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Theory and Methodology

Sequencing games

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Abstract: Sequencing situations with linear cost functions are considered. A division rule for the cost savings is introduced and characterized axiomatically. Cooperative game theory is applied to analyse these situations and expressions for division rules induced by solution concepts from cooperative game theory are derived.

1. Introduction

Sequencing problems are well known in the literature. In this paper we consider sequencing situations in which a certain number of customers has to be served by one server. Everyone of them has a cost function which depends on his completion time, i.e. the time he has to wait plus the time it takes to serve him. In these situations two problems arise, namely, how to find an optimal order in which to serve the customers so as to minimize the total cost, and how to allocate this cost among the people concerned. In this paper we handle the last problem by looking at the cost savings which are obtained when the customers are rearranged in such a way that the total cost is minimized. Rules to divide the cost savings are provided.

The paper is organized as follows. In Section 2 sequencing situations are introduced and it is shown that when the cost functions are linear it is easy to determine an optimal order. In Section 3 a rule to divide the cost savings is introduced and characterized axiomatically. In Section 4 we define sequencing games. These are cooperative games which correspond to sequencing situations and solution concepts from cooperative game theory are applied as division rules for the cost savings. Finally, in Section 5 we consider some problems which are related to sequencing situations.

2. Sequencing situations

In a *sequencing situation* there is a queue standing before a counter. The queue consists of n customers who are waiting to be served. The set of customers will be denoted by $N = \{1, 2, \ldots, n\}$. By means of a permutation σ of N we can describe the position of each customer in the queue. Specifically, $\sigma(i) = j$ indicates that customer i has the j-th position in the queue. For every $i \in N$ the

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service time equals $s_i > 0$ units of time. Every $i \in N$ has a *cost function* $c_i : \mathbb{R}_+ \to \mathbb{R}$. For every $t \in \mathbb{R}_+$, $c_i(t)$ denotes the cost for customer i if his waiting time plus service time is equal to t. In the following we will assume that c_i is linear for all $i \in N$. Hence, for every $i \in N$ there exist $\alpha_i, \beta_i \in \mathbb{R}$ such that $c_i(t) = \alpha_i t + \beta_i$. Formally, a sequencing situation is an ordered triple $(\sigma; \alpha; s)$ where $\sigma \in \Pi_N$ which is the set of permutations of N, $\alpha \in \mathbb{R}^n$ with α_i equal to the first derivative of c_i for every $i \in N$, and $s \in \mathbb{R}^n_+$ with s_i equal to the service time of customer i for every $i \in N$.

The *total cost* for the group N if served according to σ is given by the expression

$$C_\sigma := c_1\Big(\sum_{i \in P(\sigma,1)} s_i + s_1\Big) + c_2\Big(\sum_{i \in P(\sigma,2)} s_i + s_2\Big) + \cdots + c_n\Big(\sum_{i \in P(\sigma,n)} s_i + s_n\Big),$$

where $P(\sigma, i) := \{ j \in N \mid \sigma(j) < \sigma(i)\}$ is the *set of predecessors* of i with respect to the permutation σ. It is possible for the group to decrease this cost by rearranging the customers. Each rearrangement corresponds to a permutation π of N. We define the *urgency index* u_i of customer i to be equal to $\alpha_i s_i^{-1}$. The following proposition states that the total cost is minimal if the customers are arranged according to decreasing urgency indices, i.e. all customers with the highest urgency index first, then all customers with the second highest urgency index, and so forth. This is, in fact, the weighted SPT rule for scheduling and a proof of the following proposition can be found in various textbooks on sequencing and scheduling. Cf. Baker (1974, p. 21).

Proposition. *Let* $(\sigma; \alpha; s)$ *be a sequencing situation. Then* $C_\pi = \min_{\tau \in \Pi_N} C_\tau$ *iff* $u_{\pi^{-1}(1)} \geqslant u_{\pi^{-1}(2)} \geqslant \cdots \geqslant u_{\pi^{-1}(n)}$. *Here* π^{-1} *denotes the inverse permutation of* π, *i.e.* $\pi^{-1}(i) = j$ *iff* $\pi(j) = i$.

Remark. Note that if i and j are neighbours with i standing in front of j and we switch them, then the total cost changes by an amount equal to $\alpha_j s_i - \alpha_i s_j$ regardless of their position in the queue.

3. The equal gain splitting rule

In the previous section we have seen that customers standing in front of a counter waiting to be served can save money by rearranging their respective positions. In this section we will discuss a way of dividing the *cost savings* among the people involved, which has some nice properties. Formally, a *division rule* for sequencing situations is a function f which assigns to every sequencing situation $(\sigma; \alpha, s)$ for any finite set N of customers a solution

$$f(\sigma; \alpha; s) = (f_1(\sigma; \alpha; s), \ldots, f_n(\sigma; \alpha; s))$$

such that

(i) $f_i(\sigma; \alpha; s) \geqslant 0$ for every $i \in N$ (*individual rationality*),

(ii) $\sum_{i \in N} f_i(\sigma; \alpha; s) = C_\sigma - C_{\text{opt}}$ (*efficiency*).

Here $C_{\text{opt}} := \min_{\tau \in \Pi_n} C_\tau$. An example is provided by the *equal division rule* which divides the cost savings equally among the customers standing in the queue. A drawback of this method is that it does not distinguish between customers who actually contribute to the savings and those who do not. The rule we want to introduce does not have this disadvantage. It is called the *equal gain splitting rule* or shorter, the EGS rule and it works as follows. We saw that an optimal arrangement can be reached by successive switches of neighbours who are not standing in decreasing order of urgency index. Every time such a switching takes place the EGS rule divides the gain equally between the neighbours who are switched. We define

$$g_{ij} := (\alpha_j s_i - \alpha_i s_j)_+ := \max\{\alpha_j s_i - \alpha_i s_j, 0\}.$$

In other words, g_{ij} represents the gain attainable in the situation where i and j are waiting next to each other with i in front of j. From the remark in Section 2 it follows that g_{ij} can be defined in this way. In fact, if $u_j > u_i$, then this gain is equal to $\alpha_j s_i - \alpha_i s_j$ and it can be obtained by switching i and j. If $u_j \leqslant u_i$, then there is nothing to gain by switching i and j and so $g_{ij} = 0$. But then we have $\alpha_j s_i - \alpha_i s_j \leqslant 0$ as well, and hence $g_{ij} = (\alpha_j s_i - \alpha_i s_j)_+$. The EGS rule gives $\frac{1}{2} g_{ij}$ to i and $\frac{1}{2} g_{ij}$ to j. Formally,

$$\mathrm{EGS}_i(\sigma; \alpha; s) := \tfrac{1}{2} \sum_{k \in P(\sigma,i)} g_{ki} + \tfrac{1}{2} \sum_{j : i \in P(\sigma,j)} g_{ij}$$

for all $i \in N$.

It is easy to see that the EGS-rule is indeed a division rule. $\mathrm{EGS}_i(\sigma; \alpha; s) \geqslant 0$ for all $i \in N$ follows from $g_{ij} \geqslant 0$ for all $i, j \in N$. Furthermore, let $\tau_1, \tau_2, \ldots, \tau_r$ be the permutation corresponding to the arrangements arising by switching neighbours in the process of going from σ to an optimal permutation. Then

$$C_\sigma - C_{\text{opt}} = \sum_{k=0}^{r} (C_{\tau_k} - C_{\tau_{k+1}}),$$

where $C_{\tau_0} = C_\sigma$, $C_{\tau_{r+1}} = C_{\text{opt}}$ and

$$C_{\tau_q} - C_{\tau_{q+1}} = g_{ij}$$

for certain $i, j \in N$ for all $0 \leqslant q \leqslant r$. It follows that $\sum_{i \in N} \text{EGS}_i(\sigma; \alpha; s) = C_\sigma - C_{\text{opt}}$. Another advantage of dividing the cost savings according to the EGS rule is that such a division forms an incentive for customers who are not standing in decreasing order of urgency index to switch position. In the following we will give an axiomatic characterization of the EGS rule. For this characterization we introduce three properties. We call an $i \in N$ a *dummy* in a sequencing situation $(\sigma; \alpha; s)$ if it is not necessary for anybody to switch with i in order to arrive at an optimal arrangement. Formally, i is a dummy if $\sigma(j) > \sigma(i)$ implies $u_j \leqslant u_i$ and $\sigma(k) < \sigma(i)$ implies $u_k \geqslant u_i$ for all $j, k \in N$. In fact, if i is a dummy he does not contribute to the cost savings. A division rule f is said to satisfy the *dummy property* if it does not give anything to a dummy, i.e. i is a dummy in $(\sigma; \alpha; s)$ implies $f_i(\sigma; \alpha; s) = 0$.

Two sequencing situations $(\sigma; \alpha; s)$ and $(\tau; \alpha; s)$ are *i-equivalent* if $P(\sigma, i) = P(\tau, i)$. We say that f possesses the *equivalence property* if for each $i \in N$ and each pair of i-equivalent sequencing situations $(\sigma; \alpha; s)$ and $(\tau; \alpha; s)$ we have

$$f_i(\sigma; \alpha; s) = f_i(\tau; \alpha; s).$$

Let $(\sigma; \alpha; s)$ be a sequencing situation with $|\sigma(i) - \sigma(j)| = 1$ so that i and j are neighbours in this situation. We say that $(\tau; \alpha; s)$ is the *ij-inverse* of $(\sigma; \alpha; s)$ if $(\tau; \alpha; s)$ arises from $(\sigma; \alpha; s)$ by switching the positions of i and j in the queue. Then f is said to possess the switch property if for all $i, j \in N$ and all pairs of sequencing situations $(\sigma; \alpha; s)$ and $(\tau; \alpha; s)$ where $(\tau; \alpha; s)$ is the ij-inverse of $(\sigma; \alpha; s)$ we have

$$f_i(\tau; \alpha; s) - f_i(\sigma; \alpha; s) = f_j(\tau; \alpha; s) - f_j(\sigma; \alpha; s).$$

The following theorem states that these three properties are sufficient to characterize the EGS-rule axiomatically.

Theorem. *The EGS-rule is the unique rule for sequencing situations that possesses the dummy, the equivalence and the switch properties.*

Proof. Let $(\sigma; \alpha; s)$ be a sequencing situation such that i is a dummy. Then $g_{ki} = 0$ for all $k \in P(\sigma, i)$ and $g_{ij} = 0$ for all j with $i \in P(\sigma, j)$. From the definition of the EGS rule it follows that $\text{EGS}_i(\sigma; \alpha; s) = 0$ and therefore the EGS rule possesses the dummy property.

Let $(\sigma; \alpha; s)$ and $(\tau; \alpha; s)$ be two i-equivalent sequencing situations. Then $P(\sigma, i) = P(\tau, i)$ and it follows that

$$\{ j \in N \mid i \in P(\sigma, j)\} = \{ j \in N \mid i \in P(\tau, j)\}.$$

From the definition it follows that $\text{EGS}_i(\sigma; \alpha; s) = \text{EGS}_i(\tau; \alpha; s)$ and hence the EGS-rule possesses the equivalence property.

Let $(\sigma; \alpha; s)$ be a sequencing situation and $(\tau; \alpha; s)$ its ij-inverse. Without loss of generality we assume that $\sigma(i) < \sigma(j)$. Then

$$\begin{aligned} &\text{EGS}_i(\tau; \alpha; s) - \text{EGS}_i(\sigma; \alpha; s) \\ &\quad = \tfrac{1}{2} g_{ji} - \tfrac{1}{2} g_{ij} \\ &\quad = \text{EGS}_j(\tau; \alpha; s) - \text{EGS}_j(\sigma; \alpha; s). \end{aligned}$$

Thus, the EGS rule possesses the switch property.

Suppose f is a division rule for sequencing situations which also possesses these three properties. We define the set of misplaced pairs of neighbours of a sequencing situation $(\sigma; \alpha; s)$ by

$$M_\sigma := \{(i, j) \mid \sigma(i) = \sigma(j) + 1,\ u(i) > u(j)\},$$

and show by induction on the cardinality of M_σ that f is equal to the EGS rule. Let $(\sigma; \alpha; s)$ be a sequencing situation with $M_\sigma = \emptyset$, then everybody is a dummy and it follows that $f(\sigma; \alpha; s) = (0, 0, \ldots, 0) = \text{EGS}(\sigma; \alpha, s)$. Suppose $f(\sigma; \alpha; s) = \text{EGS}(\sigma; \alpha; s)$ for all sequencing situations $(\sigma; \alpha; s)$ with $|M_\sigma| \leqslant m$, where $m \geqslant 0$. Let $(\tau; \alpha; s)$ be such that $|M_\tau| = m + 1$. Then there is a sequencing situation $(\sigma; \alpha; s)$ and a pair $k, l \in M_\tau$ such that $\sigma(i) = \tau(i)$ for all $i \notin \{k, l\}$ and $\sigma(k) = \tau(l)$, $\sigma(l) = \tau(k)$. Thus, $M_\sigma = M_\tau \setminus \{(k, l)\}$ and from the equivalence property and the induction assumption it follows that

$$f_i(\tau; \alpha; s) = f_i(\sigma; \alpha; s) = \text{EGS}_i(\sigma; \alpha; s) = \text{EGS}_i(\tau; \alpha; s) \quad \text{for all } i \notin \{k, l\}.$$

Further, $C_\tau - C_\sigma = g_{lk}$ and from the efficiency, the switch property and the induction assumption it follows that

$$f_k(\tau;\ \alpha;\ s) = f_k(\sigma;\ \alpha;\ s) + \tfrac{1}{2}g_{lk}$$
$$= \mathrm{EGS}_k(\sigma;\ \alpha;\ s) + \tfrac{1}{2}g_{g_{lk}}$$
$$= \mathrm{EGS}_k(\tau;\ \alpha;\ s).$$

In the same way one can show that

$$f_l(\tau;\ \alpha;\ s) = \mathrm{EGS}_l(\tau;\ \alpha;\ s).$$

Hence

$$f(\tau;\ \alpha;\ s) = \mathrm{EGS}(\tau;\ \alpha;\ s)$$

and the proof is completed. □

We conclude this section with an example.

Example. Let $N = \{1, 2, 3\}$, $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 3$, $s = (7, 3, 5)$ and $\alpha = (10, 20, 30)$. Then $u_1 = \frac{10}{7}$, $u_2 = \frac{20}{3}$, $u_3 = \frac{30}{5}$. Since $u_2 > u_3 > u_1$ the optimal service order is 2 first, then 3 and 1 last. $C_\sigma = 720$, $C_{\text{opt}} = 450$ and therefore the total cost savings if the customers rearrange themselves according to the optimal order is 270. Further, $g_{12} = 110$, $g_{21} = 0$, $g_{13} = 160$, $g_{31} = 0$, $g_{23} = 0$, $g_{32} = 10$ and we can compute EGS(σ; α, s).

$$\mathrm{EGS}_1(\sigma;\ \alpha,\ s) = \tfrac{1}{2}(g_{12} + g_{13}) = 135,$$
$$\mathrm{EGS}_2(\sigma;\ \alpha;\ s) = \tfrac{1}{2}(g_{12} + g_{23}) = 55,$$
$$\mathrm{EGS}_3(\sigma;\ \alpha;\ s) = \tfrac{1}{2}(g_{13} + g_{23}) = 80.$$

4. Sequencing games

In this section we will apply *cooperative game theory* to sequencing situations. A *cooperative game in characteristic function form* is an ordered pair $\langle N, v\rangle$ where N is a finite set, the *set of players* and the *characteristic function* v is a function on 2^N, the set of all subsets of N, which assigns to every $S \in 2^N$ a real number with $v(\emptyset) = 0$. An $S \in 2^N$ is called a coalition and $v(S)$ can be regarded as the worth of coalition S. Such a game is called *convex* if

$$v(S \cup \{i\}) - v(S) \leqslant v(T \cup \{i\}) - v(T)$$

for all $i \in N$, and all $S \subset T \subset N \setminus \{i\}$.

The problem is now how to divide $v(N)$ when the grand coalition N is formed. A *payoff vector* is a vector $x \in \mathbb{R}^n$ with $\sum_{i\in N} x_i = v(N)$, where x_i represents the payoff to player i. *Solution concepts* assign payoff vectors or sets of payoff vectors to games. In this paper we are concerned with the following three solution concepts: the *core* introduced by Gillies (1953), the *Shapley value* introduced by Shapley (1953) and the *τ-value* introduced by Tijs (1981). Let $\langle N, v\rangle$ be a cooperative game in characteristic function form. The core of $\langle N, v\rangle$ is denoted by $C(v)$ and defined by

$$C(v) := \left\{ x \in \mathbb{R}^n \mid \sum_{i\in N} x_i = v(N) \text{ and } \sum_{i\in S} x_i \geqslant v(S) \text{ for all } S \in 2^N \right\}.$$

The core of a game can be empty. If $v(N)$ is divided according to an element of $C(v)$ no coalition has an incentive to split off the grand coalition because it cannot do better on its own.

For the Shapley value we need the following definition. Let $\pi \in \Pi_N$. Then $\Psi^\pi(v)$ is defined to be the vector with i-th coordinate equal to $v(P(\pi, i) \cup \{i\}) - v(P(\pi, i))$. The Shapley value $\Phi(v)$ is given by

$$\Phi_i(v) := \frac{1}{n!} \sum_{\pi\in\Pi_N} \psi_i^\pi(v) \quad \text{for all } i \in N.$$

The idea behind this expression is the assumption that the formation of the coalition N takes place according to a permutation π, with the players joining one after the other in the order $\pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(n)$. Player i receives his marginal contribution which is equal to the amount $\psi_i^\pi(v)$. The Shapley value assigns to player i his expected payoff in the case where all possible orders of formation are equally likely to occur.

For the τ-value we need the following definitions. Let $\langle N, v\rangle$ be a cooperative game. Then we define

$$M_i(v) := v(N) - v(N \setminus \{i\}) \quad \text{for all } i \in N.$$

$M_i(v)$ is the maximal payoff player i can expect to obtain; if he asks for more, the others will do better by working without him. Let S be a coalition with $i \in S$, we calculate what is left for player i if the other members of S get their maximal payoff. In other words,

$$R^v(S, i) := v(S) - \sum_{j\in S\setminus\{i\}} M_j(v).$$

The minimal payoff that player i will consent to get is

$$\mu_i(v) := \max_{S \ni i} R^v(S, i),$$

because he can ensure himself this payoff by offering the members of a coalition S, for which the maximum is achieved, their maximal payoff and remaining with $\mu_i(v)$. A cooperative game $\langle N, v \rangle$ is said to be *quasi-balanced* if

$$\sum_{i \in N} \mu_i(v) \leqslant v(N) \leqslant \sum_{i \in N} M_i(v).$$

For a quasi-balanced game the τ-value is defined by

$$\tau_i(v) := \lambda M_i(v) + (1-\lambda)\mu_i(v) \quad \text{for all } i \in N.$$

Here λ is uniquely determined by the fact that

$$\sum_{i \in N} \tau_i(v) = v(N).$$

In order to construct a cooperative game from a sequencing situation we take the set of players to be equal to the set of customers. As far as the characteristic function is concerned, we want to define v in such a way that the worth of a coalition is equal to the maximal cost savings the members of the coalition can ensure themselves by rearranging their positions before the counter. In such a rearrangement they may not jump ahead of customers not in the coalition, i.e. two members of the coalition who have a non-member between them may not change position. A coalition $S \subset N$ in a sequencing situation $(\sigma; \alpha; s)$ is called *connected* if for all $i, j \in S$ and $k \in N$, $\sigma(i) < \sigma(k) < \sigma(j)$ implies $k \in S$. From the proposition in Section 1 it follows that for a connected coalition an optimal arrangement is to have its members in decreasing urgency index order. This can be done by switching neighbours who are not standing in the right order. We have seen that every time such a switch takes place between neighbours i and j with $\sigma(i) < \sigma(j)$ and $u_i < u_j$ the cost decreases by g_{ij}. So, for a connected coalition T we have

$$v(T) := \sum_{i \in T} \sum_{k \in P(\sigma, i) \cap T} g_{ki}.$$

(In the summation we also have the terms g_{ki} with $\sigma(k) < \sigma(i)$ and $u_k > u_i$, but since then $g_{ki} = 0$ this does not disturb anything.) Let $S \subset N$ be a coalition which is not connected. We say that a coalition T is a *component* of S if $T \subset S$, T is connected and for every $i \in S \setminus T$, $T \cup \{i\}$ is not connected. The components of S form a partition of S which we denote by S/σ. Maximal cost savings are achieved by S when the members in all its components are rearranged in decreasing urgency index order. The total cost savings of S is the sum of the cost savings of all its components. Hence, for a non-connected coalition S,

$$v(S) := \sum_{T \in S/\sigma} v(T).$$

A cooperative game derived in the way described above from a sequencing situation will be called a *sequencing game*. The following theorem states that sequencing games are convex.

Theorem. *Sequencing games are convex games.*

Proof. Let $\langle N, v \rangle$ be the sequencing game corresponding to the sequencing situation $(\sigma; \alpha; s)$. Let $S_1 \subset S_2 \subset N \setminus \{i\}$. Then there are $T_1, U_1 \in S_1 \setminus \sigma \cup \{\emptyset\}$ and $T_2, U_2 \in S_2 \setminus \sigma \cup \{\emptyset\}$ with $T_1 \subset T_2$ and $U_1 \subset U_2$ such that

$$v(S_p \cup \{i\}) - v(S_p) = \sum_{k \in T_p} g_{ki} + \sum_{j \in U_p} g_{ij} + \sum_{k \in T_p, j \in U_p} g_{kj}$$

for $p \in \{1, 2\}$.

It follows that

$$v(S_1 \cup \{i\}) - v(S_1) \leqslant v(S_2 \cup \{i\}) - v(S_2)$$

and hence $\langle N, v \rangle$ is convex. □

It is easy to see that not all convex games are sequencing games because sequencing games are zero-normalized, i.e. all one-person coalitions have worth zero, while convex games need not be zero-normalized. Actually, not even all zero-normalized convex games are sequencing games as the following example shows.

Example. Let $N = \{1, 2, 3\}$, $v(\{i\}) = 0$ for all $i \in N$, $v(\{1, 2\}) = v(\{2, 3\}) = 1$, $v(\{1, 3\}) = 0$ and $v(\{1, 2, 3\}) = 2$. Then $\langle N, v \rangle$ is a zero-normalized convex game. Suppose that $\langle N, v \rangle$ is the sequencing game corresponding to a sequencing situation $(\sigma; \alpha; s)$. Because $v(\{1, 2\}) > 0$ and $v(\{2, 3\}) > 0$ it follows that either $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 3$

or $\sigma(1)=3$, $\sigma(2)=2$, $\sigma(3)=1$. In the first case $g_{12}>0$ and $g_{23}>0$ implying $u_3>u_2>u_1$. But then also $g_{13}>0$ and

$$\begin{aligned}\nu(N)&=g_{12}+g_{23}+g_{13}>g_{12}+g_{23}\\&=\nu(\{1,2\})+\nu(\{2,3\})=2\end{aligned}$$

leading to a contradiction. In the same way a contradiction can be derived in the second case. Therefore $\langle N, \nu\rangle$ is not a sequencing game.

In a cooperative game $\langle N, \nu\rangle$ a player i is called a *dummy player* if $\nu(S\cup\{i\})-\nu(S)=\nu(\{i\})$ for all $S\subset N\backslash\{i\}$. Note that the dummies in a sequencing situation are exactly the dummy players in the corresponding sequencing game.

Shapley (1971) proved that convex games have non-empty cores and that the core of a convex game $\langle N, \nu\rangle$ is the *convex hull* of the vectors $\psi^{\pi}(\nu)$. This means that every $x\in C(\nu)$ can be written as

$$x=\sum_{x\in\Pi_N}\lambda_{\pi}\psi^{\pi}(\nu)$$

with $\lambda_{\pi}\geqslant 0$ for all $\pi\in\Pi_N$ and $\Sigma_{\pi\in\Pi_N}\lambda_{\pi}=1$. Since sequencing games are convex games we know that the core of a sequencing game $\langle N, \nu\rangle$ is very large: it is the convex hull of the vectors $\psi^{\pi}(\nu)$ and the Shapley value of the game is the barycenter of the core. The following theorem states that the payoff vector corresponding to the division of the cost savings according to the EGS rule is an element of the core too.

Theorem. *Let* $(\sigma;\ \alpha;\ s)$ *be a sequencing situation and* $\langle N, \nu\rangle$ *the corresponding sequencing game. Then* $\mathrm{EGS}(\sigma;\ \alpha;\ s)\in C(\nu)$.

Proof. We already know that

$$\sum_{i\in N}\mathrm{EGS}_i(\sigma;\ \alpha;\ s)=C_{\sigma}-C_{\mathrm{opt}}=\nu(N).$$

Let $S\in 2^N$, then

$$\begin{aligned}&\sum_{i\in S}\mathrm{EGS}_i(\sigma;\ \alpha;\ s)\\&=\sum_{i\in S}\tfrac{1}{2}\Bigg(\sum_{k\in P(\sigma,\,i)}g_{ki}+\sum_{j:i\in P(\sigma,j)}g_{ij}\Bigg)\\&\geqslant\sum_{i\in S}\tfrac{1}{2}\Bigg(\sum_{k\in P(\sigma,i)\cap S}g_{ki}+\sum_{\substack{j\in S\\ i\in P(\sigma,j)}}g_{ij}\Bigg)\\&=\sum_{i\in S}\sum_{k\in P(\sigma,i)\cap S}g_{ki}\geqslant\nu(S).\end{aligned}$$

Hence, $\mathrm{EGS}(\sigma,\ \alpha;\ s)\in C(\nu)$. □

Let $(\sigma;\ \alpha;\ s)$ be a sequencing situation. We define the Shapley value $\Phi(\sigma;\ \alpha;\ s)$ of this sequencing situation to be the Shapley value of the corresponding sequencing game. In the following theorem we will give an expression for the Shapley value of a sequencing situation. For the proof we need the following characterizing properties of the Shapley value for cooperative games, cf. Shapley (1953).

$$\sum_{i\in N}\Phi_i(\nu)=\nu(N)\quad(\textit{Efficiency}).$$

$$\Phi_{\pi i}(\pi\nu)=\Phi_i(\nu)$$

for every $i\in N$ and $\pi\in\Pi_N$ (*Symmetry*). Here $\pi\nu$ is the game defined by $\pi\nu(\pi(S))=\nu(S)$ for every $S\in 2^N$.

$$\Phi(\nu+w)=\Phi(\nu)+\Phi(w)$$

for any two cooperative games ν and w (*Additivity*).

$$\Phi_i(\nu)=\nu(\{i\})$$

if i is a dummy player (*Dummy player property*).

Theorem. *The Shapley value of the sequencing situation* $(\sigma;\ \alpha;\ s)$ *is given by*

$$\begin{aligned}&\Phi_i(\sigma;\ \alpha;\ s)\\&=\sum_{\sigma(k)\leqslant\sigma(i)\leqslant\sigma(j)}g_{kj}(\sigma(j)-\sigma(k)+1)^{-1}\end{aligned}$$

for every $i\in N$.

Proof. Let $\langle N, \nu\rangle$ be the sequencing game corresponding to the sequencing situation $(\sigma;\ \alpha;\ s)$. For all $i,\ j\in N$ with $\sigma(j)>\sigma(i)$ let the game ν_{ij} be defined by

$$\nu_{ij}(S):=\begin{cases}g_{ij}&\text{if }\{l\mid\sigma(i)\leqslant\sigma(l)\leqslant\sigma(j)\}\subset S,\\0&\text{otherwise.}\end{cases}$$

Note that for a connected coalition T we have $\nu_{ij}(T)=g_{ij}$ iff $\{i,\ j\}\subset T$. For a non-connected coalition S we have

$$\nu_{ij}(S)=\sum_{T\in S/\sigma}\nu_{ij}(T).$$

It follows that for a connected coalition T

$$\sum_{\sigma(k)<\sigma(j)}\nu_{kj}(T)=\sum_{j\in T}\sum_{k\in P(\sigma,j)\cap T}g_{kj}=\nu(T).$$

For a non-connected coalition S we have

$$\sum_{\sigma(k)<\sigma(j)} v_{kj}(S) = \sum_{\sigma(k)<\sigma(j)} \sum_{T\in S/\sigma} v_{kj}(T) = \sum_{T\in S/\sigma} v(T) = v(S).$$

Thus $v = \Sigma_{\sigma(k)<\sigma(j)} v_{kj}$. Because of the efficiency, symmetry and dummy player property it follows that

$$\Phi_i(v_{kj}) = \begin{cases} g_{kj}(\sigma(j)-\sigma(k)+1)^{-1} \\ \quad \text{if } i \in \{l \mid \sigma(k) \leqslant \sigma(l) \leqslant \sigma(j)\}, \\ 0 \quad \text{otherwise.} \end{cases}$$

From the additivity it follows that

$$\Phi_i(\sigma;\ \alpha;\ s) = \Phi_i(v) = \sum_{\sigma(k)<\sigma(j)} \Phi_i(v_{kj}) = \sum_{\sigma(k)\leqslant\sigma(i)\leqslant\sigma(j)} g_{kj}(\sigma(j)-\sigma(k)+1)^{-1}$$

and the proof is completed. □

The Shapley value divides the gain that two players can make equally among themselves and the players standing between them.

We define the τ-value of a sequencing situation to be the τ-value of the corresponding sequencing game. Because games with a non-empty core are quasi-balanced, cf. Tijs (1981), this definition is correct.

Theorem. *Let* $(\sigma;\ \alpha;\ s)$ *be a sequencing situations. Then the* τ*-value is given by*

$$\tau_i(\sigma;\ \alpha;\ s) = \sum_{\sigma(k)\leqslant\sigma(i)\leqslant\sigma(j)} g_{kj}\lambda \quad \textit{for all } i \in N,$$

where

$$\lambda = \sum_{j\in N} \sum_{k\in P(\sigma,j)} g_{kj} \Big(\sum_{j\in N} \sum_{\sigma(k)\leqslant\sigma(j)\leqslant\sigma(l)} g_{kl} \Big)^{-1}.$$

Proof. Let $\langle N, v\rangle$ be the sequencing game corresponding to this sequencing situation. Driessen and Tijs (1985) have proved that for a convex game $\langle N, w\rangle$, $\mu_i(w) = w(\{i\})$ for all $i \in N$. Therefore,

$$\mu_i(v) = v(\{i\}) = 0 \quad \text{for all } i \in N$$

and

$$\tau_i(v) = \lambda M_i(v).$$

Further,

$$M_i(v) = v(N) - v(N\setminus\{i\}) = \sum_{\sigma(k)\leqslant\sigma(i)\leqslant\sigma(j)} g_{kj}.$$

It follows that

$$\lambda = v(N)\Big(\sum_{j\in N} M_j(v)\Big)^{-1} = \sum_{j\in N} \sum_{k\in P(\sigma,j)} g_{kj} \Big(\sum_{j\in N} \sum_{\sigma(k)\leqslant\sigma(j)\leqslant\sigma(l)} g_{kl} \Big)^{-1}$$

and the proof is completed. □

The τ-value divides the cost savings proportional to the maximal payoffs of the players.

Example. Let $(\sigma;\ \alpha;\ s)$ be the sequencing situation of the example in Section 3. Then the corresponding sequencing game $\langle N, v\rangle$ is defined by $v(\{i\}) = 0$ for all $i \in N$, $v(\{1, 2\}) = 110$, $v(\{1, 3\}) = 0$, $v(\{2, 3\}) = 0$ and $v(\{1, 2, 3\}) = 270$. The Shapley value is given by $\Phi(\sigma;\ \alpha;\ s) = (108\frac{1}{3}, 108\frac{1}{3}, 53\frac{1}{3})$, whereas the τ-value is $\tau(\sigma;\ \alpha;\ s) = \frac{27}{7}(27, 27, 16) = (104\frac{1}{7}, 104\frac{1}{7}, 61\frac{5}{7})$. The core of the game is the convex hull of the vectors (270, 0, 0), (110, 0, 160), (0, 270, 0) and (0, 110, 160). Note that

$$\text{EGS}(\sigma;\ \alpha;\ s) = \tfrac{1}{2}(270, 0, 0) + \tfrac{1}{2}(0, 110, 160).$$

Remark that EGS$(\sigma;\ \alpha;\ s)$ differs from the Shapley value, the τ-value and the nucleolus which equals (95, 95, 80) for this game. All the three game theoretic solution concepts assign the same amount to player 1 and 2 since these two play a symmetric role in the sequencing game. The EGS rule assigns different amounts to them because in the process of going from the original permutation to the optimal permutation 1 switches with 2 and 3 contributing 110 + 160 to the total gain while 2 switches only with 1.

5. Some related problems

In this last section we will consider some problems related to sequencing situations. Tijs et al. (1984) have introduced *permutation games*. These games can also be seen as corresponding to situations where customers are standing in front of a counter waiting to be served. But now everyone has the same service time and the cost functions need not be linear. Originally permutation games were defined in terms of costs, but here we will give a definition based on savings. Let

$$K = [k_{ij}]_{i=1,\ j=1}^{n\quad n}$$

be the cost matrix of a permutation situation, i.e. k_{ij} denotes the costs if player i takes the j-th position. Then the cost savings permutation game $\langle N, v\rangle$ is defined by

$$v(S) := \max_{\pi_s \in \Pi_s} \sum_{i \in S} \left(k_{ii} - k_{i\pi_s(i)}\right),$$

where for every $S \in 2^N$, Π_S denotes the set of permutations of S. Here, in rearranging its members a coalition S is allowed to jump over non-members, contrary to sequencing games where this is not permitted. Tijs et al. (1984) have proved that permutation games defined in terms of costs have a nonempty core. It follows that the permutations games defined above also have a non-empty core. These games need not be convex as the following example shows.

Example. Let $N = \{1, 2, 3\}$, $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 3$ and the cost matrix

$$K = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{pmatrix}.$$

Then $v(\{i\}) = 0$ for all $i \in N$, $v(\{1, 2\}) = 7 - 5 = 2$, $v(\{1, 3\}) = 13 - 7 = 6$, $v(\{2, 3\}) = 18 - 17 = 1$ and $v(\{1, 2, 3\}) = 19 - 13 = 6$. This game is not convex because

$$v(\{1, 3\}) - v(\{3\}) = 6 > 5 = v(\{1, 2, 3\}) - v(\{2, 3\}).$$

The core of this game is

$$C(v) = \{(x, 0, 6 - x) \mid 2 \leqslant x \leqslant 5\}.$$

Since permutation games have non-empty cores, it follows that sequencing games with general cost functions and equal service times have non-empty cores too because they are relaxations of permutation games. This means that the worth of the grand coalition stays the same while the worths of the other coalitions decrease or stay the same.

We consider now another modification of sequencing situations where the cost functions are linear and the service times differ. Here again a coalition S is allowed to jump over non-members in the process of rearranging the order of its members. The worth $v(S)$ of a coalition S is the amount that S can achieve without cooperation of $N \setminus S$. The following example shows that these games can have an empty core.

Example. Let $N = \{1, 2, 3\}$, $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 3$, $\alpha = (1, 4, 3)$, $s = (1, 2, 3)$. Then $v(\{i\}) = 0$ for all $i \in N$, because i can not gain anything by himself, $v(\{1, 2\}) = 13 - 11 = 2$, $v(\{1, 3\}) = 19 - 15 = 4$, $v(\{2, 3\}) = 0$, $v(\{1, 2, 3\}) = 31 - 29 = 2$. It is easy to see that the core is empty because it is impossible for a vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ to satisfy simultaneously the conditions

$$x_1 \geqslant 0, \qquad x_2 \geqslant 0, \qquad x_3 \geqslant 0, \qquad x_1 + x_3 \geqslant 4$$

and

$$x_1 + x_2 + x_3 = 2.$$

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