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Published in:
Game theory and mathematical economics

Publication date:
1981

Citation for published version (APA):
Jansen, M. J. M., & Tijs, S. H. (1981). Solutions of a finite arbitration game: Structure and computation. In O. Moeschlin, & D. Pallaschke (Eds.), *Game theory and mathematical economics: Proceedings of the Seminar on game theory and mathematical economics, Bonn/Hagen, 7-10 October, 1980* (pp. 53-63). North-Holland Publishing Company.

SOLUTIONS OF A FINITE ARBITRATION GAME: STRUCTURE AND COMPUTATION

by

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Report 8033
December 1980

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ABSTRACT

In this paper, it is shown that for a finite arbitration game with a regular arbitration function, the optimal (threat) strategy spaces of the players are polytopes. Furthermore a multifunction is introduced which is useful to find good approximations of the value and optimal strategies for the players.

1. INTRODUCTION AND PRELIMINARIES

In the (non-cooperative) *bimatrix game* Γ corresponding to the ordered pair (A,B) of two m×n-matrices A and B, the players 1 and 2 choose, independently of each other, a mixed strategy $x \in S^m := \{p \in \mathbb{R}^m;\ p \geq 0,\ \sum_{i=1}^m p_i = 1\}$ and $y \in S^n := \{q \in \mathbb{R}^n;\ q \geq 0,\ \sum_{j=1}^n q_j = 1\}$, respectively. In that case, the expected payoff to player 1 (2) is

$$E_1(x,y) := \sum_{i=1}^m \sum_{j=1}^n x_i a_{ij} y_j = xAy^t \quad (E_2(x,y) := xBy^t).$$

If one allows the players to cooperate, then they have the possibility of jointly choosing so-called *correlated strategies* $z \in S^{m\times n} := \{r = (r_{11},\dots,r_{mn}) \in \mathbb{R}^{mn};\ r \geq 0,\ \sum_{i=1}^m \sum_{j=1}^n r_{ij} = 1\}$. Now, the expected payoff to player 1 (2) is

$$\langle A,z\rangle := \sum_{i=1}^m \sum_{j=1}^n a_{ij} z_{ij} \qquad (\langle B,z\rangle).$$

Acting cooperatively all payoff pairs in the *payoff region* $\mathcal{R}(\Gamma) := \text{conv}\{(a_{ij},b_{ij}) \in \mathbb{R}^2;\ i \in \mathbb{N}_m,\ j \in \mathbb{N}_n\}$ are attainable, where conv(S), for $S \subset \mathbb{R}^2$, denotes the convex hull of S and where $\mathbb{N}_k := \{1,\ldots,k\}$, for $k \in \mathbb{N}$. It is clear that the *Pareto set*

$$\mathcal{P}(\Gamma) := \{p \in \mathcal{R}(\Gamma);\ \text{for each } r \in \mathcal{R}(\Gamma) \text{ with } r \geq p, \text{ we have } r = p\}$$

of undominated elements in $\mathcal{R}(\Gamma)$ contains the most attractive payoff pairs for the players. In this paper, we only consider games with more than one Pareto point. Then the problem for the players is which Pareto point to choose, or better, which correlated strategy.

We suppose, in this paper, that the players solve their problem with the aid of an arbitrator, who, in turn, makes use of a so-called *arbitration function* $\phi : \mathcal{R}(\Gamma) \to \mathcal{P}(\Gamma)$. Then the situation proceeds as follows.

Step 1. Independently of each other, the players assign an $x \in S^m$ and a $y \in S^n$ and deliver these *threat strategies* to the arbitrator.

Step 2. The arbitrator calculates the payoff $\phi(xAy^t, xBy^t)$ and chooses a correlated strategy $z \in S^{m\times n}$ such that $(\langle A,z\rangle, \langle B,z\rangle) = \phi(xAy^t, xBy^t)$.

Step 3. The players are obliged to use the correlated strategy z.

From a strategic point of view, for the players, this situation corresponds, essentially, to the non-cooperative game in normal form $\Gamma_\phi = \langle S^m, S^n, \phi_1 E, \phi_2 E\rangle$, where $\phi_i E(x,y)$ is the i-th coordinate of $\phi(E_1(x,y), E_2(x,y))$. This game is called the *arbitration game* corresponding to the game Γ and the arbitration function ϕ.

For Γ_ϕ, the expressions $v_1(\Gamma_\phi) := \sup_{x\in S^m} \inf_{y\in S^n} \phi_1 E(x,y)$ and $v_2(\Gamma_\phi) := \sup_{y\in S^n} \inf_{x\in S^m} \phi_2 E(x,y)$ are called the *security levels* for player 1 and player 2, respectively. We say that Γ_ϕ is *strictly determined* if $v(\Gamma_\phi) := (v_1(\Gamma_\phi), v_2(\Gamma_\phi)) \in \mathcal{P}(\Gamma)$. In that case, $v_i(\Gamma_\phi)$ is called the *(arbitration) value* for player i and $v(\Gamma_\phi)$ the *value* of the arbitration game.

Let $\varepsilon > 0$. If Γ_ϕ is strictly determined, then $x^* \in S^m$ is called an ε-*optimal strategy* for player 1, if

$$\phi_1 E(x^*,y) \geq v_1(\Gamma_\phi)-\varepsilon, \qquad \text{for all } y \in S^n.$$

In a similar way, one defines ε-optimal strategies for player 2. A strategy is called *optimal* if it is ε-optimal, for all $\varepsilon > 0$.
The set of ε-optimal strategies for player i in a strictly determined game Γ_ϕ will be denoted by $O_i^\varepsilon(\Gamma_\phi)$ and the set of optimal strategies for player i by $O_i(\Gamma_\phi)$.
Note that $E(\Gamma_\phi) = O_1(\Gamma_\phi) \times O_2(\Gamma_\phi)$, where $E(\Gamma_\phi)$ denotes the set of equilibrium points of the game Γ_ϕ.

We need some notation. For a bimatrix game $\Gamma = (A,B)$, let the *upper point* $\bar{p}(\Gamma)$ of $P(\Gamma)$ be the unique point of $P(\Gamma)$ with maximal second coordinate and let the *lower point* $\underline{p}(\Gamma)$ of $P(\Gamma)$ be the unique point of $P(\Gamma)$ with maximal first coordinate. Let $\overset{\circ}{P}(\Gamma) := P(\Gamma)\setminus\{\bar{p}(\Gamma),\underline{p}(\Gamma)\}$ and let $W(\Gamma) := \{w \in R(\Gamma);$ for each $r \in R(\Gamma)$ with $r \geq w$, we have $r_1=w_1$ or $r_2=w_2\}$ be the *weak Pareto set* of $R(\Gamma)$. Finally, let
$\overline{W}(\Gamma) := \{w \in W(\Gamma);\ w_2 = \bar{p}_2(\Gamma)\}$ and $\underline{W}(\Gamma) := \{w \in W(\Gamma);\ w_1 = \underline{p}_1(\Gamma)\}$.

In this paper, we only consider arbitration functions $\phi : R(\Gamma) \to P(\Gamma)$, that are *regular*, i.e.

(R.1) $\phi(r) \geq r$, for each $r \in R(\Gamma)$,

(R.2) ϕ is continuous,

(R.3) $\phi^{-1}(p) = \{r \in R(\Gamma);\ \phi(r) = p\}$ is a convex subset of $R(\Gamma)$, for each $p \in P(\Gamma)$.

Such arbitration functions were studied for the first time by Raiffa [5]. Using the fixed point theorem of Kakutani, he proved the following result.
THEOREM 1.1. *For a bimatrix game* $\Gamma = (A,B)$ *and a regular arbitration function* ϕ*, the arbitration game* Γ_ϕ *is strictly determined and both players have optimal strategies.*

In [6], the authors, inspired by a paper of Owen [4], introduced a new technique for proving the existence of value and ε-optimal strategies of arbitration games. This new approach exploits in a suitable way the zero-sum-like character of arbitration games. Basic for the theory is the following observation proved in [6].

LEMMA 1.2. *Let* Γ_ϕ *be a finite arbitration game. If* $p \in \overset{\circ}{P}(\Gamma)$ *and* $r \in \phi^{-1}(p) \setminus \{p\}$, *then* $\phi(s) = p$, *for any* s *in* $R(\Gamma)$ *of the form* $s = p+\lambda(r-p)$, *where* $\lambda \in [0,\infty)$.

This property leads to the next definition.

DEFINITION 1.3. For a finite arbitration game Γ_ϕ, we call an element $d \in \Delta := \{x \in \mathbb{R}^2;\ x \le 0,\ x_1+x_2 = -1\}$ a *suitable direction* for $p \in \overset{\circ}{P}(\Gamma)$ if $\phi(p+\alpha d) = p$, for each $\alpha > 0$ with $p+\alpha d \in R(\Gamma)$.

For $x \in \mathbb{R}^2$, $\overset{\vee}{x} := (-x_2, x_1)$.

Some properties of $D(p)$, the set of suitable directions for p, are gathered in

LEMMA 1.4. ([6] , lemma 3.3). *Let* Γ_ϕ *be a finite arbitration game and let* $p \in \overset{\circ}{P}(\Gamma)$. *Then*

(1) $D(p)$ *is a non-empty, compact and convex subset of* Δ,

(2) the multifunction $p \mapsto D(p)$ *on* $\overset{\circ}{P}(\Gamma)$ *is upper semicontinuous.*

Again, let $\Gamma = (A,B)$ be an m×n-bimatrix game. For each $p \in \overset{\circ}{P}(\Gamma)$ and each $d \in D(p)$, we introduce the *dummy zero-sum game*

$\Gamma_{p,d} = \langle S^m, S^n, E_{p,d}, -E_{p,d} \rangle$, where

$$E_{p,d}(x,y) := \langle \overset{\vee}{d}, E(x,y)-p \rangle, \text{ for all } (x,y) \in S^m \times S^n.$$

Note that, in fact, $\Gamma_{p,d}$ is the mixed extension of the m×n-matrix game

$$M_{p,d} := -d_2 A + d_1 B - \langle \overset{\vee}{d}, p \rangle J,$$

where J is the m×n-matrix with all coefficients equal to 1.

Essential in the theory is

LEMMA 1.5. ([6], lemma 5.1). *Let* Γ_ϕ *be a finite arbitration game and let* $p \in \overset{\circ}{P}(\Gamma)$, $d \in D(p)$. *Then*

(1) $E_{p,d}(x,y) \leq 0$ *implies* $\phi_2 E(x,y) \geq p_2$,

(2) $E_{p,d}(x,y) \geq 0$ *implies* $\phi_1 E(x,y) \geq p_1$.

The reason why the family $\{\Gamma_{p,d}\ ;\ p \in \overset{\circ}{P}(\Gamma),\ d \in D(p)\}$ of dummy zero-sum games plays a crucial role in the derivation of existence theorems for arbitration games, is revealed in

THEOREM 1.6. ([6], lemma 5.2). *Let* Γ_ϕ *be a finite arbitration game. Let* $p \in \overset{\circ}{P}(\Gamma)$, $d \in D(p)$ *and suppose that* $\mathrm{val}(\Gamma_{p,d}) = 0$. *Then*

(1) $p = v(\Gamma_\phi)$.

(2) $O_i(\Gamma_{p,d}) \subset O_i(\Gamma_\phi)$, *for* $i \in \{1,2\}$.

2. THE OPTIMAL STRATEGY SPACES OF A FINITE ARBITRATION GAME

Unless otherwise stated, in this section, Γ is a fixed m×n-bimatrix game (A,B), ϕ is a fixed regular arbitration function and $v := v(\Gamma_\phi)$. We will show that the optimal strategy spaces of players 1 and 2 in the arbitration game Γ_ϕ, are polytopes in S^m and S^n, respectively.

Suppose, for the present, that $v \in \overset{\circ}{P}(\Gamma)$. Then dummy games may be helpful to characterize the optimal strategy spaces. First we note that theorem 1.6 implies that optimal strategies of dummy games $\Gamma_{v,d}$ with $\mathrm{val}(\Gamma_{v,d}) = 0$ and $d \in D(v)$, are also optimal strategies in the arbitration game. But the converse statement is not necessarily true, in general, as the following example shows.

EXAMPLE 2.1. Let $\Gamma = (A,B)$, where $A := \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 3 & 0 \\ 0 & 2 & 0 \end{bmatrix}$.
Let ϕ be the Nash arbitration function [2], [3] (see also example 4.4). Then $v(\Gamma_\phi) = (2,2)$ and $D(v) = \{(\delta,-1-\delta) \in \Delta;\ -\frac{2}{3} \leq \delta \leq -\frac{1}{3}\}$.

Now, $\mathrm{val}(\Gamma_{v,(\delta,-1-\delta)}) = \begin{cases} 0 & \text{if } \delta \leq -\frac{1}{2} \\ -2-4\delta & \phantom{\text{if }} \delta > -\frac{1}{2} \end{cases}$. Consequently, all dummy games $\Gamma_{v,(\delta,-1-\delta)}$ with $\delta \le -\frac{1}{2}$ have a value equal to zero. However, $O_2(\Gamma_\phi) = \mathrm{conv}\{e_2, e_3, (\frac{1}{3}, 0, \frac{2}{3})\}$ and

$$O_2(\Gamma_{v,(\delta,-1-\delta)}) = \begin{cases} \{e_2\} & \text{if} \quad \delta < -\frac{1}{2} \\ \{q \in S^3;\ q_1 = 0\} & \phantom{\text{if}} \quad \delta = -\frac{1}{2} \end{cases}.$$

For the description of the optimal strategy spaces for the arbitration game Γ_ϕ, for the case $v \in \overset{\circ}{P}(\Gamma)$, we introduce the notion of extreme direction. As observed in lemma 1.4(1), $D(v)$ is a non-empty, compact subset of Δ. Therefore, there exist $d^+, d^- \in D(v)$ such that $d_2^+ = \max_{d \in D(v)} d_2$ and $d_2^- = \min_{d \in D(v)} d_2$. We call d^+ and d^- the *extreme directions* for v.

THEOREM 2.2. *Let Γ_ϕ be a finite arbitration game for which the underlying bimatrix game Γ has size* m×n. *If, furthermore,* $v := v(\Gamma_\phi) \in \overset{\circ}{P}(\Gamma)$, *then*

$$O_1(\Gamma_\phi) = \{x^* \in S^m;\ E_{v,d^+}(x^*, y) \ge 0, \textit{ for all } y \in S^n\}$$

and

$$O_2(\Gamma_\phi) = \{y^* \in S^n;\ E_{v,d^-}(x, y^*) \le 0, \textit{ for all } x \in S^m\}.$$

PROOF. (a) If $E_{v,d^+}(x^*, y) \ge 0$, for all $y \in S^n$, then, by lemma 1.5(2), $\phi_1 E(x^*, y) \ge v_1$, for all $y \in S^n$. So $x^* \in O_1(\Gamma_\phi)$.

(b) For $x^* \in O_1(\Gamma_\phi)$ we have, by definition,

$$\phi_1 E(x^*, y) \ge v_1, \text{ for all } y \in S^n. \tag{2.1}$$

Suppose that there exists a $\tilde{y} \in S^n$ with

$$E_{v,d^+}(x^*, \tilde{y}) < 0. \tag{2.2}$$

Then lemma 1.5(1) implies that $\phi_1 E(x^*, \tilde{y}) \le v_1$.
So, in view of (2.1), $\phi_1 E(x^*, \tilde{y}) = v_1$. Note that $E(x^*, \tilde{y}) \ne v$ by (2.2).
Put $d := \|E(x^*, \tilde{y}) - v\|_1^{-1} (E(x^*, \tilde{y}) - v)$. Then $d \in D(v)$. Furthermore, it follows from (2.2) that $-d_2^+ d_1 + d_1^+ d_2 < 0$; so $d_2 > d_2^+$ which contradicts the definition of d^+.
Consequently, $E_{v,d^+}(x^*, y) \ge 0$, for all $y \in S^n$.

(c) In a similar way, one proves the other equality in the theorem. □

COROLLARY 2.3. *Let* Γ_ϕ *be a finite arbitration game with* $v = v(\Gamma_\phi) \in \mathring{P}(\Gamma)$ *and* $d^+ = d^-$. *Then, for* $i \in \{1,2\}$,

$$O_i(\Gamma_\phi) = O_i(\Gamma_{v,d^+}).$$

In the remainder of this section, we consider finite arbitration games Γ_ϕ with $v(\Gamma_\phi) = \bar{p}(\Gamma)$. Games with $v(\Gamma_\phi) = \underline{p}(\Gamma)$ can be handled in a similar way.

LEMMA 2.4. *Let* Γ_ϕ *be a finite arbitration game where the underlying bimatrix game* $\Gamma = (A,B)$ *has size* $m\times n$. *If* $\phi^{-1}(\bar{p}(\Gamma)) \subset \overline{W}(\Gamma)$, *then*
(1) $v(\Gamma_\phi) = \bar{p}(\Gamma)$ *iff* $\mathrm{val}(-B) = -\max\{b_{ij}\}$,
(2) if $v(\Gamma_\phi) = \bar{p}(\Gamma)$, *then*

$$O_1(\Gamma_\phi) = O_1(-B) = S^m \quad \text{and} \quad O_2(\Gamma_\phi) = O_2(-B).$$

PROOF. (a) Suppose that $v(\Gamma_\phi) = \bar{p}(\Gamma)$. If $\bar{y} \in O_2(\Gamma_\phi)$, then $\phi_2(xA\bar{y}^t, xB\bar{y}^t) \geq \bar{p}_2(\Gamma)$, for all $x \in S^m$. This implies that $\phi(xA\bar{y}^t, xB\bar{y}^t) = \bar{p}(\Gamma)$, for all $x \in S^m$. Since $\phi^{-1}(\bar{p}(\Gamma)) \subset \overline{W}(\Gamma)$, we obtain that $xB\bar{y}^t = \bar{p}_2(\Gamma) = \max\{b_{ij}\}$, for all $x \in S^m$. Consequently, $\mathrm{val}(-B) = -\max\{b_{ij}\}$ and $\bar{y} \in O_2(-B)$.
(b) If $\mathrm{val}(-B) = -\max\{b_{ij}\}$, then it is easy to show that for a $\bar{y} \in O_2(-B)$, $\phi_2(xA\bar{y}^t, xB\bar{y}^t) \geq \bar{p}_2(\Gamma)$, for all $x \in S^m$. Hence, $v(\Gamma_\phi) = \bar{p}(\Gamma)$ and $\bar{y} \in O_2(\Gamma_\phi)$.
(c) Combining (a) and (b), we have proved (1) and the second statement in (2). In order to complete the proof, we note that $\phi_1(xAy^t, xBy^t) \geq \bar{p}_1(\Gamma) = v_1(\Gamma_\phi)$, for all $(x,y) \in S^m \times S^n$. Therefore, $O_1(\Gamma_\phi) = S^m = O_1(-B)$. □

The sets

$$L(d) := \{r \in \mathcal{R}(\Gamma);\ r = \bar{p}(\Gamma) + \alpha d, \text{ for some } \alpha > 0\} \quad (d \in \Delta)$$

play an important role in the following two results.

LEMMA 2.5. *Let* Γ_ϕ *be a finite arbitration game satisfying*

(i) $v(\Gamma_\phi) = \bar{p}(\Gamma)$,

(ii) $L(d) \subset \phi^{-1}(\bar{p}(\Gamma))$, *for all* $d \in \Delta$ *with* $L(d) \cap \phi^{-1}(\bar{p}(\Gamma)) \neq \emptyset$.

Then $O_1(\Gamma_\phi) = S^m$ *and there exists a* $d^* \in \Delta$ *such that*

$$O_2(\Gamma_\phi) = \{y \in S^n;\ E_{\bar{p}(\Gamma),d^*}(x,y) \leq 0, \text{ for all } x \in S^m\}.$$

PROOF. (a) Let $D(\bar{p}(\Gamma)) := \{d \in \Delta;\ \emptyset \neq L(d) \subset \phi^{-1}(\bar{p}(\Gamma))\}$. As in the proof of lemma 3.3 in [6], one can show that $D(\bar{p}(\Gamma))$ is compact. Therefore, there exists a $d^* \in \Delta$ such that $d_2^* = \min\{d_2;\ d \in D(\bar{p}(\Gamma))\}$.

(b) Now we want to show that

$$\phi^{-1}(\bar{p}(\Gamma)) = \{r \in R(\Gamma);\ \langle \check{d}^*, r-\bar{p}(\Gamma)\rangle \leq 0\}. \tag{2.3}$$

First suppose that $\langle \check{d}^*, r-\bar{p}(\Gamma)\rangle \leq 0$ for an $r \in R(\Gamma)\setminus\{\bar{p}(\Gamma)\}$. We distinguish two cases.

If $\langle \check{d}^*, r-\bar{p}(\Gamma)\rangle = 0$, then $r \in L(d^*)$ and $\phi(r) = \bar{p}(\Gamma)$ by the definition of $D(\bar{p}(\Gamma))$.

If $\langle \check{d}^*, r-\bar{p}(\Gamma)\rangle < 0$ and $\phi(r) = p \neq \bar{p}(\Gamma)$, then $[r,p] \cap L(d^*) \neq \emptyset$. However, for an $x \in [r,p] \cap L(d^*)$, $\bar{p}(\Gamma) = \phi(x) = p \neq \bar{p}(\Gamma)$, which is impossible. So $r \in \phi^{-1}(\bar{p}(\Gamma))$.

Now suppose that $r \in \phi^{-1}(\bar{p}(\Gamma))$ and that $\langle \check{d}^*, r-\bar{p}(\Gamma)\rangle > 0$. Then $r \neq \bar{p}(\Gamma)$ and $d := \|r-\bar{p}(\Gamma)\|_1^{-1}(r-\bar{p}(\Gamma)) \in D(\bar{p}(\Gamma))$, in view of (ii). As in the proof of theorem 2.2, one can show that $d_2 < d_2^*$, which contradicts the definition of d^*.

(c) The fact that $O_1(\Gamma_\phi) = S^m$ can be proved as in (c) of the proof of lemma 2.4. The remainder of the theorem is an easy consequence of (2.3). □

Now suppose that Γ_ϕ is a finite arbitration game with $v(\Gamma_\phi) = \bar{p}(\Gamma)$ while (ii) of lemma 2.5 is not satisfied. Then there exist $d \in \Delta$ and $r,s \in L(d)$ with $\phi(s) = \bar{p}(\Gamma)$ and $\phi(r) \neq \bar{p}(\Gamma)$.

We want to show that, in this case, $\phi^{-1}(\bar{p}(\Gamma)) \subset L(d) \cup \{\bar{p}(\Gamma)\}$. Since $\phi^{-1}(\bar{p}(\Gamma))$ is closed, there exists an $s^* \in L(d)$ with $s_2^* = \min_{t \in L(d)} t_2$. Now suppose there is a $t \in \phi^{-1}(\bar{p}(\Gamma))$, $t \neq \bar{p}(\Gamma)$ with $t \notin L(d)$. Note that $t_2 < s_2^*$. So $\langle \check{d}, t-\bar{p}(\Gamma)\rangle \neq 0$. We distinguish two cases.

If $\langle \check{d}, t-\bar{p}(\Gamma)\rangle < 0$, then choose, for each $k \in \mathbb{N}$, a $p^k \in \mathring{P}(\Gamma)$ with $p_2^k > \phi_2(r)$ and $\|\bar{p}(\Gamma)-p^k\|_\infty < k^{-1}$. In view of the connectedness of the line segment $[r,s^*]$ and the continuity of ϕ, there exists, for each $k \in \mathbb{N}$, an $x^k \in [r,s^*]$ with $\phi(x^k) = p^k$. For large k, the line through p^k and x^k intersects with the line segment $[r,t]$. Let y^k be the intersection point. Then, by lemma 1.2, $\phi(y^k) = p^k$, for all $k \in \mathbb{N}$. Since $\lim_{k\to\infty} y^k = r$, we obtain,

$$\bar{p}(\Gamma) = \lim_{k\to\infty} \phi(x^k) = \lim_{k\to\infty} \phi(y^k) = \phi(r) \neq \bar{p}(\Gamma).$$

This is impossible.

If $\langle \check{d}, t-\bar{p}(\Gamma)\rangle > 0$, then we can choose an $r^* \in [r,s^*]$ such that $[r^*,\phi(r^*)] \cap [\bar{p}(\Gamma),t] \neq \emptyset$. However, for an $x \in [r^*,\phi(r^*)] \cap [\bar{p}(\Gamma),t]$, $\bar{p}(\Gamma) = \phi(x) = \phi(r^*) \neq \bar{p}(\Gamma)$. This, also, is impossible.

Summarizing,

$$\phi^{-1}(\bar{p}(\Gamma)) = \{r \in R(\Gamma);\ \langle \check{d}, r-\bar{p}(\Gamma)\rangle = 0,\ r_2 \geq s_2^*\}.$$

Now the following result is immediate.

LEMMA 2.6. *Let* Γ_ϕ *be a finite arbitration game with* $v(\Gamma_\phi) = \bar{p}(\Gamma)$ *such that (ii) of lemma* 2.5 *is not satisfied. Then* $O_1(\Gamma_\phi) = S^m$ *and there are a* $d \in \Delta$ *and* $c \in \mathbb{R}$ *such that*

$$O_2(\Gamma_\phi) = \{y \in S^n;\ \textit{for all}\ x \in S^m,\ E_{\bar{p}(\Gamma),d}(x,y) = 0\ \textit{and}\ E_2(x,y) \geq c\}.$$

From theorem 2.2, lemma 2.5 and lemma 2.6, we infer

THEOREM 2.7. *For a finite arbitration game* Γ_ϕ, $O_1(\Gamma_\phi)$ *and* $O_2(\Gamma_\phi)$ *are polytopes.*

3. THE DUMMY-VALUE MULTIFUNCTION

For a finite arbitration game Γ_ϕ, we construct a multifunction m on

$\mathring{P}(\Gamma)$ for which the point where the sign changes (if this exists) is definitive for the position of the arbitration value.

DEFINITION 3.1. The multifunction $m : \mathring{P}(\Gamma) \to \mathbb{R}$ defined by

$$m(p) := \{\mathrm{val}(\Gamma_{p,d});\ d \in D(p)\}, \qquad \text{for } p \in \mathring{P}(\Gamma),$$

is called the *dummy-value multifunction* corresponding to the arbitration game Γ_ϕ.

LEMMA 3.2. *Let* Γ_ϕ *be a finite arbitration game. Then*

(1) $m(p)$ *is a non-empty, compact, convex subset of* $\mathbb{R}$, *for all* $p \in \mathring{P}(\Gamma)$,

(2) m *is a closed multifunction.*

PROOF. (a) Take a $p \in \mathring{P}(\Gamma)$. Since, by lemma 1.4(1), $D(p)$ is a non-empty, compact, convex subset of Δ, the continuity of the function $d \mapsto \mathrm{val}(\Gamma_{p,d})$ on $D(p)$ implies (1).

(b) Let $p^1, p^2, \ldots$ be a sequence in $\mathring{P}(\Gamma)$ converging to $p \in \mathring{P}(\Gamma)$. Suppose that for each $k \in \mathbb{N}$, $c_k \in m(p^k)$ and that $\lim_{k\to\infty} c_k = c$. We must prove that $c \in m(p)$. Since $c_k \in m(p^k)$, there exists, for all $k \in \mathbb{N}$, a $d^k \in D(p^k)$ such that $c_k = \mathrm{val}(\Gamma_{p^k,d^k})$. In view of the compactness of Δ, we may suppose that the sequence $d^1, d^2, \ldots$ converges, say, to $d \in \Delta$. The upper semi-continuity of the multifunction D implies that $d \in D(p)$. Since $c = \lim_{k\to\infty} \mathrm{val}(\Gamma_{p^k,d^k}) = \mathrm{val}(\Gamma_{p,d})$, $c \in m(p)$. □

We use the following notation:

$m(p) > 0$ if $t > 0$, for all $t \in m(p)$; $m > 0$ if $m(p) > 0$, for all $p \in \mathring{P}(\Gamma)$.

The next two lemmas imply that the multifunction m has at most one zero, e.g. there is at most one point $p \in \mathring{P}(\Gamma)$ with $0 \in m(p)$.

LEMMA 3.3. *Let* Γ_ϕ *be a finite arbitration game with* $v(\Gamma_\phi) \in \mathring{P}(\Gamma)$. *Then*

(1) $v(\Gamma_\phi)$ *is the only zero of* m: $0 \in m(p)$ *iff* $p = v(\Gamma_\phi)$,

(2) $m(p) > 0$ *iff* $p_1 < v_1(\Gamma_\phi)$,

(3) $m(p) < 0$ *iff* $p_1 > v_1(\Gamma_\phi)$.

PROOF. (a) By theorem 1.6, $p = v(\Gamma_\phi)$ if $0 \in m(p)$. As in the proof of theorem 5.2 of [6], one can show that $\mathrm{val}(\Gamma_{v(\Gamma_\phi),d^+}) \geq 0$ and $\mathrm{val}(\Gamma_{v(\Gamma_\phi),d^-}) \leq 0$. Let for $\delta \in [d_1^+, d_1^-]$, $f(\delta) := \mathrm{val}(\Gamma_{v(\Gamma_\phi),(\delta,-1-\delta)})$. Then f is a continuous function with $f(d_1^+) \geq 0$ and $f(d_1^-) \leq 0$. As a consequence, there is a $d \in D(v(\Gamma_\phi))$ with $\mathrm{val}(\Gamma_{v(\Gamma_\phi),d}) = 0$, that is $0 \in m(v(\Gamma_\phi))$.

(b) If $0 \notin m(p)$, then in view of (a), $p \neq v(\Gamma_\phi)$. Now suppose that $p \in \mathring{P}(\Gamma)$, $p_1 < v_1(\Gamma_\phi)$ and that $\mathrm{val}(\Gamma_{p,d}) < 0$, for some $d \in D(p)$. Then, for $x \in O_1(\Gamma_\phi)$ and $y \in O_2(\Gamma_{p,d})$, $E_{p,d}(x,y) < 0$, so that by lemma 1.5(1), $p_1 \geq \phi_1 E(x,y) \geq v_1(\Gamma_\phi) > p_1$, which is impossible. So $\mathrm{val}(\Gamma_{p,d}) > 0$, for all $d \in D(p)$. Consequently, $m(p) > 0$. Analogously, $p_1 > v_1(\Gamma_\phi)$ implies $m(p) < 0$. The other assertions in the theorem follow easily. □

LEMMA 3.4. *Let Γ_ϕ be a finite arbitration game where the underlying bimatrix game $\Gamma = (A,B)$ has size $m\times n$. Then*

(1) $m < 0$ *iff* $v(\Gamma_\phi) = \bar{p}(\Gamma)$,

(2) $m > 0$ *iff* $v(\Gamma_\phi) = \underline{p}(\Gamma)$.

PROOF. We only prove (1). Suppose that $m < 0$. Then, for $p \in \mathring{P}(\Gamma)$ and $d \in D(p)$, $\mathrm{val}(\Gamma_{p,d}) < 0$. This implies that for a $y^* \in O_2(\Gamma_{p,d})$, $E_{p,d}(x,y^*) < 0$, for all $x \in S^m$. Using lemma 5.1(1), we obtain that $\phi_2 E(x,y^*) \geq p_2$, for all $x \in S^m$. Consequently,

$$v_2(\Gamma_\phi) = \sup_{y\in S^n} \inf_{x\in S^m} \phi_2 E(x,y) \geq \inf_{x\in S^m} \phi_2 E(x,y^*) \geq p_2.$$

Since $p \in \mathring{P}(\Gamma)$ was arbitrarily chosen, this means that $v_2(\Gamma_\phi) \geq p_2$, for all $p \in \mathring{P}(\Gamma)$. So $v(\Gamma_\phi) = \bar{p}(\Gamma)$.

Now suppose that $v(\Gamma_\phi) = \bar{p}(\Gamma)$.

Then, similarly, one can show that it is impossible that $m > 0$. Therefore, if not $m < 0$, the connectedness of the set $\{t \in \mathbb{R};$ there exists a $p \in \mathring{P}(\Gamma)$ with $t \in m(p)\}$ implies the existence of a

$p \in \overset{\circ}{P}(\Gamma)$ with $0 \in m(p)$. But then theorem 1.6 implies that $v(\Gamma_\phi) = p$, which is also impossible. Hence, $m < 0$. □

EXAMPLE 3.5. Let $\Gamma = (A,B)$, where $A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and let ϕ be the Nash arbitration function. Then

$$m(p) = \begin{cases} \{2 - \frac{2}{3}p_1\} & \text{if } 0 < p_1 < 2 \\ [-\frac{2}{3}, \frac{2}{3}] & \text{if } \quad p_1 = 2. \\ \{2 - \frac{4}{3}p_1\} & \text{if } 2 < p_1 < 3 \end{cases}$$

4. THE DETERMINATION OF A ZERO OF THE DUMMY-VALUE MULTIFUNCTION

In this section, a method is given to determine a possible zero of the dummy-value multifunction corresponding to a finite arbitration game Γ_ϕ, where $\Gamma = (A,B)$ is an m×n-bimatrix game.

The first problem to be solved is the determination of the Pareto set $P(\Gamma)$ of Γ. Since $R(\Gamma)$ is a polytope, it is sufficient to give a method for finding those Pareto optimal points that are also extreme points of $R(\Gamma)$. Let $P(\Gamma) \cap \text{ext } R(\Gamma) = \{p(1), p(2), \ldots, p(\nu)\}$, where $\bar{p}(\Gamma)_1 = p(1)_1 < p(2)_1 < \ldots < p(\nu)_1 = \underline{p}(\Gamma)_1$ Now we can base an algorithm on the following scheme.

(1) $p(\nu)$ is the lexicographic maximum of $C_1 := \{(a_{ij}, b_{ij});\ i \in \mathbb{N}_m, j \in \mathbb{N}_n\}$, while $\tilde{p}(1) := (p(1)_2, p(1)_1)$ is the lexicographic maximum of $\tilde{C}_1 := \{(b_{ij}, a_{ij});\ i \in \mathbb{N}_m, j \in \mathbb{N}_n\}$.

(2) Let C_2 be the collection of those points (a_{ij}, b_{ij}) that lie above the line through $p(1)$ and $p(\nu)$. Then $p(\nu-1)$ is the lexicographic maximum of C_2 and $\tilde{p}(2)$ is the lexicographic maximum of $\tilde{C}_2 := \{(b_{ij}, a_{ij});\ (a_{ij}, b_{ij}) \in C_2\}$.

(3) It is clear how to define, for suitable k's, $C_k, \tilde{C}_k$ and $p(k)$.

(4) We are finished if, for some k, $C_k = \emptyset$ or $p(k) = p(k+1)$.

Secondly, one must be able to determine one or more elements of the direction set of a Pareto optimal point. In general, however, it is not possible to give a method for finding a $d \in D(p)$, for $p \in \mathring{P}(\Gamma)$, because this problem depends heavily upon the arbitration function under consideration. In the examples 4.3 and 4.4, we will show how one deals the cases where the arbitration functions are those of Nash and Kalai / Rosenthal.

Next, we describe an algorithm for finding the possible zero of the multifunction m. Let $\varepsilon > 0$ be the desired precision.

(0) Choose $\bar{p}(\varepsilon)$ and $\underline{p}(\varepsilon)$ in $\mathring{P}(\Gamma)$ in such a way that $\|\bar{p}(\Gamma)-\bar{p}(\varepsilon)\|_\infty < \varepsilon$ and $\|\underline{p}(\Gamma)-\underline{p}(\varepsilon)\|_\infty < \varepsilon$.
If $m(\underline{p}(\varepsilon)) \geq 0$, then take $\underline{p}(\varepsilon)$ as an estimate for $v(\Gamma_\phi)$.
If $m(\bar{p}(\varepsilon)) \leq 0$, then take $\bar{p}(\varepsilon)$ as an estimate for $v(\Gamma_\phi)$.
If, however, $m(\bar{p}(\varepsilon)) > 0$ and $m(\underline{p}(\varepsilon)) < 0$, then we proceed as follows,

(1a) take $p := \frac{1}{2}(\bar{p}(\varepsilon)+\underline{p}(\varepsilon))$ (here and in what follows, we tacitly identify p with p_1) and compute an element m(p) of $m(p)$, using some of the well-known methods for determing the value of a matrix game.

(1b) if $m(p) > 0$, replace $\bar{p}(\varepsilon)$ by p. If $m(p) < 0$, replace $\underline{p}(\varepsilon)$ by p. Repeat (1a) if $\|\bar{p}(\varepsilon)-\underline{p}(\varepsilon)\|_\infty \geq \varepsilon$.

(1c) if $\|\bar{p}(\varepsilon)-\underline{p}(\varepsilon)\|_\infty < \varepsilon$, then stop and take $\underline{p}(\varepsilon)$ as an estimate for $v(\Gamma_\phi)$.

Obviously, the algorithm terminates after a finite number of steps and produces a sequence $p(1), p(2), \ldots, p(N)$ of successive approximations of the value $v(\Gamma_\phi)$. We distinguish two cases: $v(\Gamma_\phi) \in \mathring{P}(\Gamma)$ and $v(\Gamma_\phi) \notin \mathring{P}(\Gamma)$.

(i) If $v(\Gamma_\phi) \in \mathring{P}(\Gamma)$, then, without loss of generality, we may suppose that $m(p(N-1)) \geq 0$, $m(p(N)) \leq 0$ and $\|p(N)-p(N-1)\|_\infty < \varepsilon$. Then, by lemma 3.2,

$$v_1(\Gamma_\phi) \in (p(N-1)_1, p(N)_1) \quad \text{and} \quad v_2(\Gamma_\phi) \in (p(N)_2, p(N-1)_2). \tag{4.1}$$

The following lemma describes how to find, with the help op $p(N-1)$ and $p(N)$, good approximations of the value and optimal strategies for both players in the arbitration game.

THEOREM 4.1. *Let* Γ_ϕ *be a finite arbitration game, where the underlying bimatrix game* Γ *has size* $m\times n$. *Suppose that* $v(\Gamma_\phi) \in \overset{\circ}{P}(\Gamma)$ *and let* $p(N-1)$, $p(N)$ *and* ε *be as before. Then*

(1) $p(N)$ *is a good approximation for* $v(\Gamma_\phi)$: $\|p(N)-v(\Gamma_\phi)\|_\infty < \varepsilon$.

(2) $O_1(\Gamma_{p(N-1),d}) \subset O_1^\varepsilon(\Gamma_\phi)$, *for any* $d \in D(p(N-1))$.

(3) $O_2(\Gamma_{p(N),d}) \subset O_2^\varepsilon(\Gamma_\phi)$, *for any* $d \in D(p(N))$.

PROOF. (1) is an immediate consequence of (4.1). We only prove (2). Let $x^* \in O_1(\Gamma_{p(N-1),d})$ and $d \in D(p(N-1))$. Then

$$E_{p(N-1),d}(x^*,y) \geq \mathrm{val}(\Gamma_{p(N-1),d}) \geq 0, \text{ for all } y \in S^n.$$

Then, by lemma 1.5(2), for all $y \in S^n$,

$$\phi_1 E(x^*,y) \geq p(N-1)_1 \geq v_1(\Gamma_\phi)-(p(N)_1-p(N-1)_1) \geq v_1(\Gamma_\phi)-\varepsilon.$$

So $x^* \in O_1^\varepsilon(\Gamma_\phi)$. □

(ii) If $v(\Gamma_\phi) \notin \overset{\circ}{P}(\Gamma)$, say, $v(\Gamma_\phi) = \underline{p}(\Gamma)$, then $N = 2$ and $\|v(\Gamma_\phi)-p(N)\|_\infty < \varepsilon$.

Furthermore, one can show, as in the proof of theorem 4.1, that if the underlying bimatrix game has size $m\times n$, then

$$O_1(\Gamma_{p(N),d}) \subset O_1^\varepsilon(\Gamma_\phi), \text{ for any } d \in D(p(N))$$

and

$$S^n \subset O_2^\varepsilon(\Gamma_\phi).$$

In the following example we show that, in general, it is not possible the use the method of Newton-Raphson for finding a zero of the multifunction $\mathcal{M}$.

EXAMPLE 4.2. Let $\Gamma = (A,B)$ be the bimatrix game introduced in example 3.5 and let ϕ be the Nash arbitration function. If one starts with $p(1) = (1\frac{1}{2}, 2\frac{1}{4})$, then $p(n) = \begin{cases} (3,0) & \text{if } n > 1 \text{ is even} \\ p(1) & \text{if } n > 1 \text{ is odd} \end{cases}$, where $p(n)$ is determined with the help of the Newton-Raphson method.

Finally, we give two examples in which we consider, successively, the situations in which ϕ represents the arbitration function of Kalai/Rosenthal and of Nash, respectively.

EXAMPLE 4.3. Let $\Gamma = (A,B)$ be an $m\times n$-bimatrix game. In this example, we consider the arbitration function ϕ that assigns to $r \in R(\Gamma)$ the unique element in $[r, u(\Gamma)] \cap P(\Gamma)$, where $u(\Gamma) := (\max\{a_{ij}\}, \max\{b_{ij}\})$ is the *utopia point* of Γ (cf. Kalai and Rosenthal [1]). Now let for each $p \in P(\Gamma)$, $\theta \in [0,\pi/2]$ be the angle that the line through p and $u(\Gamma)$ makes with the horizontal axis through $u(\Gamma)$. Then the mapping $p \mapsto \theta$ $(p \in P(\Gamma))$ is a one-to-one correspondence between $P(\Gamma)$ and $[0,\pi/2]$. Furthermore, if $p \in \overset{\circ}{P}(\Gamma)$ and $\theta \in (0,\pi/2)$ corresponds to p, then $D(p) = \{\alpha_\theta(-\cos\theta, -\sin\theta)\}$, where $\alpha_\theta = (\sin\theta + \cos\theta)^{-1}$. Therefore, we denote the dummy zero-sum game corresponding to $p \in \overset{\circ}{P}(\Gamma)$ with Γ_θ. Using the fact that $\tan\theta = [u_2(\Gamma)-p_2]/[u_1(\Gamma)-p_1]$, we obtain that

$$\Gamma_\theta = \alpha_\theta[A\sin\theta - B\cos\theta - J\max\{a_{ij}\}\sin\theta + J\max\{b_{ij}\}\cos\theta],$$

for $\theta \in (0,\pi/2)$. Hence,

$$\alpha_\theta^{-1} m(\theta) = \mathrm{val}[A\sin\theta - B\cos\theta] - \max\{a_{ij}\}\sin\theta + \max\{b_{ij}\}\cos\theta.$$

As we know, the arbitration value can be found by solving the equation $m(\theta) = 0$, $\theta \in (0,\pi/2)$, or, equivalently, since $\cos\theta \neq 0$, by solving

$$\tilde{m}(\theta) := (\alpha_\theta \cos\theta)^{-1} m(\theta) = 0, \qquad \theta \in (0,\pi/2).$$

One can prove that $\tilde{m}$ is a differentiable monotonically decreasing function of θ. This simplifies the computational procedure.

EXAMPLE 4.4. Let $\Gamma = (A,B)$ be an m×n-bimatrix game. Consider the arbitration function ϕ which assigns to an $r \in R(\Gamma)$ the unique element $\phi(r)$ of $P(\Gamma)$ with the property that

$$(\phi_1(r)-r_1)(\phi_2(r)-r_2) = \max_{p \in P(\Gamma)} (p_1-r_1)(p_2-r_2)$$

(cf. Nash [2]). For each $p \in \overset{\circ}{P}(\Gamma)$, let

$\Theta(p) := \{\theta \in (0,\pi/2)$; there exists a line through p and supporting to $R(\Gamma)$ which makes the angle θ with a horizontal axis$\}$.

Then $D(p) = \{\alpha_\theta(-\cos\theta, -\sin\theta);\ \theta \in \Theta(p)\}$, where $\alpha_\theta = (\sin\theta + \cos\theta)^{-1}$. Writing $\Gamma_{p,\theta}$ for the dummy zero-sum game corresponding to $p \in \overset{\circ}{P}(\Gamma)$ and $\alpha_\theta(-\cos\theta, -\sin\theta) \in D(p)$, we have to solve $\text{val}(\Gamma_{p,\theta}) = 0$, $\theta \in \Theta(p)$, or equivalently,

$$\text{val}[A\sin\theta - B\cos\theta] - p_1 \sin\theta + p_2 \cos\theta = 0 \quad (\theta \in \Theta(p)),$$

in order to find the arbitration value $v(\Gamma_\phi)$ (cf. Owen [4]).

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