde{S}_1(u) < 0$ immediately to the left of x, because then $\widetilde{p}(u) = 0$ immediately to the left of x. Because $\widetilde{S}_1$ is continuous, it follows that $\widetilde{S}_1(x) \geq 0$. For similar reasons $\widetilde{S}_1(y) \geq 0$. Figure 5.2(c) tells us that $\widetilde{q}(u) = 0$ for every u on the interval $(0,x)$ and that $\widetilde{q}(u) = 1$ on the interval $(y,1)$. Consequently, $\widetilde{S}_1$ decreases on $[0,x]$ and increases on $[y,1]$, as indicated in Figure 5.2(b).

Figure 5.2(b) enables us to tie down $\widetilde{p}$ completely. We already know that $\widetilde{p}(u) = 0$ for $u \in (x,y)$. But now we know that $\widetilde{S}_1(u) > 0$ on $[0,x)$ and $(y,1]$. Thus, $\widetilde{p}(u) = 1$ on these intervals, as Figure 5.2(d) shows.

Next, use the information about $\widetilde{p}$ and $\widetilde{q}$, together with the fact that $\widetilde{S}_1(x) = \widetilde{S}_1(y) = \widetilde{S}_2(x) = \widetilde{S}_2(y) = 0$ to see that x and y are determined by the equations

$$F(y) = 1 - \frac{2a+b}{b}F(x) \qquad \text{and} \qquad F(y) = \frac{a+b}{2a+b} + \frac{a}{2a+b}F(x).$$

Since f is assumed to be positive on its domain, we know that F is strictly increasing and continuous. Therefore, we can solve these equations to find

$$x = F^{-1}\left(\frac{ab}{(a+b)(4a+b)}\right) \qquad \text{and} \qquad y = F^{-1}\left(\frac{(2a+b)^2 - 2a^2}{(a+b)(4a+b)}\right). \tag{5.1}$$

So $\widetilde{p}$ is determined uniquely. However, $\widetilde{q}$ is not. For $x \leq v < y$, $\widetilde{q}(v)$ can be chosen freely, subject to the constraints

$$1 - F(y) = \int_x^y q(v)f(v)dv \qquad \text{and} \qquad \widetilde{S}_1(u) \le 0 \text{ for } x < u < y.$$

These constraints boil down to

$$\begin{aligned} \frac{1}{F(y) - F(x)} \int_x^y q(v)f(v)dv &= \frac{a}{a+b}, \\ \frac{1}{F(y) - F(u)} \int_u^y q(v)f(v)dv &\ge \frac{a}{a+b} \quad \text{for } x < u < y. \end{aligned}$$

So, $\widetilde{q}$ is constrained such that between x and y the average of $\widetilde{q}(v)$ is $\frac{a}{a+b}$, and on any right end of this interval the average of $\widetilde{q}(v)$ is at least $\frac{a}{a+b}$. Although there are many choices for $\widetilde{q}$ that satisfy these constraints, there is a unique admissible Nash equilibrium strategy that does this. A strategy is said to be *admissible* for a player if no other strategy for that player does better against one strategy of the opponent without doing worse against some other strategy of the opponent. This is the strategy with which player 2 folds when his hand is under a certain threshold value z and calls when his hand is above it, such that

$$\int_z^y f(v)dv = \int_y^1 f(v)dv.$$

It follows that this unique value of z is given by

$$z = F^{-1}\left(\frac{b(3a+b)}{(a+b)(4a+b)}\right). \tag{5.2}$$

This admissible strategy is already indicated in Figure 5.2(c), by the dashed line. Using the derived strategies $\widetilde{p}$ and $\widetilde{q}$, we can compute the value of the minipoker game. In Figure 5.3, all possible hands for player 1 are set out horizontally, together with the action chosen for each hand u. For player 2, the hands v and corresponding actions are set out vertically. In each of the ten areas that appear, we know the combination of actions chosen by both players and thus we can give the payoff for each possible combination of hands. By integrating over over all combinations of hand values u and v, we can compute

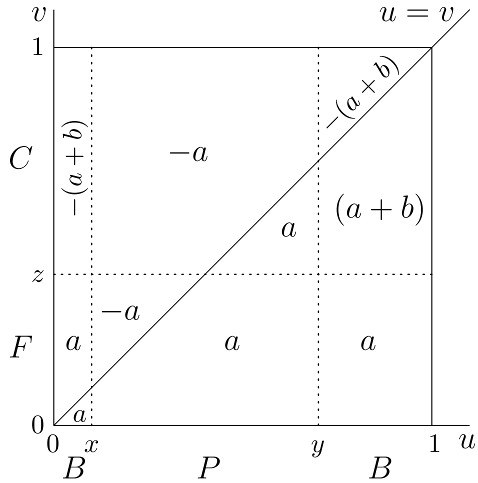


Figure 5.3: Expected payoff for all (u, v) to player 1 in the Nash equilibrium $(\widetilde{p}, \widetilde{q})$.

the value $v_{a,b}$ of the game with ante a and bet size b.

$$\begin{aligned} v_{a,b} &= a\left(\int_0^y \int_0^u f(v)dvf(u)du + \int_y^1 \int_0^z f(v)dvf(u)du\right) \\ &\quad +a\int_0^x \int_u^z f(v)dvf(u)du \\ &\quad -a\int_x^y \int_u^1 f(v)dvf(u)du + (a+b)\int_y^1 \int_z^u f(v)dvf(u)du \\ &\quad -(a+b)\left(\int_0^x \int_z^1 f(v)dvf(u)du + \int_y^1 \int_u^1 f(v)dvf(u)du\right) \\ &= 2a\int_x^1 F(u)f(u)du + 2b\int_y^1 F(u)f(u)du - \frac{a(4a+3b)}{4a+b}, \qquad (5.3) \end{aligned}$$

where x and y are as defined in equation (5.1). The results of the analysis above are summarized in Theorem 5.3.1.

Theorem 5.3.1 *If minipoker is played with ante a and bet size b and the hands u and v of the players both have cumulative distribution F with positive, continuous density f on $[0, 1]$, then the value of the game is given by*

$$v_{a,b} = 2a\int_x^1 F(u)f(u)du + 2b\int_y^1 F(u)f(u)du - \frac{a(4a+3b)}{4a+b},$$

with $x = F^{-1}\left(\frac{ab}{(a+b)(4a+b)}\right)$ *and* $y = F^{-1}\left(\frac{(2a+b)^2-2a^2}{(a+b)(4a+b)}\right)$. *In this case, the unique, admissible optimal strategies are*

$$Pr\{bet\ with\ hand\ u\} = \widetilde{p}(u) = \begin{cases} 1 & if\ Pr\{v \le u\} \le \frac{ab}{(a+b)(4a+b)} \\ & \qquad or\ Pr\{v \le u\} > \frac{(2a+b)^2-2a^2}{(a+b)(4a+b)}, \\ 0 & otherwise, \end{cases}$$

for player 1 and

$$Pr\{call\ with\ hand\ v\} = \widetilde{q}(v) = \begin{cases} 0 & if\ Pr\{u \le v\} \le \frac{b(3a+b)}{(a+b)(4a+b)}, \\ 1 & otherwise, \end{cases}$$

for player 2.

The results for the case of Von Neumann and Morgenstern (1944) where F is the uniform distribution on $[0, 1]$, follow directly from Theorem 5.3.1.

Corollary 5.3.2 *The minipoker game of Von Neumann and Morgenstern (1944), in which F was the uniform distribution, has value*

$$v_{a,b} = \frac{a^2 b}{(a+b)(4a+b)}.$$

Optimal strategies are given by

$$Pr\{bet\ with\ hand\ u\} = \widetilde{p}(u) = \begin{cases} 1 & if\ u \le \frac{ab}{(a+b)(4a+b)}\ \ or\ u > \frac{(2a+b)^2-2a^2}{(a+b)(4a+b)}, \\ 0 & otherwise, \end{cases}$$

for player 1 and

$$Pr\{call\ with\ hand\ v\} = \widetilde{q}(v) = \begin{cases} 0 & if\ v \le \frac{b(3a+b)}{(a+b)(4a+b)}, \\ 1 & otherwise. \end{cases}$$

for player 2.

So, in this simple case, the value of the game is equal to the product of the ante and the value of the hand that indicates player 1's strategic boundary between bluffing and passing. Interesting is the fact that the value is positive in this case. The game is favourable for player 1. To see for what combination of values of the ante and the bet size the game is most favourable for player 1, we fix the ante a and compute the derivative of $v_{a,b}$ with respect to b.

$$\frac{d}{db} v_{a,b} = \frac{a^2(2a-b)(2a+b)}{(a+b)^2(4a+b)^2}.$$

This derivative is zero at $b = 2a$. This is the only solution, since both a and b are positive. Since $\frac{d^2}{db^2}v_{a,b} = -\frac{1}{81a} < 0$ for these relative values of the bet size and the ante, we know that the ratio $\frac{b}{a} = 2$ is optimal for player 1. This special case is called pot-limit minipoker, since the maximal bet size (in this case the only possible bet size) equals the total size of the pot. Now we can formulate Proposition 5.3.3.

Proposition 5.3.3 *The pot-limit variant of minipoker is the unfairest variant possible with uniform hand distributions.*

Another thing that is intuitively clear, is easy to recognize now too: for minipoker with uniform distributions the strategies of the players depend only on the ratio $r = \frac{b}{a}$ of the bet size and the ante. The boundary values x, y and z given in equations (5.1) and (5.2), can be written as

$$x = \frac{r}{(r+4)(r+1)}, \quad y = \frac{r^2+4r+2}{(r+4)(r+1)} \quad \text{and} \quad z = \frac{r^2+3r}{(r+4)(r+1)}.$$

The boundary values are plotted as a function of r in Figure 5.4. In this figure, we recognize the limits for r going to infinity,

$$\lim_{r\to\infty} x = 0 \quad \text{and} \quad \lim_{r\to\infty} y = \lim_{r\to\infty} z = 1,$$

and the limiting values of the boundaries for the ratio going to zero,

$$\lim_{r\downarrow 0} x = \lim_{r\downarrow 0} z = 0 \quad \text{and} \quad \lim_{r\downarrow 0} y = \frac{1}{2}.$$

The shapes of these curves for larger values of r is intuitively clear: when betting and calling become relatively expensive, it is wise to choose these actions not too often. As the ratio goes to zero, the number of hands with which player 2 calls increases quickly. Giving up the ante by folding becomes relatively expensive. As a consequence, player 1 only folds with the higher half of the hands, for which the probability that he has the highest hand is larger than $\frac{1}{2}$. Finally, at $r = 2$, the case of pot-limit poker, the hand value below which player 1 bluffs is maximal.

5.4 A regular deck of cards

In the previous section we derived optimal strategies in the two-person poker game for both players in a general form. These strategies were given in terms

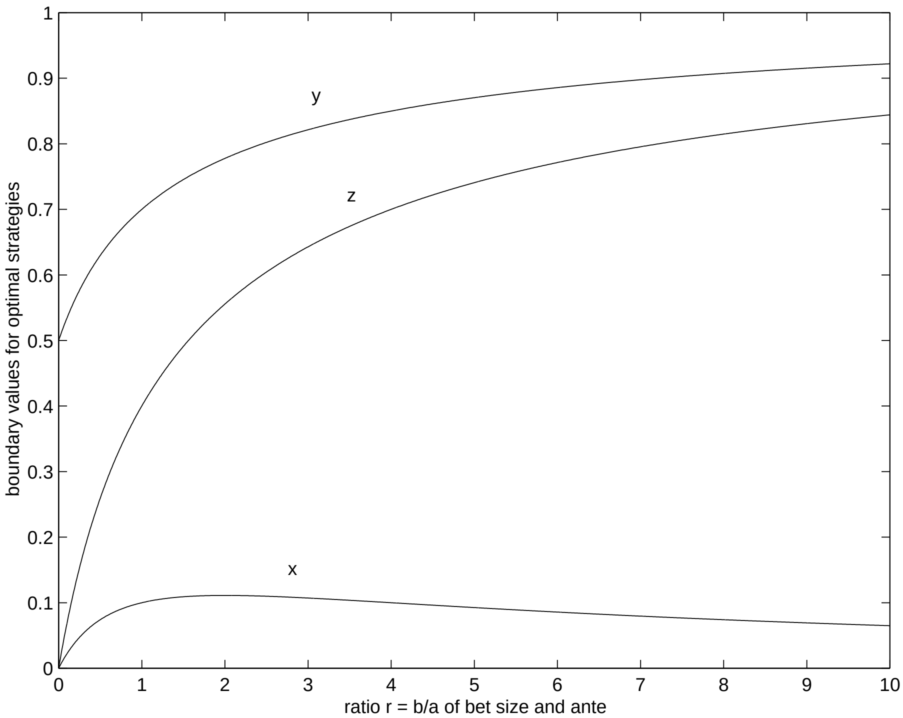


Figure 5.4: Boundary values for the optimal strategies as a function of the ratio $\frac{b}{a}$ of the bet size and the ante.

of quantiles of the continuous distribution function F, the distribution function from which the hands of the players were drawn. In this section we see what these results imply when the game is played with a regular deck of cards, from which the players draw real poker hands.

5.4.1 Classification of poker hands

Before we start translating strategies, let us first give an overview of the poker hands that can occur. A poker hand is a combination of five cards, drawn from a deck of 52 cards. The deck consists of four suits: hearts ($\heartsuit$), clubs ($\clubsuit$), diamonds ($\diamondsuit$) and spades ($\spadesuit$). All suits are equally valuable, while the 13 cards of each suit have, ranked in decreasing order, the values A(ce), K(ing), Q(ueen), J(ack), 10, 9, …, 2. All hands belong to one of the ten *classes* that are defined in decreasing order of value in Table 5.1. The order of hands within a class is determined by comparing the cards of the hands separately, starting with the most important card of a hand. The importance of the card within

	Class	Description	Example
RF	Royal Flush	five consecutive cards of one suit, starting with an ace	$(\clubsuit A, \clubsuit K, \clubsuit Q, \clubsuit J, \clubsuit 10)$
SF	Straight Flush	five consecutive cards of the same suit (an ace can have the value 1)	$(\spadesuit 5, \spadesuit 4, \spadesuit 3, \spadesuit 2, \spadesuit A)$
4K	Four of a Kind	four cards with equal values	$(\diamondsuit 4, \clubsuit 4, \heartsuit 4, \spadesuit 4, \diamondsuit Q)$
FH	Full House	a triplet of cards with the same values, together with a pair with equal values	$(\spadesuit 5, \clubsuit 5, \diamondsuit 5, \diamondsuit 10, \heartsuit 10)$
F	Flush	five cards of the same suit	$(\clubsuit K, \clubsuit J, \clubsuit 9, \clubsuit 3, \clubsuit 2)$
S	Straight	five consecutive cards	$(\heartsuit K, \spadesuit Q, \heartsuit J, \clubsuit 10, \diamondsuit 9)$
3K	Three of a Kind	three cards with the same value	$(\clubsuit Q, \heartsuit Q, \spadesuit Q, \diamondsuit J, \heartsuit 6)$
2P	Two pairs	two pairs with the same values within each pair	$(\spadesuit A, \heartsuit A, \diamondsuit 8, \spadesuit 8, \clubsuit 3)$
1P	One pair	one pair of cards with equal values	$(\heartsuit 9, \diamondsuit 9, \clubsuit K, \diamondsuit 10, \diamondsuit 4)$
HC	High Card	any combination of cards that does not fit in any of the classes above	$(\heartsuit K, \diamondsuit J, \diamondsuit 9, \clubsuit 4, \spadesuit 2)$

Table 5.1: Classification of poker hands

a hand depends on the class to which the hand belongs. In Table 5.1 the card order in the example hands is such that the most important cards are put in front.

The total number of different hands of five cards that can be drawn from a single deck of 52 cards is $\binom{52}{5} = 2,598,960$. The number of hands in each class and the probability of receiving a hand from this class is given in Table 5.2 for all ten classes. The decreasing probabilities are the reason that the order of the classes is as it is. If we pay attention to the order of the hands within the ten classes, then we obtain $7,462$ ordered *subclasses*. Within each subclass, all hands really are equal, meaning that no hand from a given subclass beats another hand from the same subclass in a showdown. In Figure 5.5 we give the frequencies with which hands of a certain subclass appear.

The small bar with high frequencies around subclass number $5,800$ corresponds to the Straights, while the somewhat wider block with frequencies of

Class	**Number**	**Prob.(%)**
RF	4	0.000
SF	36	0.001
4K	624	0.024
FH	3,744	0.144
F	5,108	0.197
S	10,200	0.392
3K	54,912	2.113
2P	123,552	4.754
1P	1,098,240	42.257
HC	1,302,540	50.118
Total	2,598,960	100.000

Table 5.2: Numbers and probabilities for all classes of poker hands

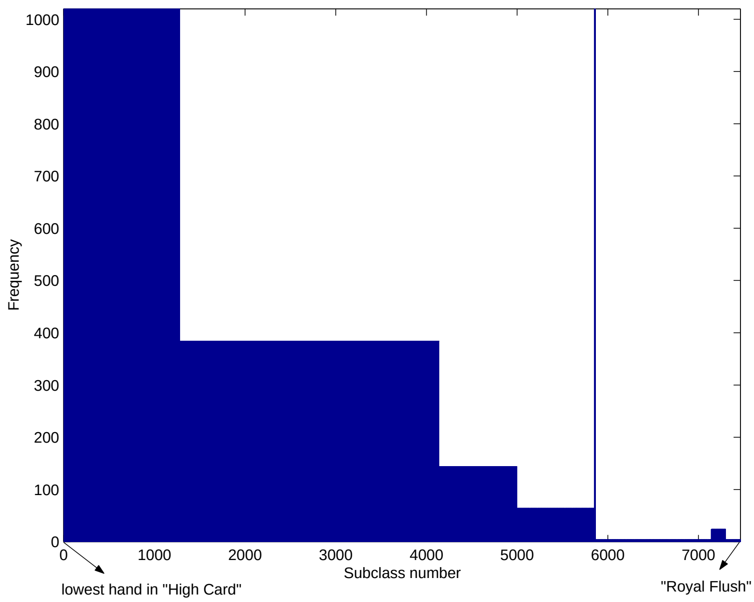


Figure 5.5: Frequencies of appearance of subclasses of poker hands in a single deck of 52 cards.

24 corresponds to Full House. Figure 5.6 gives the continuous approximation of the cumulative distribution of the poker hands, where both hand numbers and frequencies are normalized.

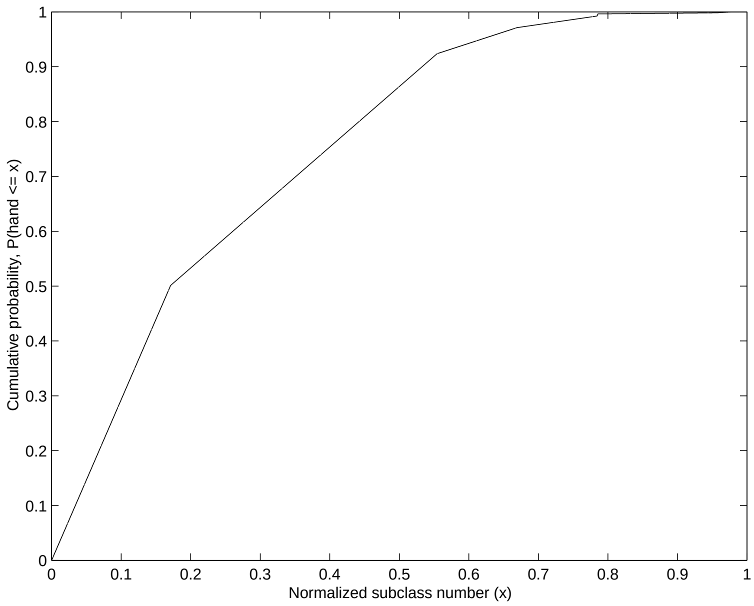


Figure 5.6: Continuous approximation of the cumulative distribution of the $7,462$ subclasses of poker hands in a single deck of 52 cards.

5.4.2 From a continuous to a discrete distribution

All results we presented so far were derived using continuous hand distributions. Now we want to take these results from the continuous situation into the discrete case, where the hands are drawn from a deck of 52 cards, without replacement. An intuitive way to approximate optimal strategies in the discrete game is the following. If the 5-card hand of a player ranks n (from the bottom) out of 2,598,960, we treat his hand as if he were dealt $\frac{n}{2{,}598{,}960}$ in the continuous game.

As Cutler (1975) remarks, there are at least three objections to this approximation. First of all, the optimal strategies for the discrete case may differ considerably from the ones derived for the continuous case. However, according to Von Neumann and Morgenstern (1944, p. 209), the maximal loss that can be incurred by playing the “continuous” strategy is not large. More precisely, the difference is only of the order $\frac{1}{2{,}598{,}960}$. Second, some different hands have an equal value, as the ordering (partly) disregards suits. This fact is taken care of by using the general distribution F in our derivation. Even if certain hands occur with higher probability than others, our results still apply. Finally, the hands are dealt from one deck without replacement. That is, the hand one player holds affects what the other may hold. As a result, increasing the rank of a hand does not necessarily increase its value. Consider the following example. If a player holds a straight flush to the five, the opponent may hold 31 higher straight flushes or three equal ones. However, when the player has four aces and a six, his opponent may only beat him with 27 different straight flushes. We do not take into account this last remark and focus on the case where minipoker is played with a separate deck of cards for each player. Or equivalently, it could be interpreted as the game in which the players' hands are drawn from a regular deck of 52 cards with replacement. We give an approximation for optimal play for this game in section 5.4.3.

5.4.3 Optimal play

In this section we tell what the optimal minipoker strategies for both players mean in terms of real poker hands. We consider the case where hands are drawn from a regular deck of cards with replacement. Unless stated otherwise explicitly, the results in this section apply to the case where the ante and the bet size are equal, i.e., $r = \frac{b}{a} = 1$. Recall from Theorem 5.3.1 that if the players are dealt the hands u and v, the optimal strategy for player 1, stated in terms of probabilities, for this ratio is

$$Pr\{\text{bet with hand } u\} = \widetilde{p}(u) = \begin{cases} 1 & \text{if } Pr\{V \leq u\} \leq \frac{1}{10} \text{ or } Pr\{V \leq u\} > \frac{7}{10}, \\ 0 & \text{otherwise,} \end{cases}$$

and that it is optimal for player 2 to play

$$Pr\{\text{call with hand } v\} = \widetilde{q}(v) = \begin{cases} 0 & \text{if } Pr\{U \leq v\} \leq \frac{2}{5}, \\ 1 & \text{otherwise.} \end{cases}$$

Using the information that is displayed in Figure 5.6, we can translate these probabilities to the probabilities of poker hands. We find that the optimal strategy for player 1 is

$$Pr\{\text{bet with hand } u\} = \begin{cases} 1 & \text{if } u \leq (Q,7,5,4,3), \\ 0 & \text{if } (Q,7,6,3,2) \leq u \leq (8,8,9,5,4), \\ 1 & \text{if } u \geq (8,8,9,6,2), \end{cases}$$

and that it is optimal for player 2 to play

$$Pr\{\text{call with hand } v\} = \begin{cases} 0 & \text{if } v \leq (A,Q,8,6,2), \\ 1 & \text{if } v \geq (A,Q,8,6,3). \end{cases}$$

To be precise, for player 1 the hands are selected such that $(Q, 7, 5, 4, 3)$ is the highest hand for which $Pr\{V \leq u\} \leq \frac{1}{10}$ and $(Q, 7, 6, 3, 2)$ is the lowest hand for which $Pr\{V \leq u\} > \frac{1}{10}$. We use this approach to keep the strategy specifications simple, but we realize that a more accurate interpretation of the optimal strategies for the continuous game in terms of real poker hands requires randomization with hands from one subclass at each boundary value. To indicate the dependency of the strategies on the ratio of bet size and ante, that was shown for the uniform case in Figure 5.4, Table 5.3 gives the boundary hands for some other relative values of a and b. In this table, x^- is the highest hand below the boundary x. The definitions for y^- and z^- are similar. The case $r = 1$ is included to compare with the results above. In Table 5.3 we can

$r = \frac{b}{a}$	x^- (player 1's upper bound for bluffing)	y^- (player 1's upper bound for passing)	z^- (player 2's upper bound for folding)
1	$(Q, 7, 5, 4, 3)$	$(8, 8, 9, 5, 4)$	$(A, Q, 8, 6, 2)$
2	$(Q, 9, 5, 4, 2)$	$(10, 10, Q, J, 2)$	$(3, 3, K, J, 2)$
3	$(Q, 8, 7, 4, 3)$	$(J, J, A, 9, 3)$	$(6, 6, J, 10, 3)$
5	$(J, 10, 9, 6, 5)$	$(K, K, 10, 9, 3)$	$(9, 9, J, 10, 6)$
10	$(J, 9, 6, 5, 4)$	$(A, A, K, J, 10)$	$(Q, Q, K, 4, 2)$
100	$(9, 7, 5, 3, 2)$	$(K, K, K, Q, 2)$	$(7, 7, 7, K, 4)$

Table 5.3: Boundary values of the optimal strategies for both players for various ratios $r = \frac{b}{a}$.

clearly see that, with a relatively high cost of betting and calling, optimal play prescribes betting and calling only for a small number of hands.

5.5 Relative skill

In this section we analyze the skill of our poker game. The basis for the computations will be the game in which the hands u and v for player 1 and 2 are drawn independently from a uniform distribution on $[0, 1]$. We focus on the case with equal ante and bet size and normalize to $a = b = 1$. We carry out the computations both for the analysis of skill that was proposed by Borm and Van der Genugten (2001) and for our alternative, which we have described in chapter 2. We refer to the first method by using the subscript BG.

From Corollary 5.3.2 we already know the strategies and the game results of optimal players. When we substitute $a = 1$ and $b = 1$ into the formulas, we see that the expected payoff for player 1 as an optimal player is $\frac{1}{10}$. Since we deal with a zero-sum game, this is the value of the game and player 2 has an expected payoff of $-\frac{1}{10}$. In sections 5.5.1 and 5.5.2 we derive the expected game results of the beginners and the fictive players. Finally, in section 5.5.3, these numbers are used to produce an overview of the results of the skill analysis.

5.5.1 Beginners

What will be the strategies of players who play this game for the first time, just after the rules are explained to them? Perhaps they heard about the famous video poker variant "Jacks or Better". In this game, as the name suggests, only hands with a pair of Jacks, Queens, Kings or Aces (and all hands from higher classes) have value for the player. As a result, naive players may be betting or calling with exactly these hands. Even if they do not know this game, this border seems to be a reasonable one. After all, poker players tend to like hands that look fancy; any hand with at least a pair of images surely satisfies this condition of prettiness.

What does this reasoning mean for the strategies of the beginners? Player 1 bets only if his hand is at least $(J, J, 4, 3, 2)$. For each player the total probability of receiving a hand up to $(J, J, 4, 3, 2)$ is $\frac{1189}{1498} \approx 0.7937$. So we can formulate the strategy for player 1 as a beginner as

$$p_0(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq 0.7937 \\ 1 & \text{if } 0.7937 \leq u \leq 1, \end{cases}$$

while the beginner's strategy for player 2 can be formulated as

$$q_0(v) = \begin{cases} 0 & \text{if } 0 \leq v \leq 0.7937 \\ 1 & \text{if } 0.7937 \leq v \leq 1. \end{cases}$$

Both strategies are displayed graphically in Figure 5.7. In Figure 5.8 the expected payoff to player 1 is given for all hand distributions (u, v), assuming that player 1 uses strategy p_0 and player two plays the strategy $\widetilde{q}$, that is given

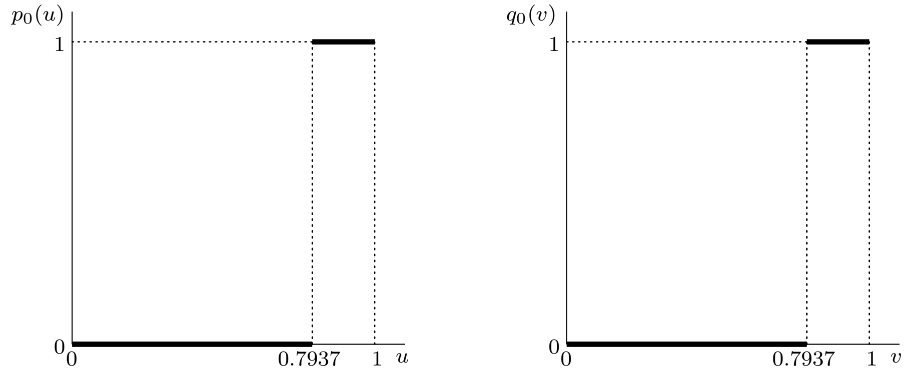


Figure 5.7: The strategies p_0 and q_0 for beginning player 1 and player 2 respectively.

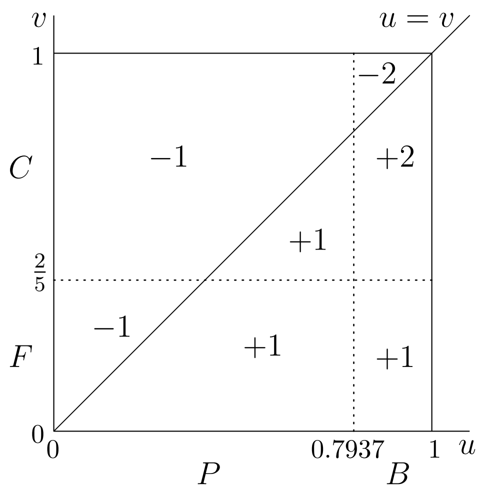


Figure 5.8: Expected payoff for all (u, v) to player 1 if he plays as a beginner against the equilibrium strategy $\widetilde{q}$ of player 2.

in Corollary 5.3.2. Using this figure, one can find that the expected payoff to player 1 as a beginner is

$$U_1(p_0, \widetilde{q}) = \frac{310}{3817} \approx 0.0812.$$

We do the same computation for player 2, using Figure 5.9 in which the expected payoffs to player 1 for the strategy combination $(\widetilde{p}, q_0)$ are shown and find that

$$U_2(\widetilde{p}, q_0) = -\frac{265}{2254} \approx -0.1176.$$

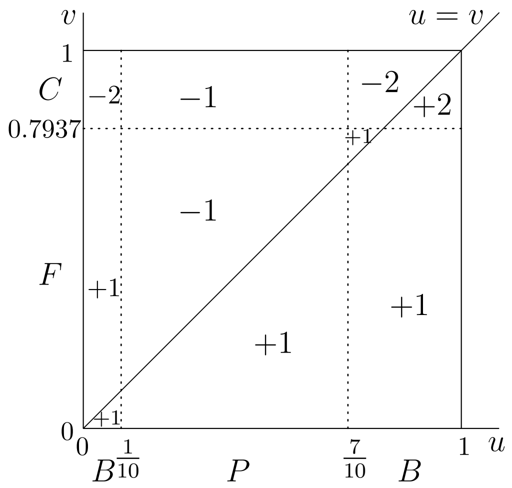
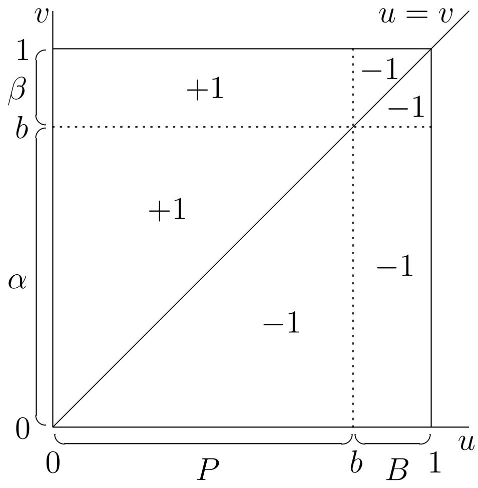


Figure 5.9: Expected payoff for all (u, v) to player 1 if player 2 plays as a beginner against equilibrium strategy $\widetilde{p}$ of player 1 that was given in Corollary 5.3.2.

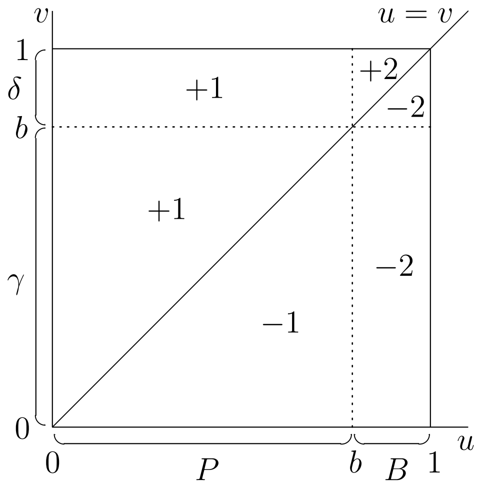
For the analysis of RS according to the definition we have introduced in section 2.8, we need to do some more work. Given the strategies for the beginners (p_0 and q_0), we have to find the optimal response of the opponent. We describe in detail how player 2 determines what will be his best strategy. For each possible value v of his hand, he has to decide whether calling or folding is optimal against p_0. Figure 5.10 displays the payoff for player 2 for each of his two actions, given a hand combination (u, v). The P and B under the horizontal axis indicate for which values of u player 1 passes or bets according to strategy p_0, while b is the boundary value 0.7937 in player 1's beginner's strategy p_0.

For each of the four marked intervals along the vertical axes (α, β, γ and δ) we can compute the expected payoff for player 2 for a specific hand value v. These expected payoffs are displayed in Table 5.4.

Player 2 should base his decisions on the numbers in this table. He has to compare the expected result for each v in α with the expectations for the same v in γ. If for a certain v the result in α is better than the result in γ,



(a) player 2 folds



(b) player 2 calls

Figure 5.10: Expected payoffs for player 2 against beginner's strategy p_0 of player 1.

Interval	Expected payoff for player 2
α	$v-(1-v)=2v-1$
β	$b-(1-b)=2b-1$
γ	$v-(b-v)-2(1-b)=2v-2+b$
δ	$b+2(v-b)-2(1-v)=4v-2-b$

Table 5.4: Expected payoffs for player 2 with a hand v.

player 2 should fold. Otherwise he should call with this hand value. A similar comparison he should make between β and δ. We find that the optimal reply against a player playing p_0 is

$$Pr\{\text{call with hand } v\} = \widetilde{q}_0(v) = \begin{cases} 0 & \text{if } v \le 0.8453, \\ 1 & \text{otherwise.} \end{cases}$$

We can now compute the resulting expected gains of player 1. A similar analysis leads to the payoff for player 2 when he acts as a beginner in the new model. The optimal strategy for player 1 against player 2 playing q_0 is

$$Pr\{\text{bet with hand } u\} = \widetilde{p}_0(u) = \begin{cases} 0 & \text{if } 0.6909 < u \le 0.8969, \\ 1 & \text{otherwise.} \end{cases}$$

The resulting expected payoffs are

$$U_1(p_0, \widetilde{q}_0) \approx -0.0053 \quad \text{and} \quad U_2(\widetilde{p}_0, q_0) \approx -0.4875.$$

5.5.2 Fictive players

In this section we compute the expected payoffs of fictive players in minipoker. Fictive players have more information than normal players. They know the outcome of the chance move in the game and they can use this information in their strategies. For minipoker, this means that the fictive player can base his actions on his own hand, but also on the hand of his opponent. Given the fact that he plays against a player who uses the minimax strategy, he can decide what will be his best action for any hand combination (u, v).

Figure 5.11 shows the payoff to player 1 for each hand combination if player 1 plays as a fictive player against player 2's equilibrium strategy $\widetilde{q}$. The payoffs in the figure are such that player 1 takes the optimal action for each pair of hands (u, v). For example, in the area above the line $v = \frac{2}{5}$ and above the line $u = v$, player 1 knows that player 2 will always call. Since player 1 has the lower card, he had better pass. This leads to the expected payoff of -1 that the figure displays for this area. The expected gains of player 1 can now be computed with help of Figure 5.11 and are equal to

$$U_1(p_f, \widetilde{q}) = \frac{17}{50}.$$

Figure 5.12 shows the payoff to player 1 for each card combination if player 2 plays as a fictive player against player 1's equilibrium strategy $\widetilde{p}$. The expected gains for player 2 as a fictive player can now be computed with help of this figure and are equal to

$$U_2(\widetilde{p}, q_f) = \frac{7}{50}.$$

We also want to compute the expected gains of the fictive players under the assumptions of our alternative *RS*. Under these assumptions, fictive players are also informed about the outcome of any randomization caused by their opponents. Therefore, when we determine optimal play for an opponent, we have to consider pure strategies only; randomizing is useless against such a fictive player.

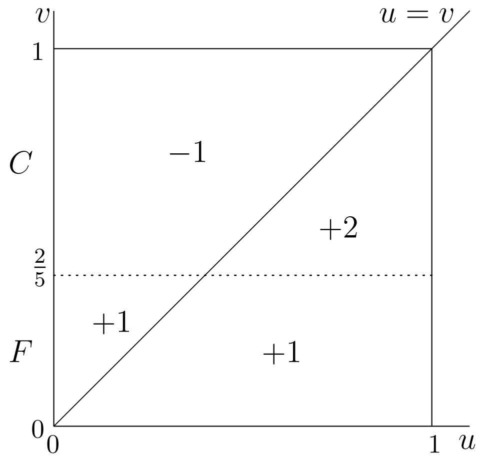


Figure 5.11: Expected payoff for all (u, v) to fictive player 1 if player 2 uses the equilibrium strategy $\widetilde{q}$.

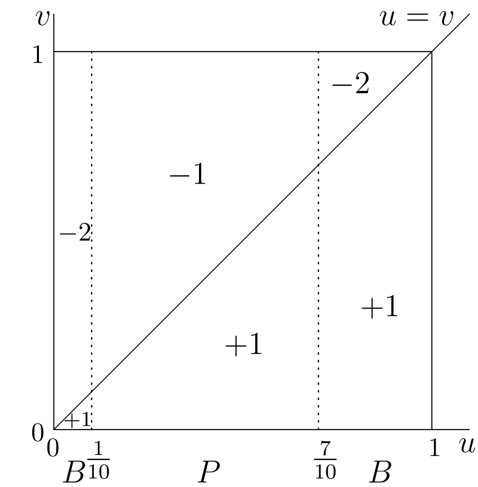


Figure 5.12: Expected payoff for all (u, v) to player 1, using his equilibrium strategy $\widetilde{p}$, if he faces a fictive player 2.

Let us first focus at player 1 as a fictive player. What is the best thing player 2 can do if player 1 does not only know both u and v, but can also anticipate player 2's actions? If player 2 calls with a specific value v, he will get 1 dollar if $u < v$ (since player 1 passes) and lose 2 dollars if $u > v$ (player 1 bets). So his expected payoff for calling will be $-(-v + 2(1 - v)) = 3v - 2$.

For folding he will get -1 on any hand value v. Therefore, it is optimal for player 2 to play

$$Pr\{\text{call with hand } v\} = \widetilde{q}_f(v) = \begin{cases} 0 & \text{if } v < \frac{1}{3}, \\ 1 & \text{otherwise.} \end{cases}$$

As a reply, it is optimal for player 1 to bet with hand u

$$Pr\{\text{bet with hand } u\} = \hat{p}_f(u) = \begin{cases} 1 & \text{if } u \geq v \text{ or } v < \frac{1}{3}, \\ 0 & \text{if } u < v \text{ and } v \geq \frac{1}{3}. \end{cases}$$

The expected gains for player 1 as a fictive player are

$$U_1(\hat{p}_f, \widetilde{q}_f) = \frac{1}{3}.$$

To see what the expected gains of player 2 as a fictive player are, consider what player 1 gets for betting and what he gets for passing, both with a hand of value u. Whereas betting will yield him a dollar if he has the more valuable hand, he will have to pay 2 dollars if his opponent has the better hand. Passing also gives him a win of one dollar if $u > v$, but with this action he will only lose one dollar in case his opponent has the better hand. Passing, therefore, is optimal for all possible values of u. After a pass of player 1 the action of player 2 becomes irrelevant, so any strategy of fictive player 2 is a best reply. Clearly, the expected payoff for player 2 as a fictive player is equal to

$$U_2(\widetilde{p}_f, \hat{q}_f) = 0.$$

5.5.3 Results of the skill analysis

In sections 5.5.1 and 5.5.2, we have computed the expected payoffs of beginners and fictive players. Together with the equilibrium information presented in section 5.4.3, these numbers form all relevant information to complete the skill analysis. Table 5.5 gives an overview. The expected payoffs of the beginners and the fictive players are lower in the new *RS* model. This is what we expected, since their opponents now try to make life as hard as possible for them. The results for optimal players are the same in both models. Therefore the learning effect in the new *RS* model is larger and the random effect is smaller than in the RS_{BG} analysis. This combination of effects leads to a

	RS_{BG}			RS		
	Player 1	Player 2	Game	Player 1	Player 2	Game
Beginner	0.0812	-0.1176	-0.0182	-0.0053	-0.4875	-0.2464
Optimal	0.1000	-0.1000	0.0000	0.1000	-0.1000	0.0000
Fictive	0.3400	0.1400	0.2400	0.3333	0.0000	0.1667
LE	0.0188	0.0176	0.0182	0.1053	0.3875	0.2464
RE	0.2400	0.2400	0.2400	0.2333	0.1000	0.1667
RS	0.0726	0.0682	0.0704	0.3110	0.7949	0.5965

Table 5.5: Results of the skill analysis.

higher skill level for the game. But of course, $RS = 0.5965$ and $RS_{\text{BG}} = 0.0704$ indicate the same skill level, since the numbers concern the same game. A comparison between the skill levels of two games on two different skill measures does not make sense.

Following the relevant jurisprudence, Van der Genugten (1997) advised a boundary between 0.05 and 0.15 to separate games of chance from games of skill. The motivation for these bounds comes from the analysis of a number of one-person games. In section 2.8 we have argued why we prefer RS to RS_{BG} as extension of the method for measuring skill to more-person games. When we compare the skill level of minipoker, 0.5965, with the suggested bounds, it is clear that this number leads to the conclusion that our poker game should be classified as a game of skill.

Another observation we wish to make is that, for both measures, $RS_{\text{game}} \neq \frac{1}{2}RS_{\text{player 1}} + \frac{1}{2}RS_{\text{player 2}}$, as we indicated in section 2.9. If we compare the skill of both players within the new RS model, we see that the skill of player 2 is relatively high. This can be explained by the beginner's strategy of player 2. This is a relatively dumb one, in the sense that player 1 can really profit from his mistakes. So against a player who gives maximal opposition, the beginner in the role of player 2 does relatively bad.

This is a distinguishing feature of the RS model, which is useful in an environment where newcomers are easily recognized and exploited by more experienced players.

We realize that the skill level depends on the strategies that we have ascribed to the beginners in minipoker. If this game would be played in practice, we could benefit from empirical information on payoffs of unexperienced players, using ideas from section 2.11. An alternative way to obtain the desired information is to offer the game as an experiment and to collect the data in a laboratory setting. The organization of such an experiment for games with a more difficult, more realistic betting structure than minipoker, is an interesting topic for further research.

Chapter 6

Poker

6.1 Introduction

In chapter 5, we have studied a continuous two-person poker model that was introduced by Von Neumann and Morgenstern (1944). The betting structure of that model is fairly simple. Both players only have to make (at most) one decision. This decision is a choice between two possible actions. In fact, this structure turned out to be sufficiently simple to find the equilibria of the game analytically. When the betting structure becomes more complicated, such a direct approach to find equilibria in the game with continuous card distributions is not always possible. In this chapter we present a way to find an equilibrium in such a game by smartly using information from an equilibrium in a related discrete game.

The study of zero-sum continuous poker games for two players with uniform hand distributions and alternate bidding goes back to Borel (1938) and Von Neumann and Morgenstern (1944). Borel (1938) discussed a game called "La Relance", while Von Neumann and Morgenstern (1944) presented the equilibrium of a game with the betting structure that we have seen in chapter 5. We display the betting tree of this game again in Figure 6.1. Player 1 chooses between P(assing) and B(etting) and player 2 chooses between F(olding) and C(alling), only after player 1 has decided to bet. For a more extensive description of the model, we refer to the explanation of the game in section 5.2. The betting tree of La Relance differs from Von Neumann's model at one place: player 1 is not allowed to pass, but has the option to fold instead. Folding does not lead to a showdown; player 1 gives up the ante if he selects this option.

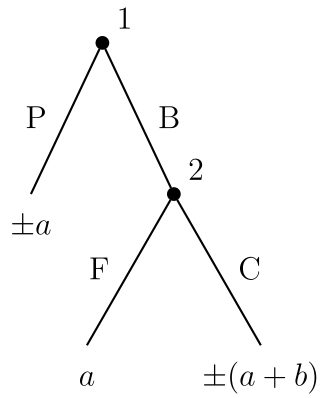


Figure 6.1: Betting tree of minipoker.

So, one of the choices player 1 can make in La Relance leads to a payoff that does not depend on the relative values of the players' hands. This makes the analysis of Borel's model easier than the analysis of Von Neumann's model. Perhaps that is the reason that the literature on continuous poker games with uniform hand distributions mainly consists of extensions of Borel's model. The simplicity of the analysis seems a more plausible explanation than the realism of the model itself. After all, a game in which player 1 can only choose between betting and folding is rather restrictive for the first player.

Both models, as well as all extensions explored in the literature discussed below, are inspired by the game of drawpoker, for which the general setting is described in section 2.10.2. Each of the models simplifies the game so that it is simple enough to analyze, thereby keeping some of the essential aspects of the original game. A detailed comparison of the poker models Von Neumann and the model of Borel is made by Ferguson and Ferguson (2003). Their paper also studies a game with a betting tree that forms a link between the models.

As already mentioned, most extensions are made to the model of Borel. An early example is the article of Bellman and Blackwell (1949), in which player 1 can choose from two different bet sizes. In the model of Bellman (1952), player 2 is allowed to raise after a bet by player 1. Karlin and Restrepo (1957) present further extensions of the betting tree. For an overview of these and other extensions made in the fifties, we refer to Karlin (1959, chapter 9). Instead of altering the betting tree, Sakaguchi (1984) discusses a different card distribution as a modification of Borel's model by studying a multi-card form of La Relance: each player receives m cards, of which he can use the highest in the game.

An interesting extension of Von Neumann's model is presented by Newman (1959), who allows player 1 to bet any nonnegative bet size. Sakaguchi (1985b) study the mathematical details of Newman's model. Sakai (1986) analyzes a restricted version of this model, in which player 1's betting choices are bounded from above by some positive real number. Cutler (1975) studies a completely different extension, allowing an unlimited number of raises under pot-limit rules.

All these extensions, of both models, have one aspect in common: the left side of the betting tree is kept very simple. Let us have another look at Von Neumann's betting tree, that is displayed in Figure 6.1. If player 1 decides to pass, then player 2 is not called to play at all. This part of the structure is crucial for the simplicity of the analysis in chapter 5. If player 1 passes with a given card u, his expected payoff does not depend on the strategy of player 2 and can therefore serve as an easy reference point when determining the optimal strategy of player 1. As soon as we add a decision for player 2 to the betting tree at the point where player 1 has passed, the complexity of the analysis increases. The extensions described above already required lengthy mathematical derivations to find equilibria analytically. And, unfortunately, even with fairly simple extensions of the left side of the betting tree, it is not possible anymore to use this approach to find optimal strategies.

In this chapter, we present an idea to solve this problem: we show that it is still possible to find equilibria of such continuous poker games, if we can make a good guess of the form of the optimal strategies. It is not difficult to check if an equilibrium of the guessed form exists, by using equalizing techniques on each player's payoff functions corresponding to different actions. The problem is then to come up with a good guess. For this aim, we use information about the equilibrium of a game with a discrete card distribution, with the same betting tree. We expect two games with the same betting tree, but with a different card distribution, to share certain features of optimal play.

Until recently, finding an equilibrium in a simple discrete poker model was a difficult task. Even analyzing a discrete version of Von Neumann's model with more than ten cards in the deck was hardly possible. The standard approach of computing equilibria, using the normal form, has the disadvantage that the normal form bimatrix grows exponentially with the size of the game tree. Since computer memory is bounded, the only way to solve large discrete games was to simplify the computations by smartly using the structure of strategies and payoffs of a specific game. Fortunately, the computational possibilities have increased by the introduction of the sequence form. This alternative way of representing games uses a matrix that grows only linearly with the size of the tree. This reduction in the size of the representation of a game makes the sequence form an appropriate tool to use in equilibrium computations. Computing an equilibrium of a discrete poker game with Von Neumann's betting tree in which, say, 100 different cards are used, is no problem anymore.

The strategy spaces in the continuous poker models are rather complex. However, the optimal strategies in these games seem to have a surprisingly simple structure. Randomization may not be needed at all. Restrepo (1964) studies a general class of continuous two-person games in which the poker games are contained. The strategies are functions of the outcome of a random move. From the characterization of the optimal strategies he concludes that in these games there must exist optimal strategies with an easy structure. We show in section 6.3 that it is at least possible to approximate the expected payoff guaranteed by any equilibrium strategy arbitrarily close using a pure strategy. Our result is a thoroughly reformulated specific case of a more general result proved in Bellman and Blackwell (1949).

The analysis of poker is interesting for a wider audience than just for poker players. The game provides an excellent domain for investigating problems of decision making under uncertainty. In particular, the role of information in the game is interesting. Sakaguchi (1985a, 1993) discusses several aspects of information in poker. Another field of research for which poker brings challenges, is artificial intelligence. Billings et al. (2002) describe the questions and problems that came across during their attempts to develop a computer program for playing two-player Texas Hold'em. This direction of research has also led to more theoretical results concerning optimal play. Billings et al. (2003) address the computation of the first complete approximations of game-theoretic optimal strategies for two-person Texas Hold'em. They combine linear programming solutions to abstracted versions of parts of the game to obtain an approximation of an optimal strategy for the complete game. In this way, a computer program is created that is able to defeat strong human players in two-person Texas Hold'em and is competitive against world-class opponents. Interesting to note is that the focus in this stream of research is not only on theoretically optimal play, but also on exploiting weaknesses of opponents.

Other, more general references on various probabilistic and strategic aspects of poker games are Zadeh (1977), Epstein (1977) and Scarne (1990).

The set up of the remainder of this chapter is as follows. In section 6.2, we present the two different approaches for finding equilibria in discrete poker games: the normal form and the sequence form. Section 6.3 discusses the structure of equilibria in poker models with a continuous distribution of cards. In section 6.4, we show how we can use equilibrium information from a discrete poker model to find an equilibrium in the continuous poker model with the same betting structure. A poker model, of which the betting tree is an extension of the betting tree of minipoker, is used to illustrate the ideas. The chapter is concluded by section 6.5, which presents an equilibrium analysis of a continuous poker game with a raising possibility for both players. For this game we also determine the relative skill level according to the framework introduced in chapter 2.

6.2 Discrete poker models

This section discusses an alternative for the continuous hand distributions in simplified models of poker games. The alternative is to decrease the number of possible hands a player can receive to a relatively small number. Unfortunately, even with a small number of hands, analyzing such a discrete poker model is a difficult task. The traditional way of finding equilibria, using the normal form, has the disadvantage that the normal form matrix grows exponentially with the number of hands. Another way of solving such a game uses the sequence form, an alternative representation of the game. The matrix of the sequence form grows only linearly with the number of hands. This reduction in size of the representation has advantages in the computations of equilibria. We use a discrete version of Von Neumann's poker model from chapter 5 to demonstrate the computations, both for the normal form (section 6.2.2) and for the sequence form (section 6.2.3). In the next section, we give the details of the game.

6.2.1 Minipoker with three cards

We use the two-person game *minipoker*, which was also subject of study in section 4.5 and in chapter 5, to illustrate the equilibrium computations of discrete poker games. Minipoker is played by two players, player 1 and player 2. We focus here on the variant of the game with three cards, namely A(ce), K(ing) and Q(ueen). As usual, A is higher than K, and Q is the lowest card of the three. Before play starts, both players donate one unit to the stakes. After (re)shuffling the deck of cards, each player is dealt one card. Each player knows his own card, but not the card of his opponent. Thus, the card which remains in the deck is not shown to either of the players. Player 1 starts the play and has to decide between P(assing) and B(etting). If he decides to pass, a showdown follows immediately. In the showdown both cards are compared and the player with the highest card gets the stakes. If player 1 decides to bet, he has to add one extra unit to the stakes. Subsequently, player 2 has to decide between F(olding) and C(alling). If he decides to fold, player 1 gets the stakes. If player 2 decides to call, he also has to add one extra unit to the stakes and a showdown follows. Figure 6.2 displays the extensive form of minipoker with three cards. The displayed payoffs are the payoffs for player 1. A dotted line between two points indicates that a player cannot distinguish these points. Since he does not know the card of the opponent, he cannot know in which node he is when he makes his decision. In game theoretical terms, the nodes that are connected by a dotted line form an information set.
In the following two sections we discuss the computation of equilibria of the game, both using the normal form and the sequence form. The description of the normal form computations will be rather brief, while we give a more extensive explanation of the use of the sequence form.

6.2.2 Normal form computations

The traditional way of finding equilibria of a game like minipoker uses the normal form. For this solution we have to construct the payoff matrix N, which lists player 1's payoffs for all possible combinations of pure strategies of

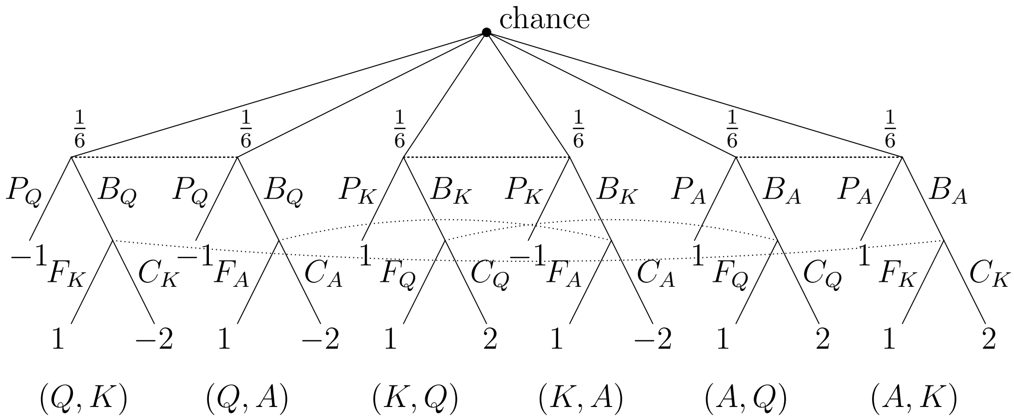


Figure 6.2: The extensive form of two-person minipoker.

the players. Both players have three information sets, with two actions in each set. This results in $2^3 = 8$ pure strategies for both players. The payoff matrix is as follows:

N	$F_QF_KF_A$	$F_QF_KC_A$	$F_QC_KF_A$	$F_QC_KC_A$	$C_QF_KF_A$	$C_QF_KC_A$	$C_QC_KF_A$	$C_QC_KC_A$
$P_QP_KP_A$	0	0	0	0	0	0	0	0
$P_QP_KB_A$	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
$P_QB_KP_A$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$P_QB_KB_A$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{3}$
$B_QP_KP_A$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
$B_QP_KB_A$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	0
$B_QB_KP_A$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{7}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{3}$
$B_QB_KB_A$	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	1	0

As in the extensive form in Figure 6.2, the subscripts in the strategy names indicate the card with which the player takes the given actions. For example, with the strategy $P_QB_KP_A$, player 1 bets with a king, but passes otherwise.

Using the payoff matrix N, we can find the equilibrium of the game as the minimum of the following linear programming problem:

$$\begin{array}{rrrcl} \min\limits_{y,v} & & v & & \\ \text{subject to} & Ny & -v & \leq & 0, \\ & y & & \geq & 0, \\ & e^\top y & & = & 1. \end{array}$$

For any combination of $\tilde{y}$ and $\tilde{v}$ that minimizes the objective function of this LP, the vector $\tilde{y}$ is an optimal (min-max) strategy $\tilde{y}$ for player 2, while the shadow prices corresponding to the inequality constraints form an optimal (max-min) strategy for player 1. The variable $\tilde{v}$ represents the amount that player 2 has to pay to player 1, the value of the game. We find that optimal strategies $\tilde{x}$ and $\tilde{y}$ (for player 1 and player 2 respectively) are given by

$$\tilde{x} = \begin{pmatrix} 0 \\ \frac{2}{3} \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{y} = \begin{pmatrix} 0 \\ \frac{2}{3} \\ 0 \\ \frac{1}{3} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Player 1 has to play $\frac{2}{3}P_Q P_K B_A + \frac{1}{3}B_Q P_K B_A$. So it is optimal for him to pass with a king, to bet with an ace and to randomize between passing and betting with a queen such that the probability of betting is equal to $\frac{1}{3}$. Player 2 should play $\frac{2}{3}F_Q F_K C_A + \frac{1}{3}F_Q C_K C_A$. So he should fold with a queen, call with an ace and with a king he should call with probability $\frac{1}{3}$. In fact, it is not difficult to check that these are the unique optimal strategies in this game.

6.2.3 Sequence form computations

The sequence form is an alternative strategic description of a game. This description can only be given for games with perfect recall: if two nodes are in the same information set for a player, this player's moves needed to arrive at any of these two nodes must be the same. Rather than planning a move for every information set, a player considers for each node in the game tree the choices he needs to make so that that node can be reached. These choices together form a *sequence* for the player. In the sequence form, S_i is the set of sequences of player i defined by all nodes of the game tree. This set replaces his set of pure strategies in the normal form. A single sequence can be represented by a set: the set of actions a player has to take to reach the node. Then the sequence needed for a player to reach the root node is the empty set. Each player has at most as many sequences as the game tree has nodes, so the number of sequences is linear in the size of the tree. Actually, the upper bound of the number of sequences for a player is determined by the number of information sets he faces. After all, the sequences leading to two different nodes in the same information sets are the same in a game with perfect recall.

In the normal form, a player can pick a pure strategy. In the sequence form, a player cannot just pick a single sequence. In Figure 6.2, for example, player 1 has to decide between P_Q and B_Q, but also between P_K and B_K and between P_A and B_A. Choosing P_Q, P_K and P_A, just like in the pure strategy $P_Q P_K P_A$ of the normal form game, assigns the *realization probabilities* $1, 1, 0, 1, 0, 1, 0$ to the sequences $\emptyset, P_Q, B_Q, P_K, B_K, P_A, B_A$ respectively. These realization probabilities can be ordered in a vector x, which we call a *realization plan.* A player can use randomization in one or more of his choices between sequences, but a realization plan for player 1 should satisfy the following equations:

$$\begin{aligned} x(\emptyset) &= 1, \\ x(\emptyset) &= x(P_Q) + x(B_Q), \\ x(\emptyset) &= x(P_K) + x(B_K), \\ x(\emptyset) &= x(P_A) + x(B_A). \end{aligned} \tag{6.1}$$

For player 2 we can also construct a (column) vector y, specifying his realization plan. This realization plan should satisfy the equations belows.

$$\begin{aligned} y(\emptyset) &= 1, \\ y(\emptyset) &= y(F_Q) + y(C_Q), \\ y(\emptyset) &= y(F_K) + y(C_K), \\ y(\emptyset) &= y(F_A) + y(C_A). \end{aligned} \tag{6.2}$$

We use the notation of Von Stengel (1996) when constructing the optimization problem of which the solution is an equilibrium of the game.

The length of a player's realization plan is equal to the sum over all information sets of this player of the number of actions in the information set plus one additional element, corresponding to the empty sequence. This last element is always equal to one. In minipoker, player 1 has three information sets. In each of them he can choose between two actions, so the length of x is $3 \times 2 + 1 = 7$. The entries of x are real numbers between zero and one. Similarly, $y \in \mathbb{R}^7$ and $0 \leq y_i \leq 1$ for all $i \in \{1, \ldots, 7\}$.

The payoff matrix A contains the expected payoff for player 1 for each pair of sequences that leads to a terminal node. Player 1's payoff at the terminal node is multiplied by the probabilities of chance moves on the path from the root to this terminal node to obtain the expected payoff. For all combinations of sequences that do not lead to a terminal node, the corresponding entry in A is zero. For minipoker with three cards, the payoff matrix A is as follows. The subscripts of the sequences indicate the information sets to which they correspond.

A	$\emptyset$	F_Q	C_Q	F_K	C_K	F_A	C_A
$\emptyset$	0	0	0	0	0	0	0
P_Q	0	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
B_Q	0	0	0	$\frac{1}{6}$	$-\frac{2}{6}$	$\frac{1}{6}$	$-\frac{2}{6}$
P_K	0	$\frac{1}{6}$	$\frac{1}{6}$	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$
B_K	0	$\frac{1}{6}$	$\frac{2}{6}$	0	0	$\frac{1}{6}$	$-\frac{2}{6}$
P_A	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0
B_A	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	0	0

Player 1 chooses a realization plan, x, for the rows of the matrix. The realization plan y, chosen by player 2, indicates the realization probabilities for the columns of A. The vector x should satisfy the equalities given in equation (6.1). These equalities can be represented by the expression $Ex = e$, with

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad e = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Similarly, following the equalities in equation (6.2), y should satisfy $Fy = f$, with $F = E$ and $f = e$. Furthermore, for both players the realization plans should be nonnegative.

We follow the derivation of Von Stengel (1996, p. 233–234) to construct the linear program with which we can compute the equilibrium. Let us first consider the problem of finding a best response of player 2 against a given realization plan x of player 1. This is equivalent to solving the following linear program, in which $B = -A$:

$$\begin{array}{rrcl} \max\limits_{y} & (x^\top B)y & & \\ \text{subject to} & Fy & = & f, \\ & y & \geq & 0. \end{array} \tag{6.3}$$

The number of variables in the dual of this LP is equal to the number of information sets of player 2 plus one. These variables are unconstrained and are represented by the vector q. The dual LP to (6.3) is

$$\begin{array}{rlcl} \min\limits_{q} & q^\top f & & \\ \text{subject to} & q^\top F & \geq & x^\top B. \end{array} \tag{6.4}$$

Analogously, finding a best response x of player 1, given that player 2 plays according to y, is equivalent to solving the following problem:

$$\begin{array}{rlcl} \max\limits_{x} & x^\top(Ay) & & \\ \text{subject to} & x^\top E^\top & = & e^\top, \\ & x & \geq & 0. \end{array} \tag{6.5}$$

The dual problem to (6.5) uses the unconstrained vector p of which the length is equal to the number of information sets of player 1 plus one and reads

$$\begin{array}{rlcl} \min\limits_{p} & e^\top p & & \\ \text{subject to} & E^\top p & \geq & Ay. \end{array} \tag{6.6}$$

In order to find an equilbrium, both x and y have to be treated as variables. In this way, the objective functions in (6.3) and (6.5) are not linear anymore. However, a zero-sum game can still be solved by a linear program. Note that the LP (6.5) and its dual (6.6) have the same optimal value $e^\top p$. If player 2 can vary y, he will try to minimize this value: an optimal choice of y will be a min-max strategy. In order to be a well-defined realization plan, y has to satisfy $y \geq 0$ and $Fy = f$. This defines the modified LP

$$\begin{array}{rrrcr} \min\limits_{y,p} & & e^\top p & & \\ \text{subject to} & -Ay & +E^\top p & \geq & 0, \\ & -Fy & & = & -f, \\ & y & & \geq & 0. \end{array} \tag{6.7}$$

Again, consider the dual of this LP:

$$\begin{array}{rllcr} \max\limits_{x,q} & & -q^\top f & & \\ \text{subject to} & x^\top(-A) & -q^\top F & \leq & 0, \\ & x^\top E^\top & & = & e^\top, \\ & x & & \geq & 0. \end{array} \tag{6.8}$$

In a zero-sum game, $-A = B$, so (6.8) is just (6.4) but with variables q and x, subject to the constraints of (6.5). This LP can be interpreted as the problem of finding a min-max strategy for player 1. Von Stengel (1996, Theorem 5.1) proves that the optimal solutions to (6.7) and (6.8) indeed define an equilibrium for the zero-sum game.

When we solve the linear program given in (6.7), we find that the game value is equal to $\frac{1}{18}$ and that optimal realization plans $\tilde{x}$ and $\tilde{y}$ are given by

$$\tilde{x} = \begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \tilde{y} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \frac{2}{3} \\ \frac{1}{3} \\ 0 \\ 1 \end{pmatrix}.$$

The realization plans are easily interpretable in terms of behavioural strategies. For player 1 it is optimal to pass with a king and to bet with an ace. With a queen, he has to bet with probability $\frac{1}{3}$. Player 2 should fold with a queen, call with an ace and with a king he should call with probability $\frac{1}{3}$. Of course, the conclusion is the same as in section 6.2.2, since the optimal strategies in this game are unique.

6.2.4 More cards in the deck

Borm and Van der Genugten (1996) computed the skill level of the discrete version of minipoker. They carried out the computations for the game with three cards, which we have studied in the preceding sections, but also for other game variants, in which the number of cards varied from four to ten. The reason that the maximum number of cards used was ten, is the complexity of the equilibrium computations using the normal form.

Since constructing the matrices that we need for the sequence form computations does not require too much effort for variants of minipoker with a larger number of cards, we can check what computational advantages this alternative approach brings us. It turns out to be not too difficult to find optimal behavioural strategies for games with, say, up to 52 cards.[1] Therefore, it would now be possible to carry out the skill analysis for minipoker with a complete deck of cards. Although that is not the aim of this section, we want to include the results of the equilibrium computations for minipoker with 52 cards. Since tables with equilibrium strategies for this game are rather large, we think one can gain the most insight by plotting the strategies for both players. This is what we have done in Figure 6.3 and Figure 6.4 for player 1 and 2, respectively. The value of the game is approximately 0.0999.

Some more computations show that the value of the game approximates $\frac{1}{10}$ as the number of cards increases. This is the value of the continuous variant of minipoker, studied in chapter 5, in which the dealing of poker hands is approximated by drawing (independently) two random variables from the uniform distribution on the interval $[0, 1]$. Also the strategies in the discrete game with a full deck of cards approach the strategies in the continuous game.

When we compare the game with 52 cards with the original game with three cards, we recognize in both games the same structure of the equilibrium strategies. Player 1 has to bet with high cards, pass with intermediate cards

[1]To be more precise, using MATLAB, the equilbrium computations for minipoker with a full deck of cards take less than four seconds on a Pentium III computer with a clock speed of 700 Mhz. With two decks of cards, it still takes less than fourteen seconds. Using the normal form, equilibrium computations for minipoker with two full decks of cards are impossible: the number of rows and columns of the game matrix N exceeds $2 \cdot 10^{31}$, whereas MATLAB already uses about 800 megabytes of memory for a matrix with 10^4 rows and 10^4 columns. For the game with 11 cards, over 33 megabytes of memory is needed to store the normal form matrix.

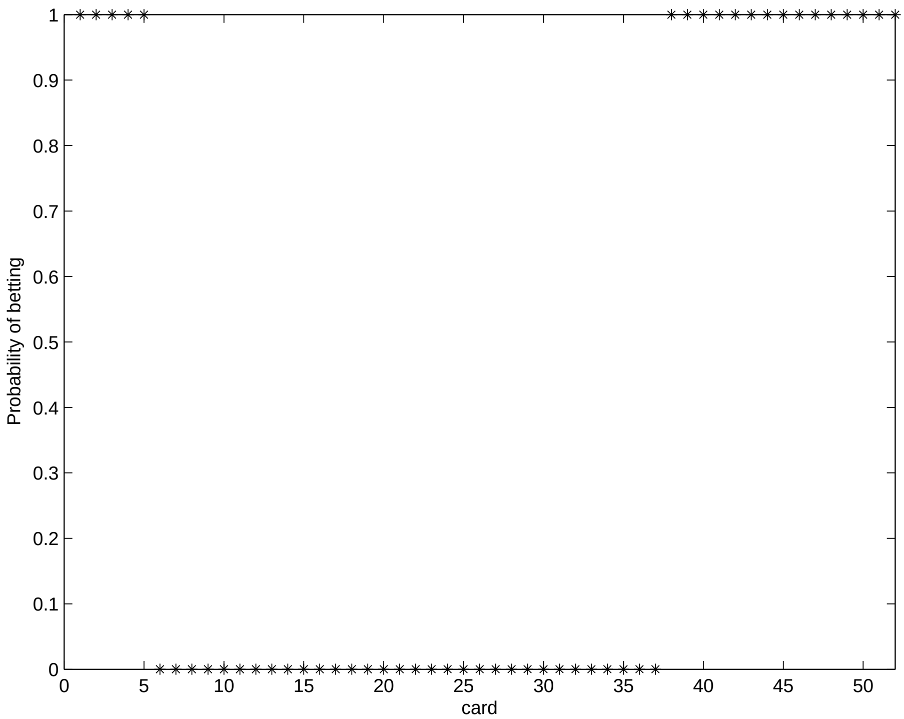


Figure 6.3: An equilibrium strategy for player 1 for minipoker with 52 cards.

and bluff (bet) with low cards. Player 2 has to fold with low cards and call with high cards. With intermediate cards, he must randomize between these two actions, in such a way that he increases his calling probability as his card gets higher.

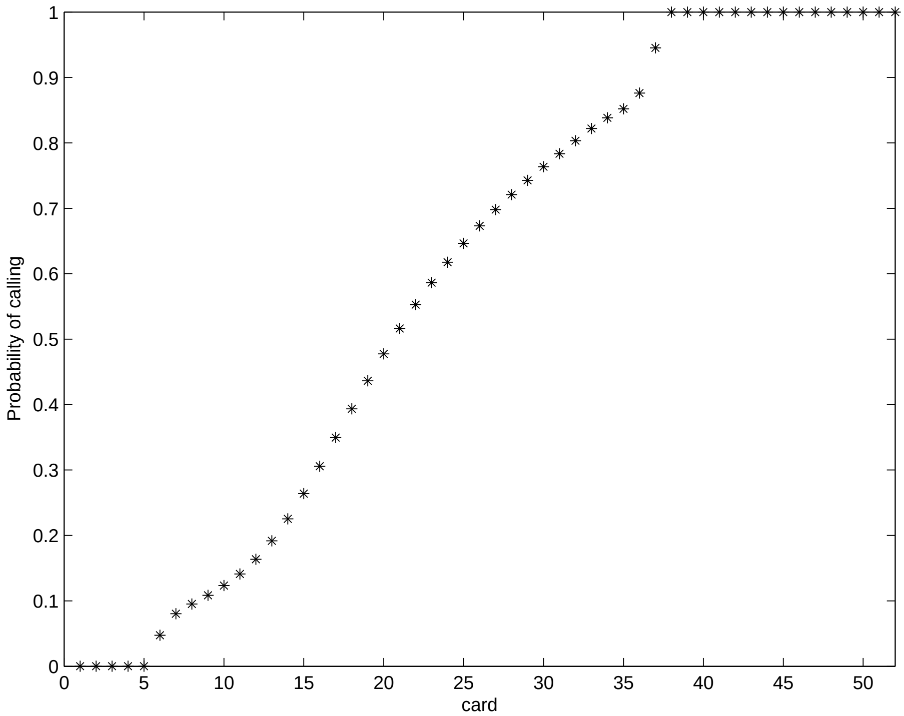


Figure 6.4: An equilibrium strategy for player 2 for minipoker with 52 cards.

6.3 Equilibrium structure

For poker games with a more complex betting structure than minipoker the equilibrium analysis of the continuous variant may be difficult. However, it may not be necessary to consider mixed or behavioural strategies in the analysis. Bellman and Blackwell (1949) have shown that pure strategies play an important role in a class of games which contains our poker games.

In the class of zero-sum continuous poker games that we discuss, the deal of cards has the same structure as in minipoker. Player 1's hand is the value u of a continuous random variable U and player 2's hand is the value v of a continuous random variable V. U and V are independently, identically distributed on $[0,1]$ according to the uniform distribution function. Player 1 does not know the value of v and player 2 does not know the value of u. A pure strategy of player 1 is a function $s : [0,1] \to S$, where $S = \{s_1, ..., s_M\}$ is a finite set. Similarly, a pure strategy for player 2 is a function $t : [0,1] \to T$, where $T = \{t_1, ..., t_N\}$ is a finite set. For elements of these sets S and T, one can think of combinations of decisions for all information sets of a player. When the number of information sets for a player is finite, and the number of actions that can be chosen in each information set is finite, then the set S (or T) is finite as well. A mixed strategy in such a game is a difficult thing. However, we only consider games with perfect recall. For these games, Aumann (1964) has shown that we can restrict attention to behavioural strategies without ignoring strategic possibilities for the players. A behavioural strategy for player 1 is a function $f : [0,1] \to \Delta(S)$, where $\Delta(S)$ is the set of all probability distributions on S. A behavioural strategy f can be seen as a combination of M functions $p_1(u), \ldots, p_M(u)$, such that for every $u \in [0,1]$,

$$\begin{aligned} p_k(u) &\geq 0 \quad \text{for } 1 \leq k \leq M, \\ \sum_{k=1}^{M} p_k(u) &= 1. \end{aligned}$$

Here, $p_k(u)$ specifies the probability of choosing $s_k \in S$ with a given u. Similarly, a behavioural strategy for player 2 is a function $g : [0,1] \to \Delta(T)$ that can be written as a combination of N functions $q_1(v), \ldots, q_N(v)$, such that for every $v \in [0,1]$,

$$\begin{aligned} q_\ell(v) &\geq 0 \quad \text{for } 1 \leq l \leq N, \\ \sum_{\ell=1}^{N} q_\ell(v) &= 1. \end{aligned}$$

$\mathcal{F}$ and $\mathcal{G}$ denote the sets of all behavioural strategies of player 1 and player 2 respectively. Pure strategies can of course be written as behavioural strategies with degenerate probability distributions.

For given u and $v \in [0,1]$, $s_k \in S$ and $t_\ell \in T$, $H(u,v,s_k,t_\ell) \in \mathbb{R}$ is the payoff for player 1. In a poker game, H is bounded and, for given s_k and t_ℓ, H is constant on the sets

$$\{(u,v) \in [0,1] \times [0,1] \mid u < v\} \text{ and } \{(u,v) \in [0,1] \times [0,1] \mid u > v\}.$$

For a fixed behavioural strategy $g \in \mathcal{G}$ of player 2, a fixed choice of $s_k \in S$ and a given card $u \in [0,1]$, we write the expected payoff for player 1 as

$$E_1(u, s_k, g) = \sum_{\ell=1}^{N} \int_0^1 H(u,v,s_k,t_\ell) q_\ell(v) dv.$$

The expected payoff $V(f;g)$ for player 1 for a given pair of behavioural strategies $f \in \mathcal{F}$ and $g \in \mathcal{G}$, can be written as

$$V(f,g) = \sum_{k=1}^{M} \int_0^1 E_1(u,s_k,g) p_k(u) du.$$

The following theorem states that the payoff for player 1 that is guaranteed by any behavioural strategy in a zero-sum continuous game where the payoffs satisfy some conditions can be approximately guaranteed by a pure strategy as well. It is a special case of a theorem of Bellman and Blackwell (1949).

Theorem 6.3.1 *For a zero-sum continuous poker game as described above, for every behavioural strategy $f \in \mathcal{F}$ for player 1 and every $\varepsilon > 0$, there is a pure strategy s, such that for every behavioural strategy $g \in \mathcal{G}$ of player 2*

$$V(f,g) \leq V(s,g) + \varepsilon. \tag{6.9}$$

Proof. Since H is uniformly bounded, we can find $K \in \mathbb{N}$, such that $|H| \leq K$. Let $\varepsilon > 0$, define $n = \lceil \frac{2K}{\varepsilon} \rceil$ and choose $\delta = \frac{1}{n}$. Divide $[0,1]$ into disjoint subintervals $I_1, \ldots, I_n$, such that the Lebesgue measure $\lambda(I_j) = \delta$ for all $1 \leq j \leq n$. Recall that $p_k(u)$ specifies the probability of choosing $s_k \in S$ with a given u. Divide I_j into M disjoint intervals $I_{j1}, \ldots, I_{jM}$, so that $\lambda(I_{jk}) = \int_{I_j} p_k(u)du$ for all $1 \leq j \leq n$ and all $1 \leq k \leq M$. This division is possible, since $\sum_{k=1}^{M} p_k(u) = 1$ for each $u \in [0,1]$. Then we claim that the pure strategy

$$\tilde{s}(u) = s_k \quad \text{if } u \in I_{jk} \quad (1 \leq j \leq n,\ 1 \leq k \leq M)$$

is the required approximation to f. Let $g \in \mathcal{G}$. If we define

$$D = V(f,g) - V(\tilde{s},g),$$

then we have to show that $D \leq \varepsilon$. We define D_{jk} as follows:

$$D_{jk} = \int_{I_j} E_1(u, s_k, g) p_k(u) du - \int_{I_{jk}} E_1(u, s_k, g) du.$$

Then we can write

$$D = \sum_{j=1}^{n} \sum_{k=1}^{M} D_{jk},$$

and clearly

$$|D| \leq \sum_{j=1}^{n} \sum_{k=1}^{M} |D_{jk}|. \tag{6.10}$$

As a result of the specific properties of the function H, we know that for any u and u' in $[0,1]$ the difference $\Delta = E_1(u', s_k, g) - E_1(u, s_k, g)$ is bounded. To see this, let $u, u' \in [0,1]$ with $u < u'$. The expected payoff $E_1(u, s_k, g)$ for player 1 can be expressed as

$$\begin{aligned}
E_1(u, s_k, g) &= \sum_{\ell=1}^{N} \int_0^1 H(u, v, s_k, t_\ell) q_\ell(v) dv \\
&= \sum_{\ell=1}^{N} \left(\int_0^u H(u, v, s_k, t_\ell) q_\ell(v) dv + \int_u^{u'} H(u, v, s_k, t_\ell) q_\ell(v) dv \right) \\
&\quad + \sum_{\ell=1}^{N} \int_{u'}^1 H(u, v, s_k, t_\ell) q_\ell(v) dv.
\end{aligned}$$

Since H is constant on the set

$$\{(u,v) \in [0,1] \times [0,1] \mid u > v\},$$

we know that

$$\sum_{\ell=1}^{N} \int_0^u H(u,v,s_k,t_\ell) q_\ell(v) dv = \sum_{\ell=1}^{N} \int_0^u H(u',v,s_k,t_\ell) q_\ell(v) dv.$$

Similarly,

$$\sum_{\ell=1}^{N} \int_{u'}^1 H(u,v,s_k,t_\ell) q_\ell(v) dv = \sum_{\ell=1}^{N} \int_{u'}^1 H(u',v,s_k,t_\ell) q_\ell(v) dv$$

and, as a result

$$\Delta = \sum_{\ell=1}^{N} \int_u^{u'} \left(H(u',v,s_k,t_\ell) - H(u,v,s_k,t_\ell)\right) q_\ell(v) dv.$$

Since $|H(u,v,s_k,t_\ell)| \leq K$, we know that $\Delta \leq 2K(u'-u)$. In particular, if $u'-u \leq \delta$, then $\Delta \leq 2K\delta$. Therefore, if we define U_{jk} and L_{jk} to be the maximum and minimum of $E_1(u,s_k,g)$ over the interval I_j, then we know the following inequality holds for all j and all k:

$$|D_{jk}| \leq \left| U_{jk} \int_{I_j} p_k(u) du - L_{jk} \int_{I_{jk}} 1 du \right| = (U_{jk} - L_{jk}) \lambda(I_{jk}) \leq 2K\delta\lambda(I_{jk}). \tag{6.11}$$

Combining equations (6.10) and (6.11) gives us the required result:

$$|D| \leq 2K\delta \sum_{j=1}^{n} \sum_{k=1}^{M} \lambda(I_{jk}) = 2K\delta = 2K \frac{1}{\lceil \frac{2K}{\varepsilon} \rceil} \leq \frac{2K}{\frac{2K}{\varepsilon}} = \varepsilon.$$

□

Crucial for the proof is that the opponent's strategy cannot depend on the value of player 1's card, u.

Repeating the steps in the proof for player 2, which can be done in a straightforward way, one can show that for any behavioural strategy of player 2 there exists a pure strategy with which he can approximately attain the same expected payoff against any behavioural strategy of player 1. So, the theorem shows that players in a poker game can approximate their equilibrium payoff up to an arbitrarily small amount using pure strategies.

The intuition behind this result is as follows. A player may want to deceive or confuse his opponent by using randomization in his decisions. However, since the number of cards he can be dealt is infinite, he can just as well confuse the other player by smartly distributing his actions over the cards. For example, instead of betting and passing with equal probability with "low" cards, he can bet with half of the low cards and pass with the other half.

Although we have not been able to prove it, our conjecture is that the special structure of the payoffs in continuous poker games ensures that mixed strategies are not needed for equilibrium play. We keep this conjecture in mind, when we search for equilibrium strategies for a particular poker game in section 6.4.

6.4 Cassidy's poker model

In this section, we study a poker game with a more symmetric betting tree than minipoker. Cassidy (1998) finds optimal strategies for the continuous variant of this game in an analytical way. The current section uses this specific game to illustrate our approach of finding optimal strategies in a continuous model using information from a related discrete model. In section 6.4.1 we describe the betting structure of the game, while in section 6.4.2 we present optimal strategies of the discrete variant of this game with 52 cards. Sections 6.4.3 and 6.4.4 are devoted to the continuous version of the game.

6.4.1 Game description

In this section we describe the game that is studied by Cassidy (1998). The betting structure of this game is related to the minipoker game that we have discussed in chapter 5 and in section 6.2. The structure of the betting in this game is displayed in Figure 6.5. Before play starts, both players add an ante a to the stakes. Then the players receive their cards. Player 1 opens the game at decision node 1_A, facing the choice between betting and passing. If he bets,

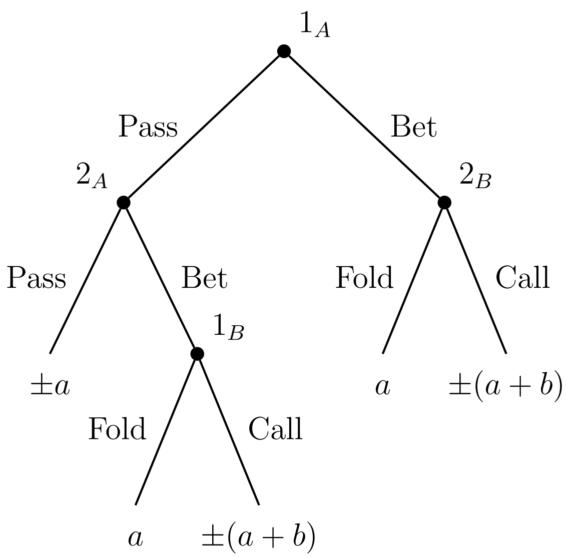


Figure 6.5: The decision tree of the poker game of Cassidy (1998).

he adds b to the stakes; moreover, player 2 then has to decide at decision node 2_B if he folds or calls. Folding means giving up the ante and calling leads to a showdown, after player 2 has put b in the stakes too. In the showdown, the player with the highest card wins the stakes. So far, it is exactly like the betting structure in minipoker. However, the two structures are different from the moment player 1 passes. In minipoker, a showdown occurs in this case. In Cassidy's model, player 2 gets the opportunity at node 2_A to bet (the same amount b that player 1 could have bet) or to pass. If he passes, the game ends in a showdown after all. If he bets, player 1 is called into play once more: at node 1_B, he has to choose between folding and calling. Since the betting in this tree happens after the dealing of the cards, different decision nodes in this betting tree correspond to different information sets.

Intuition tells us that the addition of branches to the decision tree of minipoker seems to open new perspectives for player 2. Cassidy's game is probably less favourable for player 1 than minipoker. In section 6.4.3 we will see whether this intuition is correct: there we search for optimal strategies for the continuous game with ante $a = 1$ and bet size $b = 2$, in which the hands of the players are uniformly and independently distributed on $[0, 1]$.

6.4.2 Optimal strategies in a discrete model

We have used the sequence form technique described in section 6.2.3 to find optimal strategies for a discrete version of the poker game of Cassidy (1998). The results are presented in this section.

In the discrete game, both players receive a card from a deck of 52 cards, numbered 1 to 52 and increasing in value. The cards are drawn without replacement. Inspiration for choosing the number of cards equal to 52 clearly comes from the size of a regular deck of cards. However, we want to remark here that we are not speaking about five-card poker hands. What is important, is that this number is sufficiently large to recognize the general structure of the optimal strategies. The ante is fixed at $a = 1$ and the bet size is chosen equal to $b = 2$. We present the optimal strategies we found for both players only in figures, without giving the obtained numerical details. An impression of the structure of the optimal strategies in the discrete game is enough. After all, it is this structure that is important in our search for optimal strategies in the continuous game.

Figures 6.6 and 6.7 describe optimal behaviour for player 1 in his information sets 1_A and 1_B (of Figure 6.5) respectively. Figures 6.8 and 6.9 show optimal behaviour for player 2 in the information sets 2_A and 2_B. We stress that these optimal strategies are not unique; they are just the optimal strategies we found with an algorithm that uses the sequence form. We discuss the structure of these optimal strategies by studying the results for each of the four information sets separately. In Figures 6.6 to 6.9, we distinguish two types of regions: regions where a pure action choice is played and regions where the optimal action involves randomization. In Figure 6.6, we recognize one region of the first type: player 1 has to pass with cards 10 to 35. With other cards, he has to randomize between passing and betting with certain probabilities, but there is no clear system in these betting probabilities. In information set 1_B, according to Figure 6.7 player 1 has to fold with cards 1 to 9, he has to call with cards 36 to 52 and he has to randomize with all cards in between. The probability of calling with these intermediate cards increases if the card gets higher.

The given optimal behaviour in information set 2_A, which is displayed in Figure 6.8, appears very definite. For player 2 it is clear when he has to pass

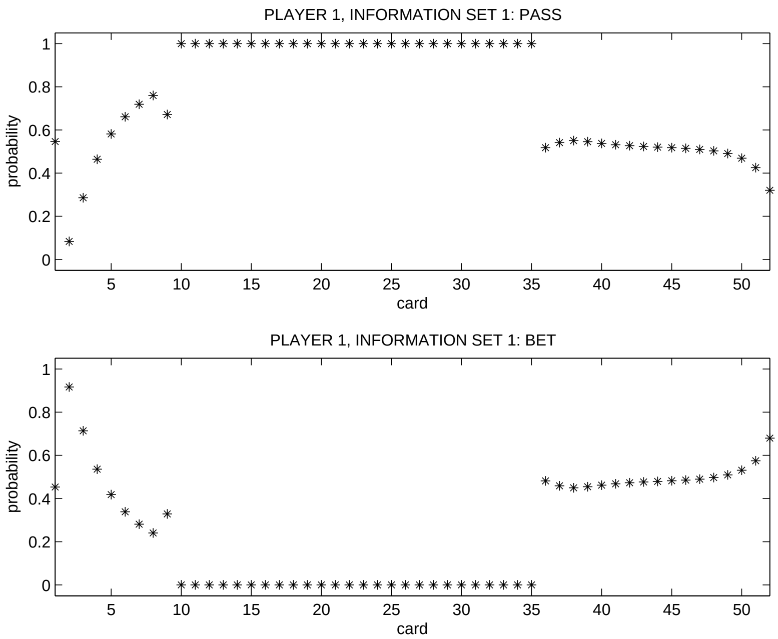


Figure 6.6: Optimal play for player 1 in his first information set (1_A) in the discrete game with 52 cards.

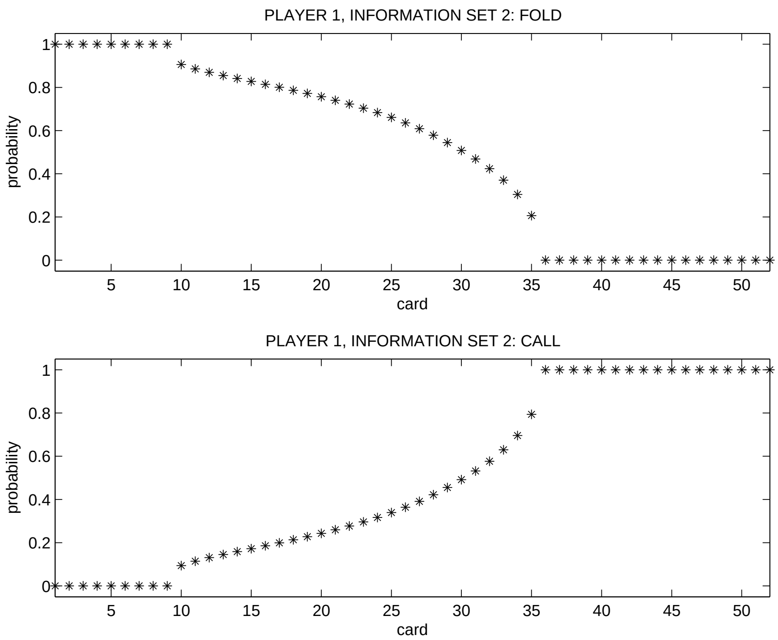


Figure 6.7: Optimal play for player 1 in his second information set (1_B) in the discrete game with 52 cards.

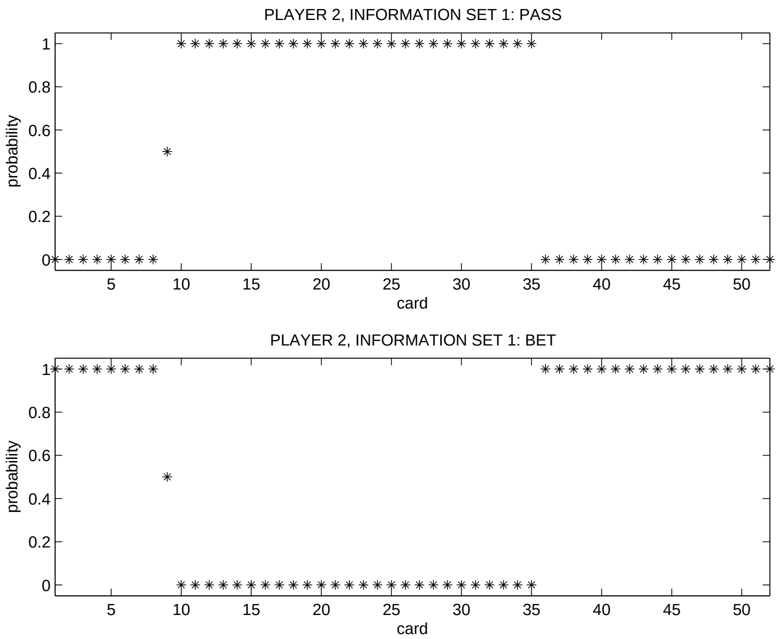


Figure 6.8: Optimal play for player 2 in his first information set (2_A) in the discrete game with 52 cards.

and when he has to bet: with low cards (1 to 8) and with high cards (36 to 52) he has to bet, with intermediate cards (10 to 35) he has to pass and with one card (9) he has to randomize with almost equal probabilities between passing and betting. The optimal behaviour for player 2 in information set 2_B is very similar to the optimal behaviour for player 1 in information set 1_B: he has to fold with cards 1 to 9, he has to call with cards 36 to 52 and he has to randomize with all cards in between. The probability of calling with these intermediate cards increases if the card gets higher.

We have seen that the optimal strategies in Figures 6.6-6.9 require randomization. Since we want to find pure strategies that are optimal in the continuous game, we have to deal with this randomization in a smart way.

Consider, for example, the information set 1_A of player 1 (see Figure 6.6). With cards 10 to 35, he plays his first action, passing, with probability 1. With the other cards he randomizes between passes and betting. For all these card values, player 1 is indifferent between the two actions. For cards 1 to 9, player 1 bets with a probability of approximately 0.47 on average. The idea is to replace the decisions for this part of the deck with a simpler combination of

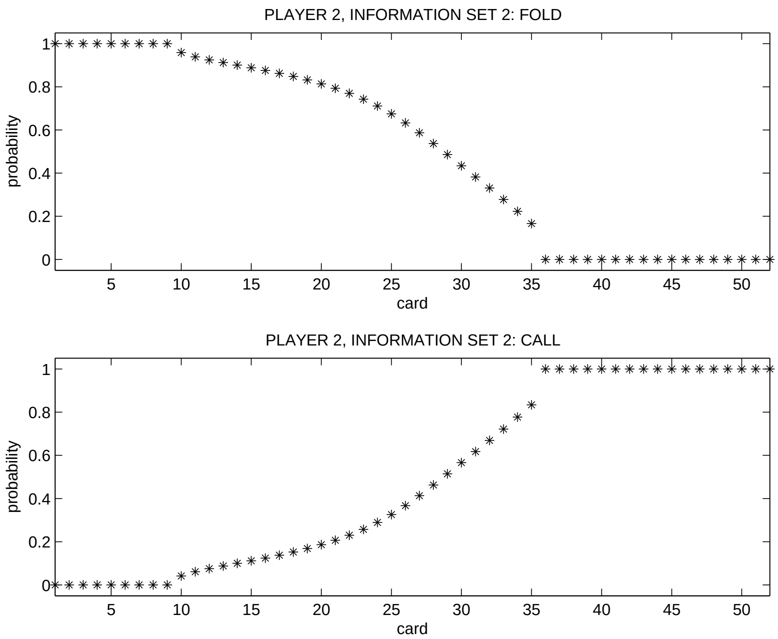


Figure 6.9: Optimal play for player 2 in his second information set (2_B) in the discrete game with 52 cards.

decisions without losing optimality of the strategy, as long as we do not change the average betting probability for this set of cards. A simple solution would be to bet with the lowest 47% of the lowest nine cards and to pass with the other 53%. In the discrete case such a solution requires randomization for at most one card within the interval under consideration. In the continuous case, we just interpret this interval as the lowest $\frac{9}{52}$ of the deck. So, in the case of draws from $U([0,1])$, it is the interval $[0, \frac{9}{52}]$. If player 1 bets with the lowest 47% of the hands in this interval, he bets roughly with all hands in $[0, 0.08]$ and he passes with all hands in $(0.08, \frac{9}{52}]$. For cards 36 to 52, the average betting probability is approximately 0.50, implying that player 1 should pass with approximately half of the cards in the interval $[\frac{35}{52}, 1]$. If we select the lowest half of the interval for passing, this implies that he passes with cards in $[\frac{35}{52}, 0.84]$ and he bets with all higher cards. Similar considerations for cards 10 to 35 in Figure 6.7, card 9 in Figure 6.8 and cards 10 to 35 in Figure 6.9 lead to the suggestions for good pure interval strategies in the continuous game that are shown in Figure 6.10.

Consider again the optimal behaviour for information set 1_B of player 1,

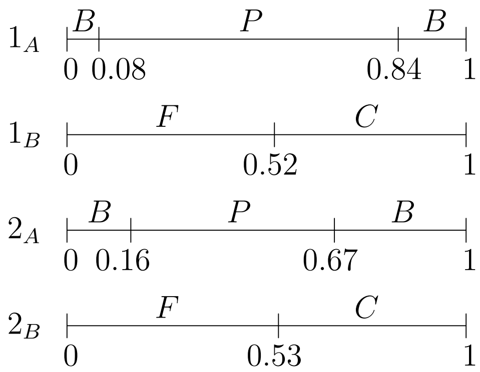


Figure 6.10: Suggestion for good pure interval strategies in the continuous game, following from the discrete game with 52 cards.

that is displayed in Figure 6.7. Player 1 has to call with cards 10 to 35 approximately 34% of the time. When we construct a pure interval equivalent, based on this information, and we have to choose between calling with the highest 34% in this interval or calling with the lowest 34%, we can use the shape of the optimal behaviour in Figure 6.7 as a guideline. The figure clearly suggests that the probability of calling should be higher with higher cards. For information set 2_B of player 2, our decision is also motivated by the fact that the probability of calling increases with the value of the cards. For information set 1_A, Figure 6.6 does not point this clearly in any direction: neither for cards 1 to 9 nor for 36 to 52 the probabilities are increasing or decreasing.

We choose the distribution of the actions over the cards in such a way that the number of "switches" in the resulting pure interval strategy is minimized. In Figure 6.10 we see that the number of switches between actions in information set 1_A is only two: from action B to P at 0.08 and back to B again at 0.84. We have chosen the cards in the interval $[0, \frac{9}{52})$ with which player 1 passes in such a way that the interval at which he passes is connected to the interval $[\frac{9}{52}, \frac{35}{52}]$, at which he has to pass anyway. A similar decision for the interval with the highest cards leads to the suggestion for the good strategy in the continuous game that is shown in Figure 6.10.

In the next section, we use the "sophisticated" guesses for optimal strategies from Figure 6.10 as a starting point for analyzing the continuous version of Cassidy's poker model.

6.4.3 A continuous card distribution

In the previous section, we have presented optimal strategies for Cassidy's poker model with ante $a = 1$ and bet size $b = 2$ for the game in which the players each get a card from a deck of 52 cards. One could expect the optimal strategies in the continuous game, in which the cards of both players are drawn independently from the uniform distribution on $[0, 1]$, to have roughly the same structure. To be more precise, one may expect that there exists an equilibrium in the continuous game in which the behaviour of the players is close to what Figure 6.10 suggests. So we return to the framework of section 6.3, but we still study the betting structure from the tree in Figure 6.5. Player 1's hand is the outcome u of a continuous random variable U and player 2's hand is the outcome v of a continuous random variable V. U and V are independently, identically distributed on $[0, 1]$ according to the uniform distribution function. A pure strategy of player 1 is a function $s : [0, 1] \to S$, where $S = \{PF, PC, B\}$. A pure strategy for player 2 is a function $t : [0, 1] \to T$, where $T = \{PF, PC, BF, BC\}$. The interpretation of the elements of S and T is straightforward: PF means passing at the first information set and folding at the second, BC means betting at the first information set and calling at the second. For player 1, the third element of S just means betting at information set 1_A. If he bets there, he will never have to make a decision at information set 1_B. Action combinations BF and BC are therefore equivalent, so instead of including these two combinations in S, we can include only B as well. The sets of behavioural strategies, $\mathcal{F}$ and $\mathcal{G}$, are defined as in section 6.3.

Aggregating the information from Figure 6.10 for each player, we find the suggestions for optimal strategies in the continuous version of Cassidy's poker model that are displayed in Figure 6.11.

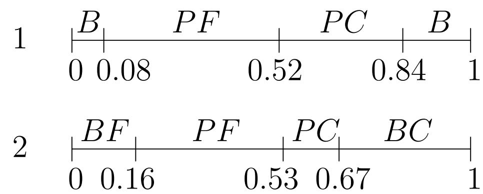


Figure 6.11: Suggestion for good pure interval strategies in the continuous game, following from the discrete game with 52 cards.

By computing the best response of player 2, we can calculate that player 1 guarantees himself an expected payoff of approximately -0.08, if he uses his strategy from Figure 6.11. If player 2 uses the strategy that this figure describes for him, he guarantees that the expected payoff for player 1 will not be higher than -0.06. Now we are going to find out if we can find slightly modified versions of the strategies in Figure 6.11, for which these guaranteed expected payoffs are equal. That is, we are looking for strategies that together form an equilibrium. The only modifications that we allow when we check this are slight movements of the bounds in the pure interval strategies, such that the joint order of the bounds for player 1 and player 2 is preserved. That is, we want to know if there exist optimal strategies of the form that is displayed in Figure 6.12, such that $u_1 \leq v_1 \leq u_2 \leq v_2 \leq v_3 \leq u_3$. In the continuous

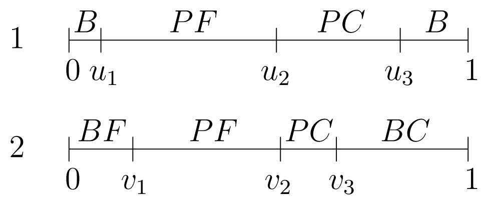


Figure 6.12: Suggestion for the structure of good pure interval strategies in the continuous game, following from the discrete game with 52 cards.

version of Cassidy's poker model, player 1's payoffs for all combinations of actions $s_k \in S$ and $t_\ell \in T$ only depend on the relative value of u and v, and can therefore be summarized in two matrices, which we call H_H and H_L. Here, $H_H(s_k, t_\ell)$ is the payoff to player 1 if $u > v$, while $H_L(s_k, t_\ell)$ is the payoff to player 1 if $u < v$. With $a = 1$ and $b = 2$, we obtain the following matrices.

H_H	PF	PC	BF	BC
PF	1	1	-1	-1
PC	1	1	3	3
B	1	3	1	3

H_L	PF	PC	BF	BC
PF	-1	-1	-1	-1
PC	-1	-1	-3	-3
B	1	-3	1	-3

From this payoff structure, one can derive that the expected payoff function $E_1(u, s_k, g)$ for player 1, defined as in section 6.3, for given $s_k \in S$ and a given behavioural strategy $g \in \mathcal{G}$, is continuous and non-decreasing in u.

We now derive the conditions that must be satisfied, if strategies with the structure of Figure 6.12 should form an equilibrium. If we call the equilibrium strategies of player 1 and player 2 $\tilde{s}$ and $\tilde{t}$, respectively, then we know that $\tilde{s}$ can only be optimal if

$$E_1(u, B, \tilde{t}) \geq E_1(u, s_k, \tilde{t}) \quad \text{for all } s_k \in S \text{ and for all } u \in (0, u_1).$$

A similar condition must hold for the intervals (u_1, u_2), (u_2, u_3) and $(u_3, 1)$. Next, consider the card u_1. The conditions for the intervals $(0, u_1)$ and (u_1, u_2), together with continuity of the functions E_1, imply that

$$E_1(u_1, B, \tilde{t}) = E_1(u_1, PF, \tilde{t}). \tag{6.12}$$

We can explicitly determine these expected payoffs for the given structure of the players' strategies, since we know the joint order of the strategy bounds and $\tilde{t}$ can be written as

$$\tilde{t}(v) = \begin{cases} BF & \text{if } v \in [0, v_1], \\ PF & \text{if } v \in (v_1, v_2], \\ PC & \text{if } v \in (v_2, v_3], \\ BC & \text{if } v \in (v_3, 1]. \end{cases}$$

We compute the expected payoff for player 1, if he bets with card u_1:

$$\begin{aligned} E_1(u_1, B, \tilde{t}) &= \int_0^{u_1} H_H(B, BF)dv + \int_{u_1}^{v_1} H_L(B, PF)dv \\ &\quad + \int_{v_1}^{v_2} H_L(B, PF)dv + \int_{v_2}^{v_3} H_L(B, PC)dv + \int_{v_3}^{1} H_L(B, BC)dv \\ &= u_1 + (v_1 - u_1) + (v_2 - v_1) - 3(v_3 - v_2) - 3(1 - v_3) \\ &= 4v_2 - 3. \end{aligned}$$

Analogously, we find the expected payoff of passing and subsequently folding with u_1:

$$E_1(u_1, PF, \tilde{t}) = -1.$$

Using these expressions, we find that equation (6.12) gives us an equilibrium condition for the bound v_2 of player 2:

$$E_1(u_1, B, \tilde{t}) = E_1(u_1, PF, \tilde{t}) \Rightarrow v_2 = \frac{1}{2}.$$

For the bounds u_2 and u_3, we can formulate similar indifference equations.

$$\begin{aligned} E_1(u_2, PF, \tilde{t}) &= E_1(u_2, PC, \tilde{t}) &\Rightarrow \quad v_1 &= \tfrac{1}{2}(1 - v_3), \\ E_1(u_3, PC, \tilde{t}) &= E_1(u_3, B, \tilde{t}) &\Rightarrow \quad 1 - v_2 &= v_1 + (1 - v_3). \end{aligned}$$

Let $E_2(v, f, t_\ell)$ denote the expected payoff of player 2 when he plays pure action t_ℓ with a given card v against strategy f of player 1. Then the indifference equations for the three bounds of player 2 are as follows.

$$\begin{array}{rclcrcl} E_2(v_1, \tilde{s}, BF) & = & E_2(v_1, \tilde{s}, PF) & \Rightarrow & (u_3 - u_2) & = & \frac{1}{2}(u_3 - v_1), \\ E_2(v_2, \tilde{s}, PF) & = & E_2(v_2, \tilde{s}, PC) & \Rightarrow & u_1 & = & \frac{1}{2}(1 - u_3), \\ E_2(v_3, \tilde{s}, PC) & = & E_2(v_3, \tilde{s}, BC) & \Rightarrow & (u_3 - u_2) & = & 2(u_3 - v_3). \end{array}$$

The indifference conditions for the bounds in the strategies form a system of six linear equations with six variables, which turns out to have a unique solution, resulting in the strategies displayed in Figure 6.13:

$$u_1 = \frac{1}{12},\ u_2 = \frac{1}{2},\ u_3 = \frac{5}{6},\ v_1 = \frac{1}{6},\ v_2 = \frac{1}{2} \text{ and } v_3 = \frac{2}{3}.$$

It is not difficult to check that these strategies are mutually best responses, so these strategies are optimal in the continuous variant of Cassidy's poker model with ante $a = 1$ and bet size $b = 2$. Observe that the bounds satisfy the joint order that we required: $u_1 \leq v_1 \leq u_2 \leq v_2 \leq v_3 \leq u_3$. The only inequality that is satisfied as an equality is $u_2 \leq v_2$. Apparently, $u_2 = v_2 = \frac{1}{2}$ in the equilibrium: the optimal bound between folding and calling is the same for both players. The value of the game is the expected payoff for player 1, when both players use the given equilibrium strategies, and is equal to $-\frac{1}{12}$.

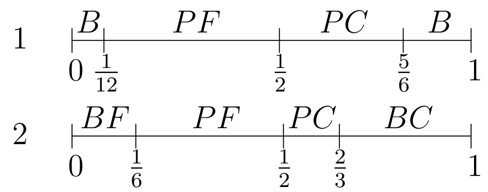


Figure 6.13: Optimal strategies in the continuous game with $a = 1$ and $b = 2$.

Now we are interested in the interpretation of these bounds in terms of real poker hands. Using the same approach as for Von Neumann's model in section 5.4, we give the optimal strategies for both players for the case that Cassidy's game is played with two separate decks of cards, from which real five-card poker hands are drawn. For the details about poker hands as well as the notation used, we refer to section 5.4. For player 1, having hand u, it is optimal to

$$\begin{cases} \text{bet} & \text{if } u \leq HC(J,10,8,3,2), \\ \text{pass and fold} & \text{if } HC(J,10,8,4,2) \leq u \leq HC(A,K,Q,J,6), \\ \text{pass and call} & \text{if } HC(A,K,Q,J,7) \leq u \leq 1P(Q,Q,9,7,4), \\ \text{bet} & \text{if } u \geq 1P(Q,Q,9,7,5). \end{cases}$$

For player 2, having hand v, it is optimal to

$$\begin{cases} \text{bet and fold} & \text{if } v \leq HC(Q,J,9,8,3), \\ \text{pass and fold} & \text{if } HC(Q,J,9,8,4) \leq v \leq HC(A,K,Q,J,6), \\ \text{pass and call} & \text{if } HC(A,K,Q,J,7) \leq v \leq 1P(7,7,8,6,5), \\ \text{bet and call} & \text{if } v \geq 1P(7,7,9,3,2). \end{cases}$$

Let us have a closer look at the form of the optimal strategies. Player 1 logically bets with the highest cards and he passes with intermediate cards. It is not surprising that he has to call with the highest of the intermediate cards and fold with the lowest of these cards. Just as we have seen in chapter 5 for the equilibrium of minipoker, we see again that player 1 has to bet with really low hands. He bluffs, thereby hoping to deceive his opponent and make him fold.

Player 2's strategy has the same form. However, for the hands with which he bets, he also has to decide whether he folds or calls, because player 1 can already decide to bet. For the high hands with which he bets, he clearly must call. But for the low hands with which he bets, his bluffing region, it is wiser to fold. There is no way he can deceive his opponent by calling, and in a showdown the probability of winning with a low hand is very small.

In the next section, we discuss the equilibrium of Cassidy's model for general values of the ante and the bet size.

6.4.4 General ante and bet size

The calculations in section 6.4.3 were based on the results of the equilibrium results for the discrete game with ante $a = 1$ and bet size $b = 2$ from section 6.4.2. For other ratios of a and b, the structure of the equilibrium strategies in the discrete game is similar. In this section, we repeat the computations of the previous section for general ante and bet size and we use Figure 6.12 (and the corresponding joint order of the strategy bounds) as our starting point again.

For general ante a and bet size b, the payoff matrices H_H and H_L are as follows.

H_H	PF	PC	BF	BC
PF	a	a	$-a$	$-a$
PC	a	a	$(a+b)$	$(a+b)$
B	a	$(a+b)$	a	$(a+b)$

H_L	PF	PC	BF	BC
PF	$-a$	$-a$	$-a$	$-a$
PC	$-a$	$-a$	$-(a+b)$	$-(a+b)$
B	a	$-(a+b)$	a	$-(a+b)$

We compute the expected payoff for player 1, if he bets with card u_1 and player 2 uses strategy $\tilde{t}$:

$$\begin{aligned} E_1(u_1, B, \tilde{t}) &= \int_0^{u_1} H_H(B, BF)dv + \int_{u_1}^{v_1} H_L(B, PF)dv \\ &\quad + \int_{v_1}^{v_2} H_L(B, PF)dv + \int_{v_2}^{v_3} H_L(B, PC)dv + \int_{v_3}^{1} H_L(B, BC)dv \\ &= au_1 + a(v_1 - u_1) + a(v_2 - v_1) - (a+b)(v_3 - v_2) \\ &\quad -(a+b)(1 - v_3) \\ &= (2a+b)v_2 - (a+b). \end{aligned}$$

Passing and folding with the same card against strategy $\tilde{t}$ of the opponent yields

$$E_1(u_1, PF, \tilde{t}) = -a.$$

The indifference equations corresponding to all six bounds now give us the following equilibrium conditions.

$$\begin{aligned} E_1(u_1, B, \tilde{t}) &= E_1(u_1, PF, \tilde{t}) &\Rightarrow&& v_2 &= \tfrac{b}{2a+b}, \\ E_1(u_2, PF, \tilde{t}) &= E_1(u_2, PC, \tilde{t}) &\Rightarrow&& v_1 &= \tfrac{b}{2a+b}(1 - v_3), \\ E_1(u_3, PC, \tilde{t}) &= E_1(u_3, B, \tilde{t}) &\Rightarrow&& 1 - v_2 &= v_1 + (1 - v_3), \\ E_2(v_1, \tilde{s}, BF) &= E_2(v_1, \tilde{s}, PF) &\Rightarrow&& (u_3 - u_2) &= \tfrac{2a}{2a+b}(u_3 - v_1), \\ E_2(v_2, \tilde{s}, PF) &= E_2(v_2, \tilde{s}, PC) &\Rightarrow&& u_1 &= \tfrac{b}{2a+b}(1 - u_3), \\ E_2(v_3, \tilde{s}, PC) &= E_2(v_3, \tilde{s}, BC) &\Rightarrow&& (u_3 - u_2) &= 2(u_3 - v_3). \end{aligned}$$

Again the system of six linear equations with six variables has a unique solution:

$$\begin{array}{rclcrcl} u_1 &=& \frac{2\,a^2 b}{(2\,a+b)^2 (a+b)}, & \qquad & v_1 &=& \frac{ab}{(a+b)(2\,a+b)}, \\ u_2 &=& \frac{b}{2\,a+b}, & & v_2 &=& \frac{b}{2\,a+b}, \\ u_3 &=& \frac{(3\,a+b)b}{(a+b)(2\,a+b)}, & & v_3 &=& \frac{b}{a+b}. \end{array}$$

The value of the game with ante a and bet size b is

$$v_{a,b} = \frac{-a^2 b^2}{(2\,a+b)^2\,(a+b)}. \tag{6.13}$$

We remark that the linear system, formed by the indifference equations, does not need to have a unique solution. There are two other possibilities: either the system is inconsistent or it has infinitely many solutions. In the first case, the guess for the optimal strategy forms is not a good guess. In the second case, there is some freedom in choosing the values of the strategy bounds. The solution set then describes the conditions the boundary values must satisfy, apart from the joint order that is used as a starting point in computing the expected payoffs.

The strategies of the players depend only on the ratio $\frac{b}{a}$ of the bet size and the ante and not on the absolute values of b and a. In Figure 6.14 and Figure 6.15 we have plotted the value of the strategy bounds (see Figure 6.12) as a function of the ratio $\frac{b}{a}$, for player 1 and player 2 respectively. Above $\frac{b}{a} = 2$, one recognizes the equilibrium strategies for the game with $a = 1$ and $b = 2$, that we have found in section 6.4.3.

The value is negative for each combination of (positive) values for a and b: Cassidy's game is favourable to player 2. The value can be plot as a function of the ratio $\frac{b}{a}$ too. We have done this in Figure 6.16. It is interesting to investigate at which ratio of ante and bet size the advantage for player 2 is maximized. For which value of $\frac{b}{a}$ is the game the most favourable to player 2? By fixing a, differentiating the expression for $v_{a,b}$ from equation (6.13) with respect to b and setting the result equal to zero, we find that the function that is shown in Figure 6.16 is minimized at $\frac{b}{a} = 1 + \sqrt{5}$. The value of the game with that ratio of b and a, is $\frac{11-5\sqrt{5}}{2} \approx -0.09$.

When poker players speak about relative sizes of bet size and ante, they often refer to the total stakes at a certain stage of the game. For example,

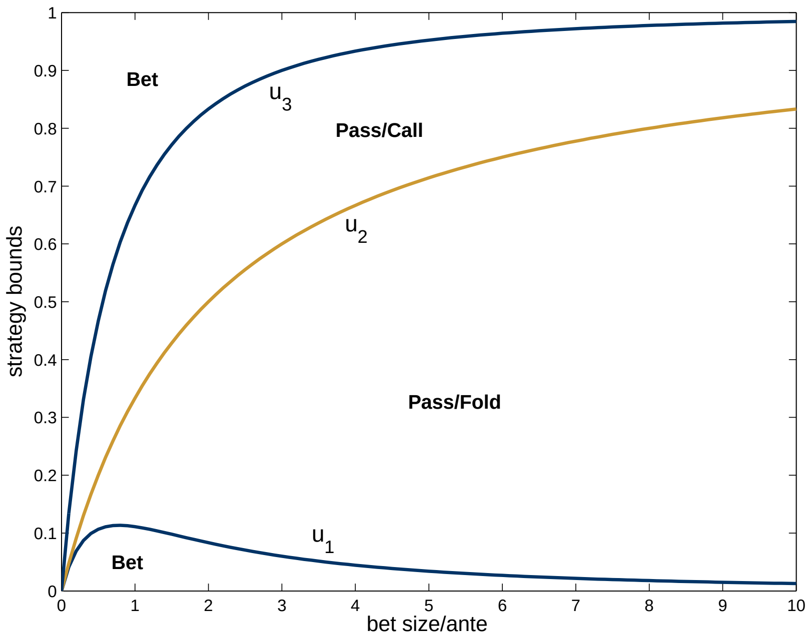


Figure 6.14: Boundary values for the optimal strategies of player 1 as a function of the ratio $\frac{b}{a}$ of the bet size and the ante.

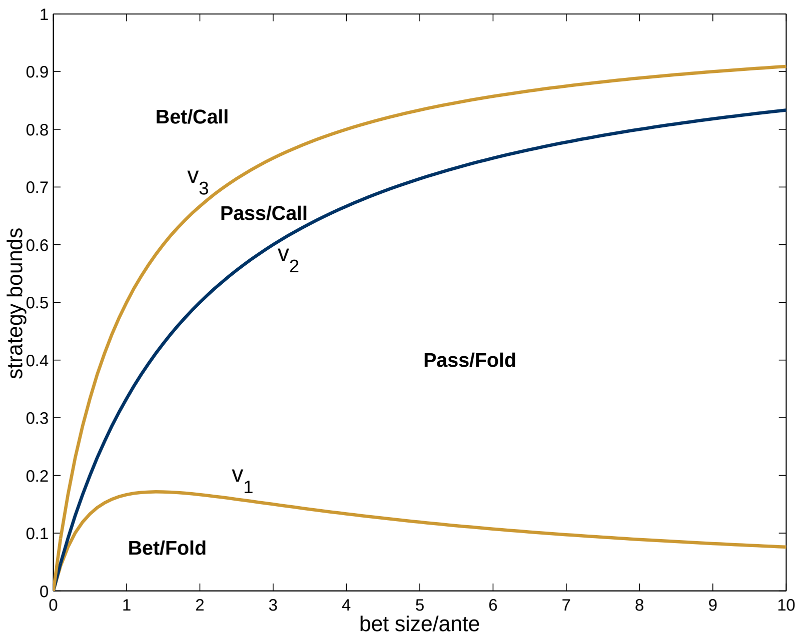


Figure 6.15: Boundary values for the optimal strategies of player 2 as a function of the ratio $\frac{b}{a}$ of the bet size and the ante.

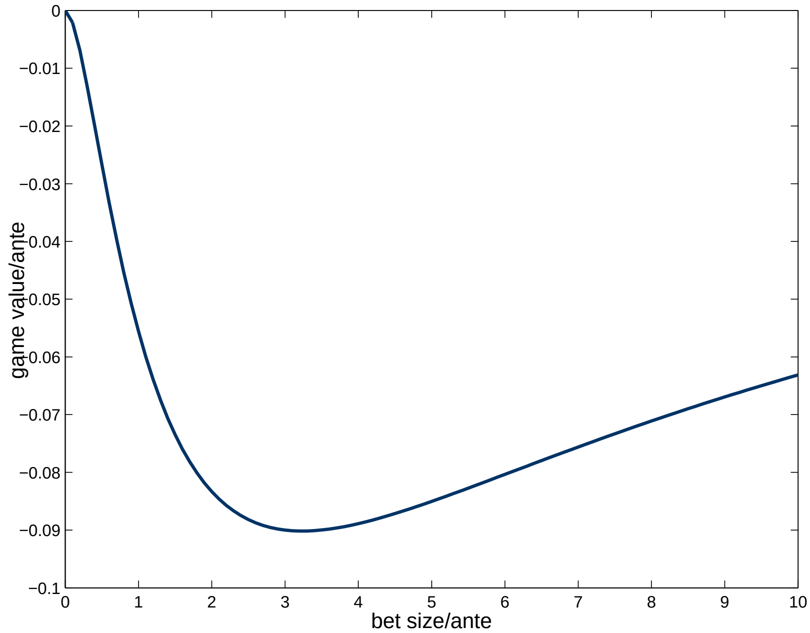


Figure 6.16: The value of the continuous version of Cassidy's poker model as a function of the ratio $\frac{b}{a}$ of the bet size and the ante.

when referring to pot-limit poker, a poker player means a game in which the bet size cannot exceed the total size of the pot. In Cassidy's poker game, at the moment a bet can be made, both players have added the ante a to the pot. The total stakes are equal to $2a$. The relative size of the bet, should therefore be expressed as $\frac{b}{2a}$. But this means that the optimal ratio, from the viewpoint of player 2, is equal to $\frac{b}{2a} = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, in which we recognize a well-known mathematical value: this is the *golden ratio*. When playing the continuous version of Cassidy's poker game, player 2 understands why this number is called the golden ratio!

6.5 A poker model allowing raising

The approach of finding equilibria that we applied to Cassidy's poker model in section 6.4 can also be applied to poker models with more complicated decision trees. In this section, we present the results for a poker model that is an extension of the model of section 6.4. The extension is formed by the possibility of raising after a bet of the opponent. Both players get this opportunity. Raising means adding the bet size b and an additional raising amount c to the stakes. After a raise, the opponent has the choice between folding and calling. As usual, folding means giving up the stakes. Calling means paying the same amount c, after which the winner is determined by means of a showdown. Figure 6.17 displays the decision tree as well as the payoffs of the game.

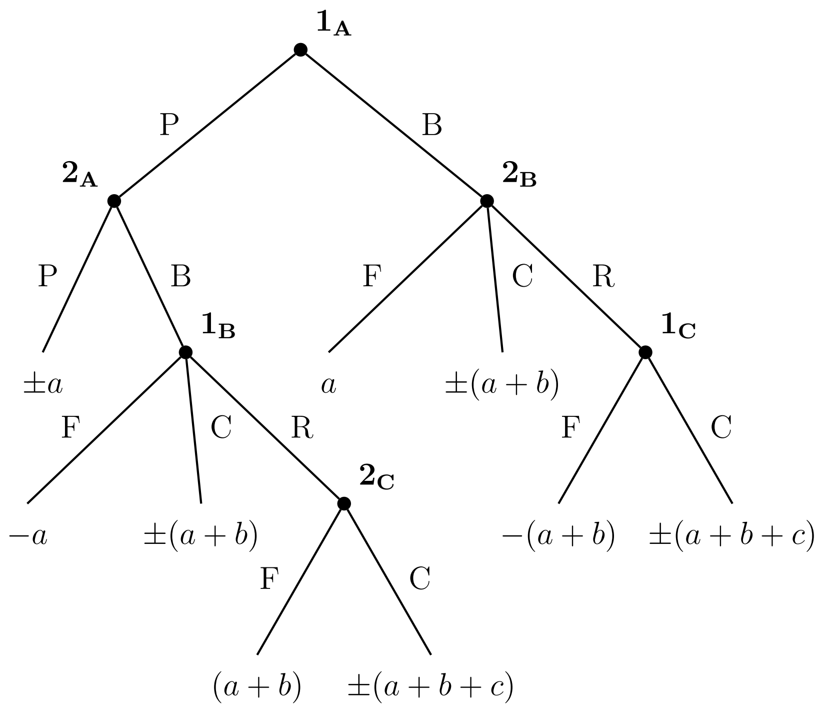


Figure 6.17: Betting tree of the poker game with the possibility of raising for both players.

Section 6.5.1 contains equilibrium results for the continuous version of this poker model with general payoffs. Section 6.5.2 interprets the results of a special case, pot-limit poker, in terms of real poker hands. We use the equilibrium results of pot-limit poker in section 6.5.3, where we present the analysis of skill for this game.

6.5.1 Equilibrium results

The continuous poker game with the betting tree of Figure 6.17, in which the hands are uniformly and independently distributed on $[0,1]$, is too complicated to be analyzed by the standard tools for finding an equilibrium. However, using the approach that was demonstrated in section 6.4, we have found optimal strategies for the players with the structure that is shown in equations (6.14) and (6.15). Stars indicate that an information set will not be reached, so that the player's decision at that information set is irrelevant. For player 1 it is optimal to play

$$\begin{cases} \text{PR*} & \text{if } u \in [0, u_1], \\ \text{B*F} & \text{if } u \in (u_1, u_2], \\ \text{PF*} & \text{if } u \in (u_2, u_3], \\ \text{PC*} & \text{if } u \in (u_3, u_4], \\ \text{B*F} & \text{if } u \in (u_4, u_5], \\ \text{B*C} & \text{if } u \in (u_5, u_6], \\ \text{PR*} & \text{if } u \in (u_6, 1]. \end{cases} \tag{6.14}$$

The three letters (or stars) indicating which actions should be taken within the specified intervals, refer to the three information sets of player 1. The first, second, and third symbol correspond to information sets 1_A, 1_B and 1_C respectively. Using the same logic of notation for player 2, we found that it is optimal for him to play

$$\begin{cases} \text{BRF} & \text{if } v \in [0, v_1], \\ \text{BFF} & \text{if } v \in (v_1, v_2], \\ \text{PF*} & \text{if } v \in (v_2, v_3], \\ \text{PC*} & \text{if } v \in (v_3, v_4], \\ \text{BCF} & \text{if } v \in (v_4, v_5], \\ \text{BCC} & \text{if } v \in (v_5, v_6], \\ \text{BRC} & \text{if } v \in (v_6, 1]. \end{cases} \tag{6.15}$$

We have found the exact expressions for all twelve strategy bounds for general values of a, b and c using the same approach as we used for Cassidy's poker game in section 6.4.4. However, we do not include them here, since some of these expressions are so long that they are hardly informative. The joint order of the bounds is as follows:

$$u_1 \leq v_1 \leq u_2 \leq v_2 \leq u_3 = v_3 \leq v_4 \leq u_4,$$
$$u_4 \leq u_5, \quad v_4 \leq v_5 \quad \text{and} \quad v_5 \leq u_5 \leq v_6 \leq u_6.$$

The order of u_4 and v_5 depends on the relative size of the ratios $\frac{b}{a}$ and $\frac{c}{a}$. For ante $a = 1$ and bet size $b = 2$, the strategy bounds for both players are displayed in Figure 6.18 as a function of c. The general expression for the value

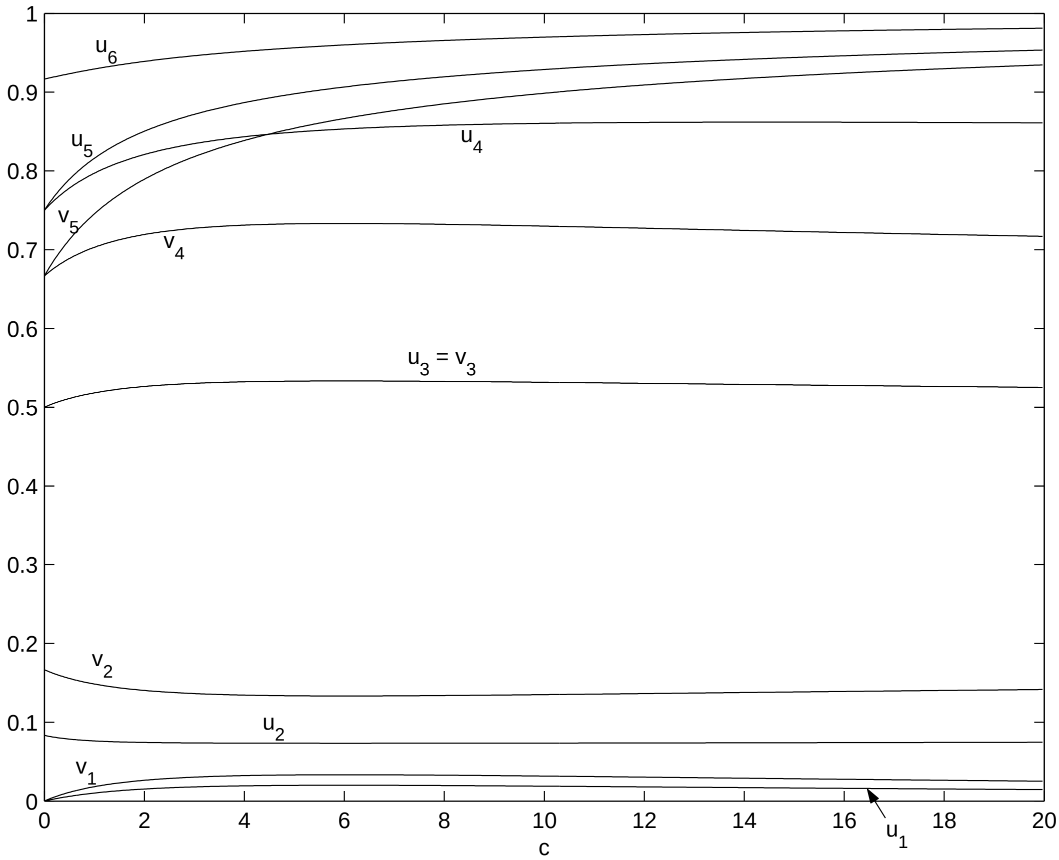


Figure 6.18: The bounds of the optimal strategies as function of the raising amount c, given ante $a = 1$ and bet size $b = 2$.

does not depend on the order of u_4 and v_5 and is given in equation (6.16).

$$v_{a,b,c} = \frac{\left(-4\,a^3c-4\,a^2b^2-8\,ca^2b-8\,b^3a-9\,acb^2-4\,b^4-5\,cb^3-c^2b^2\right)(2\,a+2\,b+c)^2a^2b^2}{(2\,a+b)^2(8\,ab^2+5\,cb^2+4\,b^3+2\,ca^2+c^2b+7\,cba+4\,a^2b)^2(b+a)} \tag{6.16}$$

When we fix the size of the ante at $a = 1$, we can plot the value as a function of b and c. This is done in Figure 6.19. We see that the value of this game is always negative. Just as was the case with Cassidy's model, this game is favourable for player 2. We also learn from Figure 6.19 that the value does not depend too much on the value of the raise size c. The intuition for this result lies in the fact that the players do not raise often when using the equilibrium strategies.

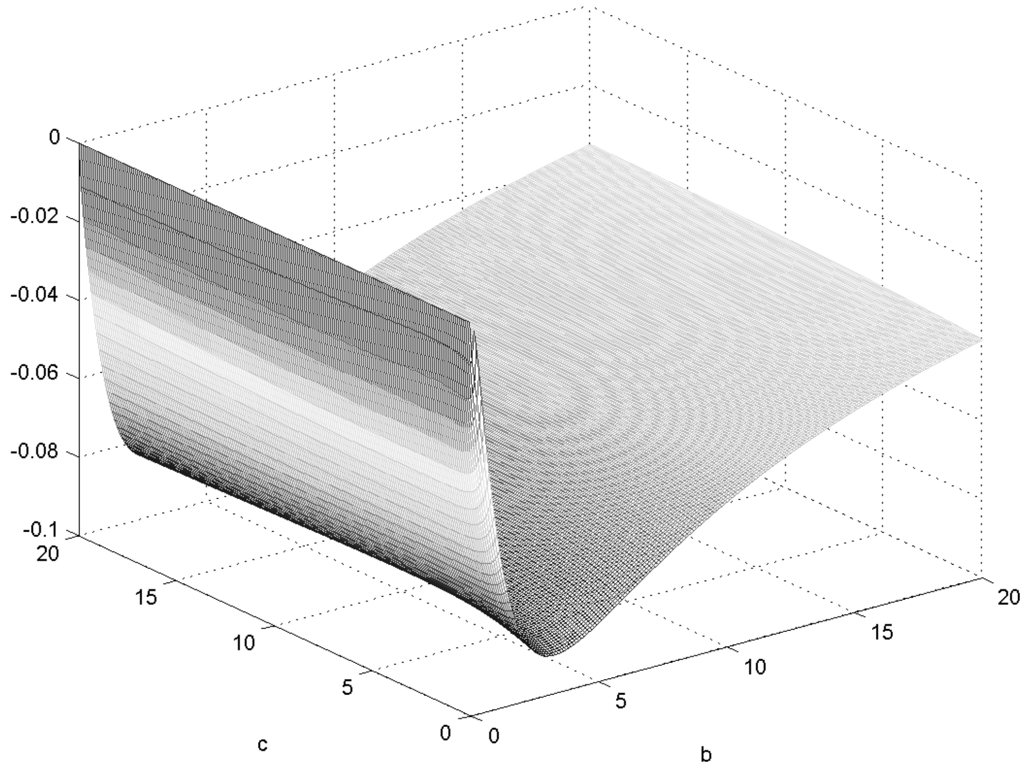


Figure 6.19: The value of the game if $a = 1$.

In fact, the higher c, the less players raise, as we can see in Figure 6.18. As a consequence, the payoffs corresponding to this action do not influence the expected payoff very much.

6.5.2 A special case: pot-limit poker

If $a = 1$ and $b = c = 2$, both the bet size and the raising amount are equal to $2a$, the size of the initial pot. We call this special case *pot-limit poker*. The bounds for the optimal strategies are as follows:

$$
\begin{array}{llllllll}
u_1 & = & \frac{11}{722} & \approx & 0.015, & v_1 & = & \frac{1}{38} \approx 0.026, \\
u_2 & = & \frac{161}{2166} & \approx & 0.074, & v_2 & = & \frac{8}{57} \approx 0.140, \\
u_3 & = & \frac{10}{19} & \approx & 0.526, & v_3 & = & \frac{10}{19} \approx 0.526, \\
u_4 & = & \frac{889}{1083} & \approx & 0.821, & v_4 & = & \frac{41}{57} \approx 0.719, \\
u_5 & = & \frac{307}{361} & \approx & 0.850, & v_5 & = & \frac{15}{19} \approx 0.790, \\
u_6 & = & \frac{339}{361} & \approx & 0.939, & v_6 & = & \frac{17}{19} \approx 0.895.
\end{array}
$$

The value $v_{1,2,2}$ of the game is then $-\frac{88}{1083} \approx -0.0813$. We can translate these strategy bounds into poker hands, thereby defining strategies that are approximately optimal in the game that is played with two separate decks of 52 cards, from which real poker hands are drawn. This gives us the results that are displayed below. For the details about poker hands as well as the notation used, we refer to section 5.4. For player 1, having hand u, it is optimal to play

$$\begin{cases} \text{PR*} & \text{if } u \leq HC(9,8,6,3,2), \\ \text{B*F} & \text{if } HC(9,8,6,4,2) \leq u \leq HC(J,9,8,7,4), \\ \text{PF*} & \text{if } HC(J,9,8,7,5) \leq u \leq 1P(2,2,A,6,4), \\ \text{PC*} & \text{if } 1P(2,2,A,6,5) \leq u \leq 1P(J,J,A,8,4), \\ \text{B*F} & \text{if } 1P(J,J,A,8,5) \leq u \leq 1P(Q,Q,K,J,8), \\ \text{B*C} & \text{if } 1P(Q,Q,K,J,9) \leq u \leq 2P(9,9,6,6,2), \\ \text{PR*} & \text{if } u \geq 2P(9,9,6,6,3). \end{cases}$$

For player 2, having hand v, it is optimal to play

$$\begin{cases} \text{BRF} & \text{if } v \leq HC(10,7,6,5,4), \\ \text{BFF} & \text{if } HC(10,8,4,3,2) \leq v \leq HC(Q,10,9,5,4), \\ \text{PF*} & \text{if } HC(Q,10,9,6,2) \leq v \leq 1P(2,2,A,6,4), \\ \text{PC*} & \text{if } 1P(2,2,A,6,5) \leq v \leq 1P(8,8,K,J,10), \\ \text{BCF} & \text{if } 1P(8,8,K,Q,2) \leq v \leq 1P(10,10,A,9,6), \\ \text{BCC} & \text{if } 1P(10,10,A,9,7) \leq v \leq 1P(A,A,8,4,3), \\ \text{BRC} & \text{if } v \geq 1P(A,A,8,5,2). \end{cases}$$

Perhaps more interesting than the details about the poker hands at the bounds is the structure of the optimal strategies. In the optimal betting behaviour of player 2 we recognize bluffing: with rather low hands, between $HC(10,8,4,3,2)$ and $HC(Q,10,9,5,4)$, player 2 must bet if player 1 passes. However, if player 1 raises after this bluffing bet, he should fold. He should also fold in case player 1 has already bet. Even more aggressive bluffing is required with the worst possible hands, at most equal to $HC(10,7,6,5,4)$. With these hands, player 2 should also bet after a pass of player 1. The aggressive bluffing follows after an initial bet of player 1, since in this situation player 2 should raise, hoping that he can frighten player 1 and make him fold. This is possible in the case that player 1 was already bluffing when making his bet.

In the optimal strategy of player 1, we also recognize bluffing: with hands between $HC(9, 8, 6, 4, 2)$ and $HC(J, 9, 8, 7, 4)$ he bets at the start of the game, although his hand is not good. However, we see another strategic possibility: with really good hands, hands that are at least as good as $2P(9, 9, 6, 6, 3)$, player 1 should begin with passing instead of betting. This way of trying to deceive the opponent is known as *sandbagging*. By passing, player 1 tries to make his opponent think that his hand is not so good. In this way, he hopes that player 2 may be brave enough to bet. Of course, after a bet, player 1 would respond with raising, leading to a fold of player 2 or a showdown with high stakes.

Player 1 should do the same, pass first but raise after a bet of player 2, also with very bad hands (at most equal to $HC(9, 8, 6, 3, 2)$). He bluff-raises, thereby hoping to make the opponent fold. In this way, player 1 ensures that his sandbagging behaviour with high hands remains profitable. If he would only pass and subsequently raise with high hands, the opponent would know that calling after such a raise is only wise with an extremely good hand, but by doing the same with low hands he forces player 2 to call every now and then to see whether player 1 is bluffing.

6.5.3 Analysis of skill

In the previous section we presented optimal strategies and the value for the continuous version of pot-limit poker with a raising possibility for both players. We can use this information to analyze the skill involved in this game. The results of the skill analysis are presented in this section. We restrict ourselves to the approach that we introduced in section 2.8, in which the opponents are assumed to give maximal opposition and a fictive player is informed about the results of any randomization in the opponent's strategy.

The expected payoff of the optimal players follows directly from the value, that we already presented in section 6.5.2. The expected payoff of player 1 as an optimal player is equal to the value $v_{1,2,2} = -0.0813$. The expected payoff of player 2 is then 0.0813.

For the beginners, we have to define reasonable strategies. We assume that a beginner in the role of player 1 plays as follows:

$$\begin{cases} \text{PF*} & \text{if } u \in [0, u_1], \\ \text{B*C} & \text{if } u \in (u_1, 1], \end{cases}$$

where $u_1 \approx 0.5012$, the bound between the classes *High Card* and *One Pair*. With at least a pair in hand, the beginner thinks his hand is sufficiently good to bet or to call. With a lower hand, he decides to pass and to fold if required. Although he may find certain high hands sufficiently attractive to raise, he will never play this action: each hand that is good enough for raising is also good enough for betting, so that player 1 will never end up with a high hand at the information set where raising is possible.

Player 2, if acting as a beginner, plays the following strategy:

$$\begin{cases} \text{PF*} & \text{if } v \in [0, v_1], \\ \text{BCC} & \text{if } v \in (v_1, v_2], \\ \text{BRC} & \text{if } v \in (v_2, 1], \end{cases}$$

where $v_1 = u_1$ and $v_2 \approx 0.9237$, the bound between the classes *One Pair* and *Two Pair*. Hands that do not even contain one pair are not good enough to do anything but passing and folding; hands with one pair are good enough to bet or to call, but for raising a beginner in the role of player 2 needs to have at least two pairs.

When player 1 plays according to the simple strategy for the beginner that is described above, and his opponent takes advantage of the weakness of this naive strategy, he has an expected payoff of -0.4585. Optimal counterplay of player 2 is as follows:

$$\begin{cases} \text{BFF / BFC} & \text{if } v \in [0, 0.5012], \\ \text{PF* / BFF / BFC} & \text{if } v \in (0.5012, 0.6675], \\ \text{PC* / BCF / BCC} & \text{if } v \in (0.6675, 0.7506], \\ \text{PR* / BRF / BRC} & \text{if } v \in (0.7506, 1]. \end{cases}$$

Next, consider player 2 as a beginner. If player 1 plays optimally, given the naive strategy that player 2 uses, the expected payoff for player 2 is -0.2074. Optimal play for player 1 is as follows:

$$\begin{cases} \text{B*F} & \text{if } u \in [0, 0.0024], \\ \text{PF*} & \text{if } u \in (0.0024, 0.6675], \\ \text{PC* / B*F} & \text{if } u \in (0.6675, 0.7506], \\ \text{PR*} & \text{if } u \in (0.7506, 1]. \end{cases}$$

If player 1 is a fictive player, then it is optimal for player 2 to play

$$\begin{cases} \text{PF*} & \text{if } v \in [0, \frac{1}{2}], \\ \text{PC*} & \text{if } v \in (\frac{1}{2}, 1]. \end{cases}$$

An optimal response of the fictive player 1 is to play

$$\begin{cases} \text{anything} & \text{if } v \in [0, \frac{1}{2}] \text{ and } u > v, \\ \text{B*F / B*C} & \text{if } v \in [0, \frac{1}{2}] \text{ and } u < v, \\ \text{B*F / B*C} & \text{if } v \in (\frac{1}{2}, 1] \text{ and } u > v, \\ \text{PF* / PC* / PR*} & \text{if } v \in (\frac{1}{2}, 1] \text{ and } u < v. \end{cases}$$

The expected payoff of the fictive player 1 is then $\frac{1}{2}$. If player 2 is a fictive player, then it is optimal for player 1 to play

$$\begin{cases} \text{PF*} & \text{if } u \in [0, \frac{1}{2}], \\ \text{PC*} & \text{if } u \in (\frac{1}{2}, 1]. \end{cases}$$

An optimal response of the fictive player 2 is to play

$$\begin{cases} \text{anything} & \text{if } u \in [0, \frac{1}{2}] \text{ and } u < v, \\ \text{BFF / BFC / BCF / BCC / BRF / BRC} & \text{if } u \in [0, \frac{1}{2}] \text{ and } u > v, \\ \text{BFF / BFC / BCF / BCC / BRF / BRC} & \text{if } u \in (\frac{1}{2}, 1] \text{ and } u < v, \\ \text{PF* / PC* / PR*} & \text{if } u \in (\frac{1}{2}, 1] \text{ and } u > v. \end{cases}$$

The expected payoff of the fictive player 2 is then $\frac{1}{2}$. Table 6.1 gives an overview of the numbers that are relevant for the skill analysis.

	RS		
	Player 1	Player 2	Game
Beginner	-0.4585	-0.2074	-0.3330
Optimal	-0.0813	0.0813	0.0000
Fictive	0.5000	0.5000	0.5000
LE	0.3772	0.2887	0.3330
RE	0.5813	0.4187	0.5000
RS	0.3935	0.4081	**0.3997**

Table 6.1: Results of the skill analysis for pot-limit poker with raising.

Comparing the skill level, 0.3997, to the highest bound between games of chance and games of skill that was advised by Van der Genugten (1997), 0.15, leads undisputedly to the classification of the poker game with raising as a game of skill. We can also compare this skill level to the result of the skill analysis for minipoker from section 5.5.3. Then we see that the relative skill involved in the poker game with raising is lower than for minipoker. However, we should be careful when making this comparison. As we have already mentioned in the comments upon the skill level of minipoker in section 5.5.3, the choice of the beginners' strategies influences the final result. Apparently, our guess for the beginners' strategies in the model with raising has lead to strategies that are relatively smart, compared to the ones we used in the analysis of minipoker. Empirical information on game results of beginners could help to confirm or adjust our guesses. Therefore, field studies or laboratory experiments to obtain this information form an interesting subject for further research.

Chapter 7

Take-and-guess games

7.1 Introduction

In this chapter two classes of take-and-guess games are studied. In both classes of games, each of the two players (I and II) has to take a number of objects out of a given private finite set of objects. After that, they both have to guess the total amount of objects taken by both players. For the objects, one can think of fingers, coins or matches. Player 1 has $m \in \mathbb{N}$ objects available: he can take any number in $\{0, 1, \ldots, m\}$. His opponent has n objects available. The values of m and n are common knowledge.

In the first class, the morra games, the objects used in general are the fingers of the players' hands. Both players have to announce their guesses simultaneously. A player wins a particular play of this game if he guesses the total number of fingers correctly, while his opponent guesses a wrong number. If both players guess correctly, the play is a draw. This is also the case if both players guess a wrong total.

In morra with an equal number of fingers for both players, the player roles are symmetric. As expected, these games turn out to be fair (i.e., their value is zero). We prove this in section 7.2 and we also show that if one player can use more fingers than his opponent ($m \neq n$), then this player has an advantage in the game.

In the other class of take-and-guess games, the so-called (m, n)-coin games, the players announce their guesses sequentially. The second player is not allowed to guess the same total as the first player. In the naming of the games, we follow Schwartz (1959), who studied the games with $m = n$. He called these games n-coin games. If a player guesses right, he wins. If neither player guesses the total correctly, the play ends in a draw.

Since coin games are not symmetric for any m and n, it is not clear at first sight whether any of these games is fair. However, Schwartz (1959) has shown that the games with $m = n$ are fair. We show in section 7.3 that a much larger class of coin games is fair: the game value is zero for any coin game in which the starting player has at least as many coins as the opponent ($m \geq n$). Furthermore, (m, n)-coin games with $m < n$ are not fair. We give an overview of the values for all these games and we describe optimal strategies for both players for all m and n.

The remainder of this chapter, which is based on Dreef and Tijs (2004), is organized as follows. Morra is discussed in section 7.2. In section 7.3, we study coin games in detail. Section 7.4 presents some concluding remarks and comparisons of morra and coin games.

7.2 Games of morra

Morra is a game that has been played since ancient Egyptian times. It is still played throughout different parts of the world, especially in Europe and Northern Africa. For a more detailed historic description we refer to Ifrah (1985, p. 67–70) and Perdrizet (1898). The game is fairly simple and can be played by two or more players, but it is usually played by two. The players face each other, each holding up a closed fist. At a given signal, they both hold up zero to five fingers and at the same time announce a number from zero to ten. If both hands are used, the number can range from zero to twenty. A player wins if the number he calls out is the total number of fingers shown by both players. However, if the opponent guesses the same number, the play ends in a draw. Also if neither of the players guesses the correct number, then there is no winner. Winning will be formally represented by getting one unit from the opponent. Payoffs in this zero-sum game can therefore only be -1, 0 and 1.

Variants of morra are a popular subject in game theory lectures (see, for example, Rector (1987)). The proof of the result that we derive in this section (or parts or variants of it), appears as an exercise in various course notes concerning non-cooperative game theory. Proposition 7.2.1 is mainly included to be able to compare morra with the coin games that are studied in section 7.3.

In the general version of morra that we study in this chapter, the first player is allowed to hold up a maximum of $m \in \mathbb{N}$ fingers, while his opponent can choose to hold up at most $n \in \mathbb{N}$ fingers. We refer to this game as (m,n)-morra, or briefly $M_{m,n}$. In the analysis of these games and the coin games that are studied in section 7.3, we will often encounter sets of integers of the form $\{a, a+1, \ldots, b-1, b\}$. It is therefore convenient to introduce a shorthand notation for such a set: $[a,b]$.

A pure strategy for player I in $M_{m,n}$ will be denoted by (x_1, y_1), where x_1 is the number of fingers he decides to hold up and y_1 is the sum he guesses. Clearly, with a strategy for which $y_1 < x_1$, player I can never win. Neither can he win with a strategy for which $y_1 > x_1+n$. Such a strategy is called *infeasible*. We restrict attention to *feasible* strategies. That is, the pure strategy space for player I is $S_1 = \{(x_1, y_1) \mid (x_1 \in [0,m]) \wedge (y_1 \in [x_1, x_1+n])\}$. Analogously, the pure strategy space for player II is given by $S_2 = \{(x_2, y_2) \mid (x_2 \in [0,n]) \wedge (y_2 \in [x_2, x_2+m])\}$. The cardinalities of the strategy spaces are equal: $|S_1| = |S_2| = (m+1)(n+1)$.

The game (m,n)-morra can be modelled as a matrix game and is then completely defined by the matrix $A = [a_{(x_1,y_1),(x_2,y_2)}]$, where

$$a_{(x_1,y_1),(x_2,y_2)} = \begin{cases} 1 & \text{if } (y_1 = x_1 + x_2) \wedge (y_1 \neq y_2), \\ -1 & \text{if } (y_2 = x_1 + x_2) \wedge (y_1 \neq y_2), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 7.2.1 *Let $m, n \in \mathbb{N}$. The value $v(M_{m,n})$ of (m,n)-morra is $\frac{m-n}{(m+1)(n+1)}$.*

Proof. Let $x_1 \in [0,m]$ and $y_1 \in [x_1, x_1+n]$. The strategy (x_1, y_1) of player I will win against all strategies $(x_2, y_2) \in S_2$ of player II for which $x_2 = y_1 - x_1$ and $y_1 \neq y_2$. Player II has exactly m strategies that fulfil these conditions. On the other hand, (x_1, y_1) will cause a victory for player II if he uses a strategy $(x_2, y_2) \in S_2$ for which $y_2 = x_2 + x_1$ and $y_2 \neq y_1$. That is, player I will lose against any of the n elements of the set $\{(x_2, x_2 + x_1) \mid x_2 \in [0,n] \backslash \{y_1 - x_1\}\}$. Against any other strategy of player II, (x_1, y_1) will cause a tie. Therefore, the elements of each row of A sum to $m - n$. Consequently, by playing all

$(x_2, y_2) \in S_2$ with equal probability, $\frac{1}{|S_2|}$, player II can guarantee that player I will not get more than $\frac{m-n}{(m+1)(n+1)}$.

In an analogous way, one can show that player I can guarantee himself $\frac{m-n}{(m+1)(n+1)}$ by playing each of his pure strategies with probability $\frac{1}{|S_1|}$. This completes the proof. □

From the proof of Proposition 7.2.1, we can see that optimal strategies in this game are rather simple. Both players just have to play all their pure strategies with equal probability. It is interesting to notice that $v(M_{m,n}) = -v(M_{n,m})$. Furthermore, one can easily derive the following results by studying the effect of varying m and n on the value $v(M_{m,n})$.

Corollary 7.2.2 *Only the (m,n)-morra games with $m = n$ are fair. For $m \neq n$, the advantage is for the player who can use more fingers.*

Corollary 7.2.3

$$\lim_{m\to\infty} v(M_{m,n}) = \lim_{m\to\infty} \frac{m-n}{(m+1)(n+1)} = \frac{1}{n+1}.$$

The intuition behind the limit of Corollary 7.2.3 is that if one of the players has extremely many objects available (in terms of fingers it becomes difficult to imagine), then his opponent will not be able to guess the number of objects he takes. The value of the game is therefore completely determined by the probability that this player guesses correctly the number of objects chosen by the other player.

7.3 Coin games

In this section we study a second class of take-and-guess games, the (m,n)-coin games. In contrast to morra, the players have to announce their guesses sequentially in these games. Schwartz (1959) studied the games with $m = n$ and called these games n-coin games. In the naming of our generalization, we also generalize the name he suggested.

The taking part of the (m,n)-coin game (or briefly $C_{m,n}$) is the same as in (m,n)-morra. The first player is allowed to take a maximum of $m \in \mathbb{N}$ objects, while his opponent can pick at most $n \in \mathbb{N}$ objects. The numbers m and n are common knowledge. When played in practice, the objects are not fingers, but things that can be hidden in a hand. As the name of the game suggests, coins are suitable. In Dutch bars the game used to be played with matches.

The difference with morra lies in the guessing part. The players have to announce their guesses sequentially instead of simultaneously. Player II hears the guess of player I and is not allowed to guess the same total as his opponent. If a player guesses right, he wins (i.e., obtains one unit of his opponent). If neither player guesses the total correctly, the play ends in a draw.

Now we can formally write down the strategy spaces of the players. Since coin games are games of perfect recall, the result of Kuhn (1953) tells us that we can restrict our analysis to behavioural strategies. A pure behavioural strategy for player I in $C_{m,n}$ is a choice $(x_1, y_1(x_1)) \in S_1$, where $S_1 = [0, m] \times [0, m+n]$. As in morra, x_1 represents the number of coins he takes in hand, while y_1 is his guess of the total number of coins taken by him and his opponent. Note that y_1 may depend on x_1.

Player II picks a combination $(x_2, y_2(x_2, y_1)) \in S_2$, where $S_2 = [0, n] \times [0, m+n]$, such that $y_2(x_2, y_1) \neq y_1$ for all $x_2 \in [0, n]$. Here, x_2 is the number of coins taken by player II and y_2 is the total that he guesses.

Notice that infeasible strategies, like guessing a total that is less than what one has taken in hand, are included in the strategy spaces. In the analysis of morra we did not take this kind of strategies into account. Here we do, and there is a reason for this difference. It is easy to see that infeasible strategies cannot help a player in morra, since the players' decisions are made simultaneously. Misleading the opponent doesn't make sense. In coin games, however, infeasible strategies could be useful for player I, at least in theory. If the game is advantageous for player II, then it may be interesting for player I to mislead his opponent by guessing a total of coins that cannot be correct, given his own hand. In this way, he could try to reduce player II's probability of guessing the right sum. Although he thereby reduces his own probability of guessing right to zero, the combined effect might be in his advantage. For this reason we include infeasible strategies in the strategy spaces. However, we show that for each $C_{m,n}$ we can find optimal behavioural strategies for both players in which the infeasible strategies are not used.

Let us give a short overview of the organization of the remainder of this section. We start by introducing a graphical model for (m,n)-coin games in section 7.3.1. In section 7.3.2, we present the equilibria for a large class of $C_{m,n}$, all games with $m \geq n$. Section 7.3.3 studies the games in which player II has one coin more available than his opponent. Sections 7.3.4 and 7.3.5 contain the equilibrium analysis of two boundary cases within the collection of games for which $n > m+2$. In section 7.3.6, the list of values is completed. This section is devoted to the games in which player II has two coins more than player I. The results of all subsections are summarized and discussed in section 7.3.7. Included in this summarizing section is Table 7.2, which illustrates the theorems of the preceding sections by listing the values for the (m,n)-coin games with small values of m and n. At some points earlier in the exposition we refer to this table.

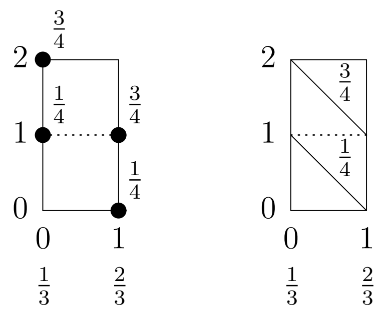
7.3.1 A graphical model of an (m,n)-coin game

The structure of coin games is more difficult than morra. We will see that for many combinations of m and n, finding the optimal strategies takes some smart construction work. To keep our arguments clear, and to make the constructions and proofs readable, we introduce a graphical representation of a coin game in (x_1,x_2)-diagrams. In such a diagram, it is not too difficult to see what a player can achieve with a specific strategy. To illustrate the interpretation of the diagrams, we compute the expected payoff that results from a specific combination of strategies. Moreover, we will show how to derive for each player a best reply against a given strategy of the opponent.

7.3.1.1 Representation of strategies in diagrams

Let us introduce the diagrams that we use to depict strategies for coin games. For the (m,n)-coin game, an (x_1,x_2)-diagram is a grid with $m+1$ columns (corresponding to $x_1 \in [0,m]$) and $n+1$ rows (corresponding to $x_2 \in [0,n]$). In the "taking" part of the game, player I picks a column and player II picks a row. Then player I guesses a sum y_1, where his guess can depend on x_1. In the (x_1,x_2)-diagram, this choice can be represented by a point in the column that was chosen by player I. On the line with slope -1 that goes through this point are all points in the grid for which $x_1+x_2=y_1$. Points on this line cannot be guessed by player II. Player II has to guess a different line with slope −1. For each combination of x_2 (the number of coins in his own hand) and y_1 (the opponent's guess) he has to make such a decision. Different choices of x_2 correspond to different rows, but for each possible value of y_1 we have to draw a separate (x_1, x_2)-diagram to represent a strategy of player II. To describe a behavioural strategy (with mixed decisions per information set), we give the conditional probability with which each of the actions is played.

Let us clarify this description with an example. The diagrams in Figure 7.1 give two graphical representations of one specific behavioural strategy of player I in $C_{1,2}$.



(a) (b)

Figure 7.1: Strategy for player I in $C_{1,2}$, represented in two ways in an (x_1, x_2)-diagram.

Figure 7.1(a) gives the general representation for the strategy. This (x_1, x_2)-diagram should be read as follows. Player I picks the left column ($x_1 = 0$) with probability $\frac{1}{3}$ and he picks the right column ($x_1 = 1$) with probability $\frac{2}{3}$. Next, he has to pick y_1. Given $x_1 = 0$, he picks the point $(0, 1)$ (corresponding to $y_1 = 0 + 1 = 1$) with probability $\frac{1}{4}$ and $(0, 2)$ (corresponding to $y_1 = 2$) with probability $\frac{3}{4}$. Similarly, given $x_1 = 1$, player I picks $(1, 0)$ and $(1, 1)$ with probabilities $\frac{1}{4}$ and $\frac{3}{4}$ respectively.

Since the conditional probabilities for the choice of y_1 are the same for $x_1 = 0$ and $x_1 = 1$, we can depict this strategy of player I also a little simpler. This is done in Figure 7.1(b). This figure gives the same probabilities for the choices of the two columns, but summarizes the probabilities for the guessed sum, y_1, in the two lines with slope −1 that are chosen with the probabilities

$\frac{1}{4}$ ($y_1 = 1$) and $\frac{3}{4}$ ($y_1 = 2$). Such a representation is only possible if the player's conditional probabilities of guessing y_1 are the same for all x_1 that are chosen with positive probability. For many values of m and n, we present equilibrium strategies for the (m, n)-coin game that can be written in this simple form.

For player II we have also depicted a strategy in $C_{1,2}$ in (x_1, x_2)-diagrams. These diagrams are given in Figure 7.2. We draw one diagram for each possible value of $y_1 \in [0, m + n]$, since the decisions of player II may depend on this value.

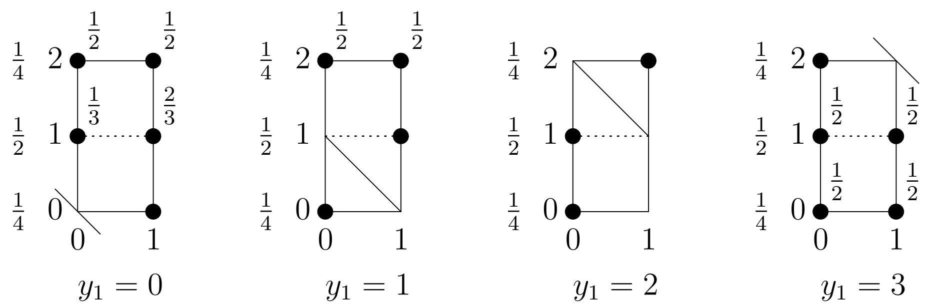


Figure 7.2: Strategy for player II in $C_{1,2}$, represented in four (x_1, x_2)-diagrams.

In the first place, player II has to pick a number of coins, i.e., he has to choose a row in the grid. A mixed decision is a probability distribution over the rows of the (x_1, x_2)-diagram. Clearly, this distribution cannot depend on y_1, so it is constant over the four diagrams in Figure 7.2. Player II takes one coin with probability $\frac{1}{2}$ and he takes zero or two coins both with probability $\frac{1}{4}$.

Next, after choosing x_2 and hearing the opponent's guess, y_1, player II has to decide what sum to guess. So for each row in each of the four diagrams, player II can give a probability distribution over the guesses that are interesting for him. In the first diagram, corresponding to $y_1 = 0$, we see that if player II has 2 coins in his hand, he chooses randomly between $y_2 = 2$ (the point $(0, 2)$) and $y_2 = 3$ (the point $(1, 2)$). If $x_2 = 1$, he picks $y_2 = 1$ with probability $\frac{1}{3}$ and $y_2 = 2$ with probability $\frac{2}{3}$. For $x_2 = 0$, player II has no choice. He is not allowed to guess the same number as his opponent and we can see in the diagram that $y_2(0, 0) = 1$. We omit the 1, the value of the conditional probability of choosing $y_2(0, 0) = 1$, since it is clear anyway. For $y_1 = 1$, we recognize two of those fixed guesses: $y_2(0, 1) = 0$ and $y_2(1, 1) = 2$. In the diagram that corresponds to $y_1 = 2$, we illustrate how we deal with probability zero: we simply don't draw the dot. Since it is clear now that the probability of choosing $y_2(0,2) = 0$ must be equal to one, we don't write this number explicitly in the figure.

Note that it is not possible to display all so-called infeasible strategies for the players in the diagrams. For example, to enable player I to guess a sum $y_1 < x_1$, we would have to extend the (x_1, x_2)-diagram at the bottom. Also, to display a strategy in which player II guesses a sum of $m + n$ while he picks $x_2 = 0$ himself, we would have to make an extension of the diagram to the right. As we already mentioned in the introduction of this section, these types of strategies are never needed in equilibrium. Therefore, this incompleteness of the diagrams is not a problem. For player II it is immediately clear that there is no point in not trying to win. For player I infeasible strategies could be useful, at least in theory, to try to deceive the opponent with his irrational guess. However, also for the first player these strategies turn out to be redundant when we look for an equilibrium for any $C_{m,n}$.

7.3.1.2 Expected payoffs

For the combination of the strategies in Figure 7.1 and Figure 7.2, we can compute the expected payoff for player I (and directly derive the expected payoff for player II in this zero-sum game) by summing over all possible combinations of takes and guesses that occur with positive probability. For example, the combination $(x_1, x_2, y_1, y_2) = (0, 0, 1, 0)$ occurs with probability

$$Pr\{x_1 = 0\}Pr\{x_2 = 0\}Pr\{y_1(0) = 1\}Pr\{y_2(0,1) = 0\} = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot 1 = \frac{1}{48}.$$

With this combination of takes and guesses player II wins, for $y_2 = x_1 + x_2 = 0$. The payoff for player I is therefore -1. Table 7.1 illustrates the computations that result in the expected payoff of $\frac{11}{288}$ for player I. Only the combinations (x_1, x_2, y_1, y_2) that occur with positive probability are included in the table.

7.3.1.3 Best replies

We have introduced our graphical representation of strategies for coin games and we have illustrated how to compute the expected payoff that results from a combination of strategies. Since we are going to study equilibria, best replies

x_1	x_2	y_1	y_2	prob	payoff	prob × payoff
0	0	1	0	$\frac{1}{48}$	-1	$-\frac{1}{48}$
0	0	2	0	$\frac{1}{16}$	-1	$-\frac{1}{16}$
0	1	1	2	$\frac{1}{24}$	1	$\frac{1}{24}$
0	1	2	1	$\frac{1}{8}$	-1	$-\frac{1}{8}$
0	2	1	2	$\frac{1}{96}$	-1	$-\frac{1}{96}$
0	2	1	3	$\frac{1}{96}$	0	0
0	2	2	3	$\frac{1}{16}$	1	$\frac{1}{16}$
1	0	1	1	$\frac{1}{24}$	1	$\frac{1}{24}$
1	0	2	0	$\frac{1}{8}$	0	0
1	1	1	1	$\frac{1}{36}$	0	0
1	1	1	2	$\frac{1}{18}$	-1	$-\frac{1}{18}$
1	1	2	2	$\frac{1}{4}$	1	$\frac{1}{4}$
1	2	1	2	$\frac{1}{48}$	0	0
1	2	1	3	$\frac{1}{48}$	-1	$-\frac{1}{48}$
1	2	2	2	$\frac{1}{16}$	0	0
1	2	2	3	$\frac{1}{16}$	-1	$-\frac{1}{16}$
total				1		$\frac{11}{288}$

Table 7.1: Computing the expected payoff of the combination of strategies in Figures 7.1 and 7.2.

will play an important role in the remainder of this chapter. Let us see how we derive best replies for each player against a given strategy of the opponent.

First, we study the possibilities of player II against the strategy of player I that is depicted in Figure 7.1. In Figure 7.3, this strategy is shown again, but this time the probabilities for taking and guessing are not separated. The probabilities that are given in the diagram are for the four combinations (x_1, y_1) that are chosen with positive probability. For example, we learn from Figure 7.3 that player I picks the combination $(x_1, y_1) = (0, 1)$ with a probability of $\frac{1}{12}$. This number was found by simply multiplying the probability of taking $x_1 = 0$ coins, $\frac{1}{3}$, and the probability of guessing $y_1 = 1$ with 0 coins in hand, $\frac{1}{4}$.

The easiest way to study the possibilities of player II is to consider each possible value of x_2 separately and see what the optimal corresponding choices

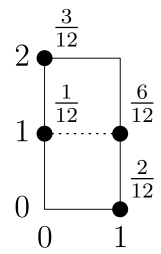


Figure 7.3: Probabilities for (x_1, y_1)-combinations for player I's strategy.

$y_2(x_2, y_1)$ are. To see what is the best reply, we compare the results for all $x_2 \in [0, n]$. Please observe the following: given the strategy of player I and the choice of x_2 by player II, the probability with which player I wins the play is fixed. Therefore, optimality regarding the selection of $y_2(x_2, y_1)$ only concerns the probability with which player II wins.

Suppose first that player II chooses a strategy with $x_2 = 0$. Then he loses if player I selects one of the points on the corresponding row, $(0, 0)$ and $(1, 0)$. According to Figure 7.3, this happens with probability $0 + \frac{2}{12} = \frac{2}{12}$. What choices of y_2 are optimal for player II, given his choice $x_2 = 0$? He must make a decision for $y_2(0, y_1)$ for each value of y_1 that player I can guess. From Figure 7.3 we know that player I guesses either $y_1 = 1$ or $y_1 = 2$.

Let us focus on the case $y_1 = 1$ first. Two of the points in Figure 7.3 that are chosen with positive probability correspond to $y_1 = 1$: $(0, 1)$ and $(1, 0)$. In the first case, the correct total number of coins taken by the players is $0+0 = 0$, in the second case the total is $1 + 0 = 1$. Since $y_1 = 1$, player II is not allowed to guess $y_2 = 1$, so the only choice for $y_2(0, 1)$ with which he can win is 0. His probability of winning is then $\frac{1}{12}$. For $y_1 = 2$, the analysis is slightly more difficult. The points in Figure 7.3 that correspond to this guess are $(0, 2)$ and $(1, 1)$. Given $x_2 = 0$, the correct totals for these points are 0 and 1 respectively. Both totals are allowed as a guess, so player II has a choice. He can select the point on the line $y_1 = 2$ for which the conditional probability that player I chooses it, given $y_1 = 2$, is maximal. This is equivalent to selecting the point on the line $y_1 = 2$ for which the probability shown in Figure 7.3 is maximal. In this case, the optimal choice is $y_2(0, 2) = 1$. With this choice, player II wins with probability $\frac{6}{12}$, the probability with which player I plays $(x_1, y_1) = (1, 2)$. The total probability with which player II wins is now $\frac{1}{12} + \frac{6}{12} = \frac{7}{12}$. Combining this with the probability of player I winning, $\frac{2}{12}$, results in an expected payoff of $\frac{5}{12}$ for player II.

We can apply similar reasoning to strategies of player II with $x_2 = 1$ and $x_2 = 2$ and find that the maximal expected payoffs for player II in these cases are $-\frac{2}{12}$ and $\frac{5}{12}$ respectively. A best reply of player II against the strategy of player I from Figure 7.1 (not unique) is therefore the strategy that we discussed, with $x_2 = 0$, $y_2(0, 1) = 0$ and $y_2(0, 2) = 1$. The corresponding expected payoff for player II is $\frac{5}{12}$.

Finding a best reply for player I against player II's strategy from Figure 7.2 is easier. We simply compute the expected payoffs for all $(x_1, y_1) \in S_2$ and compare them. Consider $(x_1, y_1) = (0, 1)$. With this strategy, player I wins with probability $\frac{1}{2}$, the probability that $x_2 = 1$, but he loses with probability $\Pr\{(x_2, y_2(x_2, 1)) = (0, 0)\} + \Pr\{(x_2, y_2(x_2, 1)) = (2, 2)\} = \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{8}$. His expected payoff with this strategy is therefore $\frac{1}{2} - \frac{3}{8} = \frac{1}{8}$. By computing the expected payoff for all his strategies, we can conclude that the unique best reply of player I is $(x_1, y_1) = (1, 2)$, for which the expected payoff equals $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

In the remainder of section 7.3, we describe equilibria for $C_{m,n}$ for all $(m, n) \in \mathbb{N}^2$. The results will be grouped into a number of classes of combinations of m and n. Within each class, the presented equilibrium strategies have a similar structure.

7.3.2 Fair coin games

Before we start with the analysis of the (m, n)-coin games for which $m \geq n$, we formulate a trivial but helpful result that enables us to use the value of $C_{m,n}$ for a certain combination (m, n) to derive bounds for the values of games with a different number of coins for one of the players. The value of $C_{m,n}$ is denoted by $v(C_{m,n})$.

Lemma 7.3.1 *For all $m, n \in \mathbb{N}$, the following two statements hold:*

(a) $v(C_{m,n}) \leq v(C_{m+1,n})$,

(b) $v(C_{m,n}) \geq v(C_{m,n+1})$.

Proof. The validity of both statements is verified by realizing that a player can ignore extra possibilities he gets by an increase of the number of coins that is available to him. By copying his equilibrium strategy from $C_{m,n}$, player I will be able to guarantee himself at least $v(C_{m,n})$ in the game $C_{m+1,n}$. This is what statement (a) says. Analogous reasoning leads to statement (b). □

As we have already mentioned, Schwartz (1959) has studied the special class of (m, n)-coin games for which $m = n$. He called the games n-coin games.

Proposition 7.3.2 (Schwartz (1959)) *Let $m \in \mathbb{N}$. Then the (m, m)-coin game is fair, i.e., $v(C_{m,m}) = 0$.*

Proof. We show that $v(C_{m,m}) \geq 0$ and postpone the other half of the proof to (the proof of) Theorem 7.3.3. Consider the behavioural strategy μ for player I that is shown in Figure 7.4 and defined by the probabilities $\mu(x_1, y_1) = \mu_1(x_1)\mu_2(y_1)$, where

$$\mu_1(x_1) = \tfrac{1}{m+1} \quad \text{for each } x_1 \in [0, m],$$

$$\mu_2(y_1) = \begin{cases} 1 & \text{if } y_1 = m, \\ 0 & \text{otherwise.} \end{cases}$$

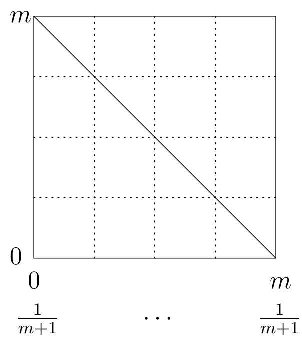


Figure 7.4: An optimal strategy for player I in $C_{m,m}$.

When player I plays according to μ, then his probability of winning is exactly $\frac{1}{1+m}$ against any strategy $(x_2, y_2) \in S_2$ of player II. Player II wins with probability $\frac{1}{m+1}$ if he uses only feasible strategies (i.e., if he puts all of his conditional probability of choosing $y_2(x_2, 1)$ inside the (x_1, x_2)-diagram) and with a lower probability otherwise. Therefore, for any $(x_2, y_2) \in S_2$ the expected payoff of player I is

$$U(\mu,(x_2,y_2)) = \Pr\{\text{I wins}\} - \Pr\{\text{II wins}\} \geq \frac{1}{1+m} - \frac{1}{1+m} = 0.$$

Therefore, $v(C_{m,m}) \geq 0$. $\square$

In the next theorem we show that a much larger class of (m,n)-coin games is fair.

Theorem 7.3.3 *The m,n-coin game is fair if $m \geq n$.*

Proof. The combination of Lemma 7.3.1(a) and Proposition 7.3.2 already shows that $v(C_{m,n}) \geq 0$. We will define a strategy ν for player II, which guarantees him that he will not have to pay more than zero. In this way we show that $v(C_{m,n}) \leq 0$. Before we can define this strategy, we have to define the sets

$$C(y_1) = [y_1 - n, y_1] \cap [0,m].$$

For a given y_1, $C(y_1)$ is the set of values for x_1 for which (x_1,y_1) is a feasible strategy. We use this set to define a set of points in $\mathbb{N}^2$, $F(x_2,y_1) = \{(a,x_2) \mid a \in C(y_1)\}$. Figure 7.5 illustrates such a set in an (x_1,x_2)-diagram.

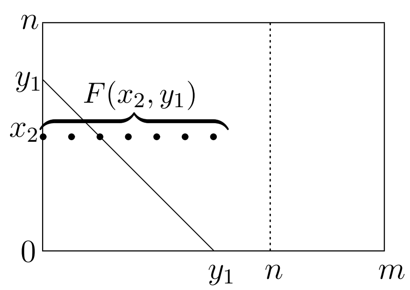


Figure 7.5: The set $F(x_2,y_1)$.

Now we are ready to define the mixed strategy ν for player II, which is determined by the probabilities $\nu(x_2,y_2|y_1) = \nu_1(x_2)\nu_2(y_2|x_2,y_1)$, where

$$\nu_1(x_2) = \frac{1}{n+1} \quad \text{for all } x_2 \in \{0,\dots,n\}$$

and

$$\nu_2(y_2|x_2,y_1) = \begin{cases} \frac{1}{|F(x_2,y_1)|-1} & \text{if } ((y_2 - x_2, x_2) \in F(x_2,y_1) \setminus \{(y_1 - x_2, x_2)\}) \\ & \quad \wedge((y_1 - x_2, x_2) \in F(x_2,y_1)), \\ \frac{1}{|F(x_2,y_1)|} & \text{if } ((y_2 - x_2, x_2) \in F(x_2,y_1)) \\ & \quad \wedge((y_1 - x_2, x_2) \notin F(x_2,y_1)), \\ 1 & \text{if } (x_2 = 0) \wedge (y_1 = 0) \wedge (y_2 = 1), \\ 1 & \text{if } (x_2 = n) \wedge (y_1 = m+n) \wedge (y_2 = m+n-1), \\ 0 & \text{otherwise.} \end{cases}$$

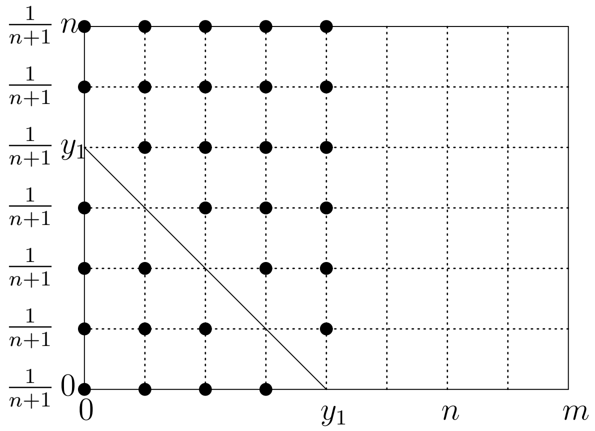


Figure 7.6: Sketch of the structure of an optimal strategy for player II in $C_{m,n}$ with $m \geq n$.

The third and fourth line of the specifications of ν_2 are arbitrary, but necessary for ν to be properly defined. Figure 7.6 shows the structure of ν for a specific y_1. Conditional probabilities for the choice of y_2 are omitted to keep the figure clear. On each x_2-row in the grid, all dots are chosen with equal probability, such that these probabilities sum to one. With infeasible strategies of the form (x_1, y_1) with $x_1 \notin C(y_1)$, player I cannot win, so his expected payoff is non-positive. With a feasible strategy, (x_1, y_1) with $x_1 \in C(y_1)$, the probability that player I wins is $\frac{1}{n+1}$. It is immediately clear from Figure 7.6 that the probability that player II wins against this strategy is

$$\begin{aligned}\Pr\{\text{II wins}\} &= \frac{1}{n+1}\left((|C(y_1)|-1)\frac{1}{|C(y_1)|-1} + ((n+1)-|C(y_1)|)\frac{1}{|C(y_1)|}\right)\\ &= \frac{1}{n+1} + \frac{(n+1)-|C(y_1)|}{(n+1)|C(y_1)|} \geq \frac{1}{n+1},\end{aligned}$$

with equality for the y_1 for which $[y_1 - n, y_1] \subseteq [0, m]$. As a result,

$$U((x_1, y_1), \nu) = \Pr\{\text{I wins}\} - \Pr\{\text{II wins}\} \leq \frac{1}{1+n} - \frac{1}{1+n} = 0.$$

□

Note that the result of Schwartz (1959), Proposition 7.3.2, can now be seen as a corollary of Theorem 7.3.3, since the case $m = n$ is included in the case $m \geq n$. In particular, the strategy ν in the proof of Theorem 7.3.3 can therefore also be used for the second half of the proof of Proposition 7.3.2.

Example 7.3.4 ($C_{3,2}$) In the $(3,2)$-coin game, the strategy shown in Figure 7.7 is optimal for player II and guarantees that the expected payoff of player I will not be positive. ◁

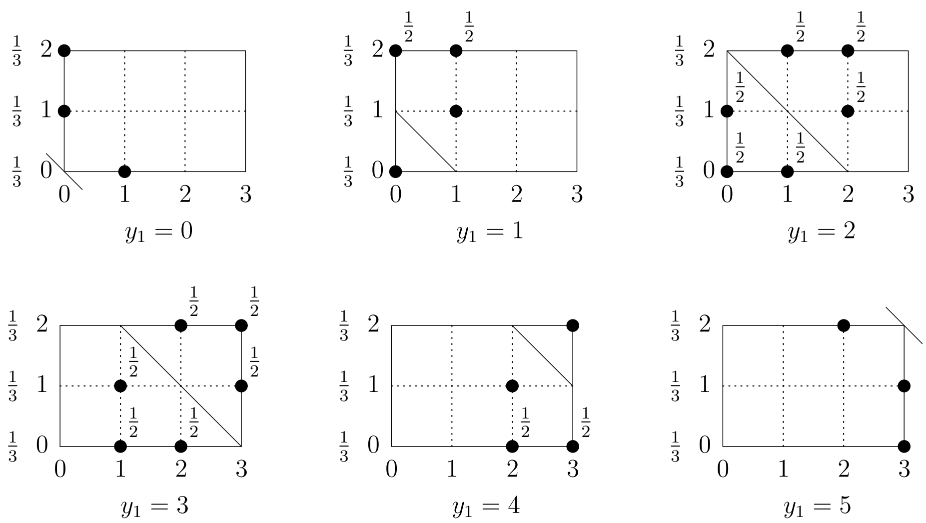


Figure 7.7: An optimal strategy for player II in $C_{3,2}$.

7.3.3 Games in which player II has one coin more

In the next theorem, we give the value of all coin games in which player II has one coin more than player I.

Theorem 7.3.5 *Let $m \in \mathbb{N}$ and let $n = m + 1$. Then $v(C_{m,n}) = -\frac{1}{2m+3}$.*

Proof. Consider the strategy μ for player I that is depicted in Figure 7.8. The strategy is defined by the following taking and guessing probabilities: $\mu(x_1, y_1) = \mu_1(x_1)\mu_2(y_1|x_1)$, where

$$\mu_1(x_1) = \begin{cases} \frac{5}{4m+6} & \text{if } x_1 \in \{0, m\}, \\ \frac{4}{4m+6} & \text{if } x_1 \in [1, m-1], \end{cases}$$

$$\mu_2(y_1|x_1) = \begin{cases} \frac{1}{2} & \text{if } (y_1 \in [m, m+1]) \wedge (x_1 \in [1, m-1]), \\ \frac{2}{5} & \text{if } (y_1 = m) \wedge (x_1 = 0), \\ \frac{3}{5} & \text{if } (y_1 = m+1) \wedge (x_1 = 0), \\ \frac{3}{5} & \text{if } (y_1 = m) \wedge (x_1 = m), \\ \frac{2}{5} & \text{if } (y_1 = m+1) \wedge (x_1 = m), \\ 0 & \text{otherwise.} \end{cases}$$

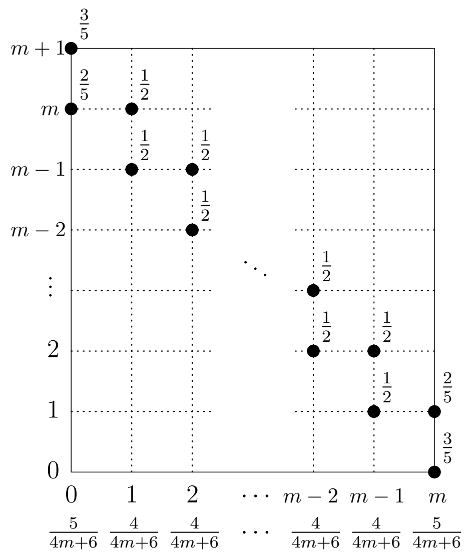


Figure 7.8: An optimal strategy for player I in $C_{m,m+1}$.

Without giving the formal proof of optimality of μ, we demonstrate how one can quickly check what player II can achieve against this strategy. In our reasoning, we follow the lines of section 7.3.1.3. First observe that one can compute the conditional probability that player I has chosen x_1, given that he has guessed a specific y_1. For example,

$$\Pr\{x_1 = 0 | y_1 = m+1\} = \frac{\frac{3}{5} \cdot \frac{5}{4m+6}}{\frac{3}{5} \cdot \frac{5}{4m+6} + (m-1) \cdot \frac{1}{2} \cdot \frac{4}{4m+6} + \frac{3}{5} \cdot \frac{5}{4m+6}} = \frac{3}{2m+4}.$$

Let us see, for example, what player II can achieve against μ by taking $x_2 = m$ and selecting his guesses optimally. Player II knows that he will lose with $x_2 = m$ if his opponent plays $(x_1, y_1) \in \{(0, m), (1, m+1)\}$. Player I will select one of these two strategies with probability $\frac{5}{4m+6} \cdot \frac{2}{5} + \frac{4}{4m+6} \cdot \frac{1}{2} = \frac{2}{2m+3}$. According to μ, player I guesses either $y_1 = m$ or $y_1 = m+1$. To maximize his winning probability, player II has to compute for which $x \in [0, m] \setminus \{(y_1 - m)\}$ the probability $\Pr\{x_1 = x | y_1 = m\}$ is maximized. He has to do the same for the probability $\Pr\{x_1 = x | y_1 = m+1\}$. For the case $y_1 = m$, this conditional probability is maximal for $x_1 = m$, $\Pr\{x_1 = m | y_1 = m\} = \frac{5}{4m+6} \cdot \frac{3}{5} = \frac{3}{4m+6}$. For player II, it is therefore optimal to choose $y_2(m, m) = 2m$. If $y_1 = m+1$, the maximal probability is assigned to $x_1 = 0$, and it is also equal to $\frac{3}{4m+6}$. So player II should choose $y_2(m, m+1) = m$. If he does this, he will win against μ (with x_2 in his hand) with probability $2 \cdot \frac{3}{4m+6} = \frac{3}{2m+3}$. So the expected payoff for player I will be $\frac{2}{2m+3} - \frac{3}{2m+3} = -\frac{1}{2m+3}$. By considering all other possible values of x_2, we can show that the expected payoff for player I is never lower than $-\frac{1}{2m+3}$.

Next, consider the strategy ν for player II that is shown in (x_1, x_2)-diagrams in Figure 7.9. The taking probabilities can be read directly from the diagrams. We don't explicitly list all underlying guessing probabilities, but we give the idea behind the construction of the strategy diagrams. Let us fix y_1 for a moment. The corresponding y_1-line crosses at least one of the rows that player II selects with positive probability, say p. The column in which this crossing occurs, corresponds to a value of x_1. With this number of coins in hand, player I wins with probability p. In order to guarantee a value $v < 0$ for player II, the strategy must imply a probability $p + v$ of winning for player II against this combination of x_1 and y_1. This probability should come from the other x_2-rows that are selected with positive probability. In this way we ensure column-wise compensations for each possible value of y_1. This guarantees the value v for player II against any choice of (x_1, y_1) by player I. $\square$

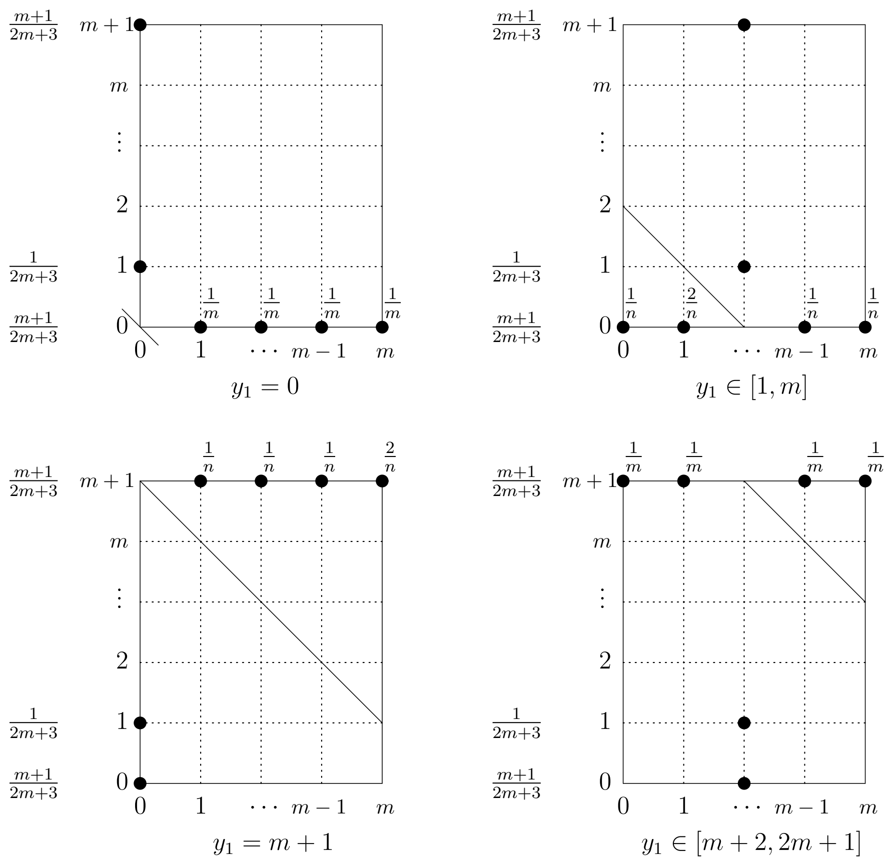


Figure 7.9: Optimal strategy for player II in $C_{m,m+1}$.

Example 7.3.6 ($C_{2,3}$) In the $(2,3)$-coin game, the strategy shown in Figure 7.10 is optimal for player I, while the strategy given in Figure 7.11 is optimal for player II. The value of $C_{2,3}$ is $-\frac{1}{7}$. $\triangleleft$

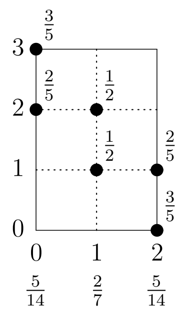


Figure 7.10: An optimal strategy for player I in $C_{2,3}$.

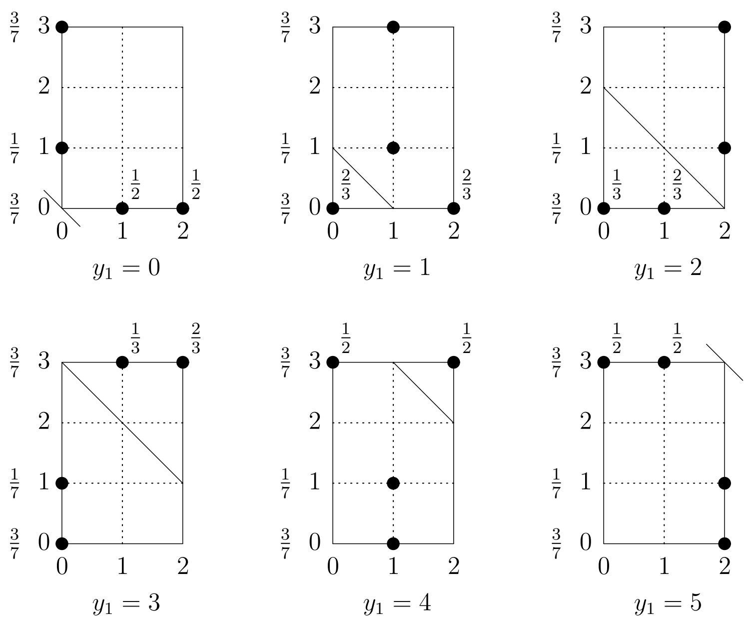


Figure 7.11: An optimal strategy for player II in $C_{2,3}$.

7.3.4 A special case: $C_{m,n(m,k)}$

The games in the following proposition turn out to be special (boundary) cases (see Table 7.2), with respect to their values, within the collection of (m,n)-coin games with $m < n$. The proposition gives lower bounds for the values for these games. These lower bounds will turn out to be tight later in the chapter.

Proposition 7.3.7 *Let $m \in \mathbb{N}$ and let $k \in \mathbb{N}$ with $k \geq 2$. Define $n(m,k) = k(m+1)-1$. Then $v(C_{m,n(m,k)}) \geq \frac{m-n(m,k)}{(m+1)(n(m,k)+1)}$.*

Proof. Consider the behavioural strategy μ for player I that is shown in Figure 7.12 and defined by the probabilities $\mu(x_1,y_1) = \mu_1(x_1)\mu_2(y_1)$, where

$$\begin{aligned}\mu_1(x_1) &= \tfrac{1}{m+1} \text{ for all } x_1 \in [0,m],\\ \mu_2(y_1) &= \begin{cases} \frac{1}{k} & \text{if } y_1 \in \{j(m+1)-1 \mid j \in [1,k]\},\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

The idea behind the strategy is that each x_2-row is covered by exactly one (x_1,y_1) combination, played with probability $\frac{1}{k(m+1)}$. We can apply the same line of reasoning as in the proof of Theorem 7.3.5, using maximum conditional probabilities of having chosen x_1, given y_1. In this way, the reader can verify that player I loses with the strategy μ with a probability that is at most equal to $\frac{1}{m+1}$, so that μ guarantees the value that is given in Proposition 7.3.7. $\square$

Example 7.3.8 ($C_{2,5}$) In the $(2,5)$-coin game, the strategy shown in Figure 7.13 is optimal for player I and guarantees that his expected payoff will not be smaller than $-\frac{1}{6}$. $\triangleleft$

7.3.5 Another special case: $C_{m,n(m,k-1)+1}$

In this section we study another class of special combinations of m and n. In the games of the next proposition, player II has (roughly speaking) one coin more than in the games of the special case of section 7.3.4. For this collection of games, which also turn out to form a boundary case (see Table 7.2), we derive an upper value.

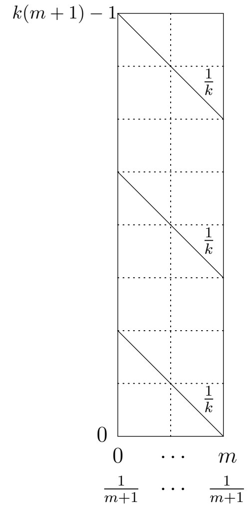


Figure 7.12: An optimal strategy for player I in $C_{m,k(m+1)-1}$ ($k \in \mathbb{N}$, $k \geq 2$).

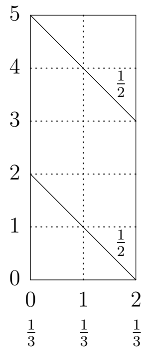


Figure 7.13: An optimal strategy for player I in $C_{2,5}$.

Proposition 7.3.9 *For all $m \in \mathbb{N}$ and all $k \in \mathbb{N}$ with $k \geq 2$, we define $n(m,k) = k(m+1)-1$. Let $k \in \mathbb{N}$ with $k \geq 3$.*[1] *Then*

$$v(C_{m,n(m,k-1)+1}) \leq \frac{m - n(m,k)}{(m+1)(n(m,k)+1)}.$$

Proof. Consider the following mixed strategy ν for player II. Define, for all $(x_2, y_2) \in [0, n(m,k-1)+1] \times [0, m+n(m,k-1)+1]$ and all $y_1 \in [0, m+$

[1]Proposition 7.3.9 concerns the games in which player II has one coin more than in the games of Proposition 7.3.7. Although we require $k \geq 3$ here, the value of n that we consider is $n(m,k-1)+1$. So, also for the case $k = 2$ in Proposition 7.3.7, the game in which player II has one coin more is included here.

$n(m, k-1)+1]$,

$$\nu(x_2, y_2|y_1) = \nu_1(x_2)\nu_2(y_2|x_2, y_1),$$

where

$$\nu_1(x_2) = \begin{cases} \frac{1}{k} & \text{if } x_2 \bmod (m+1) = 0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\nu_2(y_2|x_2, y_1) = \begin{cases} \frac{1}{m} & \text{if } (y_1 \in [x_2, x_2+m]) \wedge (y_2 \in [x_2, x_2+m] \setminus \{y_1\}), \\ \alpha & \text{if } (y_1 \notin [x_2, x_2+m]) \wedge (y_2 \in [x_2, x_2+m]) \\ & \qquad \wedge(|y_2 - y_1| \bmod (m+1) = 0), \\ \beta & \text{if } (y_1 \notin [x_2, x_2+m]) \wedge (y_2 \in [x_2, x_2+m]) \\ & \qquad \wedge(|y_2 - y_1| \bmod (m+1) \neq 0), \\ 0 & \text{otherwise.} \end{cases}$$

Here,

$$\alpha = \frac{k+m}{(k-1)(m+1)} \quad \text{and} \quad \beta = \frac{(k-1)m - (m+1)}{(k-1)m(m+1)}.$$

It is easy to check that

$$\begin{aligned} \sum_{x_2 \in [0, n(m,k-1)+1]} \nu_1(x_2) &= 1 \\ \text{and} \quad \sum_{y_2 \in [0, m+n(m,k-1)+1]} \nu_2(y_2|x_2, y_1) &= 1 \text{ for all } (x_2, y_1). \end{aligned}$$

and that $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. Thus, ν_1 and ν_2 are well-defined probability distributions. Figure 7.14 gives an illustration of the (conditional) probabilities that ν prescribes for a game $C_{m,n(m,k-1)+1}$ for a specific value of y_1. The idea behind the strategy is as follows. The given y_1-line crosses exactly one of the x_2-rows that is chosen with positive probability. The column in which this crossing occurs, indicates with which choice of x_1 player I will win. This winning probability of player I should be made up for by generating a probability of winning for player II in the same column. This compensation is taken care of by the αs. The values of α and β are chosen in such a way that

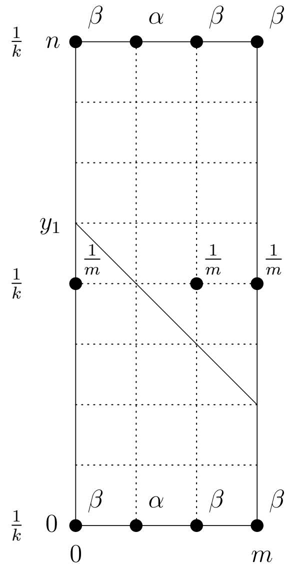


Figure 7.14: An illustration of the strategy ν, which is optimal for player II in $C_{m,n(m,k-1)+1}$.

the excess probability of winning for player II is the same in all x_1-columns. Now, let $(x_1, y_1) \in [0, m] \times [0, m + n(m, k-1) + 1]$. Then, if $U(x, y)$ denotes the expected payoff for player I for the (mixed) strategy profile (x, y), we can determine $U((x_1, y_1), \nu)$ by distinguishing two cases:

(i) $|y_1 - x_1| \bmod (m+1) = 0$ (a positive probability of winning for player I),

(ii) $|y_1 - x_1| \bmod (m+1) \neq 0$ (probability zero of winning for player I).

Case (i):

$$\begin{aligned}
U((x_1,y_1),\nu) &= \nu_1(y_1-x_1) - \sum_{x_2\in[0,(k-1)(m+1)]\setminus(y_1-x_1)} \nu_1(x_2)\nu_2(x_1+x_2|x_2,y_1) \\
&= \frac{1}{k} - \sum_{\substack{\{x_2 \mid x_2 \bmod (m+1)=0 \\ \text{and } x_2\neq y_1-x_1\}}} \nu_1(x_2)\nu_2(x_1+x_2|x_2,y_1) \\
&= \frac{1}{k} - \frac{1}{k} \sum_{\substack{\{x_2 \mid x_2 \bmod (m+1)=0 \\ \text{and } x_2\neq y_1-x_1\}}} \nu_2(x_1+x_2|x_2,y_1) \\
&= \frac{1}{k} - \frac{1}{k}(k-1)\alpha = \frac{1}{k}\left(1-(k-1)\frac{k+m}{(k-1)(m+1)}\right) \\
&= \frac{m+1}{k(m+1)} - \frac{k+m}{k(m+1)} = -\frac{k-1}{k}\frac{1}{m+1} \\
&= \frac{m-n(m,k)}{(m+1)(n(m,k)+1)}
\end{aligned}$$

Case (ii):

$$\begin{aligned}
U((x_1,y_1),\nu) &= -\sum_{x_2\in[0,(k-1)(m+1)]} \nu_1(x_2)\nu_2(x_1+x_2|x_2,y_1) \\
&= -\frac{1}{k}\sum_{\{x_2 \mid x_2 \bmod (m+1)=0\}} \nu_2(x_1+x_2|x_2,y_1) \\
&= -\frac{1}{k}\frac{1}{m} - \frac{1}{k} \sum_{\substack{\{x_2 \mid x_2 \bmod (m+1)=0 \\ \text{and } y_1\notin[x_2,x_2+m]\}}} \nu_2(x_1+x_2|x_2,y_1) \\
&= -\frac{1}{k}\frac{1}{m} - \frac{1}{k}(k-1)\beta \\
&= -\frac{1}{k}\left(\frac{1}{m} - (k-1)\frac{(k-1)m-(m+1)}{(k-1)m(m+1)}\right) \\
&= -\left(\frac{(m+1)+(k-1)m-(m+1)}{km(m+1)}\right) = -\frac{k-1}{k}\frac{1}{m+1} \\
&= \frac{m-n(m,k)}{(m+1)(n(m,k)+1)}
\end{aligned}$$

The combination of the payoffs in both cases shows that the (mixed) strategy ν guarantees an expected payoff of $U((x_1, y_1), \nu) = \frac{n(m,k)-m}{(n(m,k)+1)(m+1)}$ for player II. $\square$

Example 7.3.10 ($C_{2,6}$) In the $(2, 6)$-coin game, the strategy shown in Figure 7.15 is optimal for player II and guarantees that the expected payoff of player I will not be higher than $-\frac{2}{9}$. $\triangleleft$

The $n(m, k)$ from the definition in Proposition 7.3.9 is exactly the value of n from Proposition 7.3.7. Therefore, combining these two propositions with Lemma 7.3.1(b) yields the following result.

Theorem 7.3.11 *Let $m \in \mathbb{N}$, let $k \in \mathbb{N}$ with $k \geq 3$ and let $n \in [n(m, k-1) + 1, n(m, k)]$. Then $v(C_{m,n}) = \frac{m-n(m,k)}{(m+1)(n(m,k)+1)}$.*

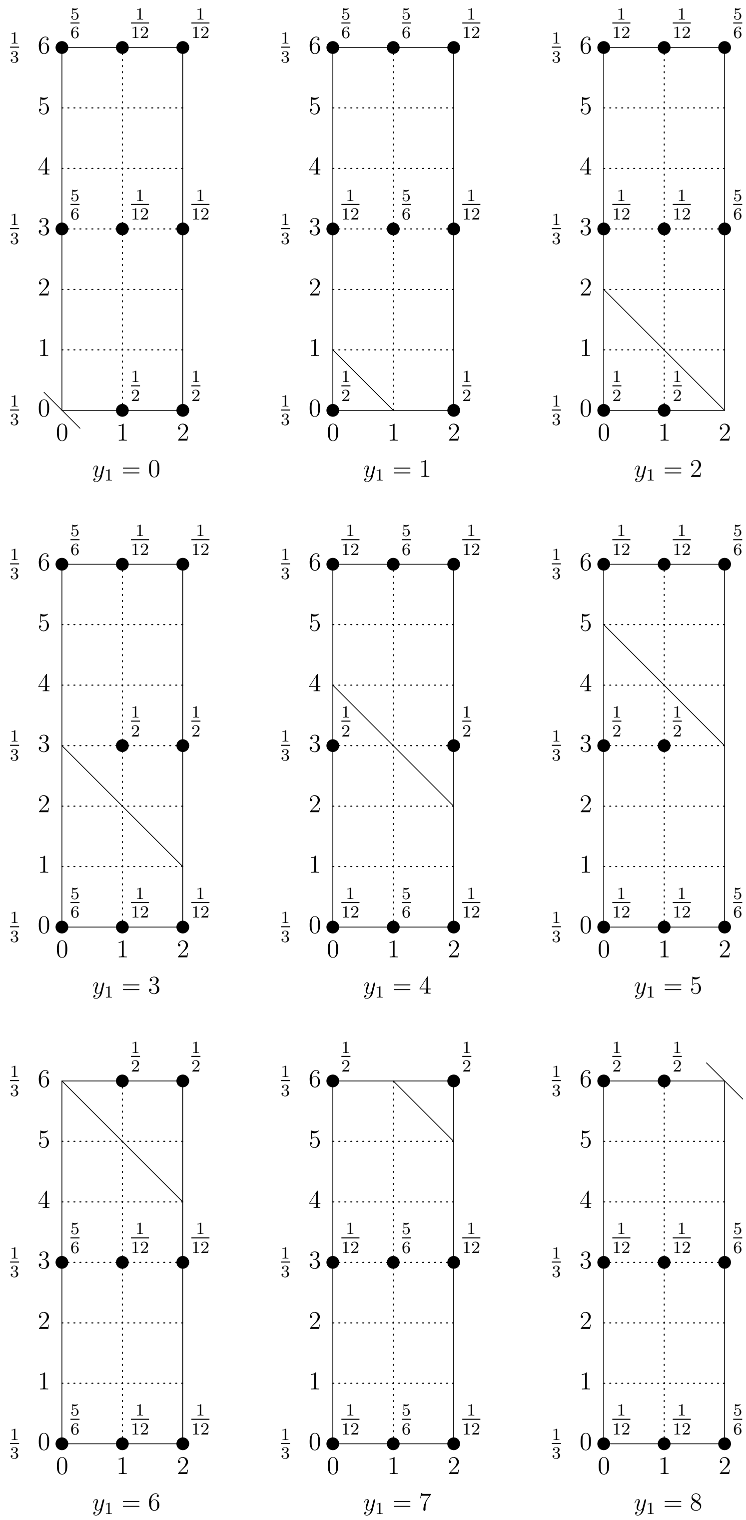


Figure 7.15: An optimal strategy for player II in $C_{2,6}$.

7.3.6 Games in which player II has two coins more

In the next proposition, we give the value of all coin games in which player II has two coins more than player I.

Proposition 7.3.12 *Let $m \in \mathbb{N}$. Then $v(C_{m,m+2}) = -\frac{1}{2(m+1)}$.*

Proof. We leave it to the reader to verify that the strategy μ for player I that is depicted in Figure 7.16 guarantees the value given in the proposition. The strategy is defined by the following taking and guessing probabilities: $\mu(x_1, y_1) = \mu_1(x_1)\mu_2(y_1)$, where

$$\mu_1(x_1) = \tfrac{1}{m+1} \text{ for all } x_1 \in [0, m],$$

$$\mu_2(y_1) = \begin{cases} \frac{1}{2} & \text{if } y_1 \in \{m, m+2\}, \\ 0 & \text{otherwise.} \end{cases}$$

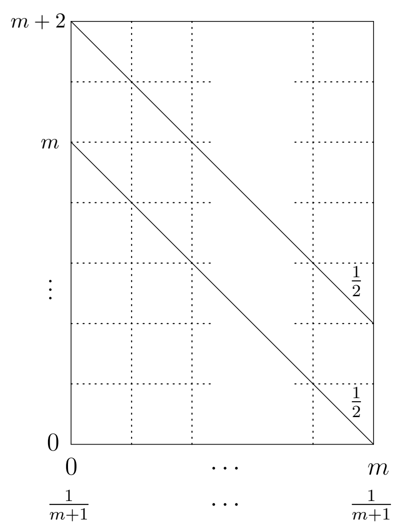


Figure 7.16: Optimal strategy for player I in $C_{m,m+2}$

Next, consider the strategy ν for player II that is shown in (x_1, x_2)-diagrams in Figure 7.17. We don't give a formal description of the taking and guessing probabilities, but we give the intuition behind the construction of the strategy. An y_1-line will intersect at most two of the four rows player II selects with positive probability. The winning probabilities for player I that result from these intersections can be compensated within these two rows (in Figure 7.17, the two dots in the rows for $x_2 \in \{m+1, m+2\}$ do the trick). The remaining rows can be used to generate an excess probability of winning for player II of at least $2 \cdot \frac{1}{4} \cdot \frac{1}{m+1} = \frac{1}{2(m+1)}$. When the y_1-line only crosses of the four rows, then any of the other three rows can be used for compensation. The remaining points on the crossed row can, for example, be selected with equal probability.

□

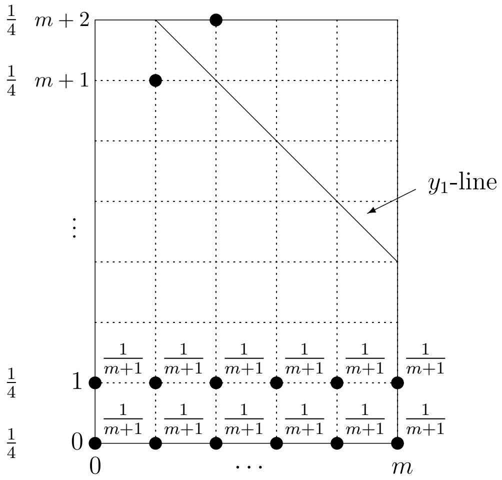


Figure 7.17: The strategy ν, which is optimal for player II in $C_{m,m+2}$.

Example 7.3.13 ($C_{2,4}$) In the $(2,4)$-coin game, the strategy shown in Figure 7.18 is optimal for player I, while the strategy given in Figure 7.19 is optimal for player II. The value of $C_{2,4}$ is $-\frac{1}{6}$.

◁

Observe that the value of $C_{m,m+2}$ is exactly the lower bound $\underline{v}$ of the value of $C_{m,2(m+1)-1}$ that we derived in section 7.3.4:

$$\begin{aligned}
\underline{v}(C_{m,2(m+1)-1}) &\overset{\text{Prop 7.3.7}}{=} \frac{m - n(m,2)}{(m+1)(n(m,2)+1)} = \frac{m-(2(m+1)-1)}{(m+1)2(m+1)} \\
&= -\frac{1}{2(m+1)} \overset{\text{Thm 7.3.12}}{=} v(C_{m,m+2}).
\end{aligned}$$

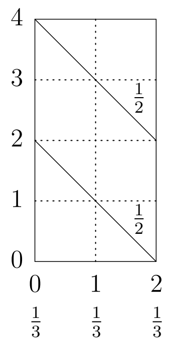


Figure 7.18: An optimal strategy for player I in $C_{2,4}$.

Therefore, we can combine the results of Theorem 7.3.12 and Proposition 7.3.7 and use Lemma 7.3.1(b) to obtain the following result.

Theorem 7.3.14 *Let $m \in \mathbb{N}$ and let $n \in [m+2, 2m+1]$. Then $v(C_{m,n}) = -\frac{1}{2(m+1)}$.*

Although Theorem 7.3.14 completes our list of values for all (m, n)-coin games (see Table 7.2), we did not yet present optimal strategies for both players for all the games. In particular, for at least one of the players we did not mention how he play optimally in the games $C_{m,n}$ with $k \in \mathbb{N}$ ($k \geq 3$) and $n \in [(k-1)(m+1), k(m+1)-2]$ and in the games $C_{m,n}$ with $m \in \mathbb{N}$ ($m \geq 3$) and $n \in [m+3, 2m+1]$. These are the games for which the values are derived in Theorems 7.3.11 and 7.3.14. Following the argument of the proof of Lemma 7.3.1, an equilibrium strategy for Player II can be copied from a game $C_{m,n}$ with a smaller value of n. Of course, this strategy is not defined for high guesses y_1, since these guesses are not allowed in the game from which player II's strategy is copied. For these values of y_1, player II has to play all feasible guesses with equal probability for each value of x_2 that he takes with positive probability.

For player I, the reader can verify that the strategy with the structure that is displayed in Figure 7.20 is optimal in all these games.

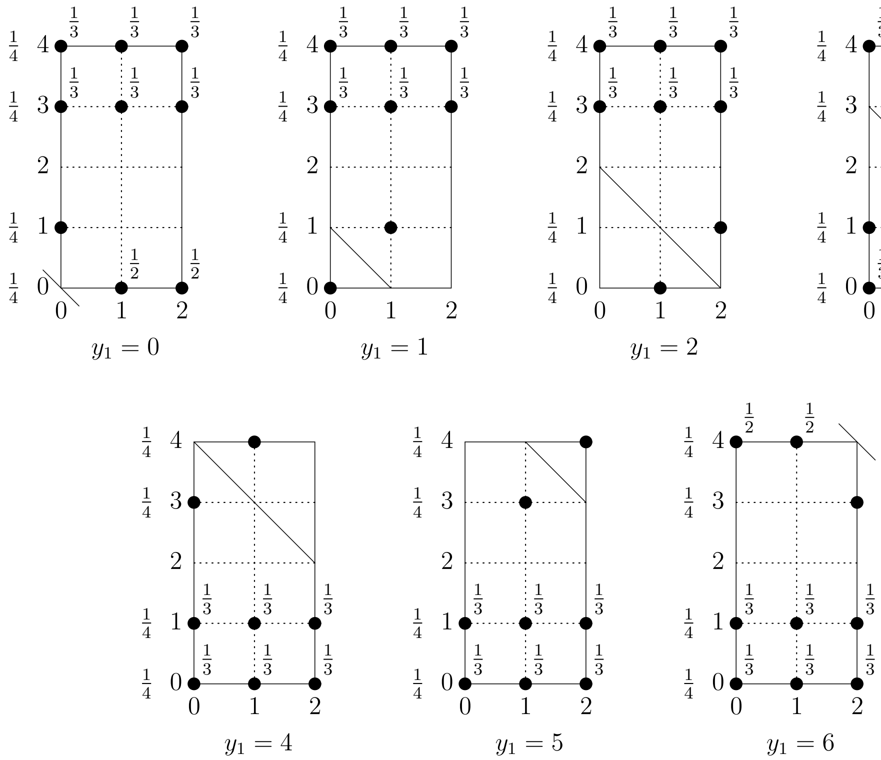


Figure 7.19: An optimal strategy for player II in $C_{2,4}$.

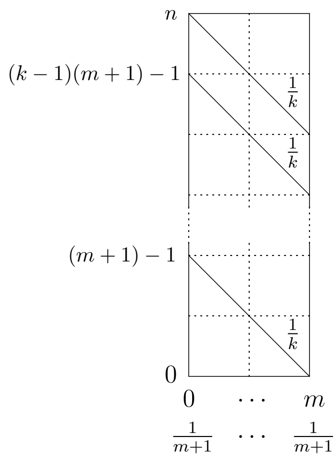


Figure 7.20: An optimal strategy for player I in the remaining (m, n)-coin games ($k = \lceil \frac{n+1}{m+1} \rceil$).

The strategy structure of Figure 7.20 is formally defined by the probabilities $\mu(x_1, y_1) = \mu_1(x_1)\mu_2(y_1)$, where

$$\begin{aligned} \mu_1(x_1) &= \frac{1}{m+1} \text{ for all } x_1 \in [0, m], \\ \mu_2(y_1) &= \begin{cases} \frac{1}{k} & \text{if } y_1 \in \{j(m+1) - 1 \mid j \in [1, k-1]\} \cup \{n\}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $k = \lceil \frac{n+1}{m+1} \rceil$. This strategy is similar to the strategy of player I in the boundary case of section 7.3.4 (see Figure 7.12). Compared to these strategies, the value of the highest guess is shifted down.

7.3.7 A summary of the results

In sections 7.3.2 to 7.3.6, we have given the values for $C_{m,n}$ for all combinations of m and n. The main results were divided over four theorems (7.3.3, 7.3.5, 7.3.11 and 7.3.14). Table 7.2 illustrates these theorems by listing the values for the (m, n)-coin games with small values of m and n.

From the table, we can get an idea about what happens if the amount of coins available to one of the players becomes extremely large. This is the subject of the following proposition.

	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
	17	$-\frac{4}{9}$	$-\frac{5}{18}$	$-\frac{1}{5}$	$-\frac{3}{20}$	$-\frac{1}{9}$	$-\frac{2}{21}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{20}$	$\cdots$
	16	$-\frac{4}{9}$	$-\frac{5}{18}$	$-\frac{1}{5}$	$-\frac{3}{20}$	$-\frac{1}{9}$	$-\frac{2}{21}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{20}$	$\cdots$
	15	$-\frac{7}{16}$	$-\frac{5}{18}$	$-\frac{3}{16}$	$-\frac{3}{20}$	$-\frac{1}{9}$	$-\frac{2}{21}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{20}$	$\cdots$
	14	$-\frac{7}{16}$	$-\frac{4}{15}$	$-\frac{3}{16}$	$-\frac{2}{15}$	$-\frac{1}{9}$	$-\frac{2}{21}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{20}$	$\cdots$
	13	$-\frac{3}{7}$	$-\frac{4}{15}$	$-\frac{3}{16}$	$-\frac{2}{15}$	$-\frac{1}{9}$	$-\frac{1}{14}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{20}$	$\cdots$
	12	$-\frac{3}{7}$	$-\frac{4}{15}$	$-\frac{3}{16}$	$-\frac{2}{15}$	$-\frac{1}{9}$	$-\frac{1}{14}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{20}$	$\cdots$
	11	$-\frac{5}{12}$	$-\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{2}{15}$	$-\frac{1}{12}$	$-\frac{1}{14}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{20}$	$\cdots$
	10	$-\frac{5}{12}$	$-\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{2}{15}$	$-\frac{1}{12}$	$-\frac{1}{14}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{21}$	$\cdots$
n	9	$-\frac{2}{5}$	$-\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{1}{10}$	$-\frac{1}{12}$	$-\frac{1}{14}$	$-\frac{1}{16}$	$-\frac{1}{19}$	0	$\cdots$
	8	$-\frac{2}{5}$	$-\frac{2}{9}$	$-\frac{1}{6}$	$-\frac{1}{10}$	$-\frac{1}{12}$	$-\frac{1}{14}$	$-\frac{1}{17}$	0	0	$\cdots$
	7	$-\frac{3}{8}$	$-\frac{2}{9}$	$-\frac{1}{8}$	$-\frac{1}{10}$	$-\frac{1}{12}$	$-\frac{1}{15}$	0	0	0	$\cdots$
	6	$-\frac{3}{8}$	$-\frac{2}{9}$	$-\frac{1}{8}$	$-\frac{1}{10}$	$-\frac{1}{13}$	0	0	0	0	$\cdots$
	5	$-\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{8}$	$-\frac{1}{11}$	0	0	0	0	0	$\cdots$
	4	$-\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{9}$	0	0	0	0	0	0	$\cdots$
	3	$-\frac{1}{4}$	$-\frac{1}{7}$	0	0	0	0	0	0	0	$\cdots$
	2	$-\frac{1}{5}$	0	0	0	0	0	0	0	0	$\cdots$
	1	0	0	0	0	0	0	0	0	0	$\cdots$
		1	2	3	4	5	6	7	8	9	$\cdots$
						m					

Table 7.2: Values for $C_{m,n}$ for small values of m and n.

Proposition 7.3.15 *Let $C_{m,n}$ be an (m,n)-coin game. Then*

$$\lim_{m\to\infty} v(C_{m,n}) = 0,$$

and

$$\lim_{n\to\infty} v(C_{m,n}) = -\frac{1}{m+1}.$$

Proof. The first part of the proposition is trivial. We prove the second part by using the expression given in Theorem 7.3.11.

$$\begin{aligned}\lim_{n\to\infty} v(C_{m,n}) &= \lim_{k\to\infty}\frac{m-n(m,k)}{(m+1)(n(m,k)+1)} = \lim_{k\to\infty}\frac{m-(k(m+1)-1)}{(m+1)((k(m+1)-1)+1)}\\ &= \lim_{k\to\infty}\frac{1-k}{k(m+1)-1} = -\frac{1}{m+1}.\end{aligned}$$

□

Comparing the result with Corollary 7.2.3, we see that the limiting value for the case where the number of coins of player II goes to infinity coincides with the limiting value for this case in morra. From Table 7.2 we can further observe the following interesting facts.

- Although coin games are never symmetric, there is a surprisingly large collection of fair (m, n)-coin games.
- For fixed values of m (and $m < n$), the value $v(C_{m,n})$ is constant for series of $m+1$ values of n. Within this series, player II is not necessarily better off with more coins available. As an example, consider the game $C_{3,5}$. The game becomes more favourable for player II, only if he gets at least three more coins available. One or two extra coins would not help him.
- On the other hand, if $m < n$, player I is always better off with one more coin if he has less coins available than his opponent. Formally, $m < n \Rightarrow v(C_{m,n}) < v(C_{m+1,n})$.
- If $n = m + 1$, i.e., if player II has only one more coin available than his opponent, player II cannot take the "regular advantage" that leads to the values for $n \geq m + 2$.

7.4 Concluding remarks

In this chapter, we have studied two classes of two-person take-and-guess games: morra and coin games. In both games, the players first have to take a number of objects and then guess the total number of objects taken by both players. In a game of morra, the players guess simultaneously, while in a coin game player II has to wait for player I's call and is not allowed to guess the same number.

The structure of coin games is less symmetric than the structure of morra. Surprisingly, all coin games in which player I has at least as many objects as player II are fair, while morra is only fair if both players have the same number of fingers available. For all other take-and-guess games in the two classes, the advantage is for the player who has more objects available than his opponent.

Unfair coin games, i.e., (m,n)-coin games with $m < n$, have the same value as (m,n)-morra only in the boundary case of section 7.3.4, where $n = k(m+1) - 1$ for some $k \in \mathbb{N}$. For all other unfair combinations of m and n, the (m,n)-coin game is more favourable for player II than (m,n)-morra: $v(C_{m,n}) < v(M_{m,n})$.

Finally, we want to mention three interesting extensions of the analysis in this chapter, which are possible subjects for further research. The first extension that deserves attention in the future is formed by take-and-guess games with more than two players. The winner of such a game receives one unit of all of his opponents. In the case of morra, where there can be multiple winners for the same play, the losers all pay one unit and the winners share the pot equally. A general difficulty in the analysis of games with more than two players, is that optimal play is not defined anymore. Multiple Nash equilibria can exist and the equilibrium strategies are not interchangeable between equilibria. Moreover, the payoffs to the players are not necessarily the same in each equilibrium; there is no such thing as a value in these games.

A second interesting modification of the game would be to make the payoffs dependent of the total number of objects taken by the players. Instead of winning one unit, the winning player receives an amount equal to this total. Guessing higher totals correctly becomes more profitable and at the same time taking higher numbers in hand becomes more risky.

The third and last extension we want to mention is one that is inspired by the way coin games were played in Dutch bars. Instead of playing one round of the take-and-guess game, the player roles are interchanged after each draw until there is a winner. Such a modification turns the game into a stochastic game, which requires a more sophisticated analysis. Especially for coin games with $m < n$ this change will probably affect the optimal strategies within a round of play too. It might become useful for player I to play infeasible strategies, since apart from winning the game it is interesting now to try to get in the advantageous role of the second player.

References

Aumann, R. (1964). Mixed and Behavior Strategies in Infinite Extensive Games. In: M. Dresher, L. Shapley, and A. Tucker (Eds.), *Advances in Game Theory*, Number 52 in Annals of Mathematics Studies, 627–650. Princeton, New Jersey: Princeton University Press.

Bassan, B., M. Scarsini and S. Zamir (1997). "I don't want to know!": Can it be rational? Discussion Paper 1997-158, The Hebrew University of Jerusalem.

Bellman, R. (1952). On Games Involving Bluffing. *Rendiconti del Circolo Matematico di Palermo*, **1**, 139–156.

Bellman, R. and D. Blackwell (1949). Some Two-Person Games Involving Bluffing. *Proceedings of the National Academy of Science*, **35**, 600–605.

Bester, H. and W. Güth (1998). Is Altruism Evolutionarily Stable? *Journal of Economic Behavior & Organization*, **34**, 193–209.

Billings, D., N. Burch, A. Davidson, R. Holte, J. Schaeffer, T. Schauenberg and D. Szafron (2003). Approximating Game-Theoretic Optimal Strategies for Full-scale Poker. In: *Proceedings of the 2003 International Joint Conference on Artificial Intelligence (IJCAI 2003)*, 661–668.

Billings, D., A. Davidson, J. Schaeffer and D. Szafron (2002). The Challenge of Poker. *Artificial Intelligence*, **134**, 201–240.

Binmore, K. (1992). *Fun and Games: A Text on Game Theory.* Lexington, Massachusetts: D.C. Heath and Company.

Borel, E. (1938). *Traité du calcul des probabilités et ses applications, Volume 4*. Paris: Gauthier-Villars.

Borel, E. (1953). On Games that Involve Chance and the Skill of the Players. *Econometrica*, **21**, 101–115.

Borm, P. (1988). Information Types: a Comparison. *Cahiers du C.E.R.O.*, **30**(1), 23–37.

Borm, P., I. García-Jurado, J. Potters and S. Tijs (1996). An Amalgamation of Games. *European Journal of Operational Research*, **89**, 570–580.

Borm, P. and B. van der Genugten (1996). On a Measure of Skill for Games with Chance Elements. Research Memorandum R7627 1996-721, Tilburg University.

Borm, P. and B. van der Genugten (1998). On the Exploitation of Casino Games: How to Distinguish between Games of Chance and Games of Skill. In: F. Patrone, I. Garcia-Jurado, and S. Tijs (Eds.), *Game Practice: Contributions from Applied Game Theory*, Theory and Decision Library, 19–33. Dordrecht: Kluwer Academic Publishers.

Borm, P. and B. van der Genugten (2001). On a Relative Measure of Skill for Games with Chance Elements. *Trabajos de Investigación Operativa*, **9**(1), 91–114.

Caillois, R. (1979). *Man, Play and Games*. New York: Schocken Books. Translation of *Les jeux et les hommes* (1961) by M. Barash.

Camerer, C. (2003). *Behavioral Game Theory: Experiments in Strategic Interaction*. Princeton: Princeton University Press.

Camerer, C. and D. Lovallo (1999). Overconfidence and Excess Entry: An Experimental Approach. *The American Economic Review*, **89**(1), 306–318.

Cassidy, J. (1998). The Last Round of Betting in Poker. *American Mathematical Monthly*, **105**(9), 825–831.

Cohen, J. (1960). *Chance, Skill, and Luck: The Psychology of Guessing and Gambling*. Harmondsworth: Penguin Books.

Costa-Gomes, M., V. Crawford and B. Broseta (2001). Cognition and Behavior in Normal-Form Games: an Experimental Study. *Econometrica*, **69**(5), 1193–1235.

Cutler, W. (1975). An Optimal Strategy for Pot-Limit Poker. *American Mathematical Monthly*, **82**(4), 368–376.

Dreef, M. and P. Borm (2005). On the Role of Chance Moves and Information in Two-Person Games. *Trabajos de Investigación Operativa*, **13**, to appear.

Dreef, M., P. Borm and B. van der Genugten (2003). On Strategy and Relative Skill in Poker. *International Game Theory Review*, **5**(2), 83–103.

Dreef, M., P. Borm and B. van der Genugten (2004a). Measuring Skill in Games: Several Approaches Discussed. *Mathematical Methods of Operations Research*, **59**(3), 375–391.

Dreef, M., P. Borm and B. van der Genugten (2004b). A New Relative Skill Measure for Games with Chance Elements. *Managerial and Decision Economics*, **25**(5), 255–264.

Dreef, M. and S. Tijs (2004). Take-and-Guess Games. CentER Discussion Paper 2004-129, Tilburg University.

Epstein, R. (1977). *The Theory of Gambling and Statistical Logic*. New York: Academic Press.

Eyster, E. (2002). Rationalizing the Past: A Taste for Consistency. Job market paper, Nuffield College, Oxford.

Ferguson, C. and T. Ferguson (2003). On the Borel and von Neumann Poker Models. In: L. Petrosjan and V. Mazalov (Eds.), *Game Theory and Applications 9*, 17–32. New York: Nova Science Publishers.

Fishburn, P. (1970). *Utility Theory for Decision Making*. New York: Wiley.

Genugten, B. van der (1997). Een keuringsmethode voor behendigheidsautomaten. EIT report, Tilburg University. (In Dutch).

Genugten, B. van der and P. Borm (1994). Een vergelijking van de behendigheid in spelen met een kanselement. Report, Department of Econometrics and Operations Research, Tilburg University. (In Dutch).

Genugten, B. van der, P. Borm and M. Dreef (2004). De toepassing van de wet op de kansspelen op de managementspelen Competitie Manager en Grand Prix Manager. Report, Department of Econometrics and Operations Research and CentER Applied Research, Tilburg University. (In Dutch).

Genugten, B. van der, J. Das and P. Borm (2001). *Behendig gokken in en rond het casino*. Schoonhoven: Academic Service. (In Dutch).

Goeree, J. and C. Holt (2001). Ten Little Treasures of Game Theory and Ten Intuitive Contradictions. *The American Economic Review*, **91**(5), 1402–1422.

Hvide, H. (2002). Pragmatic Beliefs and Overconfidence. *Journal of Economic Behavior & Organization*, **48**(1), 15–28.

Ifrah, G. (1985). *From One to Zero: A Universal History of Numbers*. New York: Viking. Translation of *Histoire universelle des chiffres* (1981) by L. Bair.

Kadane, J. (1986). Does Electronic Draw Poker Require Skill to Play? In: M. DeGroot, S. Feinberg, and J. Kadane (Eds.), *Statistics and the Law*, 333–343. New York: Wiley.

Kahneman, D. and A. Tversky (1979). Prospect Theory: An Analysis of Decision under Risk. *Econometrica*, **47**(2), 263–291.

Kamien, M., Y. Tauman and S. Zamir (1990). On the Value of Information in a Strategic Conflict. *Games and Economic Behavior*, **2**, 129–153.

Karlin, S. (1959). *Mathematical Methods and Theory in Game, Programming, and Economics II: The Theory of Inifinite Games*. Addison-Wesley Series in Statistics. Reading, Massachusetts, USA: Addison-Wesley Publishing Company, Inc.

Karlin, S. and R. Restrepo (1957). Multistage Poker Models. In: H. Kuhn and A. Tucker (Eds.), *Contributions to the Theory of Games III*, 337–363. Princeton: Princeton University Press.

Keren, G. and W. Wagenaar (1988). Chance and Skill in Gambling: A Search for Distinctive Features. *Social Behaviour*, **3**, 199–217.

Kuhn, H. (1953). Extensive Games and the Problem of Information. In: H. Kuhn and A. Tucker (Eds.), *Contributions to the Theory of Games II*, 193–216. Princeton: Princeton University Press.

Langer, E. (1975). The Illusion of Control. *Journal of Personality and Social Psychology*, **32**(2), 311–328.

Larkey, P., J. Kadane, R. Austin and S. Zamir (1997). Skill in Games. *Management Science*, **43**(5), 596–609.

Levine, P. and J. Ponssard (1977). The Values of Information in Some Nonzero Sum Games. *International Journal of Game Theory*, **6**(4), 221–229.

Luce, R. and H. Raiffa (1957). *Games and Decisions*. New York: Wiley.

Monderer, D. and L. Shapley (1996). Potential Games. *Games and Economic Behavior*, **14**(1), 124–143.

Neumann, J. von (1928). Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen*, **100**, 295–320.

Neumann, J. von and O. Morgenstern (1944). *Theory of Games and Economic Behavior*. New York: John Wiley and Sons.

Newman, D. (1959). A Model for "Real" Poker. *Operations Research*, **7**, 557–560.

Oldman, D. (1974). Chance and Skill: a Study of Roulette. *Sociology*, **8**(3), 407–426.

Palacios-Huerta, I. (2003). Professionals Play Minimax. *Review of Economic Studies*, **70**(2), 395–415.

Perdrizet, P. (1898). The Game of Morra. *The Journal of Hellenic Studies*, **18**, 129–132.

Ponssard, J. (1975). Zero-Sum Games with "Almost" Perfect Information. *Management Science*, **21**(8), 794–805.

Ponssard, J. (1976). On the Concept of the Value of Information in Competitive Situations. *Management Science*, **22**(7), 739–747.

Ponssard, J. and S. Sorin (1980). The LP Formulation of Finite Zero-Sum Games with Incomplete Information. *International Journal of Game Theory*, **9**(2), 99–105.

Ponssard, J. and S. Sorin (1982). Optimal Behavioral Strategies in 0-Sum Games with Almost Perfect Information. *Mathematics of Operations Research*, **7**(1), 14–31.

Ponssard, J. and S. Zamir (1973). Zero-Sum Sequential Games with Incomplete Information. *International Journal of Game Theory*, **2**(2), 99–107.

Rabin, M. (1993). Incorporating Fairness into Game Theory. *The American Economic Review*, **83**(5), 1281–1302.

Rabin, M. (2002). A Perspective on Psychology and Economics. *European Economic Review*, **46**(4-5), 657–685.

Rector, R. (1987). Game Theory: An Application of Probability. *Mathematics Teacher*, **80**(2), 138–142.

Reep, C., R. Pollard and B. Benjamin (1971). Skill and Chance in Ball Games. *Journal of the Royal Statistical Society. Series A (General)*, **134**(2), 623–629.

Reijnierse, J., M. Voorneveld and P. Borm (2003). Informationally Robust Equilibria. CentER Discussion Paper 2003-14, Tilburg University.

Restrepo, R. (1964). Games with a Random Move. In: M. Dresher, L. Shapley, and A. Tucker (Eds.), *Advances in Game Theory*, Number 52 in Annals of Mathematics Studies, 29–38. Princeton, New Jersey: Princeton University Press.

Sakaguchi, M. (1984). A Note on the Disadvantage for the Sente Player in Poker. *Mathematica Japonica*, **29**(3), 483–489.

Sakaguchi, M. (1985a). Information Contained in Face-up Cards in Poker. *Mathematica Japonica*, **30**(4), 535–549.

Sakaguchi, M. (1985b). On Two-Person "Real" Poker by Newman. *Mathematica Japonica*, **30**(3), 471–483.

Sakaguchi, M. (1993). Information Structures and Perfect Information in Some Two-Person Poker. *Mathematica Japonica*, **38**(4), 743–755.

Sakaguchi, M. and S. Sakai (1982). Solutions to a Class of Two-Person Hi-Lo Poker. *Mathematica Japonica*, **27**(6), 701–714.

Sakaguchi, M. and S. Sakai (1992). Solutions of some Three-Person Stud and Draw Poker. *Mathematica Japonica*, **37**(6), 1147–1160.

Sakai, S. (1986). A Model for Real Poker with an Upper Bounds of Assets. *Journal of Optimization Theory and Applications*, **50**(1), 149–163.

Scarne, J. (1990). *Guide to Modern Poker*. New York: Simon and Schuster.

Schwartz, B. (1959). Solution of a Set of Games. *American Mathematical Monthly*, **66**(8), 693–701.

Shafir, E. (1994). Uncertainty and the Difficulty of Thinking through Disjunctions. *Cognition*, **50**, 403–430.

Stengel, B. von (1996). Efficient Computation of Behavior Strategies. *Games and Economic Behavior*, **14**, 220–246.

Szép, J. and F. Forgó (1985). *Introduction to the Theory of Games*. Dordrecht: D. Reidel Publishing Company.

Thaler, R. and E. Johnson (1990). Gambling with the House Money and Trying to Break Even: The Effects of Prior Outcomes on Risky Choice. *Management Science*, **36**(6), 643–660.

Tversky, A. and D. Kahneman (1971). Belief in the Law of Small Numbers. *Psychological Bulletin*, **76**(2), 105–110.

Voorneveld, M. (1999). *Potential Games and Interactive Decisions with Multiple Criteria*. Ph. D. thesis, CentER, Tilburg University.

Ware, M. and J. Kadane (2002). Chance and Skill in Games: Electronic Draw Poker. *Jurimetrics*, **43**(1), 129–134.

Warglien, M., M. Devetag and P. Legrenzi (1999). Mental models and naive play in normal form games. Working paper, Computable and Experimental Economics Laboratory, University of Trento.

Weinstein, N. (1980). Unrealistic Optimism about Future Life Events. *Journal of Personality and Social Psychology*, **39**(5), 806–820.

Wohl, M. and M. Enzle (2002). The Deployment of Personal Luck: Sympathetic Magic and Illusory Control in Games of Pure Chance. *Personality and Social Psychology Bulletin*, **28**(10), 1388–1397.

Yu, P. and R. Cowan (1995). A Statistical Model for Duplicate Tournaments in Games of Luck and Skill. *Australian Journal of Statistics*, **37**(1), 1–17.

Zadeh, N. (1977). Computation of Optimal Poker Strategies. *Operations Research*, **25**(4), 541–562.

Behendigheid en strategie in spelen

Samenvatting

Spelletjes, of ze nu thuis of in het casino gespeeld worden, vormen een interessant discussieonderwerp, zowel voor spelers als voor toeschouwers. Bijna iedereen heeft een mening over hoe roulette of blackjack moet worden gespeeld. Daarnaast hebben veel mensen wel een idee over hoeveel behendigheid er komt kijken bij het spelen van deze spelletjes. De mate van behendigheid die aan een bepaald spel wordt toegeschreven, varieert echter sterk onder de discussianten. Als voorbeeld nemen we het pokerspel. Fanatieke pokerspelers zijn ervan overtuigd dat poker een behendigheidsspel is vanwege de blufmogelijkheden. Voor anderen, zoals liefhebbers van schaken, is alleen het feit dat het delen van kaarten onderdeel is van het spel al voldoende om poker als kansspel te kwalificeren.

Meningsverschillen over het behendigheidsniveau van spelen vormen een probleem als het gaat om de exploitatie van spelen. Wetgeving in Nederland en in andere Europese landen, maar ook in veel staten in de Verenigde Staten, maakt onderscheid tussen kans- en behendigheidsspelen. Ruwweg is een spel voor deze wetten een kansspel, wanneer toevalselementen de voornaamste factoren zijn bij het bepalen van de speluitkomst. Een spel waarin de beslissingen van de spelers van overwegende invloed zijn op de uitkomst van het spel, is een behendigheidsspel. Volgens de Nederlandse Wet op de Kansspelen is voor het exploiteren van een kansspel een vergunning vereist, terwijl zo'n vergunning niet benodigd is voor het aanbieden van een behendigheidsspel. Vanuit juridisch oogpunt is het dus belangrijk dat voor een spel objectief vastgesteld kan worden of de spelers voldoende invloed op de spelresultaten hebben om

het te spel als behendigheidsspel te classificeren. De bepaling van het relatieve behendigheidsniveau van een spel, door het vergelijken van de spelersinvloed op het spelresultaat met de invloed van de toevalselementen, is één van de twee centrale onderwerpen van dit proefschrift.

Het andere centrale onderwerp is de berekening van optimale strategieën voor tweepersoonsnulsomspelen. Hierbij is de winst van de ene speler gelijk aan het verlies van de andere speler. Voor een éénpersoonsspel als blackjack is de optimale strategie wellicht te complex om als speler te onthouden en perfect uit te voeren. De berekening ervan, gebruikmakend van kansrekening in verband met de onzekerheid die veroorzaakt wordt door de onbekende kaarten, is echter relatief eenvoudig. In spelen met twee spelers is optimaal spel nog steeds goed gedefinieerd, zolang de winsten van de spelers tot nul optellen. De berekeningen zijn echter ingewikkelder, aangezien de kwaliteit van een strategie van een speler nu afhangt van de beslissingen die de tegenstander neemt. In dit proefschrift bestuderen we het bepalen van optimale strategieën in pokervarianten voor twee spelers. Daarnaast bespreken we optimale strategieën voor een klasse van zogenaamde *take-and-guess*-spelen, die vroeger populair waren in cafés om te bepalen wie een rondje moest betalen.

Het proefschrift bestaat uit twee delen. Deel I, dat bestaat uit de hoofdstukken 2 tot en met 4, gaat voornamelijk over relatieve behendigheid en de rol van toevalsfactoren in spelen. Deel II, bestaand uit de hoofdstukken 5 tot en met 7, is gewijd aan de berekening van optimale strategieën in pokervarianten en in *take-and-guess*-spelen.

Deel I begint met hoofdstuk 2, waarin we een kwantitatieve maat voor het relatieve behendigheidsniveau van een spel presenteren en motiveren. De Wet op de Kansspelen is het uitgangspunt geweest voor de manier waarop we omgaan met het probleem van het meten van relatieve behendigheid in spelen met toevalselementen. In casinospelen leggen toevalsgeneratoren, zoals het delen van kaarten of het draaien van een cilinder in roulette, objectief de kansen op bepaalde gebeurtenissen vast. In principe zijn deze spelen onder gelijke omstandigheden en met dezelfde strategieën van spelers te herhalen. Dit maakt het mogelijk om te spreken van het gemiddelde spelresultaat van een speler op lange termijn, ofwel zijn verwachte spelresultaat. Om te bepalen of strategische keuzes van spelers de uitkomst van het spel beïnvloeden, vergelijken we het verwachte spelresultaat van beginners met dat van optimale spelers. Een vergelijking van de spelersinvloed met de invloed van de toevalsfactoren maken we door te bepalen welk spelresultaat een optimale speler naar verwachting zou kunnen behalen, als hij de uitkomst van alle toevalselementen zou weten voordat hij zijn strategie moet bepalen. De basis voor deze methode om relatieve behendigheid in spelen te meten is gelegd in het werk van Borm en Van der Genugten (1998). In dit hoofdstuk nemen we de belangrijkste aspecten van de methode onder de loep. In het bijzonder besteden we aandacht aan beginners, de definitie van het spelresultaat en aannames over tegenspel door opponenten in meerpersoonsspelen. Een kritische blik op de oorspronkelijke methode leidt uiteindelijk tot een herziene definitie van de relatieve behendigheidsmaat voor meerpersoonsspelen.

Hoofdstuk 3 bestudeert sport-gerelateerde competities die de laatste jaren aan populariteit hebben gewonnen: managementspelen. Een deelnemer aan zo'n spel vervult de rol van manager van een fictieve sportploeg. Voorbeelden van sporten waarvoor managementspelen worden georganiseerd, zijn voetbal, tennis, wielrennen en formule-I autoracen. Een deelnemer selecteert spelers en mogelijk aanvullende onderdelen voor zijn team, waarbij hij rekening moet houden met bepaalde restricties. Dit team verdient punten met de prestaties van de teamleden in de sportcompetitie waarop het managementspel is gebaseerd. Het voornaamste doel van een deelnemer aan het spel is het maximaliseren van het aantal punten van zijn team. Ruwweg komt het erop neer dat een team in het managementspel het goed doet, wanneer de teamleden het goed doen in de echte competitie. Het grote aantal deelnemers aan managementspelen op het internet heeft deze vorm van vermaak tot een commerciële aangelegenheid gemaakt. Aangezien het hier gaat om het aanbieden van spelen waarmee prijzen gewonnen kunnen worden, is het van belang om te weten of managementspelen onder de Wet op de Kansspelen vallen.

In tegenstelling tot casinospelen, zijn er bij managementspelen geen objectief gedefinieerde toevalsprocessen die de scores van de deelnemers beïnvloeden. Er is echter een andere vorm van onzekerheid die een rol speelt: de onzekerheid over de resultaten in de werkelijke sportcompetitie kan worden gezien als de toevalsfactor in het managementspel. Met dit soort toevalselementen is het niet mogelijk om het verwachte spelresultaat van een deelnemer te berekenen met behulp van door toevalsgeneratoren bepaalde kansen. Daarom moeten we de methoden uit hoofdstuk 2 aanpassen, om ze te kunnen toepassen op een managementspel voor de classificatie als kans- of als behendigheidsspel. Het verwachte spelresultaat van een bepaald type speler vervangen we hierbij door het gemiddelde spelresultaat van een groep spelers van dit type. Bij het bepalen van de gemiddelde spelresultaten voor groepen spelers maken we gebruik van de scores van deelnemers in een reeds gespeelde editie van het managementspel. Doordat we deze gegevens nodig hebben voor onze berekeningen, kunnen we het behendigheidsniveau van een managementspel pas achteraf bepalen.

In hoofdstuk 4 keren we terug naar de klasse van spelen waarin toevalsgeneratoren of toevalszetten onzekerheid creëren met objectief bepaalde kansen; denk hierbij opnieuw aan het delen van kaarten in poker of het gooien met dobbelstenen in backgammon. We concentreren ons op de beperkingen die deze toevalszetten veroorzaken voor de spelers als het gaat om hun invloed op de speluitkomst. Bij het bepalen van een strategie voor het spel moet een speler alle mogelijke uitkomsten van de toevalszetten in beschouwing nemen. De keuze van een goede strategie zou eenvoudiger zijn, wanneer de speler het resultaat van de toevalszetten van tevoren zou weten. Informatie hierover is waardevol voor spelers. Maar hoe waardevol is deze informatie? Hoeveel zou een speler ervoor willen betalen, wanneer hij de informatie kon kopen? Natuurlijk hangt dit af van de verandering in zijn verwachte winst, die hij kan realiseren met deze extra kennis. Ruwweg wordt het verschil in de verwachte uitbetaling die een speler met en zonder de informatie kan bereiken de waarde van informatie genoemd. In tegenstelling tot andere definities van de waarde van informatie in de literatuur, houdt het model in hoofdstuk 4 er rekening mee dat deze waarde kan afhangen van de tegenstander die een speler heeft. Het kan bijvoorbeeld zeer nuttig zijn om de informatie over de toevalszet te hebben, wanneer de tegenstander deze informatie niet heeft, terwijl de informatie minder waardevol is, wanneer de tegenstander ook geïnformeerd is. De berekeningen van de waarde van informatie in hoofdstuk 4 maken gebruik van het informatie-koopspel dat is geïntroduceerd door Sakaguchi (1993). In zo'n informatie-koopspel krijgen beide spelers de gelegenheid om informatie over de uitkomst van de toevalszetten te kopen voordat het echte spel begint.

Deel II van het proefschrift, dat bestaat uit de hoofdstukken 5 tot en met 7, is gewijd aan de berekening van optimale strategiën in tweepersoonsnulsomspelen. Voor eenpersoonsspelen is het soms al moeilijk om optimale strategieën te bepalen. Bij spelen voor twee spelers zijn de berekeningen nog veel complexer, aangezien de kwaliteit van een strategie nu afhangt van de strategie die de tegenstander gebruikt. Een andere factor die het bepalen van optimaal spel ingewikkeld maakt, is de onzekerheid die veroorzaakt wordt door toevalsfactoren.

Hoofdstuk 5 bevat de analyse van een eenvoudig pokermodel. Aangezien in poker geen kaarten worden uitgespeeld, zoals bij bridge, hebben alle strategische aspecten van het spel te maken met het bieden en trekken. Hoewel de strategische structuur van het spel relatief eenvoudig is, zijn echte pokerspelen moeilijk om te analyseren. Uit een stok kaarten kunnen meer dan tweeëneenhalf miljoen verschillende pokerhanden worden gevormd, zodat de dimensie van de representatie van een specifieke variant van het pokerspel snel te groot wordt om te analyseren, zelfs voor moderne, snelle computers.

Om dit probleem van het grote aantal verschillende pokerhanden aan te pakken, ordenen we de handen en stellen we ze voor als getallen tussen nul en één op de reële rechte. De hoogst mogelijke pokerhand, een *royal flush*, correspondeert dan met het getal één, terwijl de laagst mogelijke hand nul is. Om de analyse van het spel nu eenvoudiger te maken, kunnen we de kaartverdeling benaderen met een continue verdeling op het interval $[0, 1]$. Daarmee verhogen we impliciet het aantal mogelijke handen van heel veel naar oneindig. Deze aanpak volgen we in dit hoofdstuk, waarin we een pokermodel met twee spelers bestuderen dat geïntroduceerd is door Von Neumann en Morgenstern (1944, hoofdstuk 9). In het oorspronkelijke model worden de handen van de spelers getrokken uit de continue, uniforme verdeling op $[0, 1]$. Wij breiden dit model uit door het toestaan van andere dan uniforme kansverdelingen voor de pokerhanden. We berekenen analytisch de optimale strategieën voor beide spelers en de verwachte uitbetalingen die bij die strategieën horen. Vervolgens vertalen we de gevonden resultaten naar de situatie waarin het spel wordt gespeeld met een stok van 52 kaarten, waaruit pokerhanden van vijf kaarten worden getrokken. Ten slotte bepalen we het relatieve behendigheidsniveau van dit pokerspel.

Hoofdstuk 6 gaat over de berekening van optimale strategieën in poker-modellen met een biedstructuur die ingewikkelder is dan die in het model van hoofdstuk 5. In dergelijke modellen met continue kaartverdelingen is het niet altijd mogelijk om evenwichten op een analytische manier te vinden. We tonen een manier om optimale strategieën in een dergelijk spel te vinden door gebruik te maken van kennis over optimale strategieën in een gerelateerd spel met een discrete kaartverdeling. Het hoofdstuk wordt afgesloten met de beschrijving van de optimale strategieën in het meest complexe pokerspel dat we tot nu toe compleet hebben kunnen analyseren. In dit model hebben beide spelers de mogelijkheid tot verhogen, nadat de tegenstander geboden heeft. Ook voor dit spel presenteren we een analyse van het relatieve behendigheidsniveau.

In hoofdstuk 7 komt een minder bekende, maar wiskundig gezien interessante klasse van tweepersoonsspelen aan bod. Het gaat om de zogenaamde *take-and-guess*-spelen. Deze klasse kan worden opgesplitst in twee deelklassen. In beide deelklassen moeten de spelers een aantal objecten in de hand nemen. Daarna moeten ze het totaal aantal in de hand genomen objecten raden. In de eerste deelklasse, die van de morraspelen, moeten beide spelers hun gok simultaan uitspreken. In de andere deelklasse, gevormd door muntenspelen, raden de spelers na elkaar het totaal.

Take-and-guess-spelen verschillen van poker door het afwezig zijn van externe toevalsfactoren. De onzekerheid waar een speler mee te maken krijgt, wordt enkel en alleen veroorzaakt door zijn tegenstander. Dit biedt, in het bijzonder voor de muntenspelen, geen garantie dat de berekening van optimale strategieën in deze spelen eenvoudig is. Slechts voor een deelklasse van de muntenspelen waren optimale strategieën bekend. Voor de overige muntenspelen binnen deze klasse was de bepaling van deze strategieën en de spelwaarden een openstaand probleem. Dit probleem hebben we opgelost: we geven in hoofdstuk 7 een overzicht van de waarden en de optimale strategieën van morra- en muntenspelen met alle mogelijke aantallen objecten voor beide spelers.

Author index