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## A DYNAMICAL CHARACTERIZATION OF EVOLUTIONARILY STABLE STATES

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### Abstract

Evolutionary stability, the central solution concept in evolutionary game theory, is closely related to local asymptotic stability in a certain nonlinear dynamical system operating on the state space, the so-called "replicator dynamics". However, a purely dynamical characterization of evolutionary stability is not available in an elementary manner. This characterization can be achieved by investigating so-called "derived games" which consist of mixed strategies corresponding to successful states in the original game. Using well-known facts, several characterization results are obtained within this context. These also may shed light on the extremality properties of evolutionary stability.

**Keywords:** Asymptotic stability; evolutionary games; mixed strategies; polymorphisms; replicator dynamics; stable sets.

### 1. Introduction

Evolutionary stability, the central solution concept in evolutionary game theory, is closely related to local asymptotic stability in a certain nonlinear dynamical system operating on the state space, the so-called "replicator dynamics". However, a purely dynamical characterization of evolutionary stability is not available in an elementary manner. This characterization can be achieved by investigating so-called "derived games" which consist of mixed strategies corresponding to successful states in the original game. The purpose of this article is to provide a unified representation of both notation and terminology in order to maintain a coherent biological interpretation of mathematical results, the majority of which appear to be scattered in a broad spectrum of literature with similar methods of proof (for the readers' convenience, we shall present suitable variants of these arguments here), but mostly of the form "evolutionary stability implies dynamical stability".

An implication of the reverse form is theorem 2 in [16]. Very recently, an equivalence theorem [4, theorem 1] has been stated explicitly without proof. Apart from being self-contained, this paper stresses the necessity to interpret monomorphic and polymorphic models—which in some instances seem to be confused with each other—in a different way, and presents a detailed discussion of (strong) asymptotic stability versus (weak) Lyapunov stability which also may shed some light on the extremality of evolutionary stability.

The prototypical examples in the seminal papers of Maynard Smith and Price [9, 7] dealt with stability of population states with the aim to explain polymorphism of behaviour (a state where different individuals may behave differently; cf. also the forerunning article [12]). Also, the concept of evolutionary stability seems—at least from the point of view of frequency-dependent selection—to be more stringent in the context of polymorphic states than in models under a monomorphic interpretation, featuring evolutionary stable strategies (played by everyone in the population) instead of evolutionary stable population states. So we start in the present article with the basic, polymorphic model, but the results obtained below may to some extent support the monomorphistic approach, which plays a prominent role in recent literature (see, e.g. [4], focussing also on the dynamical aspects in this context). To be more precise, the paper is organized as follows: section 1 deals with (possibly) polymorphic populations consisting of individuals that display pure strategies only, while the subsequent sections treat derived games where the original states now play the role of a single (mixed) strategy, and where the dynamical outcomes (i.e. the stable equilibria) are monomorphisms centered at that strategy. In section 2, the concurring states are dimorphisms throughout (only two types of behaviour present), while section 3 deals with  $k$ -morphism setup for the derived games. Section 4 is devoted to weak (Lyapunov) stability, and section 5 characterizes evolutionary stability in terms of stable sets, focusing on the convergence of the average strategy played within the  $k$ -morphic population in the derived game.

To begin with, let us shortly recapitulate the simplest formal setting of an evolutionary game  $\Gamma$ : assume that, in a certain contest-like situation, individuals are capable to display  $n$  different types of behaviour which we for short call strategies in the sequel. With respect to this contest, the state of the population in question is then fully described by the state space

$$S^n = \{[x_1, \dots, x_n]' \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n, \sum_{i=1}^n x_i = 1\},$$

where  $x_i$  represents the relative frequency of individuals displaying strategy  $i$  (for short called  $i$ -individuals in the sequel) within the population (the symbol  $'$  denotes transposition).

There are monomorphic populations (e.g. in state  $e = [1, 0, \dots, 0]' \in S^n$ ) where all individuals display one strategy (e.g. strategy 1), but more interesting are of course polymorphic states (e.g.  $[1/n, \dots, 1/n]' \in S^n$ ), where more than one type of behaviour is present (e.g., all with equal frequency). It should be noted that in the

setting, every individual can only display one strategy throughout its lifetime, irrespective of whether the population is monomorphic or polymorphic.

In this article, we assume that the mean payoff (incremental fitness) to an  $i$ -individual contesting with individuals belonging to a population in state  $x \in S^n$ , depends linearly upon the state  $x$ . As shown in [3, pp. 61–64], this assumption essentially means that the conflict is pairwise, i.e. there are always exactly two individuals participating in a contest. Therefore, if we denote by  $a_{ij}$  the payoff of an  $i$ -individual in a confrontation with a  $j$ -individual, then

$$A = [a_{ij}]_{1 \leq i \leq n, 1 \leq j \leq n}$$

forms the  $n \times n$  payoff matrix which fully characterizes the game. Indeed, the mean payoff to an  $i$ -individual contesting with individuals belonging to a population in state  $x = [x_1, \dots, x_n]' \in S^n$ , amounts to

$$(Ax)_i = \sum_{j=1}^n a_{ij}x_j,$$

and hence the average mean payoff within a (sub-)population in state  $y = [y_1, \dots, y_n]' \in S^n$  against a population in state  $x$  is given by

$$y'Ax = \sum_{i=1}^n y_i(Ax)_i = \sum_{i=1}^n \sum_{j=1}^n y_i a_{ij}x_j.$$

Now we are in a position to formulate the notion of evolutionary stability introduced in [9] and [7]:

**DEFINITION 1**

A state  $p \in S^n$  is said to be "evolutionarily stable" in a game  $\Gamma$ , if for all different states  $q \in S^n$ ,  $q \neq p$ ,

$$\text{either } q'Ap < p'Ap \text{ or } q'Ap = p'Ap \text{ and } p'Aq > q'Aq$$

holds.

In other words, a state  $p$  is evolutionarily stable if

- the average mean payoff for a population in a different state  $q$  against  $p$  does not exceed the average mean payoff within the population in state  $p$  (equilibrium condition), and
- if, in case of equality in the equilibrium condition, the state  $q$  has a lower average mean payoff within itself than  $p$  has against  $q$  (stability condition).

Following Taylor and Jonker [13], we introduce the replicator dynamics  $\mathcal{D}(\Gamma)$  corresponding to the game  $\Gamma$ , which operates on the state space  $S^n$  (a dot  $\cdot$  denotes differentiation with respect to time  $t$ ):

$$\dot{x}_i = x_i[(Ax)_i - x'Ax], \quad 1 \leq i \leq n.$$

This dynamical system is one of the simplest models describing the evolution of the population over time: indeed, the amount  $(Ax)_i - x'_i Ax$  by which the mean payoff for  $i$ -individuals exceeds—or is exceeded by—the average mean payoff, is considered to be equal to the relative growth—or shrinking—rate  $\dot{x}_i/x_i$  of  $i$ -individuals. By means of the replicator dynamics we can now introduce a second stability notion as follows:

#### DEFINITION 2

A state  $p \in S^n$  is said to be "dynamically stable" in a game  $\Gamma$ , if and only if  $p$  constitutes a locally asymptotically stable fixed point for the replicator dynamics  $\mathcal{D}(\Gamma)$ .

In other words, a state  $p$  is dynamically stable in  $\Gamma$  if every trajectory  $x(t)$  that describes the evolution according to  $\mathcal{D}(\Gamma)$  tends to  $p$  (i.e.,  $x(t) \rightarrow p$  as  $t \rightarrow +\infty$ ), whenever it started in a state sufficiently close to  $p$  (i.e., if the distance from  $x(0)$  to  $p$  is small enough).

Taylor and Jonker showed for general  $n \times n$ -games that every evolutionarily stable state  $p$  is dynamically stable [13]. The converse is true for  $2 \times 2$ -games: here evolutionary and dynamical stability coincide (see, e.g. [2, theorem 30] or [17, p. 226]). On the other hand, some  $3 \times 3$ -games have dynamically stable equilibria that are not evolutionarily stable, as the following example due to Zeeman [19] shows:

#### EXAMPLE

For the game  $\Gamma$  given by

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix}, \quad p = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

is dynamically stable, but not evolutionarily stable, since for  $q = [6/9, 1/9, 2/9]'$  we have  $p'Aq = 10/27 < 32/81 = q'Aq$ .

#### Remark

A notion stronger than dynamical stability is that of global (dynamical) stability: a state  $p \in S^n$  is called "globally stable", if every trajectory  $x(t)$  starting in the relative interior of  $S^n$  tends to  $p$  as  $t \rightarrow +\infty$ . One readily sees that global and evolutionary stability are incompatible with each other: indeed, since there are games  $\Gamma$  with more than one evolutionarily stable state, these cannot be globally stable for  $\mathcal{D}(\Gamma)$ . On the other hand, the state  $p$  in the above example is even globally stable though not evolutionarily stable.

## 2. Mixed strategies; derived $2 \times 2$ -games

As the remarks at the end of the previous section show, a purely dynamical characterization of evolutionary stability is not available in a straightforward manner. Nevertheless, such a characterization seems to be desirable, and this is the main objective of the present paper. To achieve this, we shall pass from pure strategists' models to mixed strategists' models (see, e.g. [5, 14–16, 11], and [3, pp. 7–15]). Now it is important to distinguish between

- polymorphic (population) states describing the distribution of behaviour within a population, and
- mixed (individual) strategies, which include probabilities  $\pi_i$  for an individual to display behaviour pattern  $i$ .

Hence, (polymorphic or monomorphic) states describe a *statistical* aspect while (mixed or pure) strategies refer to an *individualistic* aspect of behaviour. The term "probability  $\pi_i$ " may have different interpretations, e.g.  $\pi_i$  may correspond to relative frequencies in sequential contests, or to a change in behaviour with relative duration  $\pi_i$  over lifetime.

A first step towards our goal is done by the following theorem, in which we use the notion of a derived  $2 \times 2$ -game which is a special case of an evolutionary mixed strategy game investigated by Sigmund [11] (following his lines, we shall treat the general case of derived  $k \times k$ -games below). The rationale for this concept is the idea that individuals belonging to a population in an evolutionarily stable state  $p$ , might adopt a mixed strategy in displaying behaviour pattern  $i$  with a probability  $\pi_i$  that equals the frequency  $p_i$  of  $i$ -individuals in state  $p$ .

Suppose there is a second state  $q \neq p$  that is adopted as a mixed strategy by some other individuals (putting  $\pi_i = q_i$ ) in the same way, perhaps because  $q$  is also evolutionarily stable. Recall there are games that have several evolutionarily stable states (e.g.  $n$  in the game with payoff matrix  $A = I$ , the  $n \times n$ -identity matrix); cf. [3, pp. 76–81]. Furthermore assume that in this new situation, every individual adopts either  $p$  or  $q$  as a mixed strategy. Hence, all population states considered here are dimorphic: there are only two types of individuals, confrontations between them would best be described in a game  $\Gamma_{p,q}$  which we call "derived game", and which has the  $2 \times 2$ -payoff matrix

$$A_{p,q} = \begin{bmatrix} p'Ap & p'Aq \\ q'Ap & q'Aq \end{bmatrix} = \begin{bmatrix} p' \\ q' \end{bmatrix} A[p, q];$$

indeed, the first row of  $A_{p,q}$  is constituted by the payoffs for  $p$ , while the second represent the payoffs for  $q$ . With respect to  $\Gamma_{p,q}$ , a state of the population is of course fully described by a vector

$$\begin{bmatrix} x \\ 1-x \end{bmatrix} \in S^2 \text{ (i.e. } 0 \leq x \leq 1),$$

where  $x$  is the relative frequency of  $p$ -individuals and  $1 - x$  is the relative frequency of  $q$ -individuals. If  $0 < x < 1$ , then the state is genuinely dimorphic, while a monomorphic population of  $p$ -individuals is given by the state  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . The next theorem now shows that a population state  $p$  (which may be a polymorphism w.r.t. the original game) is evolutionarily stable, if and only if the corresponding monomorphism, based on  $p$  played as a mixed strategy, beats any dimorphism based on both  $p$  and another mixed strategy arising from a  $q \neq p$  in the original game, provided enough individuals play  $p$ . This means that the  $p$ -monomorphism  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is dynamically stable w.r.t. the replicator dynamics  $\Gamma_{p,q}$  (see also the remark after theorem 4 below). The result is stated in verbal form by Maynard Smith in [8, app. D]; it can be derived from arguments occurring in [18], and has been written down explicitly in [6, p. 125].

#### THEOREM 1

A state  $p \in S^n$  is evolutionarily stable for  $\Gamma$  if and only if  $p$  corresponds to a dynamically stable equilibrium state  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  in any derived  $2 \times 2$ -game  $\Gamma_{p,q}$  of dimorphic populations playing  $p$  and  $q$  as mixed strategies, where  $q \neq p$  is a different state  $q \in S^n$ .

#### *Proof*

Assume that  $\Gamma$  is represented by the payoff matrix  $A$ . Then the payoff matrix  $A_{p,q}$  of the  $2 \times 2$ -game  $\Gamma_{p,q}$  equals

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} p'Ap & p'Aq \\ q'Ap & q'Aq \end{bmatrix}.$$

The corresponding replicator dynamics  $\mathcal{D}(\Gamma_{p,q})$  on  $S^2 = \{[1-x] : 0 \leq x \leq 1\}$  is then given by

$$\dot{x} = x(1-x)[(a-c)x + (b-d)(1-x)].$$

Since  $q \neq p$ , evolutionary stability of  $p$  yields

$$a - c = p'Ap - q'Ap \geq 0$$

and

$$b - d = p'Aq - q'Aq < 0 \quad \text{whenever} \quad a - c = 0.$$

Thus, for  $x$  sufficiently close to 1,  $\dot{x}$  has to be strictly positive. This implies that the state  $[1-x]$  evolves towards  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  as time goes on, if the initial state is sufficiently close to  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . Therefore, the monomorphic state where every individual displays  $p$  will be fixed. Assume, conversely, that  $p$  is dynamically stable in any game  $\Gamma_{p,q}$  with  $q \neq p$ . Since there are only two possible behavioural patterns,  $p$  and  $q$ , (local) fixation of  $p$  in turn yields  $\dot{x} > 0$  if  $x$  is close to 1, which entails, by reversion of the above arguments, evolutionary stability of the state  $p$  in the original game  $\Gamma$ .  $\square$

**EXAMPLE**

Let us investigate the example from section 1 in the light of the above results: the derived game  $\Gamma_{p,q}$  has payoff matrix

$$A_{p,q} = \begin{bmatrix} 2/3 & 10/27 \\ 2/3 & 32/81 \end{bmatrix}.$$

The corresponding replicator dynamics reads

$$\dot{x} = \frac{2}{81}x(1-x)^2 < 0, \quad 0 < x < 1,$$

so that  $x(t) \downarrow 0$ , i.e.

$$\begin{bmatrix} x(t) \\ 1-x(t) \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{as } t \rightarrow \infty.$$

Hence, the state  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  corresponding to  $p$  is not dynamically stable in the derived game, although  $p$  is dynamically stable in the original game.

**3. Characterization in terms of derived  $k \times k$ -games**

Since the original game can have more than two evolutionarily stable states, it is tempting to consider derived games of  $k$ -morphic populations. Here individuals can adopt  $k \geq 2$  mixed strategies corresponding, say, to  $p_1, \dots, p_k$ , where  $p_i \in S^n$  describe (possibly) polymorphic states of the "old" population. The state of the "new",  $k$ -morphic population comprising these  $p_i$ -individuals is then fully described by a vector  $x = [x_1, \dots, x_k]' \in S^k$ , where  $x_i$  is the relative frequency of individuals adopting a mixed strategy adapted to the state  $p_i$ . For instance, a monomorphic population consisting exclusively of  $p_1$ -individuals is described by the state  $e = [1, 0, \dots, 0]' \in S^k$ . The derived game  $\Gamma_{p_1, \dots, p_k}$  of  $k$ -morphisms has the  $k \times k$ -payoff matrix

$$A_{p_1, \dots, p_k} = \begin{bmatrix} p'_1 \\ \vdots \\ p'_k \end{bmatrix} A[p_1, \dots, p_k] = C'AC,$$

where  $C$  is an  $n \times k$ -matrix consisting of the columns  $p_1, \dots, p_k$ .

*Remark*

To emphasize that  $x$  describes the population state with respect to  $\Gamma_{p_1, \dots, p_k}$ , one could write  $\langle x, p_1, \dots, p_k \rangle$  instead of  $x$ . Formally,  $x$  corresponds to a statistical distribution (i.e. a probability measure) on  $S^n$  of the form



$$\sum_{j=1}^k x_j \delta_{p_j},$$

where  $\delta_p$  denotes Dirac measure (unit point mass) located at  $p$ . Models involving states that correspond to more general distributions on  $S^n$  are treated in [3]. To avoid lengthy notations, we however shall denote a state simply by  $x$ , if it is clear from the context that this is the description of a population with respect to the game  $\Gamma_{p_1, \dots, p_k}$ .

It is useful to introduce the notation of the "population strategy"  $m_x$  which denotes the average strategy adopted within a population in state  $x$ :

$$m_x = \sum_{j=1}^k x_j p_j = Cx.$$

In terms of the population strategy, the replicator dynamics  $\mathcal{D}(\Gamma_{p_1, \dots, p_k})$  on the new state space  $S^k$  takes the simple form

$$\dot{x}_i = x_i [p'_i A m_x - m'_x A m_x], \quad 1 \leq i \leq k.$$

Indeed, we have  $(C'ACx)_i = p'_i ACx = p'_i A m_x$  and  $x'C'ACx = m'_x A m_x$ . Of course the dynamics  $\mathcal{D}(\Gamma)$  and  $\mathcal{D}(\Gamma_{p_1, \dots, p_k})$  are strongly interrelated. For instance, in [11] it is shown that if  $\mathcal{D}(\Gamma)$  is a certain type of gradient system, then so is  $\mathcal{D}(\Gamma_{p_1, \dots, p_k})$ .

The results presented in this and the subsequent section deal with dynamical stability properties of an evolutionarily stable state  $p$  in the original game, with respect to the derived  $k$ -morphism game  $\Gamma_{p_1, \dots, p_k}$ , where  $p_1 = p$  while  $p_j$ ,  $2 \leq j \leq k$ , are different states of the original population. At first let us generalize theorem 1 (note that the converse of theorem 2 below follows by theorem 1; cf. also [4] and [18]):

#### THEOREM 2

Consider  $k$  different states  $p_1, \dots, p_k$  in  $S^n$ . If  $p_1$  is evolutionarily stable for  $\Gamma$ , and if  $p_1$  is no convex combination of  $p_2, \dots, p_k$ , then  $p_1$  corresponds to a dynamically stable state  $e$  in  $\Gamma_{p_1, \dots, p_k}$ .

#### *Proof*

(1) Let  $e = [1, 0, \dots, 0]' \in S^k$  denote the monomorphic state of the mixed strategists' population, where every individual displays strategy  $p_1$ . We at first show that, under the assumption of the theorem, the population strategy  $m_e$  determines the state  $e$  uniquely. Indeed, suppose that for some  $x \in S^k$ ,  $x \neq e$ ,

$$\sum_{j=1}^k x_j p_j = m_x = m_e = p_1$$

holds. Putting  $\lambda_j = x_j/(1 - x_1) \geq 0$ ,  $2 \leq j \leq k$ , this implies

$$p_1 = \sum_{j=2}^k \lambda_j p_j,$$

which by assumption is absurd. Hence,  $m_x \neq m_e = p_1$  if  $x \neq e$ .

(2) Now we prove  $(p_1 - m_x)Am_x > 0$  whenever  $x \neq e$  is close enough to  $e$ . Indeed, since this map  $x \mapsto m_x$  is continuous,  $m_x$  is close to  $m_e = p_1$  provided that the distance between  $x$  and  $e$  is sufficiently small. Evolutionary stability of  $p_1$  in the original game  $\Gamma$  now yields

$$(p_1 - q)' Aq > 0 \quad \text{whenever } q \neq p_1 \text{ is close enough to } p_1 \quad (*)$$

([10]; this result essentially depends upon the fact that an evolutionarily stable state in a pairwise conflict with finitely many pure strategies is uninvadable, cf. [3, p. 95]). Thus the claimed assertion follows, putting  $q = m_x$ .

(3) Finally consider the first equation of the replicator dynamics  $\mathcal{D}(\Gamma_{p_1, \dots, p_k})$ :

$$\dot{x}_1 = x_1(p_1 - m_x)' Am_x > 0,$$

whenever  $x \neq e$  is close enough to  $e$ . Hence,  $x_1(t)$  increases towards unity if  $t \rightarrow +\infty$ , implying  $x(t) \rightarrow e$  as  $t \rightarrow +\infty$ . Thus  $e$  constitutes a dynamically stable state for the dynamics  $\mathcal{D}(\Gamma_{p_1, \dots, p_k})$ .  $\square$

*Remark*

In (1) and (2) above, we in fact proved that

$$(e - x)' C' ACx = (p_1 - m_x)' Am_x > 0 \quad \text{whenever } x \neq e \text{ is close to } e,$$

which amounts to evolutionary stability of the state  $e$  in the derived game  $\Gamma_{p_1, \dots, p_k}$ . Thus, invoking the result of [13] implying dynamical stability, we obtain an alternative proof of theorem 2. However, the above proof has the advantage that it admits an immediate generalization adapted to games with nonlinear payoff; see [3, pp. 104 and 109]. Remark further that the condition on  $p_1$  in theorem 2 above to be no convex combination of the remaining states  $p_2, \dots, p_k$ , is equivalent to extremality of  $p_1$  in the convex hull of all the states  $p_1, \dots, p_k$ .

**4. Weak dynamical stability**

This section deals with the case where the extremality assumption of theorem 2 is violated. Putting  $p_2 = p_1$  one immediately sees that one cannot hope for dynamical stability of  $p_1$  in the derived game under these circumstances; moreover, we have the following general negative result:

## THEOREM 3

Consider  $k$  states  $p_1, \dots, p_k$  in  $S^n$ , where  $p_1$  is a convex combination of  $p_2, \dots, p_k$ . Then the state  $e$  in  $\Gamma_{p_1, \dots, p_k}$  corresponding to  $p_1$  cannot be dynamically stable, even if  $p_1$  were evolutionarily stable for the original game  $\Gamma$ .

*Proof*

Assume that  $p_1 = \sum_{j=2}^k \lambda_j p_j$  for some  $\lambda_j \geq 0$ ,  $2 \leq j \leq k$ , with  $\sum_{j=2}^k \lambda_j = 1$ . Put  $S = \{x \in S^k : x_1 > 0\}$  and consider the function

$$x \mapsto V(x) = \frac{1}{x_1} \prod_{j=2}^k x_j^{\lambda_j}, \quad x \in S,$$

which is continuous on  $S$  and vanishes at  $e = [1, 0, \dots, 0]' \in S$ . If  $e$  were dynamically stable, every trajectory  $x(t)$  starting in the neighbourhood of  $e$  would fulfill  $x(t) \rightarrow e$  and hence, by continuity,  $V(x(t)) \rightarrow V(e) = 0$  as  $t \rightarrow +\infty$ . If  $x(0) = [x_1, \dots, x_k]' \in S^k$  with  $x_i > 0$ , all  $i$ ,  $1 \leq i < k$ , then  $V(x(0)) > 0$ . Therefore we would obtain the relation

$$V(x(t)) < V(x(0)), \quad t \text{ large enough,}$$

which is absurd since  $V$  is a constant of motion for the dynamics  $\mathcal{D}(\Gamma_{p_1, \dots, p_k})$ : indeed, taking logarithms and differentiating with respect to time  $t$ , we get

$$\begin{aligned} [\log V(x(t))]' &= -\frac{\dot{x}_1(t)}{x_1(t)} + \sum_{j=2}^k \lambda_j \frac{\dot{x}_j(t)}{x_j(t)} \\ &= (m_{x(t)} - p_1)' Am_{x(t)} + \sum_{j=2}^k \lambda_j (p_j - m_{x(t)})' Am_{x(t)} \\ &= (m_{x(t)} - p_1)' Am_{x(t)} + \left( \sum_{j=2}^k \lambda_j p_j \right)' Am_{x(t)} - \sum_{j=2}^k \lambda_j m_{x(t)}' Am_{x(t)} \\ &= (m_{x(t)} - p_1)' Am_{x(t)} + p_1' Am_{x(t)} - m_{x(t)}' Am_{x(t)} = 0. \end{aligned}$$

Hence  $e$  cannot be dynamically stable. □

*Remark*

One could relate the above result to the proof of theorem 2 in that the population strategy  $m_x$  now does not determine the state  $x$  uniquely: more precisely, in any neighbourhood of  $e$  there are states  $x \neq e$  with  $m_x = m_e = p_1$  and therefore  $\dot{x}_i = x_i(p_i - p_1)' Ap_1 = 0$ . Hence  $e$  cannot be dynamically stable. The proof above is a variant of an argument of Sigmund [11] who showed that there is a one-dimensional foliation of  $S^k$  under the dynamics  $\mathcal{D}(\Gamma_{p_1, \dots, p_k})$ , provided  $\{p_1, \dots, p_k\}$  are linearly dependent. However, as theorem 2 shows, this foliation need not be incompatible

with dynamical stability of  $e$ . On the other hand, the existence of such a foliation might suggest that  $e$  is weakly dynamically stable in Lyapunov's sense:

#### DEFINITION 3

A state  $p \in S^n$  is said to be "weakly dynamically stable" in a game  $\Gamma$  if and only if  $p$  constitutes a Lyapunov-stable fixed point for the replicator dynamics  $\mathcal{D}(\Gamma)$ . This means that, given any (small) neighbourhood  $U$  of  $p$  in  $S^n$ , every trajectory starting in  $S^n$ , and closely enough to  $p$ , will not leave  $U$ .

In other words, a state  $p$  is weakly dynamically stable in  $\Gamma$ , if for any prescribed (small) barrier  $\varepsilon > 0$ , every state  $x(t)$  will remain within a distance from  $p$  smaller than  $\varepsilon$ , provided that  $t \geq 0$  and that the distance from  $x(0) \in S^n$  to  $p$  is small enough. Again, the trajectory  $x(t)$ ,  $t \geq 0$ , describes the evolution of the population according to  $\mathcal{D}(\Gamma)$ .

#### THEOREM 4

Let  $p_1, \dots, p_k$  be  $k$  states in  $S^n$ . If  $p_1$  is evolutionarily stable for  $\Gamma$ , then  $p_1$  corresponds to a weakly dynamically stable equilibrium in  $\Gamma_{p_1, \dots, p_k}$ .

#### *Proof*

Similarly to the proof of theorem 2, we derive from evolutionary stability of  $p_1$  that  $\dot{x}_1(t) \geq 0$  holds in the dynamics  $\mathcal{D}(\Gamma_{p_1, \dots, p_k})$ , whenever  $x(t)$  is close to  $e$ , e.g. if  $x(t) \in U_\varepsilon$ , where we put

$$U_\varepsilon = \{x \in S^k : x_1 > 1 - \varepsilon\},$$

$\varepsilon > 0$  being sufficiently small. Since this means that  $x_1(t)$  cannot decrease as time  $t \geq 0$  increases, provided that  $x(0) \in U_\varepsilon$ , we conclude  $x(t) \in U_\varepsilon$  for all  $t \geq 0$ . Because  $U_\varepsilon$  constitutes a neighbourhood of  $e$  in  $S^k$ , which shrinks to  $e$  as  $\varepsilon \downarrow 0$ , weak dynamical stability of  $e$  is thus established.  $\square$

#### EXAMPLE

Consider the meanwhile classical hawk-dove game introduced in [9]. If we rescale payoffs to reduce calculational effort, the payoff matrix for this  $2 \times 2$ - game reads

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The unique evolutionarily stable state in  $\Gamma$  then is  $p = [\frac{1}{2}]$ . Indeed, we have  $x'Ap = 1/2 = p'Ap$  and  $p'Ax = 1/2 > 2x_1(1-x_1) = x'Ax$  for all  $x \in S^2$ ,  $x \neq p$ . Taking  $p_1 = p$ ,  $p_2 = [\frac{1}{0}]$ ,  $p_3 = [\frac{0}{1}]$ , we get

$$C_1 = \begin{bmatrix} 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

and hence

$$C_1'AC_1 = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 0 & 1 \\ 1/2 & 1 & 0 \end{bmatrix},$$

yielding a dynamics  $\mathcal{D}(\Gamma_{p_1, p_2, p_3})$  on  $S^3$  where  $e \neq [1, 0, 0]'$  is Lyapunov-stable, but not locally asymptotically stable. By contrast, if we form  $C_2$  stacking  $p$ ,  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , and  $\begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$ , together, then the resulting payoff matrix

$$C_2'AC_2 = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 0 & 1/4 \\ 1/2 & 1/4 & 3/8 \end{bmatrix}$$

induces a dynamics where  $e$  is even globally asymptotically stable (see fig. 1).

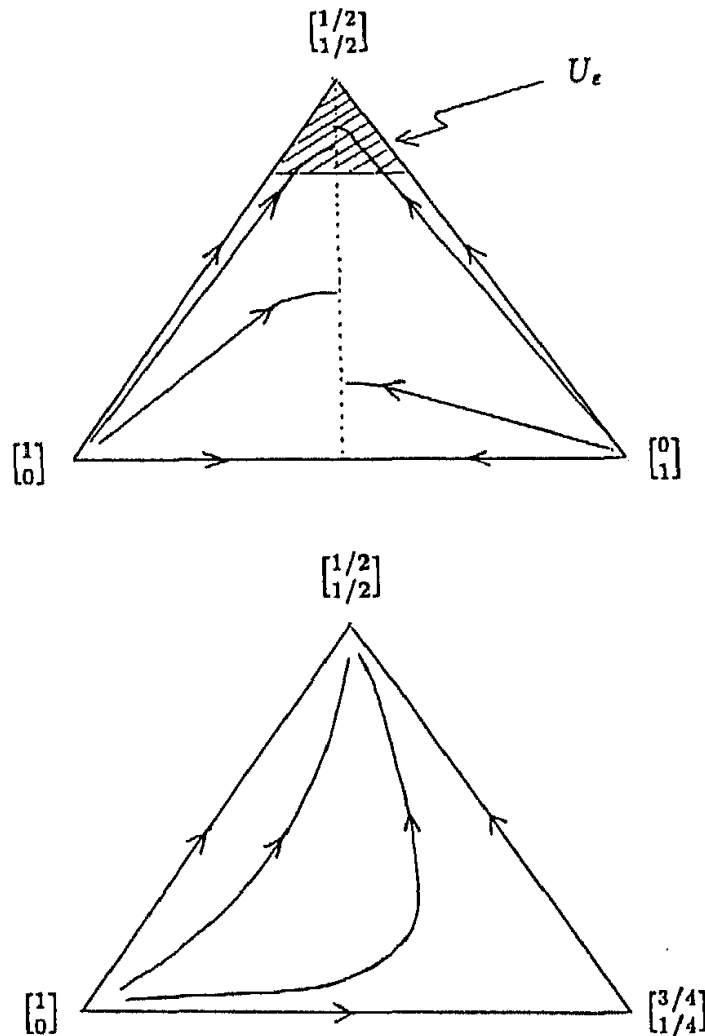


Fig. 1. Dynamics of two games derived from  $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

## 5. Characterization in terms of stable sets

This last section deals with a more general characterization result along lines similar to that followed by Thomas in [15, 16]. If one is interested only in the time evolution of observable characteristics like the population strategy  $m_{x(t)}$  rather than in the evolution of the states themselves, then the following question becomes important: suppose that  $\bar{p} \in S^n$  is an evolutionarily stable state with respect to a game  $\Gamma$ , and consider the derived game  $\Gamma_{p_1, \dots, p_k}$ , where  $\bar{p}$  is a convex combination of the states  $p_1, \dots, p_k$ ; if  $x(t)$  denotes a trajectory subject to the dynamics  $\mathcal{D}(\Gamma_{p_1, \dots, p_k})$ , will the population strategy  $m_{x(t)}$  converge to  $\bar{p}$  or not? In other words, we ask whether or not the states  $x(t)$  will approach the set

$$R_{\bar{p}} = \{x \in S^k : m_x = \bar{p}\}$$

as time goes on (observe that the map  $x \mapsto m_x = \sum_{j=1}^k x_j p_j$  is uniformly continuous on  $S^k$  so that both formulations of the question are indeed equivalent). Using a notion from the theory of dynamical systems, this question amounts to asking whether or not the set  $R_{\bar{p}}$  is stable in the following sense:

### DEFINITION 4

Consider a dynamics  $\mathcal{D}$  operating on  $S^k$ ; a set  $R \subseteq S^k$  is said to be "(dynamically) stable", if there is a set  $U \subseteq S^k$  which is open in  $S^k$  and which contains  $R$ , such that every trajectory starting in  $U$  approaches  $R$ .

In other words,  $R_{\bar{p}}$  is dynamically stable if every trajectory  $x(t)$  subject to the dynamics  $\mathcal{D}(\Gamma_{p_1, \dots, p_k})$  with  $x(0) \in U$  fulfills  $d(x(t), R_{\bar{p}}) \rightarrow 0$  as  $t \rightarrow +\infty$ , where  $U \supseteq R_{\bar{p}}$  is a suitably chosen open set in  $S^k$  and

$$d(x, R_{\bar{p}}) = \inf\{\|x - r\| : r \in R_{\bar{p}}\}$$

is the minimum Euclidean distance from  $x$  to  $R_{\bar{p}}$  ( $\|z\| = \sqrt{z'z}$  denotes the usual Euclidean norm of a vector  $z \in \mathbb{R}^s$ ).

The following result shows that the question formulated above can be answered positively, and moreover provides a general dynamical characterization of evolutionary stability that covers all cases discussed in sections 2 and 3. For the reader's convenience, we specify a proof which is a slight modification of Thomas's arguments [16]; see also [4, theorem 1].

### THEOREM 5

Let  $\bar{p} \in S^n$  be a state in the game  $\Gamma$ . Then the following assertions are equivalent:

- (1)  $\bar{p}$  is evolutionary stable (with respect to  $\Gamma$ );
- (2) for all derived games  $\Gamma_{p_1, \dots, p_k}$ , such that  $\bar{p}$  is a convex combination of the states  $p_1, \dots, p_k$ , the set  $R_{\bar{p}}$  is dynamically stable with respect to the dynamics  $\mathcal{D}(\Gamma_{p_1, \dots, p_k})$ .

*Proof*

(1)  $\Rightarrow$  (2): consider an arbitrary state  $\bar{x} \in R_{\bar{p}}$ , i.e. fulfilling  $m_{\bar{x}} = \bar{p}$ ; we first construct an open neighbourhood  $U_{\bar{x}}$  of  $\bar{x}$  in  $S^k$  that is positively invariant under  $\mathcal{D}(\Gamma_{p_1, \dots, p_k})$ , i.e. fulfills

$$x(t) \in U_{\bar{x}} \text{ for all } t \geq 0 \quad \text{whenever } x(0) \in U_{\bar{x}}$$

(similar to the proof of theorem 4, where  $\bar{x} = e$  and  $U_{\bar{x}} = U_e$ ). To this end we employ evolutionary stability of  $\bar{p}$  which guarantees the existence of an  $\varepsilon > 0$  such that

$$(\bar{p} - q)'Aq > 0 \quad \text{whenever } 0 < \|q - \bar{p}\| < \varepsilon,$$

see (\*) in proof of theorem 2. By continuity of the map  $x \mapsto m_x$ , there is a  $\rho > 0$  fulfilling

$$\|m_x - \bar{p}\| = \|m_x - m_{\bar{x}}\| < \varepsilon \quad \text{whenever } \|x - \bar{x}\| < \rho.$$

Putting these two conditions together, we obtain

$$\text{either } x \in R_{\bar{p}} \text{ or } (\bar{p} - m_x)'Am_x > 0 \quad \text{if } \|x - \bar{x}\| < \rho.$$

Now consider the function

$$V_{\bar{x}}(x) = \prod_{j=1}^k x_j^{\bar{x}_j}, \quad x \in S^k;$$

this function is continuous and strictly positive in a neighbourhood of  $\bar{x}$  in  $S^k$ . Since  $x = \bar{x}$  is the only maximizer of  $V_{\bar{x}}$  on  $S^k$ , there is an  $\alpha > 0$  such that

$$\|x - \bar{x}\| < \rho \quad \text{whenever } V_{\bar{x}}(x) > V_{\bar{x}}(\bar{x}) - \alpha.$$

We claim that

$$U_{\bar{x}} = \{x \in S^k : V_{\bar{x}}(x) > V_{\bar{x}}(\bar{x}) - \frac{\alpha}{2}\}$$

is positively invariant (for simplicity of proof, we choose  $\alpha$  smaller than  $V_{\bar{x}}(\bar{x})$  so that  $V_{\bar{x}}(x) \geq V_{\bar{x}}(\bar{x}) > 0$  for all  $x \in U_{\bar{x}}$ ): indeed, the considerations above ensure the relation

$$(\bar{p} - m_x)'Am_x \geq 0 \quad \text{for all } x \in U_{\bar{x}};$$

furthermore, the function  $V_{\bar{x}}(x(t))$  increases with time  $t$  along any trajectory  $x(t)$  starting in  $U_{\bar{x}}$ , because we obtain – similarly to the proof of theorem 3 –

$$\begin{aligned}
 [\log V_{\bar{x}}(x(t))]' &= \sum_{j=1}^k \bar{x}_j \frac{\dot{x}_j(t)}{x_j(t)} \\
 &= \sum_{j=1}^k \bar{x}_j (p_j - m_{x(t)})' Am_{x(t)} \\
 &= (m_{\bar{x}} - m_{x(t)})' Am_{x(t)} \\
 &= (\bar{p} - m_{x(t)})' Am_{x(t)} \geq 0 \quad \text{if } x(t) \in U_{\bar{x}},
 \end{aligned}$$

and hence  $x(0) \in U_{\bar{x}}$  entails  $x(t) \in U_{\bar{x}}$  for all  $t \geq 0$ . Moreover, by continuity any accumulation point  $\tilde{x} = \lim_{\nu \rightarrow \infty} x(t_\nu)$  of the trajectory  $x(t)$  (where  $t_\nu \uparrow +\infty$  as  $\nu \rightarrow \infty$ ) fulfills

$$V_{\bar{x}}(\tilde{x}) = \lim_{\nu \rightarrow \infty} V_{\bar{x}}(x(t_\nu)) \geq \frac{1}{2} V_{\bar{x}}(\bar{x}) > 0$$

as well as

$$(\bar{p} - m_{\tilde{x}})' Am_{\tilde{x}} = \lim_{\nu \rightarrow \infty} (\bar{p} - m_{x(t_\nu)})' Am_{x(t_\nu)} = \lim_{\nu \rightarrow \infty} [\log V_{\bar{x}}(x(t_\nu))]' = 0.$$

By construction of  $U_{\bar{x}}$ , this equality can prevail only if  $m_{\tilde{x}} = \bar{p}$ , i. e. if  $\tilde{x} \in R_{\bar{p}}$ . Hence we have shown  $d(x(t), R_{\bar{p}}) \rightarrow 0$  as  $t \rightarrow +\infty$ , whenever  $x(0) \in U_{\bar{x}}$ . Now observe that the set

$$U = \bigcup_{\bar{x} \in R_{\bar{p}}} U_{\bar{x}} \subseteq S^k$$

is open in  $S^k$  and contains  $\bar{p}$ . The reasoning above proves  $d(x(t), R_{\bar{p}}) \rightarrow 0$  as  $t \rightarrow +\infty$  whenever  $x(0) \in U$ .

(2)  $\Rightarrow$  (1) follows from theorem 1, taking  $p_1 = \bar{p}_2, p_2 = q$  (an arbitrary state different from  $\bar{p}$ ), and observing that in this case  $R_{\bar{p}} = \{[\frac{1}{0}]\}$  holds, so that the stability of  $R_{\bar{p}}$  is the same as asymptotic stability of  $[\frac{1}{0}]$ .  $\square$

The proof of (2)  $\Rightarrow$  (1) above already indicates why theorem 5 is a generalization of theorems 1 and 2: just take  $\bar{p} = p_1$ , and observe that the extremality condition on  $p_1$  is exactly the relation  $R_{\bar{p}} = \{e\}$ . Theorem 4 is no direct consequence of theorem 5, since the latter cannot exclude the case that a trajectory approaches  $R_{\bar{p}}$  in an "almost cycling manner", coming eventually arbitrarily close to different states  $\bar{x} \in R_{\bar{p}}$ . However, the *proof* of theorem 5 also proves theorem 4, since a trajectory starting in  $U_{\bar{x}}$  never leaves it, and because the neighbourhoods  $U_{\bar{x}}$  shrink towards  $\bar{x}$  as  $\alpha$  decreases to zero.

*Remark*

An alternative approach to theorem 5 would consist in proceeding similarly as Zeeman in [20], namely considering the dynamics for the population strategies:

$$[m_{x(t)}]' = [Cx(t)]' = C\dot{x}(t) = D_{x(t)}Am_{x(t)},$$



where  $D_x = \sum_{j=1}^k x_j p_j p_j' - m_x m_x'$  is the  $n \times n$ -dispersion matrix of the state  $x$ , i.e. the variance/covariance-matrix of the distribution  $\sum_{j=1}^k x_j d_{p_j}$  (cf. the first remark in section 3). The proof in [20, lemma 7], however, seems to work only under additional regularity assumptions both on  $\bar{p}$  in the original game and on the dispersion matrices  $D_{x(t)}$  along trajectories  $x(t)$  near  $R_{\bar{p}}$ . These assumptions are not used in the proof presented above. For instance, Zeeman's method does not apply without modifications to the setting of theorem 3, if  $k \leq n$ ; more generally, if the rank of  $C$  is less than  $n$ , then the rank of  $D_x$  is necessarily less than  $n - 1$ . Similarly, we need to assume neither fullness nor interiority, as done in [1, theorem 3.9] for a much more general setup.

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