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TOTALLY BALANCED MULTI-COMMODITY GAMES AND FLOW GAMES

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ABSTRACT

Game situations are considered, where each coalition, by cooperating, can obtain and distribute commodity bundles from a prescribed suitable subset of the commodity space. Examples of such commodity games are multi-commodity flow games arising from controlled multi-commodity flow situations. In this paper conditions are given, guaranteeing that a commodity game can be represented by a controlled multi-commodity flow situation. The results can be seen as extensions of a result of Kalai and Zemel for one-commodity flow games.

1. INTRODUCTION

In a recent paper Kalai and Zemel [7] studied one-commodity flow situations where the arcs in the network are possessed by different owners. They proved that the corresponding side payment game (SP-game) is totally balanced using the well-known max-flow-min-cut theorem of Ford and Fulkerson [4]. In [7] they also showed that a totally balanced SP-game may be seen as a flow game by proving that each non-negative totally balanced SP-game can be expressed as a minimum of a finite collection of additive SP-games.

Totally balanced SP-games occur in many other optimization problems with a control system. Examples are market games (Shapley and Shubik [10]), linear production games (Owen [9]) and permutation games (Tijs et al. [11]). In Dubey and Shapley [3] and in Kalai and Zemel [6] sufficient conditions are given to guarantee the total balancedness of SP-games arising from a controlled optimization problem.

In this paper we consider multi-commodity games (MC-games). Contrary to SP-games, the pay-offs to coalitions in an MC-game consist of commodity bundles. In Derks and Tijs [2] multi-commodity flow situations, in which arcs are controlled by owners, are considered. Such flow situations give rise to MC-games which are totally balanced. In [2] it is proved that for such games there exists a stable outcome, i.e. a distribution over the owners of a Pareto optimal commodity bundle, attainable for the grand coalition, in such a way that no subcoalition has an incentive to split off. This extends the result of Kalai and

Zemel [7] that one-commodity flow games have a non-empty core. We note here that by adapting the proof of theorem 4.1 in [2] it is possible to show that all balanced multi-commodity games have a stable outcome.

In section 3 we start elaborating the question raised in [2]: Can totally balanced MC-games be represented by controlled multi-commodity flow situations? For totally balanced one-commodity games the answer is yes as was shown by Kalai and Zemel [7]. It turns out that the answer of the above question is yes if we consider totally balanced polyhedral MC-games or if we allow infinite networks.

In section 4 we introduce strictly balanced MC-games and prove that balanced polyhedral MC-games and two-commodity flow games are strictly balanced. It is shown that the strict balancedness property for two-commodity flow games asserts that not all totally balanced MC-games are MC-flow games if we exclude the use of infinite networks.

We conclude with an open problem and summarize the obtained results in section 5.

2. PRELIMINARIES

In the following we consider a *network* D (directed graph) with *node set* $P := \{1,2,\ldots,s\}$ and *arc set* $L := \{1,2,\ldots,t\}$. In addition, $N := \{1,2,\ldots,n\}$ denotes the *set of owners (player set) of arcs* and $G := \{1,2,\ldots,m\}$ the *set of commodities*, which are involved in controlled transportations from the *source* $1 \in P$ to the *sink* $s \in P$. The *ownership function* $O : L \to N$ assigns to each arc $\ell \in L$, its controller or owner $O(\ell) \in N$. Finally, $c : L \twoheadrightarrow \mathbb{R}^m_+$ is the *capacity correspondence*, which assigns to each arc ℓ the non-empty subset $c(\ell)$ of the commodity space $\mathbb{R}^m_+$. For the arc ℓ the capacity set $c(\ell)$ consists of the commodity bundles which can be transported through arc ℓ in one unit of time. In this paper it is assumed that the capacity sets are *suitable*, i.e. are compact, convex and comprehensive. (A subset C of $\mathbb{R}^m_+$ is called comprehensive if $y \in C$ for any element y of $\mathbb{R}^m_+$, whenever there is an $x \in C$ such that $y \leq x$). For an interpretation we refer to [2]. Summarizing, a *controlled multi-commodity flow situation* (CMCF-situation) is described by the four-tuple $\Gamma = \langle D,c,N,O \rangle$.

We now give a sequence of definitions together with some comments and elementary results.

(i) A *flow* in Γ (from source 1 to sink s) is a map $f : L \to \mathbb{R}^m_+$ with the following properties:

(F.1) $f(\ell) \in c(\ell)$ for each $\ell \in L$ (*Feasibility property*).

(F.2) $\Sigma\{f(\ell) : \ell$ starts in node $p\} = \Sigma\{f(\ell) : \ell$ ends in node $p\}$ for each $p \in P-\{1,s\}$ (*Conservation property*).

(F.3) $\Sigma\{f(\ell) : \ell$ ends in the source $1\} = 0$ (*Source property*).

(F.4) $\Sigma\{f(\ell) : \ell$ starts in the sink $s\} = 0$ (*Sink property*).

(ii) The *value* $v(f)$ *of a flow* f in Γ is the amount of commodity, leaving the source 1. Hence, $v(f) := \Sigma\{f(\ell) : \ell$ starts in the source $1\}$.

(iii) For any subset A of the arc set L the *flow value set* $F_A(\Gamma)$ *corresponding to* A is the set of all possible values of flows in Γ which only use the arcs in A. Hence,

$$F_A(\Gamma) := \{y \in \mathbb{R}^m : y=v(f) \text{ for some flow f in } \Gamma \text{ with } f(\ell)=0 \text{ for all } \ell \in L-A\}.$$

The set $F(\Gamma) := F_L(\Gamma)$ is called the flow value set of Γ.

For each $A \subset L$ the set $F_A(\Gamma)$ is suitable. The proof is straightforward and therefore omitted.

For one-commodity flow situations one may use the already mentioned max-flow-min-cut theorem for characterizing the flow value set (Cf.[4], [2]). If more commodities are involved a similar theorem is not available anymore. However, some generalizations to multi-commodity networks have been obtained. For a survey of these results and of algoritmic approaches we refer to Assad [1], Hu [5] and Kennington [8].

(iv) Let $S \subset N$ be a coalition of owners. Then L(S) is the set of arcs controlled by S. Hence, $L(S) = \{\ell \in L : O(\ell) \in S\}$.

(v) The correspondence $V_\Gamma : 2^N \twoheadrightarrow \mathbb{R}^m_+$, which assigns to each coalition $S \in 2^N$ the flow value set $F_{L(S)}(\Gamma)$, is called the *multi-commodity flow game* (MC-flow game) *arising from* Γ. It is a game, where the pay-off set $V_\Gamma(S)$ of a coalition S consists of all those commodity bundles, which can be sent by the coalition S per unit of time from source 1 to sink s, without using arcs in the original network which are owned by agents outside the coalition.

(vi) A correspondence $V : 2^N \twoheadrightarrow \mathbb{R}^m_+$, which assigns to each coalition $S \in 2^N$ a suitable subset of the commodity space $\mathbb{R}^m_+$, is called a *multi-commodity game* (MC-game) if also $V(\emptyset) = \{0\}$ is satisfied.

For each $S \in 2^N$, V(S) describes the possible commodity bundles which can be obtained by S if the players in S cooperate. Examples of MC-games are MC-flow games. Note that one-commodity games can be identified with non-negative SP-games.

(vii) Let V_1 and V_2 be two MC-games with the same player set N. Then $V_1 \cap V_2$, the *intersection of* V_1 *and* V_2, is the correspondence which assigns to each coalition $S \in 2^N$ the suitable set $V_1(S) \cap V_2(S)$.

Note that $V_1 \cap V_2$ is an MC-game.

Lemma 2.1. A finite intersection of MC-flow games with the same player set is an MC-flow game.

Proof. It is sufficient to show that the intersection of two MC-flow games V_{Γ_1} and V_{Γ_2} is an MC-flow game. Consider the CMCF-situation Γ obtained by combining Γ_1 and Γ_2 as shown in figure 1, where the sink of Γ_1 and the source of Γ_2 have been melted together. It is easily verified that the MC-flow game V_Γ corresponding to Γ is the intersection of V_{Γ_1} and V_{Γ_2}. ■

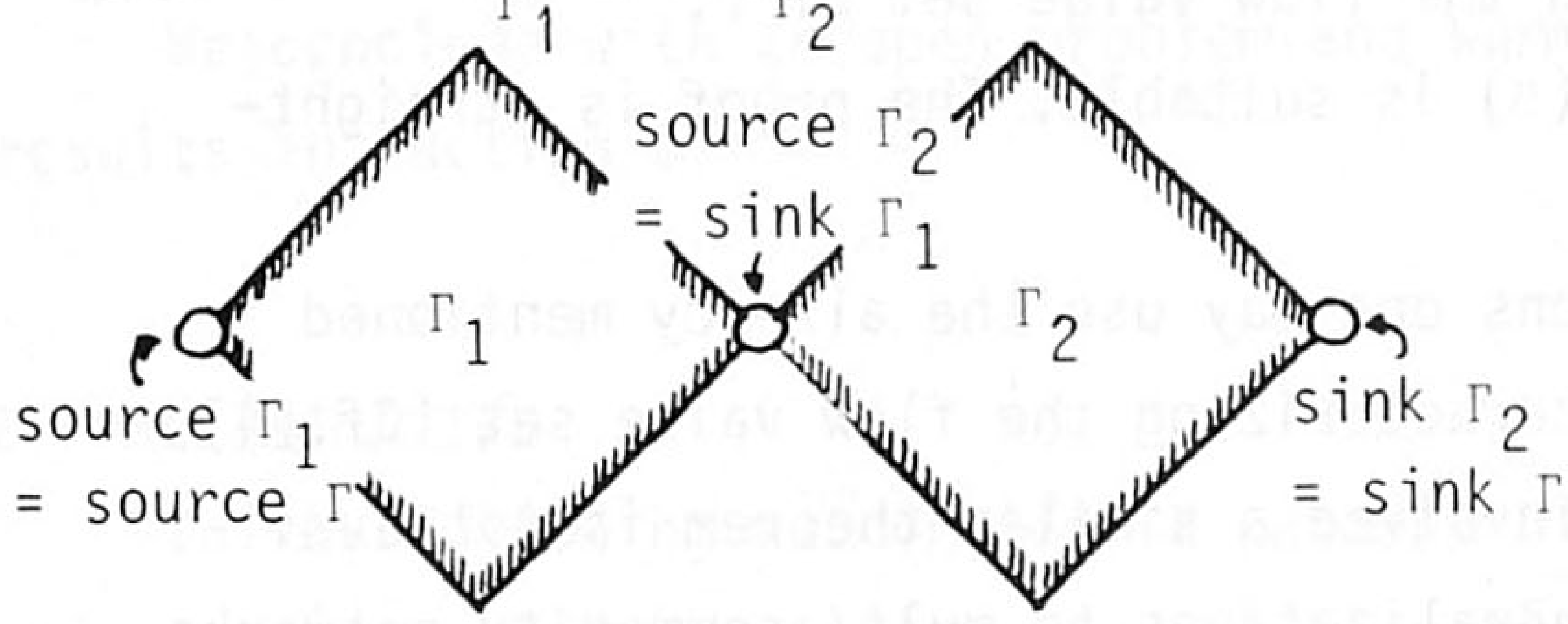

figure 1.

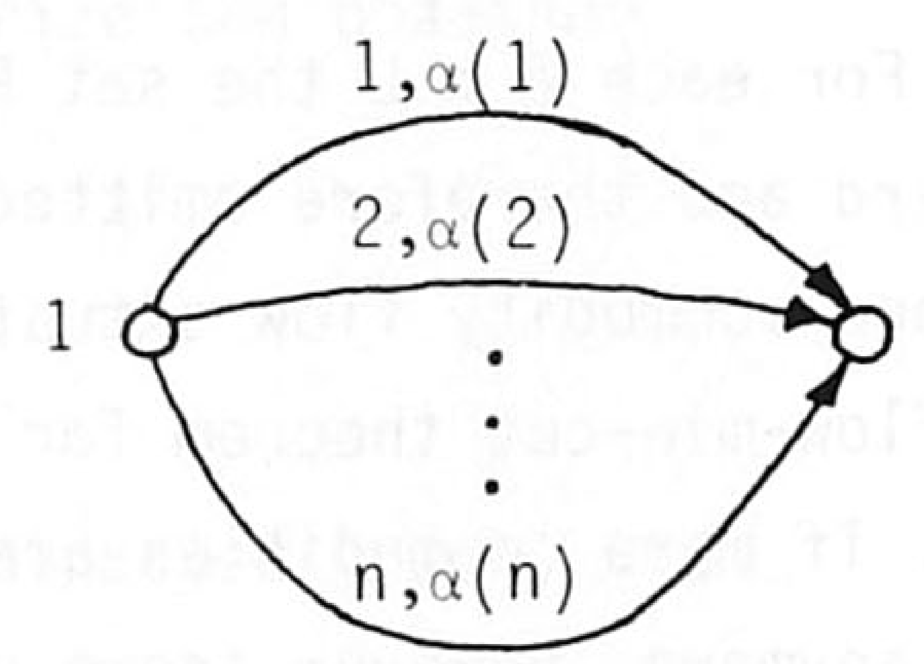

figure 2.

(viii) An MC-game V with player set N is called *additive* if there exists a correspondence $\alpha : N \twoheadrightarrow \mathbb{R}^m_+$, assigning to each player j the suitable set $\alpha(j)$ of the commodity space $\mathbb{R}^m_+$, such that

$$V(S) = \Sigma_{j \in S}\alpha(j) \text{ for all non-empty coalitions } S \in 2^N.$$

The sum on the right hand side is an algebraic sum of subsets of $\mathbb{R}^m_+$.

We often identify the correspondence α with the game V and write $\alpha(S)$ instead of $V(S)$ for all $S \in 2^N$.

Lemma 2.2. Additive MC-games are MC-flow games.

Proof. Let V be an additive MC-game corresponding to $\alpha : N \twoheadrightarrow \mathbb{R}^m_+$. Consider the CMCF-situation Γ, as shown in figure 2, with node set $P = \{1,2\}$, arc set N, capacity correspondence α and where arc j belongs to player j for each $j \in N$. Of course, the MC-flow game V_Γ equals V. ■

3. TOTAL BALANCEDNESS FOR MC-GAMES

In this section we consider the family of totally balanced MC-games and show that the set of MC-flow games is a subset of this family. Furthermore, we give in theorem 3.3 a characterization of total balancedness in terms of additive MC-games. This characterization enables us to show that all totally balanced MC-games can be identified with an MC-flow game if we allow infinite networks. For polyhedral MC-games finite networks turn out to be sufficient.

Definition 3.1. An MC-game V, with player set N, is called *balanced* if for each map $\lambda : 2^N \to \mathbb{R}_+$, with the balancedness property

$$\sum_{S\in 2^N, j\in S} \lambda(S) = 1 \text{ for all } j \in N, \tag{3.1}$$

we have

$$\sum_{S\in 2^N} \lambda(S)V(S) \subset V(N). \tag{3.2}$$

V is called *totally balanced* if each subgame of V is balanced, i.e. for each coalition $S \subset N$ the restriction of V to the family of subsets of S, which is an MC-game with player set S, is balanced.

Examples of totally balanced MC-games are additive MC-games. Also an intersection of totally balanced MC-games is totally balanced. Moreover,

Theorem 3.2. An MC-flow game is totally balanced.

Proof. Let V_Γ be an MC-flow game corresponding to the CMCF-situation Γ. Any subgame of V_Γ is, again, an MC-flow game. Therefore, it is sufficient to show that V_Γ is balanced. Let $\lambda : 2^N \to \mathbb{R}_+$ be a map with the balancedness property (3.1). We prove that V_Γ satisfies (3.2). For each $S \in 2^N$, consider an element y^S of $V_\Gamma(S)$. The set $V_\Gamma(S)$ equals the flow value set $F_{L(S)}(\Gamma)$ which implies that for each $S \in 2^N$ there exists a flow f^S with value y^S and

$$f^S(\ell) = 0 \text{ for all } \ell \in L - L(S). \tag{3.3}$$

Consider the map $f : L \to \mathbb{R}_+$ with $f(\ell) = \sum_{S\in 2^N} \lambda(S)f^S(\ell)$ for all $\ell \in L$. Trivially, f satisfies the flow conditions (F.2), (F.3) and (F.4). The only flow condition to check is the feasibility condition. Using (3.3), we obtain

$$f(\ell) = \sum_{S\in 2^N, O(\ell)\in S} \lambda(S)f^S(\ell) \text{ for all } \ell \in L.$$

Since $\Sigma_{S\in 2^N, O(\ell)\in S} \lambda(S) = 1$ and $f^S(\ell) \in c(\ell)$ for all $S \in 2^N$ with $O(\ell) \in S$, we conclude that $f(\ell)$ is a convex combination of elements of $c(\ell)$ for each arc $\ell \in L$. The convexity of $c(\ell)$ yields $f(\ell) \in c(\ell)$. Hence, f is a flow in Γ. Of course, the value of f equals $\Sigma_{S\in 2^N} \lambda(S)y^S$ implying that $\Sigma_{S\in 2^N} \lambda(S)y^S$ is an element of $V_\Gamma(N)$. ■

Theorem 3.3. An MC-game is totally balanced if and only if it is a countable intersection of additive MC-games.

Proof. The "if" statement is trivial. Therefore, we confine ourselves to the proof of the "only if" statement. Let V be a totally balanced MC-game with player set N and commodity set $\{1,2,\ldots,m\}$. Let M be a real number such that

$$V(S) \subset B_M := \{x \in \mathbb{R}^m_+ : x_i \le M \text{ for all } i \in \{1,2,\ldots,m\}\}. \tag{3.4}$$

Consider for each $z \in \mathbb{Q}^m_+$ the non-negative SP-game v_z defined by

$$v_z(S) = \max_{x\in V(S)} z\cdot x \text{ for all } S \in 2^N.$$

For each $S \in 2^N$ the comprehensiveness and closedness of V(S) yield

$$V(S) = \cap_{z\in\mathbb{Q}^m_+} \{x \in \mathbb{R}^m_+ : z\cdot x \le v_z(S)\}. \tag{3.5}$$

The SP-game v_z is totally balanced (cf.[2]) for each $z \in \mathbb{Q}^m_+$ and, therefore, it can be expressed as a minimum of a finite collection, say $\{v_{z,i} : i \in I_z\}$, of additive SP-games (see Kalai and Zemel [7]). Hence,

$$v_z(S) = \min_{i\in I_z} v_{z,i}(S) = \min_{i\in I_z} \Sigma_{j\in S} v_{z,i}(\{j\}) \text{ for all } S \in 2^N. \tag{3.6}$$

Consider for each $z \in \mathbb{Q}^m_+$ and $i \in I_z$ the additive MC-game $\alpha_{z,i} : N \twoheadrightarrow \mathbb{R}^m_+$ defined by

$$\alpha_{z,i}(j) = \{x \in \mathbb{R}^m_+ : z\cdot x \le v_{z,i}(\{j\})\} \cap B_M \text{ for all } j \in N. \tag{3.7}$$

(The set B_M assures the boundedness of $\alpha_{z,i}(j)$ whenever z has zero components. The boundedness property is needed to state that $\alpha_{z,i}$ is an additive MC-game.)

From (3.5) and (3.6) we have

$$V(S) = \cap_{z\in\mathbb{Q}^m_+} \{x \in \mathbb{R}^m_+ : z\cdot x \le \min_{i\in I_z} \Sigma_{j\in S} v_{z,i}(\{j\})\}$$

$$= \cap_{z\in\mathbb{Q}^m_+} \cap_{i\in I_z} \Sigma_{j\in S} \{x \in \mathbb{R}^m_+ : z\cdot x \le v_{z,i}(\{j\})\} \text{ for all } S\in 2^N.$$

Using (3.4) and (3.7) we obtain

$V(S) = \cap_{z \in \mathbb{Q}^m_+} \cap_{i \in I_z} \alpha_{z,i}(S)$ for all $S \in 2^N$.

Now the set $\cup_{z \in \mathbb{Q}^m_+} I_z$ is countable. Therefore, the theorem is proved. ■

Definition 3.4. An MC-game V is called *polyhedral* if for each $S \in 2^N$ the set V(S) is polyhedral.

Let V be a totally balanced polyhedral MC-game. For each $S \in 2^N$ there is a finite subset Z_S of $\mathbb{R}^m_+$ and scalars b_z of $\mathbb{R}_+$ for all $z \in Z_S$, such that

$V(S) = \cap_{z \in Z_S} \{x \in \mathbb{R}^m_+ : z \cdot x \leq b_z\}$.

By adapting the proof of theorem 3.3 such that the role of $\mathbb{Q}^m_+$ is now taken over by the finite set $\cup_{S \in 2^N} Z_S$, one shows that V is a finite intersection of additive MC-games. Using the lemmas 2.1 and 2.2, this implies

Theorem 3.5. Totally balanced polyhedral MC-games are MC-flow games.

Not each totally balanced MC-game can be expressed as a finite intersection of additive MC-games as the following example shows.

Example 3.6. Let V be the two-commodity game with 3 players where
$V(\{j\}) = \{(0,0)\}$ for each $j \in N = \{1,2,3\}$,
$V(\{1,2\}) = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, x_2 = 0\}$,
$V(\{2,3\}) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 = 0, 0 \leq x_2 \leq 1\}$,
$V(\{1,3\}) = \{(x_1,x_2) \in \mathbb{R}^2_+ : x_1^2 + x_2^2 \leq 1\}$ and
$V(N) = \frac{1}{2}(V(\{1,2\}) + V(\{1,3\}) + V(\{2,3\}))$ (see figure 3).

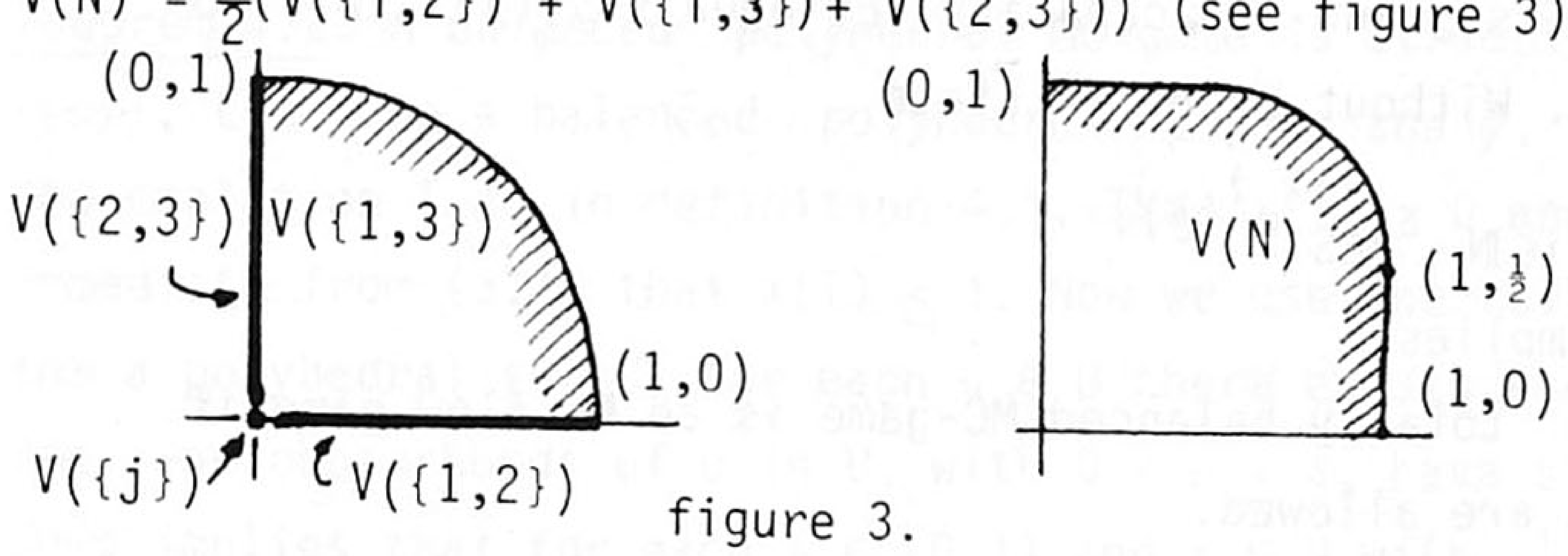

figure 3.

One can easily verify that V is totally balanced.

Claim. V is not the intersection of a finite number of additive two-commodity games.

Proof. Suppose $\{\alpha^i : i \in I\}$ is a finite collection of additive two-commodity games with

$V(S) = \cap_{i \in I} (\Sigma_{j \in S} \alpha^i(j))$ for all $S \in 2^N$.

We note first that if for an $i \in I$ the set $\alpha^i(N)$ contains an element $(1+\delta,0)$, with $\delta > 0$, then from $\{(1+\delta,0)\} \cup V(N) \subset \alpha^i(N)$ and the convexity of $\alpha^i(N)$ it follows that $\alpha^i(N)$ contains an element $(1,\frac{1}{2}+\varepsilon)$ with $\varepsilon > 0$. This implies that there is a $k \in I$ such that

$$\alpha^k(N) \cap \{x \in \mathbb{R}^2 : x_1 \geq 1,\ x_2 \geq 0\} = \{x \in \mathbb{R}^2 : x_1=1,\ 0\leq x_2\leq\tfrac{1}{2}\} \quad (3.8)$$

because $V(N) = \cap_{i\in I}\alpha^i(N)$ and I is finite. From (3.8) and $\{(1,0)\} + \alpha^k(3) \subset V(\{1,2\}) + \alpha^k(3) \subset \alpha^k(N)$ it follows that

$$\alpha^k(3) \subset \{x \in \mathbb{R}^2 : x_1 = 0,\ 0 \leq x_2 \leq \tfrac{1}{2}\}. \quad (3.9)$$

Similarly, from $\{(1,0)\} + \alpha^k(2) \subset V(\{1,3\}) + \alpha^k(2) \subset \alpha^k(N)$ we conclude that

$$\alpha^k(2) \subset \{x \in \mathbb{R}^2 : x_1 = 0,\ 0 \leq x_2 \leq \tfrac{1}{2}\}. \quad (3.10)$$

From (3.9), (3.10) and $V(\{2,3\}) \subset \alpha^k(2) + \alpha^k(3) \subset \{x \in \mathbb{R}^2 : x_1 = 0,\ 0 \leq x_1 \leq 1\} = V(\{2,3\})$ we obtain $\alpha^k(2) = \alpha^k(3) = \{x \in \mathbb{R}^2 : x_1 = 0,\ 0 \leq x_2 \leq \frac{1}{2}\}$. Since $(1,0) \in V(N) \subset \alpha^k(N) = \alpha^k(1) + \{x \in \mathbb{R}^2 : x_1 = 0,\ 0 \leq x_2 \leq 1\}$ the set $\alpha^k(1)$ contains $(1,0)$ and, therefore, $\alpha^k(N)$ contains $(1,0) + (0,1) = (1,1)$ which is in contradiction with (3.8). Hence, we have proved the claim. ∎

If we allow an infinite number of nodes and arcs in a network, the intersection of the countable collection $\{\alpha^i : i \in \mathbb{N}\}$ of additive MC-games with player set N can be seen as an MC-flow game as follows. Consider the CMCF-situation Γ with node set $\mathbb{N}$ and arc set $L = \{\ell_{ij} : i \in \mathbb{N},\ j \in N\}$, where ℓ_{ij} starts at node i and ends at i+1, its owner is j and its capacity set equals $\alpha^i(j)$. In Γ no sink is specified. Without proof we state

$$F_{L(S)}(\Gamma) = \cap_{i\in\mathbb{N}}\ \Sigma_{j\in S}\ \alpha^i(j).$$

Theorem 3.3 now implies

Theorem 3.7. Each totally balanced MC-game is an MC-flow game if infinite networks are allowed.

4. STRICTLY BALANCED MC-GAMES

In this section we introduce a new property for balanced MC-games. We show that this property holds for balanced polyhedral MC-games. It is interesting that all two-commodity flow games also turn out to have this property. From this we conclude that not all totally balanced two-commodity games are two-commodity flow games.

Definition 4.1. Let V be a balanced MC-game with player set N and y an element of V(N). V is called *strictly balanced in* y if for each map $\lambda : 2^N \to \mathbb{R}_+$ with the balancedness property (3.1) and elements y^S of V(S) for all $S \in 2^N$, such that y equals $\Sigma_{S\in 2^N}\lambda(S)y^S$, the following holds:
for each $T \in 2^N$ with $\lambda(T) > 0$, there exists an $\varepsilon > 0$ such that for all $x \in V(T)$ with $||x-y^T|| < \varepsilon$, we have

$$y + x - y^T \in V(N).$$

V is called *strictly balanced* if it is strictly balanced in each element of V(N).

It follows directly from the definition that a balanced MC-game V is strictly balanced in each element of the interior of V(N). The balanced two-commodity game V in example 3.6 doesn't possess the strict balancedness property. To show this consider $y = (1,\frac{1}{2}) \in V(\{1,2,3\})$. Consider also the map λ with $\lambda(S) = \frac{1}{2}$ if $|S| = 2$ and $\lambda(S) = 0$ otherwise, and $y^{\{1,2\}} = y^{\{1,3\}} = (1,0)$, $y^{\{2,3\}} = (0,1)$ and $y^S = 0$ otherwise. Of course, $y = \Sigma_{S\in 2^N}\lambda(S)y^S$. Let $T = \{1,3\}$ and $x^\varepsilon := (\sqrt{1-(\frac{1}{2}\varepsilon)^2},\frac{1}{2}\varepsilon) \in V(T)$ and $u^\varepsilon := \frac{1}{2}((1,0) + (\sqrt{1-\varepsilon^2},\varepsilon) + (0,1)) = (\frac{1}{2} + \frac{1}{2}\sqrt{1-\varepsilon^2},\frac{1}{2}+\frac{1}{2}\varepsilon) \in V(N)$ with $\varepsilon \in (0,1]$. For all $\varepsilon \in (0,1]$ we have $||x^\varepsilon-y^T|| < \varepsilon$ and

$$y + x^\varepsilon - y^T = (\sqrt{1 - (\tfrac{1}{2}\varepsilon)^2}, \tfrac{1}{2} + \tfrac{1}{2}\varepsilon) \geq u^\varepsilon. \tag{4.1}$$

From (4.1), $y + x^\varepsilon - y^T \neq u^\varepsilon$ and the Pareto optimality of u^ε in V(N) we conclude that $y + x^\varepsilon - y^T \notin V(N)$ for all $\varepsilon \in (0,1]$. Hence, V is not strictly balanced in $(1,\frac{1}{2})$.

Theorem 4.2. A balanced polyhedral MC-game is strictly balanced.

Proof. Let V be a balanced polyhedral MC-game and y, λ, y^S with $S \in 2^N$, and coalition T as in definition 4.1. Thus $\lambda(T) > 0$ and it is immediate from (3.1) that $\lambda(T) \leq 1$. Now we use the following property for a polyhedral set U. For each $u \in U$ there exists a $\delta > 0$ such that the ε-neighbourhoods of u in U, with $0 < \varepsilon \leq \delta$, have similar shape. This implies that for each $\mu \in (0,1]$ and $x \in U$ with $||x-u|| \leq \mu\delta$ there exists an $\tilde{x} \in U$, with $||\tilde{x} - u|| \leq \delta$, such that $x = \mu\tilde{x} + (1-\mu)u$. Applying this property to the polyhedral set V(T) and $y^T \in V(T)$ we obtain that there exists an $\varepsilon > 0$ such that for all $x \in V(T)$, with $||x-y^T|| \leq \varepsilon$, there is an $\tilde{x} \in V(T)$ such that $x = \lambda(T)\tilde{x} + (1-\lambda(T))y^T$. Hence,

$$y + x - y^T = y + \lambda(T)\tilde{x} - \lambda(T)y^T = \Sigma_{S\in 2^N-\{T\}}\lambda(S)y^S + \lambda(T)\tilde{x}. \tag{4.2}$$

From (4.2), the balancedness of V and $\tilde{x} \in V(T)$ we conclude that $y + x - y^T \in V(N)$. ■

Theorem 4.3. Each two-commodity flow game is strictly balanced.

Proof. Let V_Γ be a two-commodity flow game corresponding to the CMCF-situation Γ with player set N and commodity set $G = \{1,2\}$ and suppose that V_Γ is not strictly balanced in an element y of $V_\Gamma(N)$. Then there exist a map $\lambda : 2^N \to \mathbb{R}_+$, satisfying (3.1) and elements y^S of $V_\Gamma(S)$ for each $S \in 2^N$, such that $y = \Sigma_{S\in 2^N} \lambda(S)y^S$ and there is a coalition T with $\lambda(T) > 0$ and a sequence $x^1, x^2, \ldots$ in $V_\Gamma(T)$ with $\lim_{n\to\infty} x^n = y^T$ and

$$y + x^n - y^T \notin V(N) \text{ for all } n \in \mathbb{N}. \tag{4.3}$$

Let $f^1, f^2, \ldots$ be a sequence of flows in Γ such that $v(f^n) = x^n$ and $f^n(\ell) = 0$ for all $\ell \in L-L(T)$ and $n \in \mathbb{N}$. Without loss of generality we suppose that there is a flow f^T such that

$$f^T(\ell) = \lim_{n\to\infty} f^n(\ell) \text{ for all } \ell \in L. \tag{4.4}$$

Of course, (4.4) implies $v(f^T) = y^T$ and $f^T(\ell) = 0$ for all $\ell \in L-L(T)$. For each $S \in 2^N-\{T\}$ let f^S be a flow in Γ with value y^S and $f^S(\ell) = 0$ for all $\ell \in L-L(S)$. Then $f := \Sigma_{S\in 2^N} \lambda(S)f^S$ is a flow with value $y = \Sigma_{S\in 2^N} \lambda(S)y^S$ (see the proof of theorem 3.2).

For each $n \in \mathbb{N}$ the map $g^n := f+f^n-f^T$ on the arc set L is not a flow in Γ since otherwise $v(g^n) = v(f) + v(f^n) - v(f^T) = y + x^n - y^T \in F(\Gamma) = V_\Gamma(N)$ which is in contradiction with (4.3). However, g^n satisfies the flow conditions (F.2), (F.3) and (F.4) and it also satisfies the feasibility property (F.1) for all $\ell \in L-L(T)$ since for these arcs $g^n(\ell) = f(\ell)+f^n(\ell)-f^T(\ell) = f(\ell)+0-0 \in c(\ell)$. Therefore, for each $n \in \mathbb{N}$ there is an arc, say ℓ^n, in L(T) such that $g^n(\ell^n) \notin c(\ell^n)$. Because the arc set L(T) is finite, there is an arc, say ℓ^*, in L(T) such that $g^n(\ell^*) \notin c(\ell^*)$ infinitely often. Again, without loss of generality, we suppose that

$$g^n(\ell^*) \notin c(\ell^*) \text{ for all } n \in \mathbb{N}. \tag{4.5}$$

In the sequel of the proof we need the existence of an $M \in \mathbb{N}$ such that

$$g^n(\ell^*) \geq 0 \text{ for all } n \geq M. \tag{4.6}$$

To prove (4.6) we distinguish two cases.

(i) Suppose $(f(\ell^*))_r > 0$ for an $r \in G = \{1,2\}$. Choose $M_r \in \mathbb{N}$ such that $(f^T(\ell^*))_r-(f^n(\ell^*))_r \leq (f(\ell^*))_r$ for all $n \geq M_r$. This is possible since $\lim_{n\to\infty} f^n(\ell^*) = f^T(\ell^*)$. Hence, $(g^n(\ell^*))_r \geq 0$ for all $n \geq M_r$.

(ii) Suppose $(f(\ell^*))_r = 0$ for an $r \in G$. Then $\Sigma_{S\in 2^N} \lambda(S)(f^S(\ell^*))_r = 0$, yielding $(f^S(\ell))_r = 0$ for all $S \in 2^N$ with $\lambda(S) > 0$. Especially, $(f^T(\ell^*))_r = 0$. Hence, $(g^n(\ell^*))_r = (f^n(\ell^*))_r \geq 0$.

Choosing $M \geq \min\{M_r : r \in G, (f(\ell^*))_r > 0\}$ we obtain (4.6).
Now we proceed with the proof in such a way that we will derive two different elements u and w from $c(\ell^*)$ with the property that, for n sufficiently large, $g^n(\ell^*)$ is majorized by a convex combination of u and w. Applying now (4.6) and the convexity and comprehensiveness of $c(\ell^*)$ we obtain a contradiction with (4.5).
From (4.5), (4.6) and $\lim_{n\to\infty} g^n(\ell^*) = f(\ell^*) \in c(\ell^*)$ we conclude that $f(\ell^*)$ is weak Pareto optimal in the set $c(\ell^*)$. Because $c(\ell^*)$ is compact and convex, there exists a $z \in \mathbb{R}^2_+$, $||z|| = 1$, such that

$$z\cdot f(\ell^*) = \max_{x\in c(\ell^*)} z\cdot x. \tag{4.7}$$

Moreover, it is possible to express $f(\ell^*)$ as a strict convex combination of two weak Pareto optimal elements, say u and w, of $c(\ell^*)$. To prove this let $u := f^T(\ell^*)$ and $w := (1-\lambda(T))^{-1}\Sigma_{S\in 2^N-\{T\}}\lambda(S)f^S(\ell^*)$. Now w is properly defined because if $\lambda(T) = 1$ then $y+x^n-y^T = \Sigma_{S\in 2^N-\{T\}}\lambda(S)y^S + \lambda(T)x^n \in V_\Gamma(N)$, using the balancedness of V_Γ and $x^n \in V_\Gamma(T)$ for all $n \in \mathbb{N}$. This is in contradiction with (4.3). Therefore, $\lambda(T) \neq 1$. The balancedness property (3.1) asserts $0 \leq \lambda(T) \leq 1$. Hence,

$$0 < \lambda(T) < 1. \tag{4.8}$$

Of course, $f(\ell^*) = \lambda(T)u+(1-\lambda(T))w$ and $u \in c(\ell^*)$. Also $w \in c(\ell^*)$ because w is a convex combination of elements of the convex set $c(\ell^*)$ since $(1-\lambda(T))^{-1}\Sigma_{S\in 2^N-\{T\}}\lambda(S) = 1$ and $f^S(\ell^*) \in c(\ell^*)$ for all $S \in 2^N-\{T\}$.
Now $u \neq f(\ell^*)$ because otherwise $g^n(\ell^*) = u+f^n(\ell^*)-u = f^n(\ell^*) \in c(\ell^*)$ for all $n \in \mathbb{N}$, contradicting (4.5).
Using $u \neq f(\ell^*)$, $f(\ell^*) = \lambda(T)u+(1-\lambda(T))w$ and (4.8) we obtain $w \neq f(\ell^*)$. Furthermore, we have $z\cdot f(\ell^*) = \lambda(T)z\cdot u+(1-\lambda(T))z\cdot w \leq z\cdot f(\ell^*)$, using (4.7) and $u,w \in c(\ell^*)$. Hence, $z\cdot f(\ell^*) = z\cdot u = z\cdot w$ which implies the weak Pareto optimality of u and w in $c(\ell^*)$.
Consider $a^n := g^n(\ell^*) + (z\cdot f(\ell^*)-z\cdot g^n(\ell^*))z$ for each $n \in \mathbb{N}$. Since $||z|| = 1$, we have $z\cdot a^n = z\cdot f(\ell^*)$. Moreover,

$$g^n(\ell^*) \leq a^n \text{ for all } n \in \mathbb{N}, \tag{4.9}$$

since $z \in \mathbb{R}^2_+$ and $z \cdot f(\ell^*) - z \cdot g^n(\ell^*) = z \cdot f^T(\ell^*) - z \cdot f^n(\ell^*) = z \cdot u - z \cdot f^n(\ell^*) \geq 0$, using $z \cdot u = z \cdot f(\ell^*)$, $f^n(\ell^*) \in c(\ell^*)$ and (4.7).

From $\lim_{n\to\infty} g^n(\ell^*) = f(\ell^*)$ we obtain $\lim_{n\to\infty} a^n = f(\ell^*)$. Because $u, w, f(\ell^*)$ and a^n, for all $n \in \mathbb{N}$, are elements of the line $\{x \in \mathbb{R}^2 : z \cdot x = f(\ell^*)\}$ and $f(\ell^*)$ is a strict convex combination of u and w we conclude that a^n is a convex combination of u and w for sufficiently large n. The convexity of $c(\ell^*)$ and $u, w \in c(\ell^*)$ now imply $a^n \in c(\ell^*)$ for n sufficiently large. Combining this with (4.9), (4.6) and the comprehensiveness of $c(\ell^*)$, we conclude that $g^n(\ell^*) \in c(\ell^*)$ for n sufficiently large which is in contradiction with (4.5). Therefore, the correctness of (4.3) cannot be maintained which finishes the proof. ■

The totally balanced two-commodity game in example 3.6 is not strictly balanced as we saw earlier. So for that MC-game there doesn't exist a CMCF-situation corresponding to that MC-game. Hence

Corollary 4.4. The family of totally balanced MC-games properly contains the set of MC-flow games.

5. CONCLUSION

We have proved that multi-commodity flow games are totally balanced and that totally balanced MC-games can be seen as an intersection of countable many additive MC-games which implies that such games can be represented as MC-flow games if we allow infinite networks.

To summarize our other results let TBMC denote the family of totally balanced MC-games, FMC the family of MC-flow games on finite networks, FIA the family of MC-games which are a finite intersection of additive MC-games and PMC the family of totally balanced polyhedral MC-games. Then we have

$$PMC \underset{\neq}{\subset} FIA \subset FMC \underset{\neq}{\subset} TBMC.$$

Open is

Problem 5.1. Coincides FIA with FMC?

Or: Is it possible to express each multi-commodity flow game as a finite intersection of additive multi-commodity games?

If the answer is yes, this will imply that the strict balancedness is not a sufficient condition for an MC-game to assure that such a game is an MC-flow game. To see this, consider the two-commodity game V' with $V'(S) = V(S)$ for $S \neq \{1,3\}$, where V is defined in 3.6, and $V'(\{1,3\}) = \{x \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, x_2 = 0\}$. The MC-game V' is strictly balanced but in a similar way as in 3.6 one shows that $V' \notin$ FIA.

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