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Tijs, S.H.; Jansen, M.J.M.

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On the Existence of Values for Arbitration Games

By *S.H. Tijs*, and *M.J.M. Jansen*, Nijmegen[1])

Abstract: Two-person games in normal form are considered, where the players may use correlated strategies and where the problem arises, which Pareto optimal point in the payoff region to choose. We suppose that the players solve this problem with the aid of an arbitration function, which is continuous and profitable, and for which the inverse image of each Pareto point is a convex set. Then the existence of values and defensive ϵ-optimal strategies is discussed. Existence theorems are derived, using families of suitable dummy zero-sum games. The derived existence theorems contain all known existence results as special cases.

1. Introduction

Let $\Gamma = \langle X, Y, K_1, K_2 \rangle$ be a two-person game in normal form, where X is a non-empty set (the *strategy space of player* 1), Y is a non-empty set (the *strategy space of player* 2), and $K_i: X \times Y \to \mathbf{R}$ is, for $i \in \{1,2\}$, a real-valued function on the Cartesian product of the strategy spaces (the *payoff function for player i*). The set $X \times Y$ is called the *outcome space* and

$$\mathcal{R}_0 = \{K(x, y) = (K_1(x, y), K_2(x, y)) \in \mathbf{R}^2; (x, y) \in X \times Y\}$$

the (*non-cooperative*) *payoff space.* If no form of cooperation is allowed, such a game is played as follows: independently of each other, the players choose an $x \in X$ and a $y \in Y$; then player i obtains a payoff $K_i(x, y)$.

In this paper, we assume that the players may cooperate by mixing the outcomes in the following way: before the game is played, they are allowed to take a finite number[2]) of outcomes $(x_1, y_1), \ldots, (x_k, y_k)$ and real non-negative numbers $p_1, \ldots, p_k$ with $\sum_{i=1}^{k} p_i = 1$ and let a lottery choose one of these outcomes, where (x_s, y_s) is chosen with probability p_s. If (x_s, y_s) is chosen, then the players 1 and 2 are obliged to play x_s and y_s in the game. In such a situation the expected payoff for player i is $\sum_{s=1}^{k} p_s K_i(x_s, y_s)$.

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[1]) Drs. *S.H. Tijs* and *M.J.M. Jansen*, Department of Mathematics, Catholic University Nijmegen, Toernooiveld, NL–6525 ED Nijmegen.

[2]) It is no restriction in the following, to look only at finite mixtures of the outcomes.

The lotteries can be identified with probability measures on $X \times Y$ with finite support and we call them *correlated strategies* in the following. The expected payoff for player i, corresponding to such a probability measure μ, is denoted by $\widetilde{K}_i(\mu)$. By using correlated strategies, the payoff space $\mathcal{R}_0$ is extended to $\mathcal{R}_1 = \text{conv}(\mathcal{R}_0)$, the convex hull of $\mathcal{R}_0$.

For mathematical convenience, we will concentrate our attention not on $\mathcal{R}_1$ but on the closed, convex set $\mathcal{R}(\Gamma) = cl(\mathcal{R}_1)$. $\mathcal{R}(\Gamma)$ is called the *cooperative payoff space* or the *payoff region* of the game. Instead of $\mathcal{R}(\Gamma)$, we will often write $\mathcal{R}$. The set

$$\{p \in \mathcal{R};\ \text{for each } q \in \mathcal{R} \text{ with } q \geqslant p, \text{ we have } q = p\}$$

of undominated elements in $\mathcal{R}$ is called the *Pareto set* of $\mathcal{R}$, and is denoted by $\mathcal{P}(\Gamma)$ or by $\mathcal{P}$. We will say that $\mathcal{R}$ is a *simple region*, if $\mathcal{P} = \emptyset$ or if $\mathcal{P}$ consists of one point; in that case, we also say that the cooperative game Γ is *simple*. Let, for $i \in \{1,2\}$, $u_i = \sup\{r_i; r \in \mathcal{R}\}$. The point $u = (u_1, u_2) \in (-\infty, \infty]^2$ is called the *ideal point or utopia point* of $\mathcal{R}$.

The problem of the players is which correlated strategy to choose. If $\mathcal{R}$ is simple, then this problem is easily solved, because for each claim $c_i < u_i$ of player $i \in \{1,2\}$, there exists a correlated strategy μ such that $\widetilde{K}_i(\mu) \geqslant c_i$ [cf. *Tijs/Jansen*, 1979, p. 5]. In this case u is called the *value* of the cooperative game.

In the following, we mainly look at games Γ with a non-simple cooperative payoff space. For such a non-simple cooperative game Γ, a bargaining problem arises. We will suppose [cf. *Nash*, 1953; *Raiffa*] that, in this case, the players decide to solve their problem with the aid of an arbitrator, who helps the players with the aid of a so-called *arbitration function* ϕ: $\mathcal{R} \to \mathcal{P}$, as follows. A new game is played consisting of four steps.

Step 1: Independently of each other, the players assign an $x_0 \in X$ and a $y_0 \in Y$ and deliver it to the arbitrator. (x_0 and y_0 are called *threat strategies* and $K(x_0, y_0)$ is called the *arbitration starting point* or also the *threat point.*)

Step 2: The arbitrator calculates the payoff $\phi(K_1(x_0, y_0), K_2(x_0, y_0))$ and chooses a correlated strategy μ, such that $\widetilde{K}(\mu) = \phi(K_1(x_0, y_0), K_2(x_0, y_0))$, if that is possible; otherwise μ is chosen, such that $\widetilde{K}(\mu)$ is as close to $\phi K(x_0, y_0)$ as both players want.

Step 3: With the lottery, corresponding to μ, an outcome (x_1, y_1) is determined.

Step 4: Player i ($i \in \{1,2\}$)obtains a payoff $K_i(x, y)$.

From a strategic point of view, for the players, this new game is, essentially, the following non-cooperative game $\Gamma_\phi = \langle X, Y, \phi_1 K, \phi_2 K\rangle$, where $\phi_i K(x, y)$ is the i-th coordinate of $\phi(K_1(x, y), K_2(x, y)) \in \mathbf{R}^2$. This game Γ_ϕ is called the *arbitration game*, corresponding to the non-simple game Γ and the arbitration function ϕ.

Definition 1.1: Let $\epsilon \geqslant 0$. A point $(x^*, y^*) \in X \times Y$ will be called a *defensive ϵ-equilibrium point* of the arbitration game Γ_ϕ, if the following four conditions hold.

$$\phi_1 K(x, y^*) \leqslant \phi_1 K(x^*, y^*) + \epsilon, \text{ for each } x \in X \tag{E.1}$$

$$\phi_2 K(x^*, y) \leqslant \phi_2 K(x^*, y^*) + \epsilon, \text{ for each } y \in Y \tag{E.2}$$

$$\phi_1 K(x^*, y) \geqslant \phi_1 K(x^*, y^*) - \epsilon, \text{ for each } y \in Y \tag{D.1}$$

$$\phi_2 K(x, y^*) \geqslant \phi_2 K(x^*, y^*) - \epsilon, \text{ for each } x \in X. \tag{D.2}$$

Defensive 0-equilibrium points are also called *defensive equilibrium points.*

Note that a point (x^*, y^*), satisfying only E.1 and E.2, is an ϵ-equilibrium point in the Nash sense. D.1 implies that for player 1 strategy x^* is ϵ-defensive in the sense that player 1 can guarantee himself with the aid of this strategy a payoff of at least $\phi_1 K(x^*, y^*)$-ϵ, whatever player 2 does. D.2 can be interpretated similarly for player 2. Note, that the notions defensive ϵ-equilibrium point and ϵ-equilibrium point coincide, in the case that $\epsilon = 0$.

The central problem in this paper is that of the existence of defensive ϵ-equilibrium points for each $\epsilon > 0$, or, more generally, the problem of the existence of a value (see section 4). We will give a number of existence theorems in section 5. These include all known existence results, obtained by *Nash* [1953], *Raiffa* [1953], *Burger* [1956], *Owen* [1971], *Kalai/Rosenthal* [1978], *Tijs/Jansen* [1980]. Most of these older theorems are obtained by using fixed point theorems, which implies that in these theorems strong conditions are laid upon the strategy spaces and payoff functions. With the aid of our proof technique, in which families of suitable dummy zero-sum games play a role, it appears that we need only very mild conditions on the game parameters, to guarantee the existence of a value. The price is some technical work, done in the sections 2 and 3.

2. Pareto Points of Closed Convex Subsets of $\mathbf{R}^2$

Some well-known properties of non-simple regions, which we need in the following, are gathered in the next lemma. The proof of this lemma is left to the reader.

Lemma 2.1: [Cf. *Owen*, p. 4]. *Let $\mathcal{R}$ be a non-simple region of $\mathbf{R}^2$, with Pareto set $\mathcal{P}$. Let $I = \{x \in \mathbf{R}; \exists_{y \in \mathbf{R}}\, (x, y) \in \mathcal{P}\}$. For each $x \in I$, denote by $g(x)$ the unique element $y \in \mathbf{R}$ with $(x, y) \in \mathcal{P}$. Then the following holds:*

(1) *I is a convex subset of $\mathbf{R}$.*

(2) *The functions $g: I \to \mathbf{R}$ and $g^{-1}: g(I) \to \mathbf{R}$ are monotonically decreasing, concave, continuous functions, which are left and right differentiable in the interior points of the domains.*

(3) *$\mathcal{P} = \{(x, g(x)); x \in I\}$ and $\mathcal{P}$ is closed.*

Definition 2.1: Let $\mathcal{R}$, $\mathcal{P}$ and I be as in lemma 2.1.
Let $\alpha = \inf(I)$ and $\beta = \sup(I)$.
If $\alpha \notin I$, then define $g(\alpha) = \lim_{x \downarrow \alpha} g(x)$.
If $\beta \notin I$, then define $g(\beta) = \lim_{x \uparrow \beta} g(x)$.
Then the point $\bar{p} = (\alpha, g(\alpha))$ will be called the *left upper point* of $\mathcal{P}$ and $\underline{p} = (\beta, g(\beta))$ the *right lower point* of $\mathcal{P}$. The function g: $[\alpha, \beta] \to \bar{\mathbf{R}}$ will be called the *Pareto function* corresponding to $\mathcal{R}$.

Note that the first coordinate of the ideal point u equals the first coordinate of $\underline{p}$ and the second coordinate of u equals $\bar{p}_2$. In the following we will denote the set $\mathcal{P} \cup \{\underline{p}, \bar{p}\} \subset \bar{\mathbf{R}}^2$ by $\bar{\mathcal{P}}$ and $\mathcal{P} - \{\underline{p}, \bar{p}\}$ by $\mathring{\mathcal{P}}$. For a compact region $\mathcal{R}$ we have $\mathcal{P} = \bar{\mathcal{P}}$. Finally, we note that

$$(\bar{p}_1, \underline{p}_1) \subset I \subset [\bar{p}_1, \underline{p}_1] \text{ and } \bar{\mathcal{P}} = \{(x, g(x)); x \in [\bar{p}_1, \underline{p}_1]\}.$$

3. Regular Functions on Non-Simple Regions of $\mathbf{R}^2$

Let $\mathcal{R}$ be a non-simple region of $\mathbf{R}^2$, and let $\mathcal{P}$ be the Pareto set of $\mathcal{R}$. We will call a function ϕ: $\mathcal{R} \to \mathcal{P}$ a *regular function* if the following three conditions hold.
R.1: ϕ is continuous,
R.2: ϕ is *profitable* i.e. $\phi(r) \geqslant r$, for all $r \in \mathcal{R}$,
R.3: $\phi^{-1}(p) = \{r \in \mathcal{R}; \phi(r) = p\}$ is a convex subset of $\mathcal{R}$, for each $p \in \mathcal{P}$.
Regular functions were introduced by *Raiffa* [1953, p. 372].

The following lemma says that, for a regular function ϕ, property R.3 is equivalent with the property that ϕ_1 and ϕ_2 are quasi-affine (i.e. quasi-concave and quasi-convex).

Lemma 3.1: Let $\mathcal{R}$ be a non-simple region with Pareto set $\mathcal{P}$. Let ϕ: $\mathcal{R} \to \mathcal{P}$ be a function, satisfying the properties R.1 *and* R.2. *Then* R.3 *is satisfied, if and only if*

$$\{r \in \mathcal{R}; \phi_i(r) \geqslant c\} \text{ and } \{r \in \mathcal{R}; \phi_i(r) \leqslant c\}$$

are convex sets, for all $c \in \mathbf{R}$ and $i \in \{1,2\}$.

Proof: The implication to the right is proved by *Burger* [1956, p. 154]. For the other implication, we observe that, for each $p \in \mathcal{P}$, $\phi^{-1}(p)$ is the intersection of the convex sets $\{r \in \mathcal{R}; \phi_1(r) \geqslant p_1\}$ and $\{r \in \mathcal{R}; \phi_2(r) \geqslant p_2\}$. □

The next lemma plays an important role in the following.

Lemma 3.2: Let ϕ: $\mathcal{R} \to \mathcal{P}$ be regular. Let $p \in \mathring{\mathcal{P}}$.

Let $x \in \mathcal{R}$, $x \neq p$, $\phi(x) = p$ and $\lambda \in [0, \infty)$. Then, for $y = p + \lambda(x - p)$, we have: if $y \in \mathcal{R}$, then $\phi(y) = p$.

Proof: If $\lambda \in [0,1]$, then $y \in [x, p] \subset R$ and $\phi(y) = p$ by R.3. Suppose $\lambda \in (1, \infty)$ and $y \in R$ and put $\phi(y) = q$. Suppose $q \neq p$. Then, without loss of generality, $q_1 < p_1$. Since $p \neq \bar{p}$, we can take an $r \in P$ with $p_1 < r_1$.

Let s be the point, where the line through q and x intersects the line through y and r. Since ϕ is continuous, there is an element $z \in R$, lying on $[y, s]$, such that $z \neq y$ and $\phi_2(z) > p_2$. The line through z and x intersects the Pareto set P in a point u with $u_2 > p_2$. Put $c = \min\{u_2, \phi_2(z)\}$. Then $\phi_2(z) \geqslant c$, $\phi_2(u) = u_2 \geqslant c$, $\phi_2(x) = p_2 < c$, and x is a convex combination of z and u. This is in contradiction with lemma 3.1. Hence $q = p$. □

Let $S = \{x \in \mathbf{R}^2; x \leqslant 0, x_1 + x_2 = -1\}$ and let, for each $x = (x_1, x_2) \in \mathbf{R}^2$, $\check{x}$ be the element $(-x_2, x_1)$ in $\mathbf{R}^2$ and $\| x \|_1 = |x_1| + |x_2|$.

Definition 3.1: For a non-simple region R in $\mathbf{R}^2$ and a regular function ϕ: $R \to P$, we call an element $d \in S$ a *suitable direction* for $p \in \mathring{P}$, if the following holds: for each $\alpha > 0$ with $p + \alpha d \in R$, we have $\phi(p + \alpha d) = p$.

In the following the set of suitable directions for p is denoted by $D(p)$.

Lemma 3.3: Let R, P and ϕ be as above and let $p \in \mathring{P}$. Then

(1) *$D(p)$ is a non-empty, compact, convex subset of S.*

(2) *The multifunction $p \to D(p)$ $(p \in \mathring{P})$ is upper semicontinuous* [cf. *Berge*, 1959].

Proof:

a) If int $(R) = \emptyset$, then $P = R$ and $D(p) = S$, for all $p \in \mathring{P}$, and S is non-empty, compact and convex. Also (2) holds in that case. In the following, we suppose that there is an interior point $i \in R$.

b) Take $q, r \in P$ with $q_1 < p_1 < r_1$. Then $K = [q, i] \cup [i, r]$ is a connected subset of R and $\phi_1 : K \to \mathbf{R}$ is a continuous function. Hence, $p_1 \in [q_1, r_1] = [\phi_1(q), \phi_1(r)] \subset \phi_1(K)$. So there is a $k \in K$ with $\phi(k) = p$. Then $(\| k - p \|_1)^{-1} (k - p) \in D(p) \neq \emptyset$, by lemma 3.2.

c) We show that, for each $d \in D(p)$, there is an $\alpha > 0$ such that $p + \alpha d \in K$. Consider the function f: $K \to \mathbf{R}$ with $f(x) = \langle \check{d}, x - p \rangle$. ($\langle .\,, . \rangle$ is the usual inner product in $\mathbf{R}^2$). Since $f(q) < 0, f(r) > 0$ and f is continuous on the connected set K, there is an $x \in K$ such that $f(x) = 0$. This implies that $x - p = \alpha d$, for some $\alpha \in \mathbf{R}$. Since $p \in P$, $p \neq x$, $x = p + \alpha d \in R$ and $d \leqslant 0$, we have $\alpha > 0$.

d) Now we prove that $D(p)$ is a convex set. Let $d^1, d^2 \in D(p)$ and $t \in (0,1)$. In view of c), there are positive numbers α_1 and α_2, such that $x^1 = p + \alpha_1 d^1 \in K$ and $x^2 = p + \alpha_2 d^2 \in K$. Suppose, without loss of generality, that $\alpha_1 \leqslant \alpha_2$. Then $y = p + \alpha_1 d^2 \in R$ and $\phi(y) = p$ by R.3. Hence, $tx^1 + (1-t)y \in R$ and $\phi(tx^1 + (1-t)y) = p$ by R.3, because $\phi(x^1) = p$ and $\phi(y) = p$. Since $tx^1 + (1-t)y = p + \alpha_1(td^1 + (1-t)d^2)$, we conclude that $td^1 + (1-t)d^2 \in D(p)$, by using lemma 3.2.

e) Now we prove that D is upper semicontinuous in $p \in \mathring{P}$. For each $k \in \mathbf{N}$, let $p^k \in \mathring{P}$, $d^k \in D(p^k)$ and suppose that $\lim_{k\to\infty} p^k = p$ and $\lim_{k\to\infty} d^k = d$. We have to prove that $d \in D(p)$. It follows from c), that, for each $k \in \mathbf{N}$, there exists an $\alpha_k > 0$ such that $y^k = p^k + \alpha_k d^k \in K$. Since K is compact, there is a limit point $y \in K$ of the sequence $y^1, y^2, \ldots$ Since $\phi(y^k) = p^k$, for each $k \in \mathbf{N}$, we have $\phi(y) = p$, by R.1. Now $\alpha_k = \| \alpha_k d^k \|_1 = \| y^k - p^k \|_1$, $d^k = \| y^k - p^k \|_1^{-1} (y^k - p^k)$. So $d = \| y - p \|_1^{-1} (y - p) \in S$. Hence, $y = p + \| y - p \|_1 d$ and $\phi(y) = p$ imply, in view of lemma 3.2, that $d \in D(p)$.

f) It follows immediately from the upper semicontinuity of D that $D(p)$ is a closed subset of the compact set S, for each $p \in \mathring{P}$. Hence, $D(p)$ is compact. □

Lemma 3.4: Let ϕ: $R \to P$ be regular and let $p \in \mathring{P}$, $d \in D(p)$ and $\epsilon > 0$. Then there exists a $\delta > 0$ such that

(1) *for each $r \in R$ with $\langle \check{d}, r - p \rangle > -\delta$, we have $\phi_1(r) \geq p_1 - \epsilon$,*

(2) *for each $r \in R$ with $\langle \check{d}, r - p \rangle < \delta$, we have $\phi_1(r) \leq p_1 + \epsilon$.*

Proof: We only show that there is a $\delta > 0$, such that (1) holds. Take $q \in \mathring{P}$ with $p_1 - \epsilon < q_1 < p_1$. In view of lemma 3.3, there is a $u \in D(q)$ with $u_1 = \max_{d \in D(q)} d_1$. Let

$$L = \{r \in R; r = q + \alpha u, \text{ for some } \alpha \in [0, \infty)\}.$$

Let ℓ: $\mathbf{R}^2 \to \mathbf{R}$ be the affine function with $\ell(r) = \langle \check{d}, r - p \rangle$. If we can show that

$$\sup_{r \in L} \ell(r) = \sup \{\ell(r); r \in \phi_1^{-1}(-\infty, q_1]\} < 0 \tag{3.1}$$

then we can take $\delta = -\sup \{\ell(r); r \in \phi_1^{-1}(-\infty, q_1]\}$ and then $\langle \check{d}, r - p \rangle = \ell(r) > -\delta$ implies $r \notin \phi_1^{-1}(-\infty, q_1]$ or $\phi_1(r) > q_1 > p_1 - \epsilon$, and then the proof is complete.

a) We want to show that for each $y \in \phi_1^{-1}(-\infty, q_1]$, there is an $s \in L \cap [y, p]$. First we prove that $\langle y - q, \check{u} \rangle \leq 0$ for all $y \in \phi_1^{-1}(-\infty, q_1]$. For $y = q$, there is nothing to prove. If $\phi(y) = q \neq y$, then $w = \| y - q \|_1^{-1} (y - q) \in D(q)$ and then $w_1 \leq u_1$, by definition of u_1, which implies that $\langle w, \check{u} \rangle \leq 0$, and so $\langle y - q, \check{u} \rangle \leq 0$. Now, let $\phi_1(y) < q_1$ and suppose that $\langle y - q, \check{u} \rangle > 0$. Since $\langle \phi(y) - q, \check{u} \rangle < 0$, there is a $z \in [y, \phi(y)] \subset R$ with $\langle z - q, \check{u} \rangle = 0$. Then $\phi(z) = q$, because $u \in D(q)$. On the other hand, by R.3, $\phi(z) = \phi(y)$. Hence $\phi(y) = q$, which is impossible. So $\phi_1(y) < q_1$ implies $\langle y - q, \check{u} \rangle \leq 0$. Hence, for all $y \in \phi_1^{-1}(-\infty, q_1]$, $\langle y - q, \check{u} \rangle \leq 0$. Now $\langle p - q, \check{u} \rangle > 0$. So there is an $s \in [p, y]$ with $\langle s - q, \check{u} \rangle = 0$. Then $s \in L$.

b) Now we prove (3.1). We distinguish two cases.

Case 1. $\langle u, \check{d}\rangle \leqslant 0$. Then, for all $\alpha \geqslant 0$, $\ell(q+\alpha u) \leqslant \ell(q)$.
So $\sup_{r\in L} \ell(r) = \ell(q) < 0$. Take $y \in \phi_1^{-1}(-\infty, q_1]$. In view of part a) of the proof, we can find an $s \in L \cap [y, p]$. Then $\ell(s) \leqslant \ell(q) < 0 = \ell(p)$. Since ℓ is affine, $s \in [y, p]$ and $\ell(s) < \ell(p)$, we may conclude that $\ell(y) \leqslant \ell(s)$. Then $\ell(y) \leqslant \ell(q)$ and we have proved (3.1) for this case.

Case 2. $\langle u, \check{d}\rangle > 0$. We show that there is a $b \in R$ such that $L = [b, q]$ and $\ell(b) = \sup_{r\in L} \ell(r)$. Note that $\ell(q) < 0$ and $\lim_{\alpha\to\infty} \ell(q+\alpha u) = \infty$. First we note that, for all $\alpha \in [0, \infty)$ with $q + \alpha u \in L$, we have $\ell(q+\alpha u) < 0$; if $\ell(q+\alpha_1 u) \geqslant 0$ and $q + \alpha_1 u \in L$, then there is $\alpha_2 \in [0, \alpha_1]$ with $\ell(q+\alpha_2 u) = 0$. Therefore, there is a $\beta \geqslant 0$ with $q + \alpha_2 u = p + \beta d$ and, consequently, $q = \phi(q+\alpha_2 u) = \phi(p+\beta d) = p$, which is a contradiction. Let $\alpha_0 = \sup\{\alpha \geqslant 0; q+\alpha u \in L\} \in \mathbf{R}$ and take $b = q + \alpha_0 u \in L$. Then $\ell(r) \leqslant \ell(b)$, for all $r \in L$. For $y \in \phi_1^{-1}(-\infty, q_1]$ and $s \in [y, p] \cap [b, q]$ we have

$$\ell(s) \leqslant \ell(b) < 0 = \ell(p) \text{ and then } \ell(y) \leqslant \ell(s) \leqslant \ell(b).$$

Hence, (3.1) also holds in this case. □

The following corollary is immediate.

Corollary 3.1: Let ϕ: $R \to P$ be regular and let $p \in \mathring{P}$, $d \in D(p)$. Then

(1) *for each $r \in R$ with $\langle \check{d}, r-p\rangle \geqslant 0$, we have $\phi_1(r) \geqslant p_1$,*

(2) *for each $r \in R$ with $\langle \check{d}, r-p\rangle \leqslant 0$, we have $\phi_1(r) \leqslant p_1$.*

4. Values for Arbitration Games

For a non-cooperative two-person game in normal form $\Gamma = \langle X, Y, K_1, K_2\rangle$, we denote the security levels $\sup_{x\in X} \inf_{y\in Y} K_1(x, y)$ and $\sup_{y\in Y} \inf_{x\in X} K_2(x, y)$ of player 1 and 2 by v_1 and v_2, respectively.

We suppose, for the moment, that Γ is a zero-sum game (i.e. $K_2 = -K_1$). Then $v_2 \leqslant -v_1$. If $v_2 = -v_1$, then we say that the game has a value v_1, which is denoted by val (Γ). For a zero-sum game Γ and an arbitration function ϕ, satisfying R.2, we have $P = R = cl$ conv $\{(K_1(x, y), -K_1(x, y)); (x, y) \in X \times Y\}$ and $\Gamma_\phi = \Gamma$. Furthermore, the following three assertions are equivalent.

(1) Γ possesses a real value (val $(\Gamma) \in \mathbf{R}$),

(2) $(v_1, v_2) \in P$,

(3) Γ possesses, for each $\epsilon > 0$, an ϵ-equilibrium point.

(The equivalence of the assertions (1) and (2) is trivial; the equivalence of (1) and (3) is proved in *Tijs* [1977, p. 756].)

Also equivalent are the following two assertions:

(1) Γ possesses a value (val $(\Gamma) \in [-\infty, \infty]$),

(2) $(v_1, v_2) \in P$.

Now, we consider, for an arbitration game $\Gamma_\phi = \langle X, Y, \phi_1 K, \phi_2 K \rangle$, the expressions $v_1 = \sup_x \inf_y \phi_1 K(x, y)$ and $v_2 = \sup_y \inf_x \phi_2 K(x, y)$.

In (1) and (2) of the following lemma, the position of $v = (v_1, v_2) \in \bar{\mathbf{R}}^2$ is described.

Lemma 4.1: Let Γ_ϕ be an arbitration game with Pareto set P. Then,

(1) $v_1 \geqslant \bar{p}_1, v_2 \geqslant \underline{p}_2$,

(2) $v_2 \leqslant g(v_1)$,

(3) *if $v \notin \bar{P}$, then there is a $p \in \overset{\circ}{P}$ with $p > v$.*

Proof: The inequalities in (1) follow immediately from the definitions of v_1 and v_2.

For the proof of (2), suppose that $v_2 > g(v_1)$. Then there exists a $b \in \mathbf{R}$ with $v_2 > b > g(v_1)$. Now $v_2 > b$ implies that there is an $\hat{y} \in Y$ such that

$$\phi_2 K(x, \hat{y}) > b, \text{ for all } x \in X.$$

Likewise, $b > g(v_1)$ implies the existence of an $\hat{x} \in X$ such that

$$\phi_1 K(\hat{x}, y) > g^{-1}(b), \text{ for all } y \in Y.$$

Hence, $\phi K(\hat{x}, \hat{y}) > (b, g^{-1}(b)) \in P$, which is a contradiction. Consequently, $v_2 \leqslant g(v_1)$. Now suppose that $v \notin \bar{P}$. Then, in view of (2) and the last remark in section 2, $v_2 < g(v_1)$. So there is a $c \in \mathbf{R}$ with $v_2 < c < g(v_1)$. Take $p = (g^{-1}(c), c)$. Then $p \in \overset{\circ}{P}$, because $g^{-1}(c) \in (v_1, g^{-1}(v_2)) \subset (\bar{p}_1, \underline{p}_1)$, and $p > v$. □

Inspired by the foregoing remarks about zero-sum games, we now give the following definition.

Definition 4.1: We say that Γ_ϕ has *arbitration value* (v_1, v_2), if $(v_1, v_2) \in \bar{P}$. In that case, v_i is called the *arbitration value for player i* ($i \in \{1,2\}$).

Theorem 4.1: Let $\langle X, Y, \phi_1 K, \phi_2 K \rangle$ be an arbitration game. Then the following two assertions are equivalent:

(1) $(v_1, v_2) \in \bar{P}$.

(2) *For each $\epsilon > 0$, there exists a defensive ϵ-equilibrium point.*

(Note, that we put no conditions on ϕ.)

Proof:

a) Suppose that $(v_1, v_2) \in P$. Let $\epsilon > 0$. By lemma 2.2, the Pareto function g and its inverse g^{-1} are continuous in v_1 and v_2, respectively. Hence, there is a $\delta \in (0, (1/2)\epsilon]$ such that

$$g^{-1}(t) \in (-\infty, v_1 + (1/2)\epsilon], \text{ for each } t \in [v_2 - \delta, \infty) \cap g^{-1}(I) \tag{4.1}$$

$$g(s) \in (-\infty, v_2 + (1/2)\epsilon], \text{ for each } s \in [v_1 - \delta, \infty) \cap I \tag{4.2}$$

Choose $x^* \in X$ such that

$$\phi_1 K(x^*, y) \geqslant v_1 - \delta, \text{ for all } y \in Y, \tag{4.3}$$

and $y^* \in Y$ such that

$$\phi_2 K(x, y^*) \geqslant v_2 - \delta, \text{ for all } x \in X. \tag{4.4}$$

Then (4.2) and (4.3) imply

$$\phi_2 K(x^*, y) \leqslant v_2 + (1/2)\epsilon, \text{ for each } y \in Y, \tag{4.5}$$

and (4.1) and (4.4) imply

$$\phi_1 K(x, y^*) \leqslant v_1 + (1/2)\epsilon, \text{ for each } x \in X. \tag{4.6}$$

Then it follows from (4.6) and (4.3) that, for each $x \in X$,

$$\phi_1 K(x, y^*) \leqslant v_1 + (1/2)\epsilon \leqslant \phi_1 K(x^*, y^*) + (1/2)\epsilon + \delta \leqslant \phi_1 K(x^*, y^*) + \epsilon$$

and from (4.4) and (4.5) that

$$\phi_2 K(x, y^*) \geqslant v_2 - \delta \geqslant \phi_2 K(x^*, y^*) - \delta - (1/2)\epsilon \geqslant \phi_2 K(x^*, y^*) - \epsilon.$$

Hence, E.1 and D.2 are proved.
Similarly, E.2 and D.1 can be proved. Consequently, (x^*, y^*) is a defensive ϵ-equilibrium point. We have proved that assertion (1) implies (2).

b) Now suppose that (2) holds. Let $\epsilon > 0$ and take $(x^*, y^*) \in X \times Y$ such that E.1, E.2, D.1 and D.2 hold. It follows with the aid of E.1 that

$$v_1 \leqslant \inf_y \sup_x \phi_1 K(x, y) \leqslant \sup_x \phi_1 K(x, y^*) \leqslant \phi_1 K(x^*, y^*) + \epsilon$$

and it follows from D.1 that

$$v_1 = \sup_x \inf_y \phi_1 K(x, y) \geqslant \inf_y \phi_1 K(x^*, y) \geqslant \phi_1 K(x^*, y^*) - \epsilon.$$

So $|v_1 - \phi_1 K(x^*, y^*)| \leqslant \epsilon$. Analogously, E.2 and D.2 imply that $|v_2 - \phi_2 K(x^*, y^*)| \leqslant \epsilon$. Since $\phi(K(x^*, y^*)) \in P$, the distance between v and P is at most ϵ. Because $\epsilon > 0$ was arbitrary and P is closed (lemma 2.1(3)), we may conclude that $v \in P$. Hence, (2) implies (1). □

The proof of the following theorem is left to the reader.

Theorem 4.2: For an arbitration game $\langle X, Y, \phi_1 K, \phi_2 K\rangle$ the following two assertions are equivalent:

(1) $(v_1, v_2) \in P$ *and* $\max_x \inf_y \phi_1 K(x, y)$ *and* $\max_y \inf_x \phi_2 K(x, y)$ *exist.*

(2) *The arbitration game possesses a defensive equilibrium point.*

The next theorem can be useful for the calculation of the arbitration value and defensive equilibrium points.

Theorem 4.3: Suppose that the arbitration game $\Gamma_\phi = \langle X, Y, \phi_1 K, \phi_2 K\rangle$ possesses an arbitration value v. Then the zero-sum games $\Gamma_1 = \langle X, Y, \phi_1 K, -\phi_1 K\rangle$ and $\Gamma_2 = \langle X, Y, -\phi_2 K, \phi_2 K\rangle$ possess a value. Furthermore, $v_1 = \text{val}(\Gamma_1)$, $-v_2 = \text{val}(\Gamma_2)$ and the sets of saddle points $S(\Gamma_1)$ and $S(\Gamma_2)$ of Γ_1 and Γ_2 coincide with the set $E(\Gamma_\phi)$ of defensive equilibrium points of Γ_ϕ.

$$[S(\Gamma_1) = \{(x^*, y^*) \in X \times Y;\ \sup_{x \in X} \phi_1 K(x, y^*) = \phi_1 K(x^*, y^*) = \inf_{y \in Y} \phi_1 K(x^*, y)\}].$$

Proof:

a) Suppose, firstly, that $v \in P$. Let $\epsilon > 0$. In view of theorem 4.1, there is a defensive ϵ-equilibrium point (x^*, y^*), satisfying E.1, E.2, D.1, D.2. From E.1 it follows that $\inf_y \sup_x \phi_1 K(x, y) \leqslant \sup_x \phi_1 K(x, y^*) \leqslant \phi_1 K(x^*, y^*) + \epsilon$ and from D.1 it follows that $\sup_x \inf_y \phi_1 K(x, y) \geqslant \inf_y \phi_1 K(x^*, y) \geqslant \phi_1 K(x^*, y^*) - \epsilon$. But then $0 \leqslant \inf_y \sup_x \phi_1 K(x, y) - \sup_x \inf_y \phi_1 K(x, y) \leqslant 2\epsilon$, for each $\epsilon > 0$, which implies that val (Γ_1) exists and is equal to v_1.

b) Suppose, now, that $v = \bar{p} \notin P$. Then $v_2 = \sup_y \inf_x \phi_2 K(x, y)$ implies that, for each $c \in (\underline{p}_2, \bar{p}_2)$, there is an $\hat{y} \in Y$ such that $\phi_2 K(x, \hat{y}) \geqslant c$, for all $x \in X$. Then $\phi_1 K(x, \hat{y}) \leqslant g^{-1}(c)$, for all $x \in X$. Hence, $v_1 = \sup_x \inf_y \phi_1 K(x, y) \leqslant \inf_y \sup_x \phi_1 K(x, y) \leqslant \sup_x \phi_1 K(x, \hat{y}) \leqslant g^{-1}(c)$, for all $c \in (\underline{p}_2, \bar{p}_2)$. Since $\lim_{c \uparrow \bar{p}_2} g^{-1}(c) = g^{-1}(\bar{p}_2) = \bar{p}_1 = v_1$, we may conclude that val $(\Gamma_1) = v_1$ in this case, as well.

c) Suppose that $\mathrm{v} = \underline{p} \notin P$. Since $\phi_1 K(x, y) \leqslant \underline{p}_1$, for all $(x, y) \in X \times Y$, we have $\inf_{y \in Y} \sup_{x \in X} \phi_1 K(x, y) \leqslant \underline{p}_1 = \mathrm{v}_1$. Hence $\mathrm{val}(\Gamma_1) = \mathrm{v}_1$ in this case, also.

d) Now we have proved that, in all cases, $\mathrm{val}(\Gamma_1) = \mathrm{v}_1$. By interchanging the roles of the players one obtains $\mathrm{val}(\Gamma_2) = -\mathrm{v}_2$.

e) The proof of $S(\Gamma_1) = S(\Gamma_2) = E(\Gamma_\phi)$ is straightforward and is left to the reader. □

The next result is an immediate consequence of the foregoing theorem.

Theorem 4.4: [Cf. *Kalai/Rosenthal*, p. 67; *Rauhut/Schmitz/Zachow*, p. 234.] *Let* $\Gamma_\phi = \langle X, Y, \phi_1 K, \phi_2 K\rangle$ *be an arbitration game with arbitration value* v. *Then we have*

(1) *if* $(x^1, y^1) \in E(\Gamma_\phi)$ *and* $(x^2, y^2) \in E(\Gamma_\phi)$, *then* $(x^1, y^2), (x^2, y^1) \in E(\Gamma_\phi)$ *((defensive) equilibria are interchangeable),*

(2) $\phi_i K(x^1, y^1) = \phi_i K(x^2, y^2) = \mathrm{v}_i$, *for* $i \in \{1,2\}$ *(the payoffs are equal for all defensive equilibrium points).*

Remark: Suppose Γ_ϕ has a value $\mathrm{v} \in P$ and let $\epsilon \geqslant 0$. Let us call a point $x^* \in X$ ($y^* \in Y$) an *ε-optimal strategy for player 1 (player 2) in the zero-sum game* Γ_1 if $\inf_y \phi_1 K(x^*, y) \geqslant \mathrm{v}_1 - \epsilon$ ($\sup_x \phi_1 K(x, y^*) \leqslant \mathrm{v}_1 + \epsilon$). Let us call an $x^* \in X$ ($y^* \in Y$) an *ε-optimal strategy in the arbitration game* Γ_ϕ, if $\inf_y \phi_1 K(x^*, y) \geqslant \mathrm{v}_1 - \epsilon$ and $\sup_y \phi_2 K(x^*, y) \leqslant \mathrm{v}_2 + \epsilon$ ($\inf_x \phi_2 K(x, y^*) \geqslant \mathrm{v}_2 - \epsilon$ and $\sup_x \phi_1 K(x, y^*) \leqslant \mathrm{v}_1 + \epsilon$). Then we have

(1) a pair of 0-optimal strategies in Γ_ϕ is a defensive 0-equilibrium point, and also the converse statement is true.

(2) An ε-optimal strategy for player 1 in the game Γ_1 is a δ-optimal strategy for player 1 in Γ_ϕ, where $\delta = \max\{\epsilon, -\epsilon D_\varrho g(\mathrm{v}_1)\}$ and $D_\varrho g(\mathrm{v}_1) = \lim_{h \uparrow 0} h^{-1}(g(\mathrm{v}_1 + h) - g(\mathrm{v}_1))$ ($\delta < \infty$, if $\mathrm{v} \in P - \{\underline{p}\}$ or if $\mathrm{v} = \underline{p}$ and $D_\varrho g(\underline{p}) \neq -\infty$).

5. Existence Theorems

Let $\Gamma = \langle X, Y, K_1, K_2\rangle$ be a game in normal form with non-simple payoff region R and Pareto set P. Let ϕ: $R \to P$ be a regular arbitration function. For each $p \in \overset{\circ}{P}$ and each $d \in D(p)$, we introduce the *dummy zero-sum game* $\Gamma_{p,d} = \langle X, Y, K_{p,d}, -K_{p,d}\rangle$, where

$$K_{p,d}(x, y) = \langle \check{d}, K(x, y) - p\rangle, \text{ for all } (x, y) \in X \times Y.$$

For these dummy games, we derive two lemmas.

Lemma 5.1: Let $p \in \mathring{P}$, $d \in D(p)$. *Then:*

(1) $K_{p,d}(x,y) \leqslant 0$ *implies* $\phi_2 K(x,y) \geqslant p_2$,
(2) $K_{p,d}(x,y) \geqslant 0$ *implies* $\phi_1 K(x,y) \geqslant p_1$.

Proof: The lemma follows immediately from the definition of $K_{p,d}$ and corollary 3.1. □

Lemma 5.2: Let $p \in \mathring{P}$, $d \in D(p)$ *and suppose that* val $(\Gamma_{p,d}) = 0$. *Then p is the arbitration value of the arbitration game* Γ_ϕ. *If, moreover,* $\Gamma_{p,d}$ *possesses optimal strategies* x^* *and* y^* *for player 1 and player 2, respectively, then* (x^*, y^*) *is a defensive equilibrium point of* Γ_ϕ.

Proof:

a) Take $\epsilon \in (0, \underline{p}_1 - p_1)$. By lemma 3.4, there is a $\delta > 0$, such that

$$\langle \check{d}, r - p\rangle > -\delta,\ r \in R \text{ imply } \phi_1(r) \geqslant p_1 - \epsilon,$$

$$\langle \check{d}, r - p\rangle < \delta,\ r \in R \text{ imply } \phi_1(r) \leqslant p_1 + \epsilon.$$

Since val $(\Gamma_{p,d}) = 0$, we can take $x(\epsilon) \in X$, such that

$$\langle \check{d}, K(x(\epsilon), y) - p\rangle > -\delta, \text{ for all } y \in Y$$

and $y(\epsilon) \in Y$, such that

$$\langle \check{d}, K(x, y(\epsilon)) - p\rangle < \delta, \text{ for all } x \in X.$$

Consequently, $\phi_1 K(x(\epsilon), y) \geqslant p_1 - \epsilon$, for all $y \in Y$. Hence, $v_1 \geqslant p_1 - \epsilon$, for each $\epsilon \in (0, \underline{p}_1 - p_1)$, or $v_1 \geqslant p_1$. Similarly, $\phi_1 K(x, y(\epsilon)) \leqslant p_1 + \epsilon$, for all $x \in X$. Then $\phi_2 K(x, y(\epsilon)) \geqslant g(p_1 + \epsilon)$, for each $x \in X$. This implies, that $v_2 \geqslant g(p_1 + \epsilon)$, for each $\epsilon \in (0, \underline{p}_1 - p_1)$. Hence $v_2 \geqslant g(p_1)$ by lemma 2.2. Since, by lemma 4.1(2), $g(v_1) \geqslant v_2$, we have $g(v_1) \geqslant g(p_1)$ and $v_1 \geqslant p_1$. This implies that $g(v_1) = v_2$ and $v = p$. Hence $v \in P$, so $p = v$ is the arbitration value of Γ_ϕ.

b) Now suppose that (x^*, y^*) is an optimal pair for the game $\Gamma_{p,d}$. From $\phi(K(x^*, y^*)) = p$ and

$$K_{p,d}(x, y^*) \leqslant K_{p,d}(x^*, y^*) = 0 \leqslant K_{p,d}(x^*, y) \text{ for all } x \in X, y \in Y$$

we obtain, by lemma 5.1:

$$\phi_2 K(x, y^*) \geqslant p_2 = \phi_2 K(x^*, y^*),\ \phi_1 K(x^*, y) \geqslant p_1 = \phi_1 K(x^*, y^*)$$

for all $x \in X$, $y \in Y$.

Then (x^*, y^*) is a defensive equilibrium point of Γ_ϕ. □

The next three theorems play a crucial role in the derivation of existence theorems for arbitration games.

Theorem 5.1: Let Γ_ϕ be an arbitration game, where $\Gamma = \langle X, Y, K_1, K_2 \rangle$ and ϕ satisfy the following conditions:

(1) *$\phi: R \to P$ is regular,*

(2) *for each $\alpha \in [0,1]$, the zero-sum game $\langle X, Y, \alpha K_1 - (1-\alpha) K_2 \rangle$ possesses a value.*

Then the arbitration game possesses an arbitration value.

Proof: We have to show that $v \in \bar{P}$. Suppose that $v \notin \bar{P}$. Then, by (3) of lemma 4.1, there exists a $p \in \mathring{P}$ with $p > v$. Choose $d \in D(p)$. Since $\sup_y \inf_x \phi_2 K(x, y) = v_2 < p_2$, for each $y \in Y$, there is an $x_y \in X$ such that $\phi_2 K(x_y, y) < p_2$. Then (1) of lemma 5.1 implies that, for each $y \in Y$, $K_{p,d}(x_y, y) > 0$. Condition (2) of this theorem implies that $\Gamma_{p,d}$ has a value. However,

$$\text{val}(\Gamma_{p,d}) = \inf_y \sup_x K_{p,d}(x, y) \geqslant \inf_y K_{p,d}(x_y, y) \geqslant 0.$$

Similarly, the inequality $v_1 < p_1$ implies that $\text{val}(\Gamma_{p,d}) \leqslant 0$. Consequently, $\text{val}(\Gamma_{p,d}) = 0$ and, by lemma 5.1, p is the arbitration value of Γ_ϕ. Hence $v = p > v$, which is a contradiction. So $v \in \bar{P}$. Thus, we have proved that Γ_ϕ possesses an arbitration value. □

Theorem 5.2: Let Γ_ϕ be an arbitration game, where $\Gamma = \langle X, Y, K_1, K_2 \rangle$ and ϕ satisfy the following conditions:

(1) *$\phi: R \to P$ is regular,*

(2) *for each $\alpha \in [0,1]$ the zero-sum game $\langle X, Y, \alpha K_1 - (1-\alpha) K_2 \rangle$ possesses a value and optimal strategies for both players,*

(3) *$K_1 + K_2$ is a bounded function on $X \times Y$,*

(4) *the arbitration value v of Γ_ϕ is unequal to $\underline{p}$ and $\bar{p}$.*

Then *Γ_ϕ possesses a defensive equilibrium point.*

Proof: Since $v \in \mathring{P}$, by lemma 3.3 (1) there exist $a, b \in \mathbf{R}$ with $-1 \leqslant a \leqslant b \leqslant 0$, such that $D(v) = \{(\xi, -1-\xi) \in S; a \leqslant \xi \leqslant b\}$.

a) Firstly, we show that $\text{val}(\Gamma_{v,(a,-1-a)}) \geqslant 0$.
Suppose that $\gamma = \text{val}(\Gamma_{v,(a,-1-a)}) < 0$. Then there is, by (2), an $\hat{y}$ such that

$$\langle (1+a, a), K(x, \hat{y}) - v \rangle \leqslant \gamma, \text{ for all } x \in X.$$

Let $L = \{r \in R; \langle (1+a, a), r - v \rangle = \gamma\}$ and $T = \{r \in R; \langle (1+a, a), r - v \rangle \leqslant \gamma\}$.
Then condition (3) implies that L is a compact set. In view of the definition of a, we have $\delta = \max_{r \in L} \phi_1(r) < \phi_1(v) = v_1$. Furthermore, it is not difficult to prove that

$$\sup_{r\in T} \phi_1(r) = \max_{r\in L} \phi_1(r).$$

Hence, $\phi_1 K(x, \hat{y}) \leqslant \delta$, for each $x \in X$. Consequently, $v_2 = \sup_{y\in Y} \inf_{x\in X} \phi_2 K(x, y) \geqslant$ $\geqslant \inf_{x\in X} \phi_2 K(x, \hat{y}) \geqslant g(\delta) > g(v_1) = v_2$, which is a contradiction. So $\text{val}(\Gamma_{v,(a,-1-a)}) \geqslant 0$.

b) Similarly, as in a), one can prove that $\text{val}(\Gamma_{v,(b,-1-b)}) \leqslant 0$.

c) Let $f: [a, b] \to \bar{\mathbf{R}}$ be the map with $f(\xi) = \text{val}(\Gamma_{v,(\xi,-1-\xi)})$. Then, for $\xi, \eta \in [a, b]$, we have

$$|f(\xi) - f(\eta)| \leqslant |\xi - \eta|\ |v_1 + v_2| + |\xi - \eta| \sup_{(x,y)\in X\times Y} |K_1(x, y) + K_2(x, y)|$$

Then, however, condition (3) implies, that f is a continuous function on the connected set $[a, b]$. By a) and b), we have $f(a) \geqslant 0, f(b) \leqslant 0$. Consequently, there is a $\rho \in [a, b]$, with $f(\rho) = 0$. Take an optimal pair (x^*, y^*) of strategies in the game $\Gamma_{v,(\rho,-1-\rho)}$. By Lemma 5.2, (x^*, y^*) is a defensive equilibrium point of Γ_ϕ. □

Theorem 5.3: Let Γ_ϕ be an arbitration game, where $\Gamma = \langle X, Y, K_1, K_2\rangle$ and ϕ satisfy the following conditions:

(1) *ϕ: $\mathcal{R} \to \mathcal{P}$ is regular,*

(2) *X and Y are compact sets and K_1 and K_2 continuous functions,*

(3) *for each $\alpha \in [0, 1]$, the zero-sum game $\langle X, Y, \alpha K_1 - (1-\alpha) K_2\rangle$ possesses a value.*

Then Γ_ϕ possesses a defensive equilibrium point.

Proof: By theorem 5.1, Γ_ϕ possesses an arbitration value v. Now, $v \in \mathbf{R}^2$, by (2). The sets O_1^ϵ and O_2^ϵ of ϵ-optimal strategies in Γ_ϕ for the players 1 and 2, respectively, are, by (1) and (2), closed subsets of the compact sets X and Y, respectively. Then $O_1 = \bigcap_{\epsilon>0} O_1^\epsilon \neq \emptyset$ and $O_2 = \bigcap_{\epsilon>0} O_2^\epsilon \neq \emptyset$, and $O_1 \times O_2$ is the set of defensive equilibrium points of Γ_ϕ. □

Now we are able to derive, with the aid of theorems 5.1 – 5.3 a string of existence theorems. We start with reproving some old results. In *Nash* [1950], a map ϕ^N was introduced, which assigns to a compact convex set $\mathcal{R}$ and a point $r \in \mathcal{R}$, the (unique) point $\phi^N(r)$ of the Pareto set $\mathcal{P}$ of $\mathcal{R}$, satisfying

$$(\phi_1^N(r) - r_1)(\phi_2^N(r) - r_2) = \max_{p\in\mathcal{P}} (p_1 - r_1)(p_2 - r_2).$$

It is well known that ϕ^N is an example of a regular function. The next theorem was first proved by *Nash* [1953], using the fixed point theorem of Kakutani for multifunctions.

Another proof was given by *Owen* [1971], who, for each point $p \in \mathcal{P}$, constructed a suitable family of zero-sum games. For this special case, it appears that these zero-sum games coincide, for the points $p \in \mathring{\mathcal{P}}$, with our dummy games.

Theorem 5.4: Let $\widetilde{\Gamma} = \langle \widetilde{X}, \widetilde{Y}, \widetilde{K}_1, \widetilde{K}_2 \rangle$ be the mixed extension of the game $\Gamma = \langle X, Y, K_1, K_2 \rangle$ with finite strategy spaces X and Y. If Γ is nonsimple, then the arbitration game $\widetilde{\Gamma}_{\phi^N}$ possesses a value and a defensive equilibrium point.

Proof: For each $\alpha \in [0, 1]$, the game $\langle \widetilde{X}, \widetilde{Y}, \alpha\widetilde{K}_1 - (1-\alpha)\widetilde{K}_2 \rangle$ is the mixed extension of the finite zero-sum game $\langle X, Y, \alpha K_1 - (1-\alpha) K_2 \rangle$, and possesses, therefore, a value and optimal strategies, in view of the minimax theorem of J. von Neumann. Then, however, $\widetilde{\Gamma}_{\phi^N}$ satisfies all conditions in theorem 5.3. □

Similarly, the following extension of *Raiffa* [1953] of the Nash theorem can be proved with the aid of theorem 5.3. We note that Raiffa used the Kakutani fixed point theorem in his proof and that *Burger* [1956] gave a proof, in which the fixed point theorem of Brouwer plays a role.

Theorem 5.5: Let $\widetilde{\Gamma} = \langle \widetilde{X}, \widetilde{Y}, \widetilde{K}_1, \widetilde{K}_2 \rangle$ be the mixed extension of the finite game $\Gamma = \langle X, Y, K_1, K_2 \rangle$, and suppose that the region $\mathcal{R}$ of $\widetilde{\Gamma}$ is non-simple. Then, for each regular ϕ: $\mathcal{R} \to \mathcal{P}$, the arbitration game $\widetilde{\Gamma}_\phi$ possesses a value and a defensive equilibrium point.

In *Kalai/Rosenthal* [1978] and in *Rauhut/Schmitz/Zachow* [1979], special attention is paid to the monotone arbitration function ϕ^M: $\mathcal{R} \to \mathcal{P}$, which assigns to a point r of a non-simple region $\mathcal{R}$ with Pareto set $\mathcal{P}$, the unique point of $\mathcal{P}$ lying on $[r, u]$, where u is the ideal point of $\mathcal{R}$. Obviously, ϕ^M is regular. Hence, it follows immediately from theorem 5.3, that for the mixed extension $\widetilde{\Gamma}$ of a finite game Γ, the arbitration game $\widetilde{\Gamma}_{\phi^M}$ possesses a defensive equilibrium point, a result which was also obtained in the two papers, mentioned above, with the aid of fixed point theorems. *Kalai/Rosenthal* [1978] also considered arbitration games Γ_ϕ, where Γ is the mixed extension of a finite game and where ϕ_1 and ϕ_2 are quasi-concave functions and ϕ satisfies the properties R.1, R.2. They proved the existence of equilibria for such games. In view of lemma 3.1, this result coincides with the existence result in theorem 5.5.

Now we can obtain new results, with the aid of our crucial theorems 5.1 – 5.3. The policy, which we follow, is to look at well-known minimax theorems for zero-sum games (for a survey, see chapter 5 of *Parthasarathy/Raghavan* [1971] or *Yanovskaya* [1974]) and to formulate conditions for arbitration games Γ_ϕ, where $\Gamma = \langle X, Y, K_1, K_2 \rangle$, in such a way that these conditions imply, for each game $\alpha K_1 - (1-\alpha) K_2$, $\alpha \in [0, 1]$, the conditions in the minimax theorems. We start with a theorem, which can be derived from the minimax theorem of *Nikaido* [1954] and theorem 5.3, and also includes the foregoing theorems 5.4 and 5.5.

Theorem 5.6: Let Γ_ϕ be an arbitration game, where $\Gamma = \langle X, Y, K_1, K_2 \rangle$ and ϕ satisfy the following conditions:

(1) *ϕ: $\mathcal{R} \to \mathcal{P}$ is regular,*
(2) *X and Y are compact and convex subsets of linear topological spaces,*
(3) *K_1 and K_2 are continuous functions,*

(4) *For each* $\alpha \in [0, 1]$, *the function* $\alpha K_1 - (1-\alpha) K_2$ *is quasi-concave in the first coordinate, and quasi-convex in the second coordinate.*

Then Γ_ϕ *possesses a defensive equilibrium point.*

A direct consequence of theorem 5.6 is

Theorem 5.7: Let $\Gamma = \langle X, Y, K_1, K_2 \rangle$ *be a game in normal form, where X and Y are compact Hausdorff spaces and where the payoff functions* K_1 *and* K_2 *are continuous functions. Let* $\Gamma = \langle \tilde{X}, \tilde{Y}, \tilde{K}_1, \tilde{K}_2 \rangle$ *be the mixed extension of* Γ, *where* $\tilde{X}$ $(\tilde{Y})$ *is the family of probability measures on the Borel sets of X (Y) and*

$$\tilde{K}_i (\mu, \nu) = \int \int K_i (x, y)\, d\mu (x)\, d\nu (y), \text{ for } (\mu, \nu) \in \tilde{X} \times \tilde{Y}.$$

Then, for each regular function ϕ: $R(\tilde{\Gamma}) \to P(\tilde{\Gamma})$, *the arbitration game* $\tilde{\Gamma}_\phi$ *possesses a defensive equilibrium point.*

Now we present some existence theorems for arbitration games, where both strategy spaces are not necessarily compact. See also *Tijs/Jansen* [1980].

Theorem 5.8: Let $\tilde{\Gamma}_\phi$ *be an arbitration game with regular* ϕ *and where* $\tilde{\Gamma}$ *is a mixed extensions of a semi-infinite game* $\langle X, Y, K_1, K_2 \rangle$, *where X is a finite set and Y a countably infinite set. Then* $\tilde{\Gamma}_\phi$ *possesses an arbitration value.*

Proof: The theorem follows from theorem 5.1 and the fact that all mixed extensions of the semi-infinite zero-sum games $\langle X, Y, \alpha K_1 - (1-\alpha) K_2 \rangle$ possess a value [cf. *Tijs*, 1975, 29–31]. □

The minimax theorem of *Fan* [1953] and theorem 5.1 imply

Theorem 5.9: Let Γ_ϕ *be an arbitration game, where* $\Gamma = \langle X, Y, K_1, K_2 \rangle$ *and* ϕ *satisfy the following conditions*:

(1) ϕ *is regular,*
(2) *X is a compact Hausdorff space,*
(3) K_1 *and* $-K_2$ *are semicontinuous (i.e. upper semicontinuous in the first coordinate and lower semicontinuous in the second coordinate),*
(4) K_1 *and* $-K_2$ *form a concave-convex-like pair i.e. for each* $x^1, x^2 \in X$ *and* $t \in (0, 1)$, *there is an* $x^0 \in X$, *such that*

$$K_1(x^0, y) \geqslant tK_1(x^1, y) + (1-t) K_1(x^2, y), -K_2(x^0, y) \geqslant -tK_2(x^1, y) - (1-t) K_2(x^2, y), \textit{ for each } y \in Y,$$

and for each $y^1, y^2 \in Y$ *and* $t \in (0, 1)$, *there is a* $y^0 \in Y$ *such that*

$$K_1(x, y^0) \leqslant tK_1(x, y^1) + (1-t) K_1(x, y^2), -K_2(x, y^0) \leqslant -tK_2(x, y^1) - (1-t) K_2(x, y^2), \textit{ for each } x \in X.$$

Then Γ_ϕ *possesses an arbitration value.*

The next theorem follows from theorem 5.1 and the minimax theorem of *Sion* [1958].

Theorem 5.10: Let Γ_ϕ *be an arbitration game, where* $\Gamma = \langle X, Y, K_1, K_2 \rangle$ *and* ϕ *satisfy the following conditions*:

(1) ϕ *is regular,*
(2) X *is a compact and convex subset of a linear topological space,*
(3) Y *is a convex set,*
(4) K_1 *and* $-K_2$ *are semicontinuous,*
(5) *for each* $\alpha \in [0, 1]$, *the function* $\alpha K_1 - (1-\alpha) K_2$ *is quasi-concave in the first coordinate, and quasi-convex in the second coordinate.*

Then Γ_ϕ *possesses an arbitration value.*

Also with the aid of theorem 5.1 and the minimax theorems of *König* [1968] and *Terkelsen* [1972], interesting existence theorems for arbitration games can be derived, but we will not state them here, explicitly.

Now we extend a result of *Wald* [1950] for zero-sum games to arbitration games.

Theorem 5.11. Let $\Gamma = \langle X, Y, K_1, K_2 \rangle$ *be a two-person game in normal form, where*

(1) K_1 *and* K_2 *are bounded Borel measurable functions (with respect to the Borel sets corresponding with the* intrinsic *metrics* d_1 *and* d_2 *corresponding to* K_1 *and* K_2, *resp.* [*cf. Wald,* 1950, p. 33],
(2) $\langle X, d_1 \rangle$ *is a conditionally compact metric space.*

Let $\tilde{\Gamma}$ *be the mixed extension of* Γ *and let* ϕ: $\mathcal{R}(\tilde{\Gamma}) \to \mathcal{P}(\tilde{\Gamma})$ *be a regular function.*

Then the arbitration game $\tilde{\Gamma}_\phi$ *possesses an arbitration value.*

In an obvious way, the extensions of *Teh-Tjoe Tie* [1963] and *Parthasarathy* [1965] of Wald's result also imply existence theorems for arbitration games.

6. Some Final Remarks

1. In *Tijs/Jansen* [1979] and *Jansen* [1981] examples are given to show that an arbitration game Γ_ϕ does not necessarily have an arbitration value or a defensive equilibrium point if ϕ is not regular or if Γ does not satisfy one of the conditions in theorem 5.2.
2. In *Jansen/Tijs* [1981] the dummy game approach is used to show that for an arbitration game, where the underlying non-cooperative game is a bimatrix game, the optimal threat strategy spaces for the players are polytopes. Furthermore an algorithm is introduced for the approximation of the arbitration value and optimal threat strategies of such arbitration games.

3. In *Tijs/Jansen* [1982], the authors have studied the effect on the arbitration value and the ϵ-defensive equilibrium points of perturbations of game parameters such as payoff functions and arbitration functions.
4. An extension of Theorem 2 in *Kalai/Rosenthal* [1978] concerning arbitration games with incomplete information for the arbitrator, can be extended to infinite games as can be seen in *Tijs/Jansen* [1979].

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